EE2703 : Applied Programming Lab
Week8 : The Digital Fourier Transform

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EE20B004

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AIM

- To Find fast Fourier transform for given functions.
- Plot and understand magnitude and phase of the corresponding responses.
- To use fft function of numpy to find Fourier transform of non periodic function $e^{-t^2/2}$

Theory

Discrete Fourier Transform

The discrete Fourier transform (DFT) converts a finite sequence of equally-spaced samples of a function into a same-length sequence of equally-spaced samples of the discrete-time Fourier transform (DTFT), which is a complex-valued function of frequency.

Formula

A finite energy function of the type $f(t)$ has a Fourier Transform (and its inverse)

$$F(j\omega) = \int f(t)e^{jwt}dx \tag{1}$$

If f(t) is periodic with period 2, the Fourier Transform collapses to the Fourier Series

$$f(t) = \frac{1}{2\pi}\int F(j\omega)e^{j\omega t}d\omega \tag{2}$$

$$f(t) = \sum c_n e^{jnt} \tag{3}$$

$$c_n = \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} f(t)e^{-jnt} \tag{4}$$

We can invert this picture and say, suppose f [n] are the samples of some function f(t), then we define the Z transform as

$$F(z) = \sum f[n]z^{-n}$$

Replacing z with $e^{j\theta}$we get

$$F(e^{j\theta}) = \sum f[n]e^{-jn\theta}$$

So clearly F(z) is like the periodic time function that gives rise to the fourier series whose coefficients are the samples f [n]
$F(e^{j\theta})$ **is called the Digital Spectrum of the samples f [n]. It is also called the DTFT of f [n]** Suppose now $f[n]$ is itself periodic with a period N, i.e.

$$f[n+N] = f[n]\,\forall n$$

Then, it should have samples for its DTFT. This is true, and leads to the Discrete Fourier Transform or the DFT: Suppose f [n] is a periodic sequence of samples, with a period N. Then the DTFT of the sequence is also a periodic sequence F[k] with the same period N.

$$F[k] = \sum_{n=0}^{N-1} f[n]e^{-2\pi\frac{nk}{N}j} = \sum_{n=0}^{N-1} f[n]W^{nk}$$

$$f[n] = \frac{1}{N}\sum_{n=0}^{N-1} F[K]W^{-nk}$$

Assignment Questions

Question1: Spectrum of sin5t

The Fourier transform of $sin(5t)$

$$Y(\omega) = \frac{1}{2j}[\delta(\omega-1) - \delta(\omega+1)]$$

The corresponding magnitude and phase plot is:

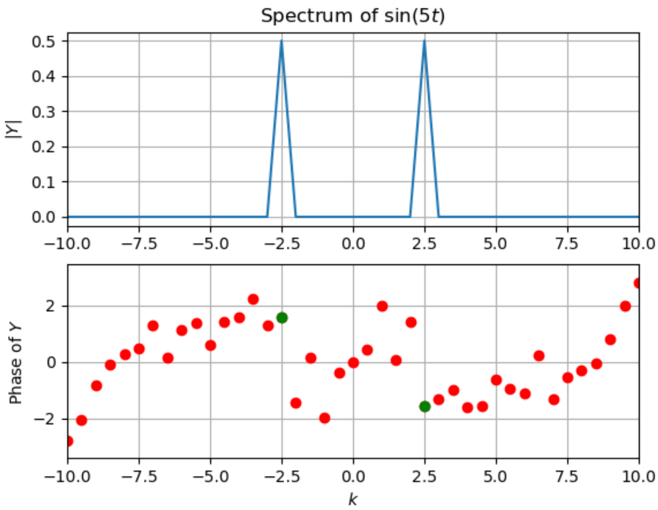

Figure 1: DFT of sin5t

Question 1 : Spectrum response of (1+0.1cos(t))cos(10t)

1. The fourier transfrom of $(1 + 0.1cos(t))cos(10t)$ is :

$$0.025e^{-11jt} + 0.5e^{-10jt} + 0.025e^{-9jt} + 0.025e^{9jt} + 0.5e^{10jt} + 0.025e^{11jt}$$

2. The magnitude and phase response of $(1 + 0.1cos(t))cos(10t)$ is :

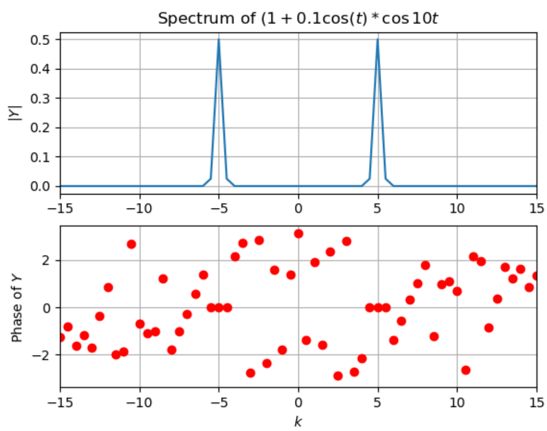

Figure 2: System step response

3. As can be seen from above the graph the magnitude response is incorrect as we expect well defined spikes at ω = [-11,-10,-9,9,10,11].This can be solved by stretching the t axis to $[-4\pi, 4\pi]$. We correspodingly obtain the following :

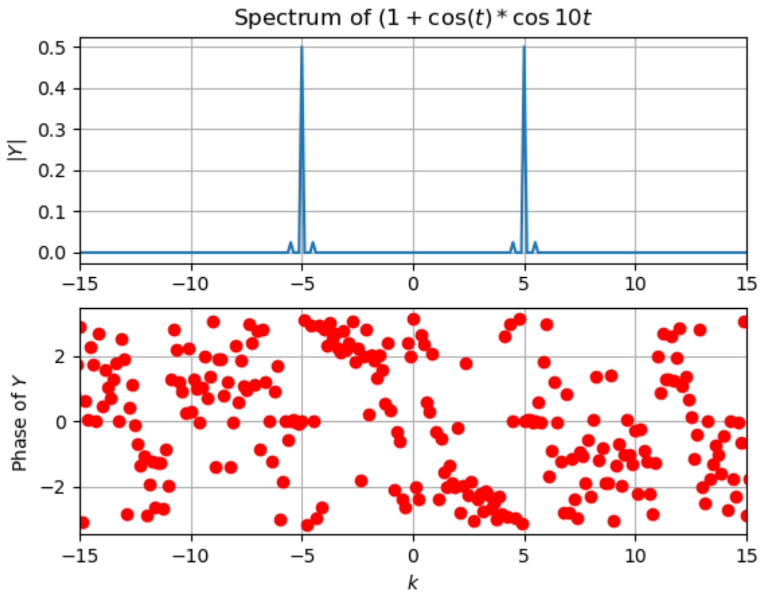

Figure 3: System step response

Question 2: Spectrum of sin^3t and cos^3t

1. The Fourier transform of sin^3t is

$$Y(\omega) = \frac{1}{j}(0.125e^{-3jt} - 0.375e^{-jt} + 0.375e^{jt} - 0.125e^{3jt})$$

2. The magnityde and phase plot of sin^3t is

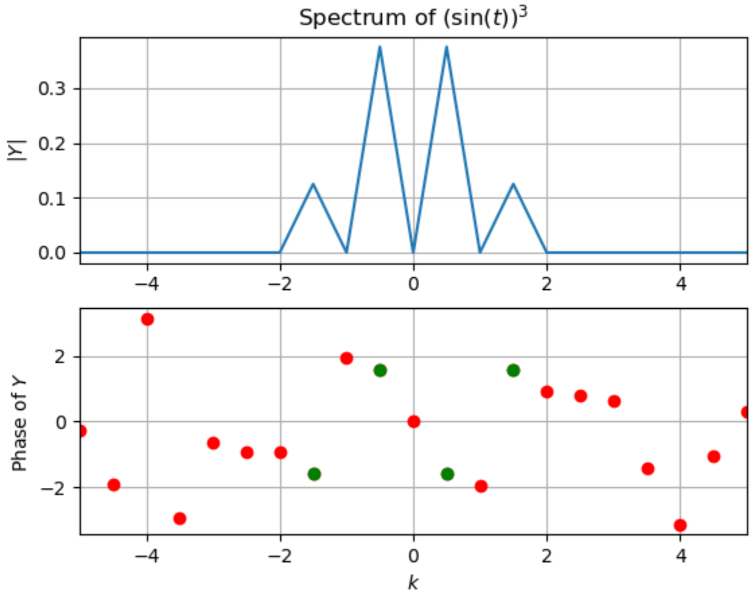

Figure 4: DFT of $sin^3 t$

3. From The above graph its clear that we get spikes at [-3,-1,1,3] (as expected) with phase angles at them being[$\frac{-\pi}{2}$, $\frac{\pi}{2}$,$\frac{-\pi}{2}$, $\frac{\pi}{2}$] respectively as can be seen from the green points in phase plots

4. Similarly the Fourier transform of $\cos^3 t$ is:

$$Y(\omega) = 0.125e^{-3jt} + 0.375e^{-jt} + 0.375e^{jt} + 0.125e^{3jt}$$

5. The magnitude and phase plot of $\cos^3 t$ is

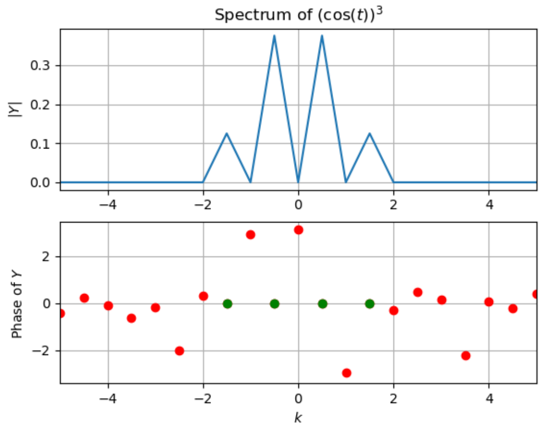

Figure 5: DFT of sin^3t

6. From The above graph its clear that we get spikes at [-3,-1,1,3] (as expected) with phase angles at them being 0 at those points as can be seen from the green points in phase plots

Question 3: Spectrum of f $cos(20t + 5cos(t))$

1. The function $cos(20t + 5cos(t))$ is frequency modulated wave and the different spikes in the magnitude plot can be justified with the help of the section under Bessel function in the wiki article : $https://en.wikipedia.org/wiki/Frequ$

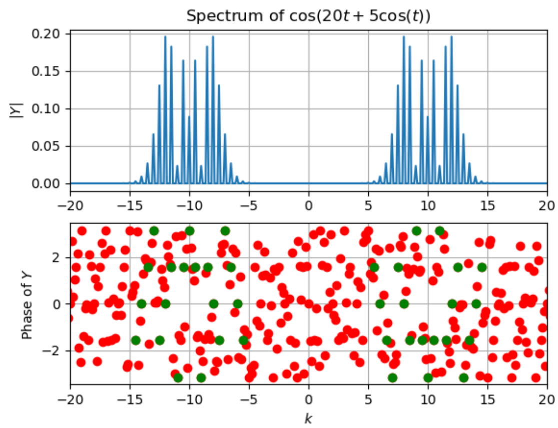

Figure 6: Magnitude and phase plots of $cos(20t + 5cos(t))$

Question 4: Fourier transform of Gaussian function $e^{-t^2/2}$

1. Since the above function is a periodic , unlike the rest we have tries fft on , we must readjust the time axis and scaling factor in order to achieve closeness to Fourier Transform.

2. Following is the data frame generated for varying *timeranges* and samples is given below.

	Interval	Samples Taken	Frequency	Max Error
0	[-6.283185307179586, 6.283185307179586]	128	20.530988	3.705838e-01
1	[-6.283185307179586, 6.283185307179586]	256	40.902820	1.369916e+00
2	[-6.283185307179586, 6.283185307179586]	512	81.646486	2.029686e+00
3	[-6.283185307179586, 6.283185307179586]	1024	163.133817	2.335002e+00
4	[-12.566370614359172, 12.566370614359172]	128	10.265494	1.345270e+00
5	[-12.566370614359172, 12.566370614359172]	256	20.451410	1.884431e-01
6	[-12.566370614359172, 12.566370614359172]	512	40.823243	1.276122e+00
7	[-12.566370614359172, 12.566370614359172]	1024	81.566908	1.991453e+00
8	[-25.132741228718345, 24.740042147019622]	128	5.092958	1.951763e+00
9	[-25.132741228718345, 24.936391687868984]	256	10.185916	1.181112e+00
10	[-25.132741228718345, 25.03456645829366]	512	20.371833	1.010437e-15
11	[-25.132741228718345, 25.083653843506006]	1024	40.743665	1.183046e+00
12	[-37.69911184307752, 37.69911184307752]	128	5.132747	2.240726e+00
13	[-37.69911184307752, 37.69911184307752]	256	10.225705	1.743840e+00
14	[-37.69911184307752, 37.69911184307752]	512	20.411621	7.901990e-01
15	[-37.69911184307752, 37.69911184307752]	1024	40.783454	5.688963e-01

3. Out of the above table the interval $[-8\pi, 8pi]$ with 512 samples has least maximum error.The corresponding spectrum plot of Fourier transform is-

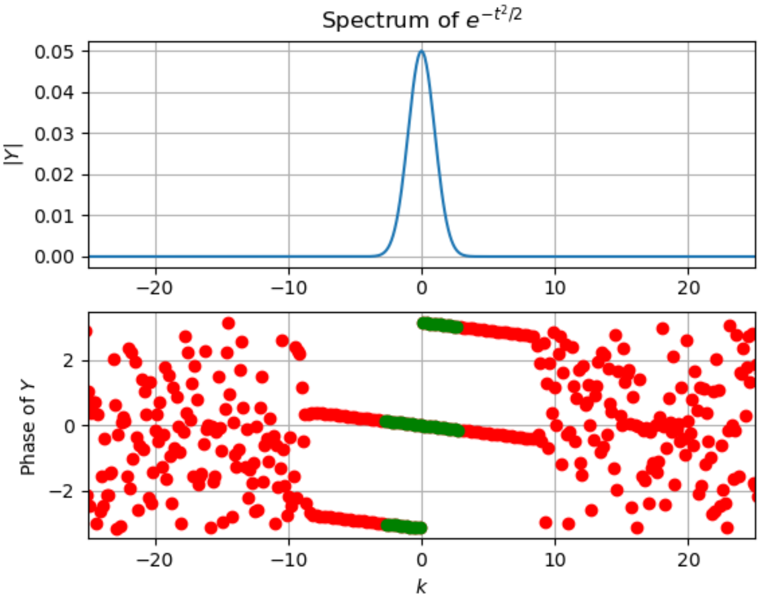

Figure 7: Magnitude and phase plots of $e^{-t^2/2}$

4. The corresponding maximum error (as can be seen from table) is : $\mathbf{1.01}\mathbf{x}10^{-15}$

Conclusions

We have successfully calculated DFT's of various given function with the help of fft library in numpy module. In case of pure sinusoids we had magnitude response consisting of spikes at countable number of points.The DFT of Frequency modulated signal consisted of infinitely many number of side band frequencies. The Gaussian signal had a Gaussian DFT, whose spectrum sharpens as we increase sampling rates.