Problem 3. Numerical Integration

Derive the local *Three-point Gauss Quadrature Formula* for integrating a function $f(x)$ over the reference interval $[-1, 1]$. (This formula uses the roots of the orthogonal polynomial $e_3(x)$.)

Use the corresponding *composite formula* to integrate the function $f(x) = \sin(\pi x)$ over the interval $[0, 1]$, using $N = 2, 4, 8, 16$, equally spaced subintervals in $[0, 1]$.

List the observed errors (the difference between the numerical integral and the exact integral) in a Table.

How many function evaluations are needed for the error to be less than 10^{-7}?

Solution

We will use the "Legendre" polynomial to the 3rd degree, given by:

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

We can compute the nodes by computing the roots of P_3, which are:

$$x = 0,\ x = -\sqrt{\frac{3}{5}},\ x = \sqrt{\frac{3}{5}}$$

Then we have:

$$I = \int_{-1}^{1} f(x)dx = \omega_1 f(0) + \omega_2 f\left(-\sqrt{\frac{3}{5}}\right) + \omega_3 f\left(\sqrt{\frac{3}{5}}\right)$$

Computing weights ω_i:

We can get the weights by the following formula:

$$\omega_i = \frac{2}{(1 - x_i^2)(P_3'(x_3))^2}$$

This, we get:

$$\omega_1 = \frac{5}{9},\ \omega_2 = \frac{8}{9},\ \omega_3 = \frac{5}{9}$$

As a result, we have:

$$I = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$$

By the linear transformation, for $x = \dfrac{1+t}{2}$, we get:

$$I = \int_0^1 sin(\pi x)dx = \frac{1}{2}\int_{-1}^{1} sin\left(\left(\frac{1+t}{2}\right)\pi\right) dt$$

We determine I by Gauss Quadrature Formula at 3 points:

$$\begin{aligned}
I &= \frac{1}{2}\int_{-1}^{1} sin\left(\left(\frac{1+t}{2}\right)\pi\right) dt \\
&= \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \frac{1}{2} sin\left(\left(\frac{1+t}{2}\right)\pi\right) dt \\
&= \sum_{i=0}^{N-1} (\omega_1 g(x_{i,1}) + \omega_2 g(x_{i,2}) + \omega_3 g(x_{i,3}))
\end{aligned}$$

Where $x_{i,j} = \frac{i}{N} + \frac{1+x_j}{2N}$**, for** $j \in \{1,2,3\}$**, and** $g(x) = \frac{1}{2} sin\left(\left(\frac{1+x}{2}\right)\pi\right)$**.**