Classroom notes of Laplace Transforms

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Contents

1 Introduction

Definition 1 *A function $F(s)$ given as*

$$F(s) = \int_0^\infty k(s,t) f(t)\, dt$$

is said to be ***integral transform*** *of $f(t)$ with* ***kernel*** *$k(s,t)$.*

Definition 2 *Laplace transform of $f(t)$, is an integral transform with kernel $k(s,t) = e^{-st}$, denoted as $L(f)$. Symbolically*

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t)\, dt$$

The given function $f(t)$ in a Laplace Transform expression is called inverse transform of $F(s)$ and is denoted by $L^{-1}(F)$, more explicitely $f(t) = L^{-1}(F)$.

Remark 3 *Laplace transforms are of immense practical utility for solution of linear ODEs. They transform a differential equation into an algebraic equation, thus making the solution very easy.*

Remark 4 *PIERRE SIMON MARQUIS DE LAPLACE (1749-1827), great French mathematician, was a professor in Paris. He developed the foundation of potential theory and made important contributions to celestial mechanics, astronomy in general, special functions and probablity theory. Napoleon Bonaparte was his student for a year.*

Example 5 *Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant. Find $L(f)$?*

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Solution 6

$$\begin{aligned}
L\left(e^{at}\right) &= \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{(a-s)t}dt \\
&= \left[\frac{1}{a-s}e^{-(s-a)t}\right]_0^\infty \quad \textit{by using} \int e^{ax}dx = \frac{1}{a}e^{ax} \\
&\therefore \; L\left(e^{at}\right) = \frac{1}{s-a} \quad \blacksquare
\end{aligned}$$

Example 7 *(9ed-6.1-2) For* $f(t) = \left(t^2-3\right)^2$ *we have*

$$\begin{aligned}
L\left(t^2-3\right)^2 &= \int_0^\infty e^{-st}\left(t^2-3\right)^2 dt = \int_0^\infty \left(\frac{t^4}{e^{st}} - 6\frac{t^2}{e^{st}} + \frac{9}{e^{st}}\right) dt \\
&= \int_0^\infty \frac{t^4}{e^{st}}dt - 6\int_0^\infty \frac{t^2}{e^{st}}dt + \int_0^\infty \frac{9}{e^{st}}dt \\
&= \Bigg[\left(-\frac{1}{s^5}e^{-st}\left(s^4t^4 + 4s^3t^3 + 12s^2t^2 + 24st + 24\right)\right) - \\
&\quad 6\left(-\frac{1}{s^3}e^{-st}\left(s^2t^2+2st+2\right)\right) + \left(-\frac{9}{s}e^{-st}\right)\Bigg]_0^\infty \\
&= \left[-\frac{1}{s^5}e^{-st}\left(s^4t^4 - 6s^4t^2 + 9s^4 + 4s^3t^3 - 12s^3t + 12s^2t^2 - 12s^2 + 24st + 24\right)\right]_0^\infty \\
&= \left[-\frac{1}{s^5}e^{-st}\left(s^4t^4 - 6s^4t^2 + 9s^4 + 4s^3t^3 - 12s^3t + 12s^2t^2 - 12s^2 + 24st + 24\right)\right]_{t=\infty} \\
&\quad - \left[-\frac{1}{s^5}e^{-st}\left(s^4t^4 - 6s^4t^2 + 9s^4 + 4s^3t^3 - 12s^3t + 12s^2t^2 - 12s^2 + 24st + 24\right)\right]_{t=0} \\
&= 0 - \left(-\frac{1}{s^5}\left(9s^4 - 12s^2 + 24\right)\right) \\
&= \frac{9}{s} - \frac{12}{s^3} + \frac{24}{s^5} \quad \blacksquare
\end{aligned}$$

Table of Important Laplace Transforms

$f(t)$	$L(f)$	$f(t)$	$L(f)$
1	$\frac{1}{s}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
t^2	$\frac{2!}{s^3}$	$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$t^a, a \geq 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

Example 8 *(9ed-6.1-4) For* $f(t) = \sin^2(4t)$, *we have*

$$\begin{aligned}
L\left(\sin^2(4t)\right) &= \int_0^\infty e^{-st}\sin^2(4t)\,dt \\
L\left(\sin^2(4t)\right) &= \int_0^\infty e^{-st}\left(\frac{1}{2} - \frac{1}{2}\cos(8t)\right) dt \\
&= \left[-\frac{1}{128se^{st} + 2s^3e^{st}}\left(8s\sin 8t - s^2\cos 8t + s^2 + 64\right)\right]_0^\infty \\
&= 0 - \left(-\frac{64}{2s^3+128s}\right) \\
&= \frac{32}{s\left(s^2+64\right)} \quad \blacksquare
\end{aligned}$$

Example 9 *(9ed-6.1-7) For* e^{3a-2bt}*, we have*

$$\begin{aligned}
\int \left(e^{3a-2bt}e^{-st}\right)dt &= \int e^{3a-2bt-st}dt = -\frac{e^{3a-2bt-st}}{2b+s} \\
\therefore \quad L\left(e^{3a-2bt}\right) = \int_0^\infty \left(e^{3a-2bt}e^{-st}\right)dt &= \left[-\frac{e^{3a-2bt-st}}{2b+s}\right]_0^\infty \\
&= \left[-\frac{e^{3a-2bt-st}}{2b+s}\right]_{t=\infty} - \left[-\frac{e^{3a-2bt-st}}{2b+s}\right]_{t=0} \\
&= 0 - \left(-\frac{e^{3a}}{2b+s}\right) \\
&= \frac{e^{3a}}{2b+s} \quad \blacksquare
\end{aligned}$$

Example 10 *(9ed-6.1-11) We find* $L(\sin t \cos t)$

$$\begin{aligned}
L(\sin t\cos t) &= \int_0^\infty (\sin t\cos t)\,e^{-st}dt \\
&= \left[-\frac{1}{8e^{st}+2s^2e^{st}}(2\cos 2t + s\sin 2t)\right]_0^\infty \\
&= 0 - \left(-\frac{2}{2s^2+8}\right) \\
&= \frac{1}{s^2+8} \quad \blacksquare
\end{aligned}$$

2 Important Properties of Laplace Transform

- Linearity: $L\{af(t)+bg(t)\} = a\ L\{f(t)\} + b\ L\{g(t)\}$
- s-Shifting (aka first shifting theorem): $L\{e^{at}f(t)\} = F(s-a)$
  - equivalently: $L\{e^{at}f(t)\} = F(s)|_{(s-a)}$ i.e. $F(s)$ at $(s-a)$
  - equivalently: $e^{at}f(t) = L^{-1}\{F(s-a)\}$

Example 11 *Find Laplace of* $\cosh at$ *and* $\sinh at$ *.*

Solution 12 *Since* $\cosh at = \frac{1}{2}\left(e^{at}+e^{-at}\right)$ *and* $\sinh at = \frac{1}{2}\left(e^{at}-e^{-at}\right)$*. Using* $L\left(e^{at}\right) = \frac{1}{s-a}$ *and linearity property we get*

$$\begin{aligned}
L(\cosh at) &= \frac{1}{2}\left(L\left(e^{at}\right)+L\left(e^{-at}\right)\right) = \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right) = \frac{s}{s^2-a^2} \\
L(\sinh at) &= \frac{1}{2}\left(L\left(e^{at}\right)-L\left(e^{-at}\right)\right) = \frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right) = \frac{a}{s^2-a^2} \quad \blacksquare
\end{aligned}$$

Example 13 *(9ed-6.1-29)* $L^{-1}\left(\frac{4s-3\pi}{s^2+\pi^2}\right) =?$

$$\begin{aligned}
L^{-1}\left(\frac{4s-3\pi}{s^2+\pi^2}\right) &= L^{-1}\left(4\frac{s}{s^2+\pi^2} - 3\frac{\pi}{s^2+\pi^2}\right) \\
&= 4L^{-1}\left(\frac{s}{s^2+\pi^2}\right) - 3L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) \\
&= 4(\cos \pi t) - 3(\sin \pi t) \quad \blacksquare
\end{aligned}$$

Example 14 *(9ed-6.1-36)* $L^{-1}\left(\sum_{k=1}^{4}\frac{(k+1)^2}{s+k^2}\right) =?$

$$\begin{aligned}
L^{-1}\left(\sum_{k=1}^{4}\frac{(k+1)^2}{s+k^2}\right) &= L^{-1}\left(\frac{4}{s+1}+\frac{9}{s+4}+\frac{16}{s+9}+\frac{25}{s+16}\right) \\
&= L^{-1}\left(\frac{4}{s+1}\right) + L^{-1}\left(\frac{9}{s+4}\right) + L^{-1}\left(\frac{16}{s+9}\right) + L^{-1}\left(\frac{25}{s+16}\right) \\
&= 4e^{-t} + 9e^{-4t} + 16e^{-9t} + 25e^{-16t} \\
&= \frac{4}{e^t} + \frac{9}{e^{4t}} + \frac{16}{e^{9t}} + \frac{25}{e^{16t}} \quad \blacksquare
\end{aligned}$$

Example 15 *(9e-6.1-43) Using properties find* $L\left(5e^{-at}\sin(\omega t)\right)$?

Solution 16

$$\begin{aligned}
L\left(5e^{-at}\sin(\omega t)\right) &= 5L\left(e^{-at}\sin(\omega t)\right) \quad \textit{using linearity} \\
&= 5\left[L\left(\sin(\omega t)\right)\right]_{s-(-a)} \quad \textit{using s-shifting} \\
&= 5\left[\frac{\omega}{s^2+\omega^2}\right]_{s+a} \\
&= 5\left(\frac{\omega}{(s+a)^2+\omega^2}\right) \quad \blacksquare
\end{aligned}$$

Example 17 *(9e-6.1-46) Find* $L\left(e^{-t}\left(a_0+a_1t+...+a_nt^n\right)\right)=$?

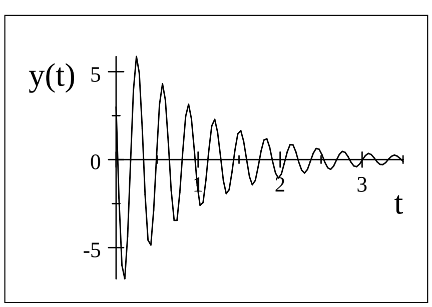
Solution 18 *We have* $e^{-t}\left(a_0+a_1t+...+a_nt^n\right)=e^{-t}\left(\sum_{k=0}^{n}a_kt^k\right)=\sum_{k=0}^{n}a_kt^ke^{-t}$. *Hence*

$$\begin{aligned}
L\left(e^{-t}\left(a_0+a_1t+...+a_nt^n\right)\right) &= L\left(\sum_{k=0}^{n}a_kt^ke^{-t}\right) \\
&= \sum_{k=0}^{n}a_kL\left(t^ke^{-t}\right) \quad \textit{using linearity} \\
&= \sum_{k=0}^{n}a_k\left[L\left(t^k\right)\right]_{s-(-1)} \\
&= \sum_{k=0}^{n}a_k\left[\frac{k!}{s^{k+1}}\right]_{s+1} \quad \textit{using } L\left(t^n\right)=\frac{n!}{s^{n+1}} \\
&= \sum_{k=0}^{n}a_k\left(\frac{k!}{(s+1)^{k+1}}\right) \quad \blacksquare
\end{aligned}$$

Example 19 *We find* $L^{-1}\left(\frac{3\,s-137}{s^2+2\,s+401}\right)$ *using s-shifting. Completing the square in denominator, we have*

$$\begin{aligned}
\frac{3\,s-137}{s^2+2\,s+401} &= \\
&= \frac{3\,s-137}{s^2+2\,s+401+\frac{2}{2}-\frac{2}{2}} \\
&= \frac{3\,s-137}{s^2+2\,s+1+400} \\
&= \frac{3\,s-137}{(s+1)^2+(20)^2} \\
&= 3\frac{s}{(s+1)^2+(20)^2}-137\frac{1}{(s+1)^2+(20)^2} \\
&= 3e^{-t}\cos(20t)-137e^{-t}\sin(20t) \\
&= e^{-t}\left(3\cos(20t)-7\sin(20t)\right)
\end{aligned}$$

The damped osciliator of above example is plotted as



Example 20 *Find* $L^{-1}\left(\frac{\sqrt{8}}{\left(s+\sqrt{2}\right)^3}\right)$?

Solution 21 *In the Inverse Laplace Table, we look for the nearest similar entry to* $\frac{\sqrt{8}}{\left(s+\sqrt{2}\right)^3}$. *This is* $\frac{2}{s^3}$. *Next we guess an application of s-shifting i.e.* $L\left\{e^{at}f(t)\right\} = F(s-a)$.
Suppose $a = -\sqrt{2}$ *in* e^{at}, *so tentatively we take* $F(s) = \frac{1}{s^3}$ *and* $a = -\sqrt{2}$. *Thus in first shifting theorem* ($e^{at}f(t) = L^{-1}\left\{F(s-a)\right\}$) *we set* $f(t) = L^{-1}\left(\frac{\sqrt{8}}{s^3}\right) = \sqrt{2}L^{-1}\left(\frac{2}{s^3}\right) = \sqrt{2}t^2$

$$\Rightarrow e^{at}f(t) = e^{-\sqrt{2}t}\left(\sqrt{2}t^2\right) = \sqrt{2}t^2e^{-\sqrt{2}t}$$

Example 22 *(10e-6.1-44) Find* $L^{-1}\left(\frac{a(s+k)+b\pi}{(s+k)^2+\pi^2}\right)$?

Solution 23 *Since* $\frac{a(s+k)+b\pi}{(s+k)^2+\pi^2} = a\frac{s+k}{(s+k)^2+\pi^2} + b\frac{\pi}{(s+k)^2+\pi^2} = a\frac{s+k}{(s+k)^2+\pi^2} + b\frac{\pi}{(s+k)^2+\pi^2}$. *Nearest similars in Table of Inverse Laplce are* $L(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ *and* $L(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$. *In s-shifting theorem we put* e^{-kt}, *then*

$$\begin{aligned} L^{-1}\left(\frac{a(s+k)+b\pi}{(s+k)^2+\pi^2}\right) &= ae^{-kt}L^{-1}\left(\frac{s}{s^2+\pi^2}\right) + be^{-kt}L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) \\ &= ae^{-kt}\cos(\pi t) + be^{-kt}\sin(\pi t) \\ &= e^{-kt}\left(a\cos(\pi t) + b\sin(\pi t)\right) \end{aligned}$$

3 Laplace Transform of the Derivative and Integral

Theorem 24 ***(Laplace Transform of the Derivative*** $f^{(n)}$ ***of Any Order)***

$$L\left(f^{(n)}\right) = s^nL(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

from which we have

$$\begin{aligned} L(f') &= sL(f) - f(0) \\ L(f'') &= s^2L(f) - sf(0) - f'(0) \end{aligned}$$

Example 25 *To find* $L^{-1}(t\sin\omega t)$, *let* $f(t) = t\sin\omega t$. *We note that second derivative of* $f(t)$ *involves the* $f(t)$ *as one of its terms due to osciliating derivative of* $\sin$ *and* $\cos$. *That is,* $f'(t) = \sin\omega t + \omega t\cos\omega$ *and* $f''(t) = 2\omega\cos\omega t - \omega^2t\sin\omega t$. *Also* $f(0) = 0,\ f'(0) = 0$. *We have*

$$\begin{aligned} L(f'') &= 2\omega L(\cos\omega t) - \omega^2L(t\sin\omega t) \\ \textit{also } L(f'') &= s^2L(f) - sf(0) - f'(0) \\ \textit{implies } s^2L(f) - sf(0) - f'(0) &= 2\omega L(\cos\omega t) - \omega^2L(t\sin\omega t) \\ s^2L(t\sin\omega t) - 0 - 0 &= 2\omega\frac{s}{s^2+\omega^2} - \omega^2L(t\sin\omega t) \\ \textit{thus } L(t\sin\omega t) &= \frac{2\omega s}{\left(s^2+\omega^2\right)^2} \quad \blacksquare \end{aligned}$$

Example 26 *Let* $f(t) = \cos\omega t$ *Then* $f(0) = 1,\ f'(0) = 0,\ f''(t) = -\omega^2\cos\omega t$. *From this and formula for* $L(f'')$ *we have*

$$\begin{aligned} s^2L(f) - sf(0) &= L(f'') = L\left(-\omega^2\cos\omega t\right) \\ s^2L(\cos\omega t) - s &= -\omega^2L(\cos\omega t) \\ \textit{solving } L(\cos\omega t) &= \frac{s}{s^2+\omega^2} \end{aligned}$$

Similarly let $g(t) = \sin\omega t$. *Then* $g(0) = 0,\ g'(t) = \omega\cos\omega t$, *which gives*

$$\begin{aligned} sL(g) - g(0) &= L(g') = \omega L(\cos\omega t) \\ \textit{Hence } L(\sin\omega t) &= \frac{\omega}{s}L(\cos\omega t) = \frac{\omega}{s^2+\omega^2} \quad \blacksquare \end{aligned}$$

Theorem 27 ***(Laplace Transform of Integral)*** *We have*

$$L\left(\int_0^t f(\tau)\,d\tau\right)=\frac{1}{s}F(s), \quad \textit{thus} \ \int_0^t f(\tau)\,d\tau = L^{-1}\left(\frac{1}{s}F(s)\right).$$

Example 28 *(10e-6.2-24) Find* $L^{-1}\left(\frac{20}{s^3-2\pi s^2}\right)$ *using LT of Integral?*
Note that simplification yields

$$\frac{20}{s^3-2\pi s^2}=\frac{20}{s^2(s-2\pi)}=20\frac{1}{s^2}\frac{1}{s-2\pi}$$

The presence of term $\frac{1}{s^2}$ *indicates that it is double integration of a function whose* $L(f(t))=F(s)=\frac{1}{s-2\pi}$*. By table of LT we know* $f(t)=L^{-1}\left(\frac{1}{s-2\pi}\right)=e^{2\pi t}$*. Hence we proceed as follows:*

$$\begin{aligned}
L^{-1}\left(\frac{20}{s^3-2\pi s^2}\right) &= L^{-1}\left(\frac{20}{s^2(s-2\pi)}\right)\\
&= 20\ L^{-1}\left(\frac{1}{s^2}\frac{1}{s-2\pi}\right)\\
&= 20\int_0^{\tau}\left(\int_0^{\tau} e^{2\pi\tau}d\tau\right)d\tau\\
&= 20\int_0^{\tau}\left(\frac{1}{2\pi}\left(e^{2\pi\tau}-1\right)\right)d\tau\\
&= 20\left(-\frac{1}{4\pi^2}\left(2\pi\tau-e^{2\pi\tau}+1\right)\right)\\
&= \frac{5}{\pi^2}e^{2(\pi\tau)}-\frac{10}{\pi}\tau-\frac{5}{\pi^2} \quad \blacksquare
\end{aligned}$$

Example 29 *Find inverse Laplace transform of* $\frac{1}{s(s^2+\omega^2)}$ *and* $\frac{1}{s^2(s^2+\omega^2)}$.

Solution 30 *Using* $L^{-1}\left(\frac{1}{s^2+\omega^2}\right)=\frac{\sin\omega t}{\omega}$, $L^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right)=\int_0^t\frac{\sin(\omega\tau)}{\omega}d\tau=\frac{1}{\omega^2}(1-\cos\omega t)$.
Next we have $L^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right)=\frac{1}{\omega^2}\int_0^t(1-\cos\omega\tau)\,d\tau=\left[\frac{\tau}{\omega^2}-\frac{\sin\omega\tau}{\omega^3}\right]_0^t=\frac{t}{\omega^2}-\frac{\sin\omega t}{\omega^3}$ $\blacksquare$

Example 31 *Using Laplace transform of integral find the inverse of* $\frac{1}{s(s^2+\omega^2)}$ *and* $\frac{1}{s^2(s^2+\omega^2)}$.

Solution 32 *From the table we have*

$$L^{-1}\left(\frac{1}{(s^2+\omega^2)}\right)=\frac{\sin\omega t}{\omega}, \qquad L^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right)=\int_0^t\frac{\sin\omega\tau}{\omega}d\tau=\frac{1}{\omega^2}(1-\cos\omega t)$$

Integrating this result again, we obtain

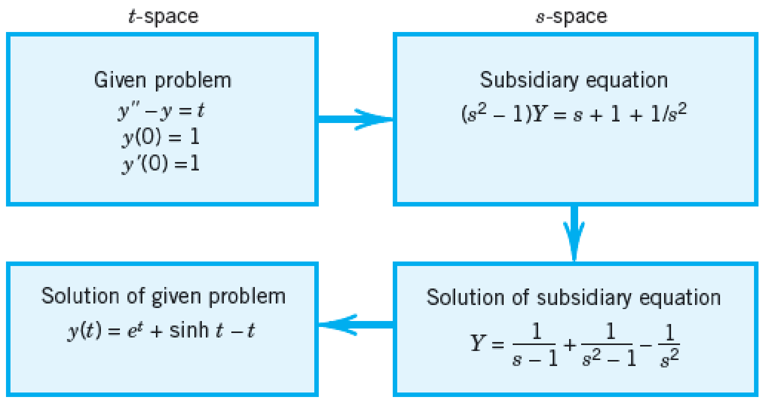
$$\begin{aligned}
L^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right) &= \frac{1}{\omega^2}\int_0^t(1-\cos\omega t)\\
&= \left[\frac{\tau}{\omega^2}-\frac{\sin\omega\tau}{\omega^3}\right]_0^t\\
&= \frac{t}{\omega^2}-\frac{\sin\omega t}{\omega^3} \quad \blacksquare
\end{aligned}$$

4 Solving Initial Value Problems by LT

Differential Equations, Initial Value Problems (IVP)

Laplace transform method solves ODEs and initial value problems using following steps:

1. Setting up the subsidiary equation.
2. Solution of the subsidiary equation by algebra.
3. Inversion of Y to obtain $y=L^{-1}(Y)$.



Example 33 *Solve following IVP using Laplace Transforms*

$$\begin{aligned} y'' - y &= t \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

Solution 34

$$\frac{d^2}{dt^2} y(t) - y(t) = t \; ; \qquad y(0) = 1, y'(0) = 1$$
$$s^2 L(y(t)) - y'(0) - sy(0) - L(y(t)) = \frac{1}{s^2}$$
$$L(y(t)) = \frac{\frac{1}{s^2} + y'(0) + sy(0)}{s^2 - 1}$$
$$L(y(t)) = \frac{\frac{1}{s^2} + 1 + s}{s^2 - 1}$$
$$L(y(t)) = \frac{s^3 + s^2 + 1}{s^2(s^2 - 1)}$$
$$L(y(t)) = -\frac{1}{s^2} - \frac{1}{2(s+1)} + \frac{3}{2(s-1)}$$
$$y(t) = -t - \frac{1}{2} e^{-t} + \frac{3}{2} e^t$$
$$y(t) = -t + \cosh(t) + 2 \sinh(t) \quad \blacksquare$$

$-t - \frac{1}{2} e^{-t} + \frac{3}{2} e^t = -t + \cosh(t) + 2\sinh(t)$ is true

Example 35 *Solve the IVP* $y'' + 9y = 10e^{-t}$; $y(0) = y'(0) = 0$ *using Laplace Transforms.*

Solution 36

$$\begin{aligned} \frac{d^2}{dt^2} y(t) + 9\, y(t) &= 10\, e^{-t} \\ s^2 L(y(t)) - y'(0) - sy(0) + 9\, L(y(t)) &= \frac{10}{s+1} \\ \textit{Isolating Laplace term} \qquad L(y(t)) &= \frac{1}{s^2+9} \left(\frac{10}{s+1} + y'(0) + sy(0) \right) \\ L(y(t)) &= \frac{10}{(s+1)(s^2+9)} \qquad \textit{by using ICs} \\ L(y(t)) &= \frac{1}{s+1} + \frac{-s+1}{s^2+9} \qquad \textit{by partial fractions (DIY)} \\ L(y(t)) &= \frac{1}{s+1} - \frac{s}{s^2+9} - \frac{1}{s^2+9} \\ y(t) &= e^{-t} - \cos(3\,t) + \frac{1}{3} \sin(3\,t) \quad \blacksquare \end{aligned}$$

Example 37 *(10e-6.1-10) $y''+3y+2.25y=9t^3+64$ with ICs $\quad y(0)=1, y'(0)=31.5$. rewrite and use Laplace Transforms.*

$$\frac{d^2}{dt^2}y(t)+3\,\frac{d}{dt}y(t)+2.25\,y(t)=9\,t^3+64 \quad ; \qquad y(0)=1, y'(0)=31.5$$

$$s^2Y-y'(0)-sy(0)+3sY-3y(0)+2.25\,Y=\frac{54}{s^4}+\frac{64}{s}$$

$$Y=\frac{\frac{54}{s^4}+\frac{64}{s}+y'(0)+sy(0)+3y(0)}{s^2+3s+2.25}$$

$$Y=\frac{\frac{54}{s^4}+\frac{64}{s}+34.5+s}{s^2+3s+2.25} \qquad \textit{using ICs}$$

$$Y=\frac{54+64s^3+34.5s^4+s^5}{s^6+3s^5+2.25s^4}=\frac{54+64s^3+34.5s^4+s^5}{\frac{1}{4}s^4(2s+3)^2}$$

By partial fractions(DIY) $\qquad Y=\dfrac{2}{2s+3}+\dfrac{4}{(2s+3)^2}+\dfrac{32}{s^2}-\dfrac{32}{s^3}+\dfrac{24}{s^4}$

$$Y=L(y(t))=\frac{1}{s+1.5}+\frac{1}{(s+1.5)^2}+\frac{32}{s^2}-\frac{32}{s^3}+\frac{24}{s^4}$$

$$y(t)=L^{-1}\left(\frac{1}{s+1.5}+\frac{1}{(s+1.5)^2}+\frac{32}{s^2}-\frac{32}{s^3}+\frac{24}{s^4}\right)$$

$$y(t)=e^{-1.5t}+te^{-1.5t}+32t-16t^2+4t^3$$

$$y(t)=4t^3-16t^2+32t+(t+1)\,e^{-1.5t} \quad \blacksquare$$

4.1 Shifted Data Problem

Example 38 *(Shifted Data Problem: where ICs are given at $t\neq 0$) Find solution of IVP*

$$y''+y=2t \quad ; \quad y\left(\frac{1}{4}\pi\right)=\frac{1}{2}\pi\,,\ y'\left(\frac{1}{4}\pi\right)=2-\sqrt{2}$$

Solution 39 *Put $\widetilde{t}=t-\frac{\pi}{4}$ to obtain*

$$\frac{d^2}{d\widetilde{t}^2}y\left(\widetilde{t}\right)+y\left(\widetilde{t}\right)=2\widetilde{t}+\frac{\pi}{2} \qquad ; \quad y(0)=\frac{\pi}{2}\ ,\ y'(0)=2-\sqrt{2}$$

Taking LT $\qquad s^2\widetilde{Y}-y'(0)-sy(0)+\widetilde{Y}=\dfrac{\pi}{2}\dfrac{1}{s}+2\,\dfrac{1}{s^2} \qquad$ *where* $\widetilde{Y}=L\left(y\left(\widetilde{t}\right)\right)$

Isolating $\quad \widetilde{Y}=\dfrac{1}{s^2+1}\left(\dfrac{\pi}{2}\dfrac{1}{s}+2\,\dfrac{1}{s^2}+y'(0)+sy(0)\right)$

Using ICs $\quad \widetilde{Y}=\dfrac{1}{s^2+1}\left(\dfrac{\pi}{2}\,\dfrac{1}{s}+2\,\dfrac{1}{s^2}+\left(2-\sqrt{2}\right)+\dfrac{\pi}{2}\,s\right)$

$$=\frac{1}{2}\,\frac{s^3\pi-2\sqrt{2}s^2+s\pi+4\,s^2+4}{s^2(s^2+1)}$$

By partial fractions(DIY) $\quad \widetilde{Y}=L\left(y\left(\widetilde{t}\right)\right)=\dfrac{\pi}{2}\,\dfrac{1}{s}+\dfrac{2}{s^2}-\dfrac{\sqrt{2}}{s^2+1}$

$$y\left(\widetilde{t}\right)=L^{-1}\left(\frac{\pi}{2}\,\frac{1}{s}+\frac{2}{s^2}-\frac{\sqrt{2}}{s^2+1}\right)$$

$$y\left(\widetilde{t}\right)=\frac{\pi}{2}+2\,\widetilde{t}-\sqrt{2}\sin\left(\widetilde{t}\right)$$

Putting back $\widetilde{t}=t-\dfrac{\pi}{4} \qquad y(t)=2\,t+\sqrt{2}\cos\left(t+\dfrac{\pi}{4}\right) \qquad \blacksquare$

Example 40 *(10e-6.2-14) The IVP*

$$y''+2y'+5y=50t-100 \quad ; \quad y(2)=-4,\ y'(2)=14$$

is a shifted data problem. So pu $\widetilde{t} = t - 2$ *and obtain*

$$\frac{d^2}{d\widetilde{t}^2} y\left(\widetilde{t}\right) + 2\frac{d}{d\widetilde{t}} y\left(\widetilde{t}\right) + 5\,y\left(\widetilde{t}\right) = 50\,\widetilde{t} \quad ; \quad y\left(0\right) = -4,\; y'\left(0\right) = 14$$

$$s^2\widetilde{Y} - y'\left(0\right) - sy\left(0\right) + 2\,s\widetilde{Y} - 2y\left(0\right) + 5\widetilde{Y} = 50\frac{1}{s^2} \qquad \textit{where } \widetilde{Y} = L\left(y\left(\widetilde{t}\right)\right)$$

$$\widetilde{Y} = \frac{1}{s^2 + 2\,s + 5}\left(50\frac{1}{s^2} + y'\left(0\right) + sy\left(0\right) + 2\,y\left(0\right)\right)$$

$$\widetilde{Y} = \frac{1}{s^2 + 2\,s + 5}\left(50\,\frac{1}{s^2} + 6 - 4s\right)$$

$$\widetilde{Y} = -2\,\frac{2\,s^3 - 3s^2 - 25}{s^2\left(s^2 + 2s + 5\right)}$$

$$\textit{By partial fractions(DIY)} \quad \widetilde{Y} = -\frac{4}{s} + \frac{4}{\left(s+1\right)^2 + 4} + \frac{10}{s^2}$$

$$y\left(\widetilde{t}\right) = -4 + 2e^{-\widetilde{t}}\sin\left(2\widetilde{t}\right) + 10\widetilde{t}$$

$$\textit{Replacing back } \widetilde{t} = t - 2 \qquad y\left(t\right) = -24 + 2\,e^{-t+2}\sin\left(2\,t - 4\right) + 10t \qquad \blacksquare$$

Example 41 *(10e-6.2-15) For*

$$y'' + 3y' - 4y = 6e^{2t-3} \quad ; \quad y\left(1.5\right) = 4, y'\left(1.5\right) = 5$$

put $\widetilde{t} = t - 1.5$ *to transform the IVP as follows:*

$$\frac{d^2}{d\widetilde{t}^2} y\left(\widetilde{t}\right) + 3\,\frac{d}{d\widetilde{t}} y\left(\widetilde{t}\right) - 4\,y\left(\widetilde{t}\right) = 6e^{2\,\widetilde{t}} \quad ; \quad y\left(0\right) = 4,\; y'\left(0\right) = 5$$

$$s^2\widetilde{Y} - y'\left(0\right) - sy\left(0\right) + 3s\widetilde{Y} - 3y\left(0\right) - 4\widetilde{Y} = \frac{6}{s-2} \qquad \textit{where } \widetilde{Y} = L\left(y\left(\widetilde{t}\right)\right)$$

$$\widetilde{Y} = \frac{1}{s^2 + 3s - 4}\left(\frac{6}{s-2} + y'\left(0\right) + sy\left(0\right) + 3y\left(0\right)\right)$$

$$\textit{Using ICs} \quad \widetilde{Y} = \frac{1}{s^2 + 3s - 4}\left(\frac{6}{s-2} + 17 + 4s\right)$$

$$\widetilde{Y} = \frac{4s - 7}{\left(s-1\right)\left(s-2\right)}$$

$$\textit{By partial fractions} \quad \widetilde{Y} = \frac{3}{s-1} + \frac{1}{s-2}$$

$$y\left(\widetilde{t}\right) = 3e^{\widetilde{t}} + e^{2\widetilde{t}}$$

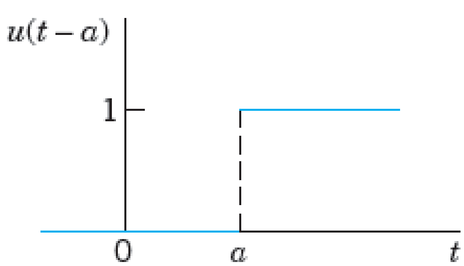
$$\textit{Replacing back } \widetilde{t} = t - 1.5, \qquad y\left(t\right) = 3e^{\left(t-1.5\right)} + e^{2\left(t-1.5\right)} \qquad \blacksquare$$

5 Heaviside Unit Step Function

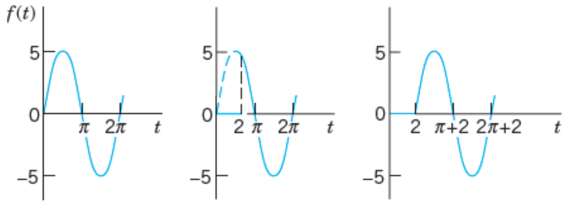
Definition 42 *(Heaviside Unit Step Function) The unit step function or Heaviside function* $u\left(t-a\right)$ *is defined as:*

$$u\left(t-a\right) = \begin{cases} 0 & if \quad t < a \\ 1 & if \quad t > a \end{cases} \qquad ; \quad a \geq 0$$

It graph is given as



Remark 43 *Let $f(t) = 0$ for all negative t. Then $f(t-a)\,u(t-a)$ with $a > 0$ is $f(t)$ shifted (translated) to the right by the amount a.*



(A) $f(t) = 5\sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

Effects of the unit step function: (A) Given function.
(B) Switching off and on. (C) Shift.

(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4\sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Use of many unit step functions.

Remark 44 *LT of Heaviside unit step function may be found as*

$$\begin{aligned} L(u(t-a)) &= \int_0^\infty e^{-st} u(t-a)\,dt = \int_a^\infty e^{-st}(1)\,dt \qquad \textit{since } a \ge 0 \\ &= \left[\frac{e^{-st}}{s}\right]_{t=a}^{\infty} \\ &= \frac{e^{-as}}{s} \qquad \blacksquare \end{aligned}$$

5.1 Second Shifting Theorem: t-Shifting

Theorem 45 *(**Second Shifting Theorem; t-Shifting**) If $f(t)$ has the transform $F(s)$, then the 'shifted function'*

$$\widetilde{f}(t) = f(t-a)\,u(t-a) = \begin{cases} 0 & \textit{if } \; t < a \\ f(t-a) & \textit{if } \; t > a \end{cases}$$

has the transform $e^{-as}F(s)$. Equivalently we have

$$L(f(t-a)\,u(t-a)) = e^{-as}F(s) \qquad \Rightarrow \qquad L^{-1}\left(e^{-as}F(s)\right) = f(t-a)\,u(t-a)$$

Remark 46 *Practically the t-shifting theorem amounts to this: if we know $F(s)$, we can obtain the transform of $f(t-a)\,u(t-a)$ by just multiplying $F(s)$ by e^{-as}.*

Remark 47 *If the conversion of $f(t)$ to $f(t-a)$ is difficult, we may use following form as well:*

$$L(f(t)\,u(t-a)) = e^{-as} L(f(t+a))$$

Example 48 *(Application of t-shifting theorem) Find LT of the piecewise function*

$$f(t) = \begin{cases} 2 & if \quad 0 < t < 1 \\ \frac{t^2}{2} & if \quad 1 < t < \frac{\pi}{2} \\ \cos t & if \quad t > \frac{\pi}{2} \end{cases}$$

Solution 49 *We first transform $f(t)$ in terms of unit-step functions as*

$$\begin{aligned} f(t) &= 2(u(t) - u(t-1)) + \frac{t^2}{2}\left(u(t-1) - u\left(t - \frac{\pi}{2}\right)\right) + (\cos t)\,u\left(t - \frac{\pi}{2}\right) \\ f(t) &= 2u(t) - 2u(t-1) + \frac{1}{2}t^2 u(t-1) - \frac{1}{2}t^2 u\left(t - \frac{\pi}{2}\right) + \cos(t)\,u\left(t - \frac{\pi}{2}\right) \end{aligned}$$

$$\Rightarrow L(f(t)) = 2L(u(t)) - 2L(u(t-1)) + \frac{1}{2}L\left(t^2u(t-1)\right) - \frac{1}{2}L\left(t^2u\left(t-\frac{\pi}{2}\right)\right) + L\left(\cos(t)\,u\left(t-\frac{\pi}{2}\right)\right)$$

Write each term in $f(t)$ in the form $f(t-a)$, so that the LT of the form $f(t-a)\,u(t-a)$ may be applied. Thus

$$\begin{aligned}
\text{for } \frac{1}{2}t^2u(t-1), \quad \frac{1}{2}t^2 &= \frac{1}{2}\left((t-1)^2+2t-1\right) = \frac{1}{2}(t-1)^2+(t-1)+1-\frac{1}{2} \\
&= \frac{1}{2}(t-1)^2+(t-1)+\frac{1}{2} \\
\text{for} \frac{1}{2}t^2u\left(t-\frac{\pi}{2}\right), \quad \frac{1}{2}t^2 &= \frac{1}{2}\left(t-\frac{\pi}{2}\right)^2+\frac{1}{2}\pi\left(t-\frac{\pi}{2}\right)+\frac{1}{8}\pi^2 \\
\text{for } \cos(t)\,u\left(t-\frac{\pi}{2}\right), \quad \cos(t) &= -\sin\left(t-\frac{1}{2}\pi\right)
\end{aligned}$$

$$L(f(t)) = \frac{2}{s} - \frac{2}{s}e^{-s} + \frac{1}{2}L\left(t^2\right)e^{-s} - \frac{1}{2}L\left(t^2\right)e^{-\frac{\pi}{2}s} + L(\cos(t))\,e^{-\frac{\pi}{2}s}$$

$$L(f(t)) = \frac{2}{s} - \frac{2}{s}e^{-s} + \frac{1}{2}L\left[\frac{1}{2}(t-1)^2+(t-1)+\frac{1}{2}\right]e^{-s} - \frac{1}{2}L\left[\frac{1}{2}\left(t-\frac{\pi}{2}\right)^2+\frac{1}{2}\pi\left(t-\frac{\pi}{2}\right)+\frac{1}{8}\pi^2\right]e^{-\frac{\pi}{2}s}$$
$$+L\left[-\sin\left(t-\frac{1}{2}\pi\right)\right]e^{-\frac{\pi}{2}s}$$

$$L(f(t)) = \frac{2}{s} - \frac{2}{s}e^{-s} + \frac{1}{2}\frac{\left(s^2+2s+2\right)}{s^3}e^{-s} - \frac{1}{8}\left(\frac{\pi^2}{s}+\frac{4\pi}{s^2}+\frac{8}{s^3}\right)e^{-\frac{\pi}{2}s} - \frac{1}{s^2+1}e^{-\frac{\pi}{2}s} \qquad \blacksquare$$

Example 50 *(DIY, hints are given) Writing the Heaviside form of*

$$f(t) = \begin{cases} \frac{1}{2}t & if \quad 0<t<\frac{1}{2} \\ 3t-2 & if \quad \frac{1}{2}<t<\frac{\pi}{2} \\ e^t & if \quad t>\frac{\pi}{2} \end{cases}$$

gives

$$f(t) = \frac{1}{2}t\ u(t) + \frac{5}{2}t\ u\left(t-\frac{1}{2}\right) - 3t\ u\left(t-\frac{\pi}{2}\right) + 2\,u\left(t-\frac{\pi}{2}\right) - 2\,u\left(t-\frac{1}{2}\right) + e^t\ u\left(t-\frac{\pi}{2}\right)$$

Applying t-shifting theorem to obtain the LT as

$$L(f(t)) = \frac{1}{2}s^{-2} + \frac{5}{4}\frac{(s+2)}{s^2}e^{\frac{-1}{2}s} - \frac{3}{2}\left(\frac{2}{s^2}+\frac{\pi}{s}\right)e^{-\frac{\pi}{2}s} + \frac{2}{s}e^{-\frac{\pi}{2}s} - \frac{2}{s}e^{-\frac{1}{2}s} + \frac{e^{-\frac{\pi}{2}(s-1)}}{s-1} \qquad \blacksquare$$

Example 51 *(10ed-6.3-10)*

$$\begin{aligned}
f(t) &= \begin{cases} \sinh t & if \quad 0<t<2 \\ 0 & if \quad t>2 \end{cases} \\
f(t) &= \sinh(t)\,u(t) - \sinh(t)\,u(t-2) \\
f(t) &= \sinh(t)\,u(t) - \sinh((t-2)+2)\,u(t-2) \\
f(t) &= \sinh(t)\,u(t) - (\cosh 2\sinh t - \sinh 2\cosh t)\,u(t-2) \\
L(f(t)) &= L(\sinh(t)\ \ u(t)) - \cosh(2)\ \ L(\sinh(t)\ \ u(t-2)) + \sinh(2)\ \ L(\cosh(t)\ \ u(t-2)) \\
&= \frac{1}{s^2-1} - \cosh(2)\left(\frac{1}{s^2-1}\right)e^{-2s} + \sinh(2)\left(\frac{s}{s^2-1}\right)e^{-2s} \\
&= \frac{1}{s^2-1} - \left(\frac{\sinh(2)\,s+\cosh(2)}{s^2-1}\right)e^{-2s} \qquad \blacksquare
\end{aligned}$$

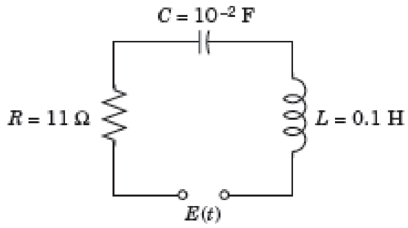
Example 52 *(10ed-6.3-14)*

$$\begin{aligned}
L^{-1}\left(\frac{4\left(e^{-2s}-2e^{-5s}\right)}{s}\right) &= L\left(\frac{1}{s}\left(4e^{-2s}-8e^{-5s}\right)\right) \\
&= 4L\left(\frac{1}{s}e^{-2s}\right) - 8L\left(\frac{1}{s}e^{-5s}\right) \\
f(t) &= 4\ u(t-2) - 8\ u(t-5) \qquad \blacksquare
\end{aligned}$$

Though not required, but this would be beneficial if student transforms above $f(t)$ into a piecewise representation. For this $f(t)$ it is given as

$$f(t) = 4\ u(t-2) - 8\ u(t-5) = \begin{cases} 4 & if \quad 2<t<5 \\ -4 & if \quad t>5 \end{cases}$$

Example 53 *Find the response (the current) of the RLC-circuit given in figure, where $E(t)$ is sinuosoidal, acting for a short time interval only, say $E(t) = 100\sin(400t)$ if $0 < t < 2\pi$ and $E(t) = 0$ if $t > 2\pi$.*



The forcing function $E(t) = \begin{cases} 100\sin(400t) & \text{if} \quad 0 < t < 2\pi \\ 0 & \text{if} \quad t > 2\pi \end{cases}$ *may be written in terms of unit-step function as* $E(t) = -100\,\sin(400\,t)\;\; u(t - 2\,\pi) + 100\,\sin(400\,t)\;\; u(t)$. *Model for current* $i(t)$ *is the integro-differential equation* $0.1\,\frac{d}{dt}i(t) + 11\,i(t) + 100\int_0^t i(\tau)\,d\tau$

$$0.1\,\frac{d}{dt}i(t) + 11\,i(t) + 100\int_0^t i(\tau)\,d\tau = -100\,\sin(400\,t)\;\; u(t - 2\,\pi) + 100\,\sin(400\,t)\;\; u(t) \qquad ; \qquad i(0) = 0, i'(0) = 0$$

$$0.1sY - 0.1i(0) + 11Y + 100\frac{1}{s}Y = \frac{40000}{(s^2 + 160000)}\left(1 - e^{-2\pi\,s}\right)$$

$$Y = -\frac{\left(-i(0)\,s^2 + 400000\;e^{-2\pi\,s} - 160000\;i(0) - 400000\right)s}{(s^2 + 160000)(s^2 + 110\,s + 1000)}$$

$$Y = -\frac{\left(-400000 + 400000\,e^{-2\pi\,s}\right)s}{(s^2 + 160000)(s^2 + 110\,s + 1000)} = \frac{400000\;s - 400000\,se^{-2\pi\,s}}{(s^2 + 160000)(s^2 + 110\,s + 1000)}$$

$$Y = \frac{400000\;s}{(s^2 + 160000)(s^2 + 110\,s + 1000)} - \frac{400000\,se^{-2\pi\,s}}{(s^2 + 160000)(s^2 + 110\,s + 1000)}$$

By partial fractions of first term

$$Y = \left(\frac{400}{153\,(s + 100)} - \frac{4000}{14\,409\,(s + 10)} - \frac{\frac{63\,600}{27\,217}s - \frac{7040\,000}{27\,217}}{s^2 + 160\,000}\right) - \left(\frac{400000\,s}{(s^2 + 160000)(s^2 + 110\,s + 1000)}\right)e^{-2\pi\,s}$$

takinging Inverse LT $\qquad i(t) = \left(\frac{400}{153}e^{-100t} - \frac{4000}{14\,409}e^{-10t} - \left(\frac{63\,600}{27\,217}\cos 400t - \frac{17\,600}{27\,217}\sin 400t\right)\right)$

$$-\left(\text{Heaviside}(t - 2\pi)\left(\frac{400}{153}e^{200\pi - 100t} - \frac{4000}{14\,409}e^{20\pi - 10t} - \frac{63\,600}{27\,217}\cos 400t + \frac{17\,600}{27\,217}\sin 400t\right)\right) \qquad \blacksquare$$

Example 54 *(10ed-6.3-23)*

$$\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) - 2\,y(t) = f(t) = \begin{cases} 3\sin(t) - \cos(t) & \text{if} \quad 0 < t < 2\pi \\ 3\sin(2t) - \cos(2t) & \text{if} \quad t > 2\pi \end{cases} \qquad ; \qquad y(0) = 1,\; y'(0) = 0$$

Transforming the forcing function into Heaviside form:

$$\begin{aligned} f(t) \;\; &= \;\; -3\sin(t)\,u(t - 2\pi) + 3\sin(t)\,u(t) + \cos(t)\,u(t - 2\pi) \\ & \quad\;\; - \cos(t)\,u(t) + 3u(t - 2\pi)\sin(2t) - u(t - 2\pi)\cos(2t) \end{aligned}$$

Taking LT on both sides of the given DE

$$s^2Y - y'(0) - sy(0) + sY - y(0) - 2Y = \frac{1}{s^2+1}\left(3 - s + 3\frac{e^{-2s\pi}(s+2)(s-1)}{s^2+4}\right)$$

$$Y = \frac{1}{s^2+s-2}\left(\frac{1}{s^2+1}\left(3 - s + 3\frac{e^{-2s\pi}(s+2)(s-1)}{s^2+4}\right) + y'(0) + sy(0) + y(0)\right)$$

$$Y = \frac{1}{s^2+s-2}\left(\frac{1}{s^2+1}\left(3 - s + 3\frac{e^{-2s\pi}(s+2)(s-1)}{s^2+4}\right) + 1 + s\right)$$

$$Y = \frac{s^4 - s^3 + 3e^{-2s\pi}s + 6s^2 - 3e^{-2s\pi} - 4s + 8}{(s-1)(s^2+1)(s^2+4)}$$

$$L(y(t)) = Y = \frac{s^4 - s^3 + 6s^2 + \left(3e^{-2s\pi} - 4\right)s - 3e^{-2s\pi} + 8}{(s-1)(s^2+1)(s^2+4)}$$

$$y(t) = e^t - \sin(t) + \frac{1}{2}\left(2\sin(t) - \sin(2t)\right)\ u(t - 2\pi)$$

$$y(t) = \begin{cases} e^t - \sin(t) & if \quad 0 < t < 2\pi \\ e^t - \frac{1}{2}\sin(2t) & if \quad t > 2\pi \end{cases} \qquad \blacksquare$$

Example 55 *(10e-6.3-27) Put $\widetilde{t} = t - 1$ in $E(t)$ and writing it in Heaviside terms:*

$$E(t) = \begin{cases} 8t^2 & 0 < t < 5 \\ 0 & t > 5 \end{cases} \Rightarrow E\left(\widetilde{t}\right) = \begin{cases} 8\left(\widetilde{t}+1\right)^2 & -1 < \widetilde{t} < 4 \\ 0 & \widetilde{t} > 4 \end{cases}$$

$$E\left(\widetilde{t}\right) = 8\,u\left(\widetilde{t}+1\right)\widetilde{t}^2 - 8\,u\left(\widetilde{t}-4\right)\widetilde{t}^2 + 16\,u\left(\widetilde{t}+1\right)\widetilde{t} - 16\,u\left(\widetilde{t}-4\right)\widetilde{t} + 8\,u\left(\widetilde{t}+1\right) - 8\,u\left(\widetilde{t}-4\right)$$

$$\frac{d^2}{d\widetilde{t}^2}y\left(\widetilde{t}\right) + 4\,y\left(\widetilde{t}\right) = E\left(\widetilde{t}\right) \qquad ; \quad y(0) = 1 + \cos(2)\,,\ y'(0) = 4 - 2\ \sin(2)$$

$$\frac{d^2}{d\widetilde{t}^2}y\left(\widetilde{t}\right) + 4\,y\left(\widetilde{t}\right) = 8\,u\left(\widetilde{t}+1\right)\widetilde{t}^2 - 8\,u\left(\widetilde{t}-4\right)\widetilde{t}^2 + 16\,u\left(\widetilde{t}+1\right)\widetilde{t} - 16\,u\left(\widetilde{t}-4\right)\widetilde{t} + 8\,u\left(\widetilde{t}+1\right) - 8\,u\left(\widetilde{t}-4\right)$$

$$s^2\widetilde{Y} - y'(0) - sy(0) + 4\,\widetilde{Y} = 16\,\frac{1}{s^2} + 8\,\frac{1}{s} + 8\,\frac{-e^{-4\,s}\left(25\,s^2 + 10\,s + 2\right) + 2}{s^3}$$

$$s^2\widetilde{Y} - 4 + 2\ \sin(2) - s\left(1 + \cos(2)\right) + 4\,\widetilde{Y} = 16\,\frac{1}{s^2} + 8\,\frac{1}{s} + 8\,\frac{-e^{-4\,s}\left(25\,s^2 + 10\,s + 2\right) + 2}{s^3} \qquad \textit{using ICs}$$

$$\widetilde{Y} = \frac{1}{s^2+4}\left(16\,\frac{1}{s^2} + 8\,\frac{1}{s} + 8\,\frac{-e^{-4\,s}\left(25\,s^2 + 10\,s + 2\right) + 2}{s^3} + 4 - 2\ \sin(2) + s\left(1 + \cos(2)\right)\right)$$

$$\widetilde{Y} = -\frac{-s^4\cos(2) + 2\ \sin(2)\,s^3 - s^4 + 200\,e^{-4\,s}s^2 - 4\,s^3 + 80\,e^{-4\,s}s - 8\,s^2 + 16\,e^{-4\,s} - 16\,s - 16}{s^3\left(s^2+4\right)}$$

$$\widetilde{Y} = -\frac{-s^4\cos(2) + 2\ \sin(2)\,s^3 - s^4 + 200\,e^{-4\,s}s^2 - 4\,s^3 + 80\,e^{-4\,s}s - 8\,s^2 + 16\,e^{-4\,s} - 16\,s - 16}{s^3\left(s^2+4\right)}$$

$$y\left(\widetilde{t}\right) = 1 + \cos(2)\cos\left(2\,\widetilde{t}\right) - \sin(2)\sin\left(2\,\widetilde{t}\right) + 2\,\widetilde{t}^2 + 4\,\widetilde{t} -$$
$$u\left(\widetilde{t}-4\right)\left(100\ \left(\sin\left(\widetilde{t}-4\right)\right)^2 + 2\,\widetilde{t}^2 + \cos\left(2\,\widetilde{t}-8\right) - 10\ \sin\left(2\,\widetilde{t}-8\right) + 4\,\widetilde{t} - 49\right)$$

$$y\left(\widetilde{t}\right) = \begin{cases} 1 + \cos(2)\cos\left(2\,\widetilde{t}\right) - \sin(2)\sin\left(2\,\widetilde{t}\right) + 2\,\widetilde{t}^2 + 4\,\widetilde{t} & t < 4 \\ -50 + \cos(2)\cos\left(2\,\widetilde{t}\right) - \sin(2)\sin\left(2\,\widetilde{t}\right) + 100\ \left(\cos\left(\widetilde{t}-4\right)\right)^2 - \cos\left(2\,\widetilde{t}-8\right) + 10\ \sin\left(2\,\widetilde{t}-8\right) & t > 4 \end{cases}$$

putting back $\widetilde{t} = t - 1$,

$$y(t) = \begin{cases} -3 + \cos(2)\cos(2\,t-2) - \sin(2)\sin(2\,t-2) + 2\ (t-1)^2 + 4\,t & t < 5 \\ -50 + \cos(2)\cos(2\,t-2) - \sin(2)\sin(2\,t-2) + 100\ \left(\cos(t-5)\right)^2 - \cos(2\,t-10) + 10\ \sin(2\,t-10) & t > 5 \end{cases}$$

$$y(t) = \begin{cases} \cos(2t) + 2t^2 - 1 & if \quad t < 5 \\ \cos(2t) + 49\cos(2t-10) + 10\sin(2t-10) & if \quad t > 5 \end{cases}. \qquad \blacksquare$$

Example 56 *(10ed-6.3-30)* $0.5\frac{d}{dt}i\left(t\right)+10i\left(t\right)=E\left(t\right)=\left\{\begin{array}{ccc} 200t & , & 0<t<2 \\ 0 & , & t>2 \end{array}\right. \quad ; \quad i\left(0\right)=0$

$$E\left(t\right)=200t\ u\left(t\right)-200t\ u\left(t-2\right)$$

$$0.5\frac{d}{dt}i\left(t\right)+10i\left(t\right)=200t\ u\left(t\right)-200t\ u\left(t-2\right)$$

$$0.5sY-0.5i\left(0\right)+10Y=200\left(1-e^{-2s}\left(2s+1\right)\right)s^{-2}$$

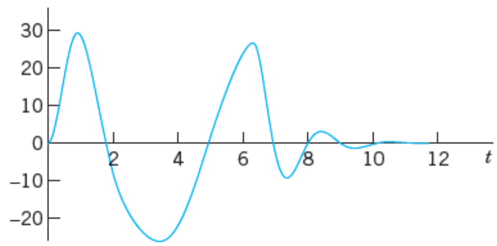
$$0.5sY+10Y=200\left(1-e^{-2s}\left(2s+1\right)\right)s^{-2}$$

$$Y=-400\frac{2e^{-2s}s+e^{-2s}-1}{s^{2}\left(s+20\right)}$$

$$i\left(t\right)=-1+20t\ u\left(2-t\right)+e^{-20t}+2\ u\left(t-2\right)\left(e^{\left(20-10t\right)}\sinh\left(10t-20\right)+20\ e^{\left(-20t+40\right)}\right)$$

$$i\left(t\right)=\left\{\begin{array}{ccc} -1+e^{-20t}+20t & , & t<2 \\ -1+e^{-20t}+2\ e^{\left(20-10t\right)}\sinh(10t-20)+40\ e^{\left(-20t+40\right)} & , & t>2 \end{array}\right. \qquad \blacksquare$$

Example 57 *(10ed-6.3-40)* $\frac{d}{dt}i\left(t\right)+2i\left(t\right)+10\int_{0}^{t}i\left(\tau\right)d\tau=E\left(t\right)=\left\{\begin{array}{cc} 255\sin(t) & 0<t<2\pi \\ 0 & t>2\pi \end{array}\right. \quad ; \quad i\left(0\right)=0$



$$E\left(t\right)=255\sin\left(t\right)\ u\left(t\right)-255\sin\left(t\right)\ u\left(-2\pi+t\right)$$

$$\frac{d}{dt}i\left(t\right)+2i\left(t\right)+10\int_{0}^{t}i\left(\tau\right)d\tau=255\sin\left(t\right)\ u\left(t\right)-255\sin\left(t\right)\ u\left(-2\pi+t\right)$$

$$s\ Y-i\left(0\right)+2Y+10\frac{Y}{s}=255\frac{1-e^{-2s\pi}}{s^{2}+1}$$

$$Y=1\left(255\frac{1-e^{-2s\pi}}{s^{2}+1}+i\left(0\right)\right)\left(s+2+10s^{-1}\right)^{-1}$$

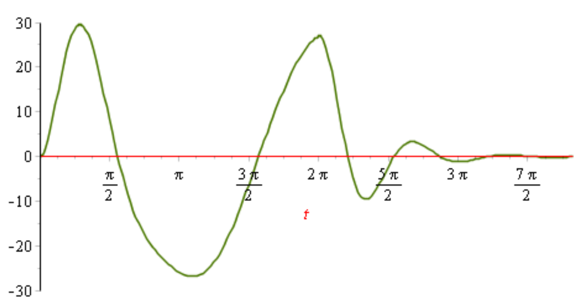
$$Y=255\frac{1-e^{-2s\pi}}{\left(s^{2}+1\right)\left(s+2+\frac{10}{s}\right)}$$

$$Y=-255\frac{\left(-1+e^{-2s\pi}\right)s}{\left(s^{2}+1\right)\left(s^{2}+2s+10\right)}$$

$$i\left(t\right)=3\ u\left(2\pi-t\right)\left(9\cos\left(t\right)+2\sin\left(t\right)\right)+\left(e^{2\pi-t}\ u\left(-2\pi+t\right)-e^{-t}\right)\left(27\cos\left(3t\right)+11\sin\left(3t\right)\right)$$

$$i\left(t\right)=\left\{\begin{array}{cc} 27\cos(t)+6\sin(t)-\left(27\cos(3t)+11\sin(3t)\right)e^{-t} & t<2\pi \\ \left(27\cos(3t)+11\sin(3t)\right)(-e^{-t}+e^{(2\pi-t)}) & t>2\pi \end{array}\right.$$

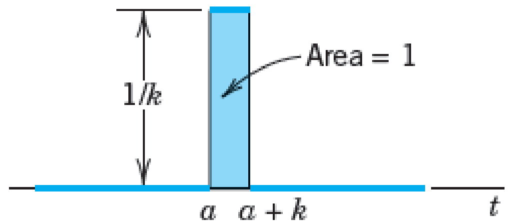
On plotting the current $i\left(t\right)$ *we have*



$\blacksquare$

6 Dirac's Delta Function

Definition 58 *Consider the function* $f_k(t-a) = \begin{cases} \frac{1}{k} & , \quad a \le t \le a+k \\ 0 & , \quad otherwise \end{cases}$ *, which has unit area as* $\int_0^\infty f_k(t-a)\ dt = \int_a^{a+k} \frac{1}{k} dt = 1.$



The function $f_k(t-a)$

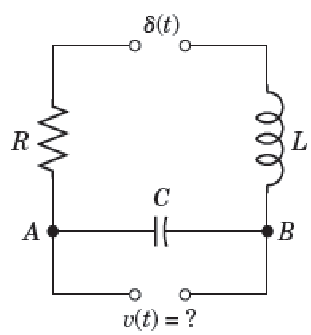
Then Dirac's delta function (aka, unit impulse function) is defined as

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a).$$

Remark 59 *LT of Dirac delta function may be obtained by writing it in Heaviside terms as*

$$\begin{aligned}
\delta(t-a) &= \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right] \\
L(f_k(t-a)) &= \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\frac{1-e^{-ks}}{ks} \\
L\left(\lim_{k\to 0} f_k(t-a)\right) &= \lim_{k\to 0} e^{-as}\frac{1-e^{-ks}}{ks} = e^{-as}\lim_{k\to 0}\frac{\frac{d}{dk}\left(1-e^{-ks}\right)}{\frac{d}{dk}(ks)} \qquad \textit{(l'Hopital)} \\
&= e^{-as}\left[\frac{se^{-ks}}{s}\right]_{k=0} = e^{-as} \\
L(\delta(t-a)) &= e^{-as} \qquad \blacksquare
\end{aligned}$$

Example 60 *Find output voltage response of the four-terminal RLC circuit given in figure if* $R = 20\Omega, L = 1H, C = 10^{-4}F$*, the input is an impulse, current and charge are zero at time* $t = 0$*?*



Solution 61 *Since for thic circuit* $Li' + Ri + \frac{q}{C} = 1\ i' + 20\ i + 10000\ q = \delta(t)$*, but the question is about voltage, hence the*

equation may be re-written as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} q(t) + 20\,\frac{\mathrm{d}}{\mathrm{d}t} q(t) + 10000\,q(t) = \delta(t) \qquad ; \quad q(0) = 0, q'(0) = 0$$

$$s^2 Q - q'(0) - s q(0) + 20\,s\ Q - 20\,q(0) + 10000\,Q = 1$$

$$Q = \frac{1 + q'(0) + s\ q(0) + 20\,q(0)}{s^2 + 20\,s + 10000}$$

$$Q = \frac{1}{s^2 + 20\,s + 10000}$$

$$Q = \frac{1}{(s+10)^2 + 9900}$$

$$q(t) = \frac{\sqrt{11} e^{-10\,t} \sin\left(30\sqrt{11} t\right)}{330}$$

$$v(t) = \frac{q(t)}{C}|_{C=10^{-4}} = \frac{1000}{33}\sqrt{11} e^{-10\,t} \sin\left(30\sqrt{11} t\right) \qquad \blacksquare$$

Example 62 *(10e-6.4-10)*

$$y'' + 5y' + 6y = \delta\left(t - \frac{1}{2}\pi\right) + u(t-\pi)\cos t \qquad ; \quad y(0) = y'(0) = 0$$

$$\frac{d^2}{dt^2} y(t) + 5\,\frac{d}{dt} y(t) + 6\,y(t) = \delta(t - \pi/2) + u(t-\pi)\cos(t)\,, y(0) = 0, y'(0) = 0$$

$$s^2 Y - y'(0) - s y(0) + 5\,s\ Y - 5\,y(0) + 6\,Y = e^{-\frac{\pi}{2}s} - \frac{s}{s^2+1} e^{-s\pi}$$

$$s^2 Y + 5\,sY + 6\,Y = e^{-\frac{\pi}{2}s} - \frac{s}{s^2+1} e^{-s\pi}$$

$$Y = \frac{1}{s^2 + 5\,s + 6}\left(e^{-\frac{\pi}{2}s} - \frac{s}{s^2+1} e^{-s\pi} \right)$$

$$Y = \frac{1}{(s+3)(s+2)}\left(e^{-\frac{\pi}{2}s} - \frac{s}{s^2+1} e^{-s\pi} \right)$$

$$Y = \left(\frac{1}{(s+3)(s+2)}\right) e^{-\frac{\pi}{2}s} - \left(\frac{s}{(s+3)(s+2)(s^2+1)}\right) e^{-s\pi}$$

$$Y = \left(\frac{1}{(s+2)} - \frac{1}{(s+3)}\right) e^{-\frac{\pi}{2}s} - \left(\frac{-2}{5}\,\frac{1}{(s+2)} + \frac{1}{10}\,\frac{s+1}{s^2+1} + \frac{3}{10}\,\frac{1}{(s+3)}\right) e^{-s\pi}$$

$$y(t) = L^{-1}\left(\frac{e^{-\frac{\pi}{2}s}}{s+2}\right) - L^{-1}\left(\frac{e^{-\frac{\pi}{2}s}}{s+3}\right) + \frac{2}{5} L^{-1}\left(\frac{1}{s+2} e^{-s\pi}\right) - \frac{1}{10} L^{-1}\left(\frac{s}{1+s^2} e^{-s\pi}\right) - \frac{1}{10} L^{-1}\left(\frac{1}{1+s^2} e^{-s\pi}\right) - \frac{3}{10} L^{-1}\left(\frac{1}{s+3} e^{-s\pi}\right)$$

$$y(t) = e^{\pi - 2t} u\left(t - \frac{1}{2}\pi\right) - e^{\frac{3}{2}\pi - 3t} u\left(t - \frac{1}{2}\pi\right) + \frac{2}{5} e^{2\pi - 2t} u(t-\pi) - \frac{1}{10}\left(-(\cos t)\,u(t-\pi)\right) - \frac{1}{10}\left(-(\sin t)\,u(t-\pi)\right) - \frac{3}{10} e^{3\pi - 3t} u(t -$$

$$y(t) = \frac{1}{10} u(t-\pi)\left(-3\,e^{-3\,t+3\,\pi} + \sin(t) + \cos(t) + 4\,e^{-2\,t+2\,\pi}\right) + u(t-\pi/2)\left(-e^{-3\,t+3/2\,\pi} + e^{-2\,t+\pi}\right) \qquad \blacksquare$$

Example 63 *(10ed-6.4-12)*

$$\begin{aligned}
\frac{d^2}{dt^2}y(t)+2\,\frac{d}{dt}y(t)+5\,y(t)=25\,t-100\,\delta(t-\pi) \qquad &, \qquad y(0)=-2,\quad y'(0)=5\\
s^2Y-y'(0)-sy(0)+2s\;Y-2\,y(0)+5\,Y=25\,\frac{1}{s^2}-100\,e^{-s\pi}\\
Y=\frac{1}{s^2+2\,s+5}\left(25\,\frac{1}{s^2}-100\,e^{-s\pi}+y'(0)+sy(0)+2\,y(0)\right)\\
Y=\frac{1}{s^2+2\,s+5}\left(25\,\frac{1}{s^2}-100\,e^{-s\pi}+1-2\,s\right)\\
Y=-\frac{100\,e^{-s\pi}s^2+2\,s^3-s^2-25}{s^2\left(s^2+2\,s+5\right)}\\
Y=-100\,\frac{1}{s^2+2\,s+5}e^{-s\pi}-2\,\frac{s}{s^2+2\,s+5}+\frac{1}{s^2+2\,s+5}+\frac{25}{s^2\left(s^2+2\,s+5\right)}\\
y(t)=-50\,u(t-\pi)\,e^{-t+\pi}\sin(2\,t)+5\,t-2\\
y(t)=\begin{cases} 5t-2 & ,\quad t<\pi\\ 5t-2-50e^{\pi-t}\sin(2t) & ,\quad t>\pi\end{cases} \qquad \blacksquare
\end{aligned}$$

7 Convolution: Product of Transforms

For two functions f and g, it must be noted that $L(f)\,L(g)\neq L(fg)$, instead we have $L(f)\,L(g)=L(f\star g)$, where $f\star g$ is defined as

$$(f\star g)(t)=\int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

In words one says: product of transforms is not the transform of product, instead product of transforms is the transform of convolution.

Example 64 *Find* $L^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right)$?

Solution 65 *We note that given entity seems a product of two familiar LT i.e.* $\frac{1}{(s^2+\omega^2)^2}=\frac{1}{(s^2+\omega^2)}\times\frac{1}{(s^2+\omega^2)}$. *We also recall* $L^{-1}\left(\frac{1}{s^2+\omega^2}\right)=\frac{\sin(\omega t)}{\omega}$. *Hence we write*

$$\begin{aligned}
L^{-1}\left(\frac{1}{s^2+\omega^2}\right) &= \frac{\sin(\omega t)}{\omega}\star\frac{\sin(\omega t)}{\omega}\\
&= \int_0^t \frac{\sin(\omega\tau)}{\omega}\frac{\sin(\omega(t-\tau))}{\omega}d\tau\\
&= \frac{1}{\omega^2}\int_0^t \sin(\omega\tau)\sin(\omega(t-\tau))\,d\tau\\
&= \frac{1}{\omega^2}\int_0^t \frac{1}{2}\left[-\cos\omega t+\cos(2\omega\tau-\omega t)\right]d\tau\\
&= \frac{1}{2\omega^2}\left[-\tau\cos\omega t+\frac{\sin(\omega\tau)}{\omega}\right]_{\tau=0}^{t}\\
&= \frac{1}{2\omega^2}\left[-t\cos\omega t+\frac{\sin(\omega t)}{\omega}\right] \qquad \blacksquare
\end{aligned}$$

7.1 Properties of Convolution:

$$\begin{aligned}
f\star g &= g\star f\\
f\star(g+h) &= f\star g+f\star h\\
f\star(g\star h) &= (f\star g)\star h\\
f\star 0 &= 0\star f=0\\
f\star 1 &\neq f
\end{aligned}$$

Example 66 *(10ed-6.5-6) Find convolution* $e^{at}\star e^{bt}$, *($a\neq b$)?*

Solution 67 *Since we have* $(f \star g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau,$ *we take* $f(t) = e^{at}$ *and* $g(t) = e^{bt}$:

$$
\begin{aligned}
\int_0^t e^{a\tau} e^{b(t-\tau)} d\tau &= \int_0^t e^{bt-b\tau} e^{a\tau}\, d\tau \\
&= \int_0^t e^{a\tau\, \ln(e)} e^{b(t-\tau)\ln(e)}\, d\tau \quad , \quad \textit{by rewriting} \quad e^{a\tau} e^{b(t-\tau)} = e^{a\tau\, \ln(e)} e^{b(t-\tau)\ln(e)} \\
&= \int_0^t e^{a\tau\, \ln(e) - b(\tau - t)\ln(e)}\, d\tau \\
\textit{now change} \quad u &= a\tau \ln(e) - b(\tau - t)\ln(e) \\
\textit{at } \tau &= 0,\ u = bt \textit{ and } \tau = t, \quad u = at \\
du &= \frac{d}{d\tau}\left(a\tau\, \ln(e) - b(\tau - t)\ln(e)\right) = a - b \Rightarrow \frac{du}{a-b} = d\tau \\
&= \int_{\ln(e)bt}^{at\ln(e)} \frac{e^u}{(a-b)\ln(e)}\, du \\
&= \frac{e^{at\ln(e)} - e^{\ln(e)bt}}{(a-b)\ln(e)} = \frac{e^{at} - e^{bt}}{(a-b)} \qquad \blacksquare
\end{aligned}
$$

Example 68 *(10ed-6.5-13)*

$$
\begin{aligned}
y(t) + 2e^t \int_0^t y(\tau)\, e^{-\tau} d\tau &= te^t \\
y(t) + 2\int_0^t y(\tau)\, e^{t-\tau} d\tau &= te^t \\
y(t) + 2\left(y(t) \star e^t\right) &= te^t \\
Y + 2\left(Y \frac{1}{s-1}\right) &= \frac{1}{(s-1)^2} \\
Y &= \frac{1}{s^2-1} \\
y(t) &= L^{-1}\left(\frac{1}{s^2-1}\right) \\
y(t) &= \sinh t \qquad \blacksquare
\end{aligned}
$$

Example 69 *(10ed-6.5-14)*

$$
\begin{aligned}
y(t) - \int_0^t y(\tau)(t-\tau)\, d\tau &= 2 - \frac{1}{2}t^2 \\
y(t) - (y(t) \star t) &= 2 - \frac{1}{2}t^2 \\
Y - \left(Y \frac{1}{s^2}\right) &= \frac{2}{s} - \frac{1}{s^3} \\
Y &= -\frac{2s^2-1}{s-s^3} \\
Y &= \frac{1}{2(s-1)} + \frac{1}{2(s+1)} + \frac{1}{s} \\
y(t) &= \frac{1}{2}e^t + \frac{1}{2}e^{-t} + 1 \\
y(t) &= 1 + \cosh t \qquad \blacksquare
\end{aligned}
$$

Example 70 *(10ed-6.5-22)*

$$\begin{aligned}
\frac{e^{-as}}{s(s-2)} &= \frac{1}{s-2}\frac{e^{-as}}{s} = L\left(e^{2t}\right)L\left(u\left(t-a\right)\right)\\
&= \left(e^{2t}\right)\star u\left(t-a\right)\\
\textit{choose } f\left(t\right) &= e^{2t} \textit{ and } g\left(t\right)=u\left(t-a\right) \textit{ in convolution integral } \left(f\star g\right)\left(t\right)=\int_0^t f\left(\tau\right)g\left(t-\tau\right)d\tau\\
\left(e^{2t}\right)\star u\left(t-a\right) &= \int_0^t e^{2\tau}u((t-\tau)-a)d\tau\\
&= \left[\frac{1}{2}u\left(t-a-\tau\right)\left(e^{2\tau}+e^{2t-2a}\right)\right]_{\tau=0}^{t}\\
&= \left[\frac{1}{2}u\left(t-a-\tau\right)\left(e^{2\tau}+e^{2t-2a}\right)\right]_{\tau=t}-\left[\frac{1}{2}u\left(t-a-\tau\right)\left(e^{2\tau}+e^{2t-2a}\right)\right]_{\tau=0}\\
&= \frac{1}{2}u\left(-a\right)\left(e^{2t-2a}+e^{2t}\right)-\frac{1}{2}u\left(t-a\right)\left(e^{2t-2a}+1\right) \qquad \blacksquare
\end{aligned}$$

Example 71 *(9ed-6.5-20) Using convolution theorem solve the following IVP*

$$y''+5y'+4y=2e^{-2t} \quad ; \quad y\left(0\right)=0,\ y'\left(0\right)=0$$

$$s^2Y-y'\left(0\right)-sy\left(0\right)+5\,sY-5\,y\left(0\right)+4\,Y=\frac{2}{s+2}$$

$$Y=\frac{1}{s^2+5\,s+4}\left(\frac{2}{s+2}+y'\left(0\right)+sy\left(0\right)+5\,y\left(0\right)\right)$$

$$Y=2\frac{1}{\left(s+2\right)\left(s^2+5\,s+4\right)}=2\frac{1}{\left(s+2\right)\left(s+4\right)\left(s+1\right)}$$

$$Y=2\frac{1}{\left(s+2\right)}\left(\frac{1}{\left(s+4\right)\left(s+1\right)}\right)$$

$$Y=\frac{2}{\left(s+2\right)}\left[L\left(e^{-4t}\right)L\left(e^{-t}\right)\right]$$

$$Y=2L\left(e^{-2t}\right)\left[L\left(e^{-4t}\right)L\left(e^{-t}\right)\right]$$

$$Y=2e^{-2t}\star\left[e^{-4t}\star e^{-t}\right]$$

$$Y=2e^{-2t}\star\int_0^t e^{-4\tau}e^{-(t-\tau)}d\tau$$

$$Y=2e^{-2t}\star\left[-\frac{1}{3}e^{-t-3\tau}\right]_{\tau=0}^{\tau=t}$$

$$Y=2e^{-2t}\star\left(\frac{1}{3}e^{-t}-\frac{1}{3}e^{-4t}\right)$$

$$Y=\frac{2}{3}e^{-2t}\star\left(e^{-t}-e^{-4t}\right)$$

$$Y=\frac{2}{3}\int_0^t e^{-2\tau}\left(e^{-(t-\tau)}-e^{-4(t-\tau)}\right)d\tau$$

$$y\left(t\right)=\frac{2}{3}\left(\frac{1}{2}e^{-t}\left(e^{-t}-1\right)^2\left(e^{-t}+2\right)\right)$$

$$y\left(t\right)=\frac{1}{3}e^{-t}\left(e^{-t}-1\right)^2\left(e^{-t}+2\right) \qquad \blacksquare$$

8 Differentiation and Integeration of LT

Theorem 72 *If $F\left(s\right)$ is the Laplace transform of $f\left(t\right)$, then*

$$F'\left(s\right)=-\int_0^\infty e^{-st}tf\left(t\right)dt$$

from which, also note that

$$L\left(tf\left(t\right)\right)=-F'\left(s\right) \quad \textit{and} \quad L^{-1}\left(F'\left(s\right)\right)=-tf\left(t\right)$$

Corollary 73 *By mathematical induction we also have*

$$L\left(t^n f(t)\right) = (-1)^n F^{(n)}(s)$$

Theorem 74 *If $F(s)$ is the Laplace transform of $f(t)$, then*

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\widetilde{s})\, d\widetilde{s} \quad \text{and} \quad L^{-1}\left(\int_s^\infty F(\widetilde{s})\, d\widetilde{s}\right) = \frac{f(t)}{t}$$

Example 75 $L^{-1}\left(\ln\left(1+\frac{\omega^2}{s^2}\right)\right) = ?$

We observe that $\ln\left(1+\frac{\omega^2}{s^2}\right) = \ln\left(\frac{s^2+\omega^2}{s^2}\right) = \ln\left(s^2+\omega^2\right) - \ln s^2$. *This indicates that a derivative of* $\ln$ *may bring terms like* $\frac{s}{s^2+\omega^2}$ *or* $\frac{1}{s}$. *Hence we set* $F(s) = \ln\left(\frac{s^2+\omega^2}{s^2}\right)$ *and proceed for an application of differentiation of LT theorem i.e.*

$$\begin{aligned}
F'(s) &= \frac{d}{ds}\ln\left(s^2+\omega^2\right) - \frac{d}{ds}\ln s^2 = 2\frac{s}{s^2+\omega^2} - \frac{2}{s} \\
L^{-1}\left(F'(s)\right) &= L^{-1}\left(2\frac{s}{s^2+\omega^2} - \frac{2}{s}\right) = 2\cos(\omega t) - 2 = 2\left(\cos(\omega t) - 1\right) \\
\textit{Since} \quad L^{-1}\left(F'(s)\right) &= -tf(t) \\
&\Rightarrow -tf(t) = 2\left(\cos(\omega t) - 1\right) \\
f(t) &= \frac{2}{t}\left(1 - \cos(\omega t)\right) \qquad \blacksquare
\end{aligned}$$

Example 76 *(10e-6.6-2)* $L\left(3t\sinh(4t)\right) = ?$

$$\begin{aligned}
\textit{We know } L\left(tf(t)\right) &= -F'(s) \\
L\left(3t\sinh(4t)\right) &= 3L\left(t\sinh(4t)\right) \\
&= 3\left(-\frac{d}{ds}L\left(\sinh(4t)\right)\right) \\
&= -3\left(\frac{d}{ds}\left(\frac{4}{s^2-16}\right)\right) \\
&= 24\frac{s}{\left(s^2-16\right)^2} \qquad \blacksquare
\end{aligned}$$

Example 77 *For finding* $L\left(t^n e^{kt}\right)$, *we note*

$$\begin{aligned}
L\left(e^{kt}\right) &= \frac{1}{s-k} \quad , \quad (= F(s)) \\
\textit{by corollary } L\left(t^n f(t)\right) &= (-1)^n F^{(n)}(s), \textit{ so we take } n \textit{ differentiations of } F(s) \\
\frac{d}{ds}\left(\frac{1}{s-k}\right) &= \frac{-1}{(s-k)^2}, \quad \frac{d^2}{ds^2}\left(\frac{1}{s-k}\right) = \frac{2}{(s-k)^3} \\
\frac{d^3}{ds^3}\left(\frac{1}{s-k}\right) &= \frac{-6}{(s-k)^4}, \quad \textit{hence} \\
\frac{d^n}{ds^n}\left(\frac{1}{s-k}\right) &= \frac{(-1)^n n!}{(s-k)^{n+1}} \\
L\left(t^n e^{kt}\right) &= \frac{(-1)^n n!}{(s-k)^{n+1}} \qquad \blacksquare
\end{aligned}$$

Example 78 *(10e-6.7-6)*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}y_1(t) &= 5\,y_1(t) + y_2(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}y_2(t) &= y_1(t) + 5\,y_2(t) \\
y_1(0) &= 1, y_2(0) = -3
\end{aligned}$$

Taking Laplace Transforms

$$\begin{aligned}
s\,Y_1 - y_1(0) &= 5\,Y_1 + Y_2 \\
sY_2 - y_2(0) &= Y_1 + 5\,Y_2
\end{aligned}$$

Using ICs

$$\begin{aligned} s\;Y_1 - 1 &= 5\,Y_1 + Y_2 \\ s\;Y_2 + 3 &= Y_1 + 5\,Y_2 \end{aligned}$$

Solving simultaneously and applying partial fractions

$$\begin{aligned} Y_1 &= \frac{s-8}{s^2-10\,s+24} = 2\;\;(s-4)^{-1} - (s-6)^{-1} \\ Y_2 &= -\frac{3\,s-16}{s^2-10\,s+24} = -2\;\;(s-4)^{-1} - (s-6)^{-1} \end{aligned}$$

Taking the inverse LT

$$\begin{aligned} y_1\,(t) &= -\,\mathrm{e}^{6\,t} + 2\,\mathrm{e}^{4\,t} \\ y_2\,(t) &= -\,\mathrm{e}^{6\,t} - 2\,\mathrm{e}^{4\,t} \qquad \blacksquare \end{aligned}$$

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$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} y_1\,(t) &= -2\,y_1\,(t) + 2\,y_2\,(t) \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} y_2\,(t) &= 2\,y_1\,(t) - 5\,y_2\,(t) \\ y_1\,(0) &= 1, y_2\,(0) = 3, y_1'\,(0) = 0, y_2'\,(0) = 0 \end{aligned}$$

Taking Laplace Transforms

$$\begin{aligned} s^2 Y_1 - y_1'\,(0) - s y_1\,(0) &= -2\,Y_1 + 2\,Y_2 \\ s^2 Y_2 - y_2'\,(0) - s y_2\,(0) &= 2\,Y_1 - 5\,Y_2 \end{aligned}$$

Using ICs

$$\begin{aligned} s^2 Y_1 - s &= -2\,Y_1 + 2\,Y_2 \\ s^2 Y_2 - 3\,s &= 2\,Y_1 - 5\,Y_2 \end{aligned}$$

Solving simultaneously and applying partial fractions

$$\begin{aligned} Y_1 &= \frac{s\left(s^2+11\right)}{s^4+7\,s^2+6} = 2\,\frac{s}{s^2+1} - \frac{s}{s^2+6} \\ Y_2 &= \frac{s\left(3\,s^2+8\right)}{s^4+7\,s^2+6} = \frac{s}{s^2+1} + 2\,\frac{s}{s^2+6} \end{aligned}$$

Taking the inverse LT

$$\begin{aligned} y_1\,(t) &= 2\,\cos(t) - \cos\left(\sqrt{6}t\right) \\ y_2\,(t) &= \cos(t) + 2\,\cos\left(\sqrt{6}t\right) \qquad \blacksquare \end{aligned}$$