Problem 1. Approximation of Functions

Consider the unique interpolating polynomial $p_n(x)$ of degree n or less that interpolates a function $f(x)$ at $n+1$ equally spaced interpolation points

$$\{x_0, x_1, x_2, \cdots, x_n\}$$

on an interval $[a, b]$, taking $x_0 = a$ and $x_n = b$. Write a program to do the following: Use the Lagrange basis functions $\ell_i(x), i = 0, 1, 2, \cdots n$ to evaluate $p_n(x)$ at $M+1$ equally spaced sampling points

$$\{y_0, y_1, y_2, \cdots, y_M\}$$

where $y_0 = a$ and $y_M = b$, and where M is much larger than n. Estimate the maximum interpolation error

$$\max_{[a,b]} |f(x) - p_n(x)|$$

by computing the approximate maximum interpolation error

$$\max_{0 \le i \le M} |f(y_i) - p_n(y_i)|.$$

Specifically, do the above for each of the following cases:

$f_1(x) = \sin(\pi x)$, on the interval $[-1, 1]$, (i.e., $a = -1$ and $b = 1$),

$f_2(x) = \frac{1}{1+x^2}$, on the interval $[-2, 2]$,

$f_3(x) = \frac{1}{1+x^2}$, on the interval $[-5, 5]$,

successively using $n = 2, 4, 8, 16.$, and in each case use $M = 500$.
For each of these 12 cases print the approximate maximum interpolation error.

Also, for each of the three functions, give a graph that shows $f(x)$ and the polynomials $p_n(x), n = 2, 4, 8, 16$.

In addition, for the case $f(x) = \sin(\pi x)$ on the interval $[-1, 1]$, use the Lagrange Interpolation Theorem to derive an upper bound on the maximum interpolation error for $n = 2, 4, 8, 16$.

Compare this upper bound to the actual (approximate) maximum interpolation error found above.

Note:
Lagrange interpolation polynomial for a function f is given by:

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, \ell_i(x)$$

Where ℓ_i is the i^{th} Lagrange basis function, given by:

$$\ell_i(x) \equiv \prod_{k=0, k\neq i}^{n} \frac{(x - x_k)}{(x_i - x_k)}, \quad i = 0, 1, \cdots, n$$

Summary and Discussions

- The function $f_1(x) = sin(\pi x)$, on interval $[-1, 1]$, is close to be approached by a polynomial of second degree ($n = 2$). And starting from a polynomial of 4^{th} degree ($n = 4$), we can see that the function and its approximation are somehow identical.

  While increasing the degree of Lagrange polynomial, we reach the identical curve of the function f.

- On the other hand, $f_2(x)$ and $f_3(x)$ are hard to approximate especially the function $f_3(x)$ which has been defined on a larger interval $[-5, 5]$.

  The issue here is the approximation of the extreme values close to the edges. Even with a higher deree of Lagrange polynomial ($n = 16$), we still have a large value of the error observed.