

Classroom notes of Vector Differential Calculus

based on Chapter 10 of

Advanced Engineering Mathematics, E. Kreyszig, 10th Edition

Dr. Athar Kharal*

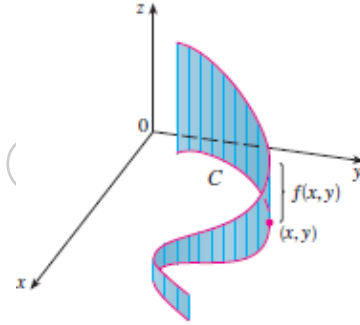
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1 Line Integrals

Definition 1 A line integral of a vector function $\mathbf{F}(\mathbf{r})$ over a curve $C: \mathbf{r}(t)$ is defined as $I = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$.



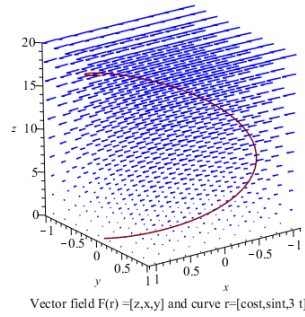
Question: Find the value of line integral when $\mathbf{F}(\mathbf{r}) = [z, x, y]$ and $C = [\cos t, \sin t, 3t]; 0 \leq t \leq 2\pi$ is a helix.

Solution: Note that $\mathbf{F}(\mathbf{r})$ is not in terms of parameter t , hence from C we substitute $x = \cos t, y = \sin t$ and $z = 3t$ into it and get $\mathbf{F}(\mathbf{r}) = [3t, \cos t, \sin t]$. Then

$$\begin{aligned} I &= \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} \left([3t, \cos t, \sin t] \cdot \frac{d}{dt} [\cos t, \sin t, 3t] \right) dt \\ &= \int_0^{2\pi} ([3t, \cos t, \sin t] \cdot [-\sin t, \cos t, 3]) dt \\ &= \int_0^{2\pi} (-3t \sin t + \cos^2 t + 3 \sin t) dt = 7\pi = 21.99 \end{aligned}$$

*email: atharkharal@gmail.com cell:0092 323 7263699 LinkedIn: <https://pk.linkedin.com/in/atharkharal>, Website: atharkharal.github.io

Sketch of the vector field and the curve is given below:



Remark 2 1. We have following natural properties of a line integral: $\int_C k\mathbf{F}\cdot d\mathbf{r} = k \int_C \mathbf{F}\cdot d\mathbf{r}$, $\int_C (\mathbf{F} + \mathbf{G})\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{G}\cdot d\mathbf{r}$ and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ where C_1 and C_2 are subdividing curves of C with same orientation as that of C .

2. At times, without taking dot product we may obtain a line integral whose value is a vector rather than a scalar, as follows:

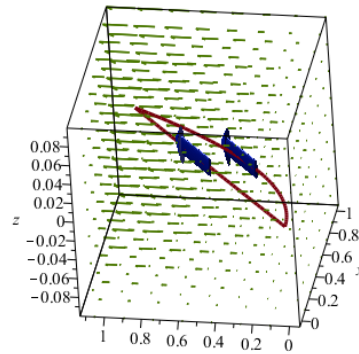
$$\int_C \mathbf{F}(\mathbf{r}) dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) dt = \int_a^b [\mathbf{F}_1(\mathbf{r}(t)), \mathbf{F}_2(\mathbf{r}(t)), \mathbf{F}_3(\mathbf{r}(t))] dt$$

For the special case when $\mathbf{F}_1 = f$ and $\mathbf{F}_2 = \mathbf{F}_3 = 0$, we have

$$\int_C f(\mathbf{r}) dt = \int_a^b f(\mathbf{r}(t)) dt$$

3. In general, a line integral's value would change by the change of curve C e.g.

$$\begin{aligned} \int_{C:[t,t^2,0];0\leq t\leq 1} [0,xy,0] d\mathbf{r} &= \int_0^1 ([0,t^2,0] \cdot [1,1,0]) dt = \int_0^1 t^2 dt = \frac{1}{3} \\ \int_{C:[t,t^2,0];0\leq t\leq 1} [0,xy,0] d\mathbf{r} &= \int_0^1 ([0,t^3,0] \cdot [1,2t,0]) dt = \int_0^1 2t^4 dt = \frac{2}{5} \end{aligned}$$



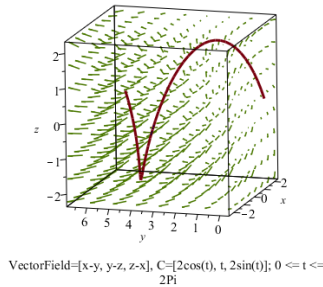
Question: (10ed-10.1-6) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} = [x - y, y - z, z - x]$ and $\mathbf{r} = [2 \cos t, 2 \sin t]$ from $(2, 0, 0)$ to $(2, 2\pi, 0)$. Show the details.

Solution: We first find variation of t : By putting $\mathbf{r} = [2 \cos t, 2 \sin t] = (2, 0, 0) \Rightarrow t = 0$ and $2 \cos t = 2 \Rightarrow t = 2\pi$. Validity of these values of t may be checked by putting final value of t i.e. 2π and getting the terminal point, already given to us i.e. $(2, 2\pi, 0)$. Next we re-write \mathbf{F} in terms of t by using $x = 2 \cos t, y = t$ and $z = 2 \sin t$ into

$\mathbf{F} = [2 \cos t - t, t - 2 \sin t, 2 \sin t - 2 \cos t]$. Hence

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left([2 \cos t - t, t - 2 \sin t, 2 \sin t - 2 \cos t] \cdot \frac{d}{dt} [2 \cos t, t, 2 \sin t] \right) dt \\ &= \int_0^{2\pi} ([2 \cos t - t, t - 2 \sin t, 2 \sin t - 2 \cos t] \cdot [-2 \sin t, 1, 2 \cos t]) dt \\ &= \int_0^{2\pi} (t - 2 \sin t + 2t \sin t - 4 \cos^2 t - 4 \cos t \sin t + 4 \sin t \cos t) dt \\ &= \int_0^{2\pi} (t - 2 \sin t - 2 \cos 2t + 2t \sin t - 2) dt \\ &= 2\pi (\pi - 4) \end{aligned}$$

Sketch:

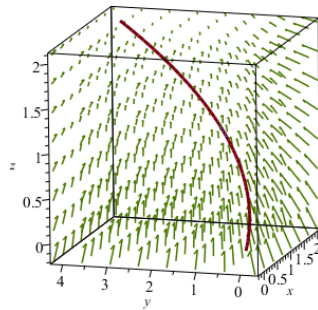


Question: (10ed-10.1-11) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = [e^{-x}, e^{-y}, e^{-z}]$ and $\mathbf{r} = [t, t^2, t]$ from $(0, 0, 0)$ to $(2, 4, 2)$. Show the details.

Solution: Initial and terminal points indicate variation of t as $0 \leq t \leq 2$. Next rewriting \mathbf{F} in terms of t only, by using $x = t, y = t^2$ and $z = t \Rightarrow \mathbf{F} = [e^{-t}, e^{-t^2}, e^{-t}]$. Hence

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \left([e^{-t}, e^{-t^2}, e^{-t}] \cdot \frac{d}{dt} [t, t^2, t] \right) dt \\ &= \int_0^2 ([e^{-t}, e^{-t^2}, e^{-t}] \cdot [1, 2t, 1]) dt \\ &= \int_0^2 (2e^{-t} + 2te^{-t^2}) dt \\ &= 3 - \frac{2}{e^2} - \frac{1}{e^4} \end{aligned}$$

Sketch



Question: (10ed-10.1-15) Without taking dot product, evaluate the line integral with $\mathbf{F} = [y^2, z^2, x^2]$ and $C : [3 \cos t, 3 \sin t, 2t]; 0 \leq t \leq 4\pi$?

Solution: From the curve we have $x = 3 \cos t$, $y = 3 \sin t$ and $z = 2t$. By using the form

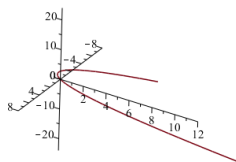
$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) dt &= \int_a^b \mathbf{F}(\mathbf{r}(t)) dt = \int_a^b [\mathbf{F}_1(\mathbf{r}(t)), \mathbf{F}_2(\mathbf{r}(t)), \mathbf{F}_3(\mathbf{r}(t))] dt \\ &= \int_0^{4\pi} [(3 \sin t)^2, (2t)^2, (3 \cos t)^2] dt \\ &= \left[\int_0^{4\pi} (3 \sin t)^2 dt, \int_0^{4\pi} (2t)^2 dt, \int_0^{4\pi} (3 \cos t)^2 dt \right] \\ &= \left[18\pi, \frac{256}{3}\pi^3, 18\pi \right] \quad \blacksquare \end{aligned}$$

Question: (10ed-10.1-19) Without taking dot product, evaluate the line integral with $f = xyz$ and $C : [4t, 3t^2, 12t] ; -2 \leq t \leq 2$?

Solution: Using the special case of non-dot-product line integral $\int_C f(\mathbf{r}) dt = \int_a^b f(\mathbf{r}(t)) dt$ we have

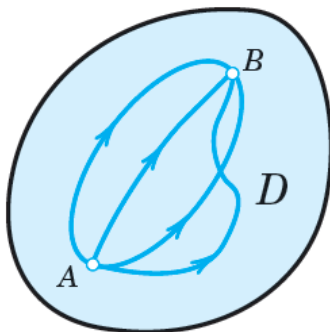
$$\begin{aligned} \int_C f(\mathbf{r}) dt &= \int_a^b f(\mathbf{r}(t)) dt \\ &= \int_{-2}^2 f(4t, 3t^2, 12t) dt \\ &= \int_{-2}^2 (4t)(3t^2)(12t) dt \\ &= 144 \int_{-2}^2 t^4 dt = \frac{64}{5} \end{aligned}$$

Curve C is sketched as



2 Path Independence of Line Integrals

Definition 3 The line integral $I = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is said to be path independent in a domain D if for every pair of endpoints A, B in domain D the integral I has same value for all paths in D that begin with A and end at B .



Path Independence

Remark 4 Path independence of a line integral in a domain D holds if and only if

1. $\mathbf{F} = \text{grad } f$, and in such case we have $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$, Or

2. Integration around any closed curve C in D always gives 0, Or

3. $\text{curl } \mathbf{F} = \mathbf{0}$, provided D is simply connected i.e. every closed curve in D can be continuously shrunk to any point in D without leaving D .

Question: Evaluate the integral $I = \int (3x^2 dx + 2yz dy + y^2 dz)$ from $A(0, 1, 2)$ to $B(1, -1, 7)$ by showing that \mathbf{F} has a potential.

Solution: If \mathbf{F} has a potential f , we should have

$$f_x = 3x^2, \quad f_y = 2yz, \quad f_z = y^2$$

By integration $\int f_x dx = f = x^3 + g(y, z)$. This further implies $f_y = g_y = 2yz \Rightarrow g = y^2 z + h(z)$. This implies $f = x^3 + y^2 z + h(z)$. Again $f_z = y^2 + h_z = y^2 \Rightarrow h_z = 0$ i.e. $h(z)$ is a constant. Suppose it is 0, then $f = x^3 + y^2 z + 0 = x^3 + y^2 z$. Thus \mathbf{F} is gradient of a potential, namely, the function f . Hence the line integral is evaluated as

$$\begin{aligned} \int_{(0,1,2)}^{(1,-1,7)} (3x^2 dx + 2yz dy + y^2 dz) &= f(B) - f(A) \\ &= f(1, -1, 7) - f(0, 1, 2) \\ &= 8 - 2 = 6 \quad \blacksquare \end{aligned}$$

Definition 5 In the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, the term $\mathbf{F} \cdot d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$ is said to be a **differential form**. It is further to be said exact iff there is a function $f(x, y, z)$ in domain D such that $F_1 = \frac{\partial f}{\partial x}$, $F_2 = \frac{\partial f}{\partial y}$ and $F_3 = \frac{\partial f}{\partial z}$.

Theorem 6 For a line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, we have following:

1. Differential form of I is exact iff I is path independent.
2. If differential form is exact then $\text{curl } \mathbf{F} = \mathbf{0}$
3. If $\text{curl } \mathbf{F} = \mathbf{0}$ in D and D is simply connected, then differential form of I is exact and hence, I is path independent.

Pictorially, above results are summed up as follows:

| | | |
|--|-------------------|-------------------------|
| Differential form of I is exact | \leftrightarrow | I is path independent |
| \downarrow | \nwarrow | |
| $\text{curl } \mathbf{F} = \mathbf{0}$ | $+$ | D is simply connected |

Remark 7 Note that $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0 \iff \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \text{ and } \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$. (The mnemonic for remembering is 21, 31, 32)

Question: If $I = \int_C (2xyz^2 dx + (x^2 z^2 + z \cos(yz)) dy + (2x^2 yz + y \cos(yz)) dz)$ is path independent, then find its value from $A(0, 0, 1)$ to $B(1, \frac{\pi}{4}, 2)$.

Solution: The question requires us to first establish the path independence of I . For this we systematically find potential function $f(x, y, z)$ as follows: we have

$$f_x = 2xyz^2, \quad f_y = x^2 z^2 + z \cos(yz), \quad f_z = 2x^2 yz + y \cos(yz)$$

$\int f_x dx = \int 2xyz^2 dx = x^2 yz^2 \Rightarrow f = x^2 yz^2 + g(y, z)$
 $f_y = g_y = x^2 z^2 + g_y = x^2 z^2 + z \cos(yz) \Rightarrow \int g_y dy = \int z \cos(yz) dy = \sin yz + h(z) \Rightarrow f = x^2 yz^2 + \sin yz + h(z)$
 $f_z = \frac{d}{dz} (x^2 yz^2 + \sin yz) = 2x^2 yz + y \cos yz + h_z = 2x^2 yz + y \cos(yz) \Rightarrow h_z = 0$ i.e. $h(z)$ is constant, say 0. Hence we have $f = x^2 yz^2 + \sin yz$. As \mathbf{F} is the gradient of $f = x^2 yz^2 + \sin yz$ hence the line integral may be evaluated as

$$\begin{aligned} \int_{(0,0,1)}^{(1,\frac{\pi}{4},2)} (2xyz^2 dx + (x^2 z^2 + z \cos(yz)) dy + (2x^2 yz + y \cos(yz)) dz) \\ = f(B) - f(A) = [x^2 yz^2 + \sin yz]_{x=1, y=\frac{\pi}{4}, z=2} - [x^2 yz^2 + \sin yz]_{x=0, y=0, z=1} \\ = (\pi + 1) - 0 = \pi + 1 \quad \blacksquare \end{aligned}$$

Question: (10ed-10.2-4) Show that the form under the integral $\int_{(4,0)}^{(6,1)} e^{4y} (2x dx + 4x^2 dy)$ is exact in the plane and evaluate the integral. Show the details of your work?

Solution: Here we have $F_1 = 2xe^{4y}$, $F_2 = 4e^{4y}x^2$ and no third component. Hence for exactness of the differential form we see $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(2xe^{4y}) = 8xe^{4y} = \frac{\partial}{\partial x}(4e^{4y}x^2) = \frac{\partial F_2}{\partial x}$, hence the form is exact and the integral is path independent. So we may evaluate it by using its potential only i.e. f . For finding f we set $f_x = 2xe^{4y}$ and $f_y = 4e^{4y}x^2$. Now $\int f_x dx = \int (2xe^{4y}) dx = x^2e^{4y} + g(y) \Rightarrow f = x^2e^{4y} + g(y)$ and $f_y = 4x^2e^{4y} + g_y = 4e^{4y}x^2 \Rightarrow g(y) = 0$ (say). Potential is $f = x^2e^{4y}$. Integral is computed as

$$\int_{(4,0)}^{(6,1)} e^{4y} (2x dx + 4x^2 dy) = f(B) - f(A) = [x^2e^{4y}]_{x=4,y=0} - [x^2e^{4y}]_{x=4,y=0} = 1949.5 \quad \blacksquare$$

Question: (10ed-10.2-9) Show that the form under the integral $\int_{(0,1,0)}^{(1,0,1)} (e^x \cosh y dx + (e^x \sinh y + e^z \cosh y) dy + e^z \sinh y dz)$ is exact in the space and evaluate the integral. Show the details of your work?

Solution: Here we have $F_1 = e^x \cosh y$, $F_2 = e^x \sinh y + e^z \cosh y$ and $F_3 = e^z \sinh y$. We have to show $\frac{\partial F_3}{\partial x} = \frac{\partial}{\partial x}(e^z \sinh y) = 0 = \frac{\partial}{\partial z}(e^x \cosh y) = \frac{\partial F_1}{\partial z}$
 $\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}(e^z \sinh y) = e^z \cosh y = \frac{\partial}{\partial z}(e^x \sinh y + e^z \cosh y) = \frac{\partial F_2}{\partial z}$ and
 $\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(e^x \sinh y + e^z \cosh y) = e^x \sinh y = \frac{\partial}{\partial y}(e^x \cosh y) = \frac{\partial F_1}{\partial y}$. Hence the integral is path independent. For finding the potential we set $f_x = e^x \cosh y$, $f_y = e^x \sinh y + e^z \cosh y$ and $f_z = e^z \sinh y$. A mere close examination of the terms indicates $f(x, y, z) = e^x \cosh y + e^z \sinh y$. The integral now may be evaluated as

$$\begin{aligned} & \int_{(0,1,0)}^{(1,0,1)} (e^x \cosh y dx + (e^x \sinh y + e^z \cosh y) dy + e^z \sinh y dz) \\ &= f(B) - f(A) = [e^x \cosh y + e^z \sinh y]_{x=1,y=0,z=1} - [e^x \cosh y + e^z \sinh y]_{x=1,y=0,z=1} \\ &= e - (\cosh 1 + \sinh 1) = -0.0001 = -\frac{1}{10000} \quad \blacksquare \end{aligned}$$

Question: (10ed-10.2-13) Check path independence of $2e^{x^2} (x \cos 2y dx - \sin 2y dy)$ and if independent, integrate from $(0, 0, 0)$ to (a, b, c) ?

Solution: We have $F_1 = 2e^{x^2} x \cos 2y$, $F_2 = -2e^{x^2} \sin 2y$ and $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(2e^{x^2} x \cos 2y) = -4x(\sin 2y)e^{x^2} = \frac{\partial}{\partial x}(-2e^{x^2} \sin 2y) = \frac{\partial F_2}{\partial x} \Rightarrow$ Exact i.e. path independence holds. So we find potential as $f(x, y) = (\cos 2y)e^{x^2}$ by guess work! Integral is given as

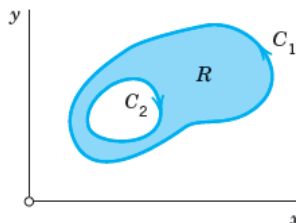
$$\begin{aligned} \int_{(0,0,0)}^{(a,b,c)} (2e^{x^2} x \cos 2y dx - 2e^{x^2} \sin 2y dy) &= f(a, b, c) - f(0, 0, 0) \\ &= [(\cos 2y)e^{x^2}]_{x=a,y=b,z=c} - [(\cos 2y)e^{x^2}]_{x=0,y=0,z=0} \\ &= e^{a^2} \cos(2b) - 1 \quad \blacksquare \end{aligned}$$

3 Green's Theorem in the Plane

Theorem 8 (Green's Theorem in the Plane) Let R be a closed bounded region in the xy -plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R . Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \oint_C (F_1 dx + F_2 dy)$$

Remark 9 1. In Green's Theorem, we integrate along the entire boundary C of R in such sense that R is on the left as we advance in the direction of integration.



2. Green's Theorem facilitates a transformation between double integrals and line integrals. Thus it is a great computation saver.

3. Two vectorial forms of Green's Theorem are obtained as

$$\iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dxdy = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

and

$$\iint_R \text{div } \mathbf{F} \, dxdy = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds$$

where \mathbf{k} is the unit vector in z -direction and \mathbf{n} is outer unit normal vector of C .

Question: Verify Green's Theorem for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C the circle $x^2 + y^2 = 1$?

Solution: From the theorem we have $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \iint_R ((2y + 2) - (2y - 7)) dxdy = 9 \iint_R dxdy = 9 \left(\pi (1)^2 \right) = 9\pi$, using the fact that $\iint_R dxdy$ is the area of a unit disk whose boundary is C .

For evaluating the RHS of theorem, we need to orient C counterclockwise i.e. $\mathbf{r}(t) = [\cos t, \sin t]$. Then $\mathbf{r}'(t) = [-\sin t, \cos t]$ and on C , $F_1 = y^2 - 7y = \sin^2 t - 7 \sin t$, $F_2 = 2xy + 2x = 2 \cos t \sin t + 2 \cos t$. Hence

$$\begin{aligned} \oint_C (F_1 dx + F_2 dy) &= \oint_C (F_1 x' + F_2 y') dt \\ &= \int_0^{2\pi} [(\sin^2 t - 7 \sin t)(-\sin t) + (2 \cos t \sin t + 2 \cos t)(\cos t)] dt \\ &= \int_0^{2\pi} [7 \sin^2 t - \sin^3 t + 2 \cos^2 t \sin t + 2 \cos^2 t] dt \\ &= 7 \int_0^{2\pi} \sin^2 t dt - \int_0^{2\pi} \sin^3 t dt + 2 \int_0^{2\pi} \cos^2 t \sin t dt + 2 \int_0^{2\pi} \cos^2 t dt \\ &= 7\pi - 0 + 0 + 2\pi \\ &= 9\pi \quad \blacksquare \end{aligned}$$

Definition 10 For a continuous function $w(x, y)$ with continuous first and second partial derivatives, $\frac{\partial w}{\partial n} = (\text{grad } w) \cdot \mathbf{n}$ is called the normal derivative of $w(x, y)$.

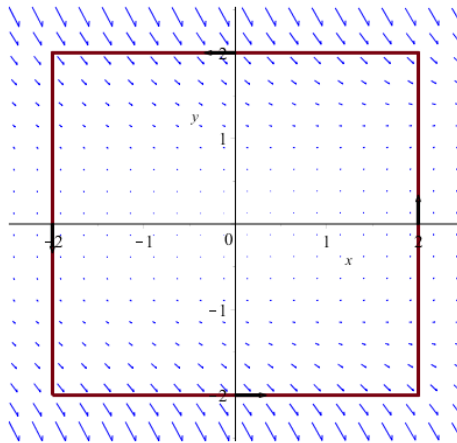
Note that $(\text{grad } w) \cdot \mathbf{n} = \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right] \cdot \left[\frac{dy}{ds}, -\frac{dx}{ds} \right] = \frac{\partial w}{\partial x} \frac{dy}{ds} - \frac{\partial w}{\partial y} \frac{dx}{ds} = F_2 \frac{dy}{ds} + F_1 \frac{dx}{ds} = F_1 dx + F_2 dy$. Hence now Green's Theorem may be re-stated:

$$\iint_R \nabla^2 w \, dxdy = \oint_C \frac{\partial w}{\partial n} ds$$

Question: (10ed-10.4-2) Evaluate $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's Theorem, where $\mathbf{F} = [6y^2, 2x - 2y^4]$ and R : the square with vertices $\pm(2, 2), \pm(2, -2)$.

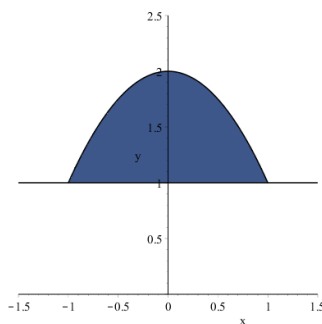
Solution:

$$\begin{aligned} \oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \oint_C [6y^2, 2x - 2y^4] \cdot d\mathbf{r} = \int_{-2}^2 \int_{-2}^2 \left(\frac{\partial}{\partial x} (2x - 2y^4) - \frac{\partial}{\partial y} (6y^2) \right) dxdy \\ &= \int_{-2}^2 \int_{-2}^2 (2 - 12y) dxdy = \int_{-2}^2 (8 - 48y) dy = 32 \end{aligned}$$



Question: (10ed-10.4-7) Evaluate $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's Theorem, where $\mathbf{F} = \text{grad}(x^3 \cos^2(xy))$ and $R: 1 \leq y \leq 2 - x^2$?

Solution: We have the region of integration as



$$\begin{aligned}\mathbf{F} &= \text{grad}(x^3 \cos^2(xy)) = [3x^2 \cos^2(xy) - 2x^3 y \cos xy \sin xy, -2x^4 \cos xy \sin xy] \\ F_1 &= 3x^2 \cos^2(xy) - 2x^3 y \cos xy \sin xy, \quad F_2 = -2x^4 \cos xy \sin xy\end{aligned}$$

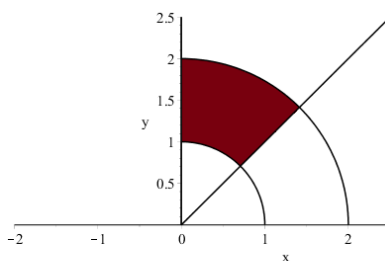
$\oint_C \text{grad}(x^3 \cos^2(xy)) \cdot d\mathbf{r} = \oint_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$ by Green's Theorem.

$$\begin{aligned}&= \iint_R \left(\frac{\partial}{\partial x} (-2x^4 \cos xy \sin xy) - \frac{\partial}{\partial y} (3x^2 \cos^2 xy - 2x^3 y \cos xy \sin xy) \right) dx dy \\&= \iint_R ((-4x^3 \sin 2xy - 2x^4 y \cos 2xy) - (-4x^3 \sin 2xy - 2x^4 y \cos 2xy)) dx dy \\&= \iint_R (0) dx dy = 0\end{aligned}$$

Alternately as \mathbf{F} is gradient of the potential $f(x, y) = x^3 \cos^2(xy)$ and $\frac{\partial F_1}{\partial y} = -4x^3 \sin 2xy - 2x^4 y \cos 2xy = \frac{\partial F_2}{\partial x}$ Hence the differential form under this integral is exact. ■

Question: (10ed-10.4-10) Evaluate $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's Theorem, where $\mathbf{F} = [x^2 y^2, -\frac{x}{y^2}]$ and $R: 1 \leq x^2 + y^2 \leq 4, x \geq 0, y \geq x$. Sketch R .

Solution: R is sketched as:

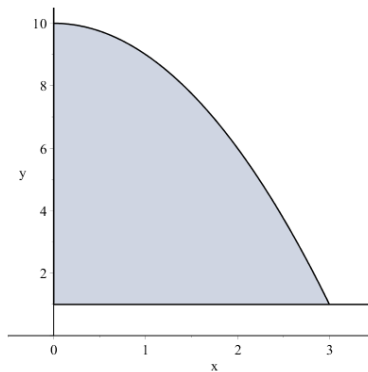


This is a portion of a circular ring (annulus) bounded by the circles of radii 1 and 2 centered at the origin, in the first quadrant bounded by $y = x$ and the y -axis. The integrand is $\frac{-1}{y^2} - 2x^2 y$. We use polar coordinates, obtaining

$$\begin{aligned}&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_1^2 \left(\frac{-1}{r^2 \sin^2 \theta} - 2r^3 \cos^2 \theta \sin \theta \right) r dr d\theta \\&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[-\frac{1}{40 \sin^2 \theta} (40 \ln r + r^5 \sin 3\theta - r^5 \sin 5\theta + 2r^5 \sin \theta) \right]_1^2 d\theta \\&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(-\frac{1}{40 \sin^2 \theta} (31 \sin 3\theta - 31 \sin 5\theta + 40 \ln 2 + 62 \sin \theta) \right) d\theta \\&= -\ln 2 - \frac{31}{30} \sqrt{2} = -2.155 \quad \blacksquare\end{aligned}$$

Question: (10ed-10.4-15)

Solution: The region is given as



Question: (10ed-10.4-16) Using the form $\iint_R \nabla^2 w \, dx dy = \oint_C \frac{\partial w}{\partial n} ds$, of Green's Theorem, find the value of $\oint_C \frac{\partial w}{\partial n} ds$ taken counterclockwise over the boundary $C : x^2 + y^2 = 4$ of the region R where $w = x^2 + y^2$. Confirm the answer by direct integration.

Solution: $\nabla^2 w = \nabla^2 (x^2 + y^2) = 4$, Answer: 8π

Confirmation: $\mathbf{r} = [2 \cos s, 2 \sin s]$, $\mathbf{r}' = \left[\frac{d}{ds} (2 \cos s), \frac{d}{ds} (2 \sin s) \right] = [-2 \sin s, 2 \cos s]$. Outer normal vector $\mathbf{n} = [2 \cos s, 2 \sin s]$

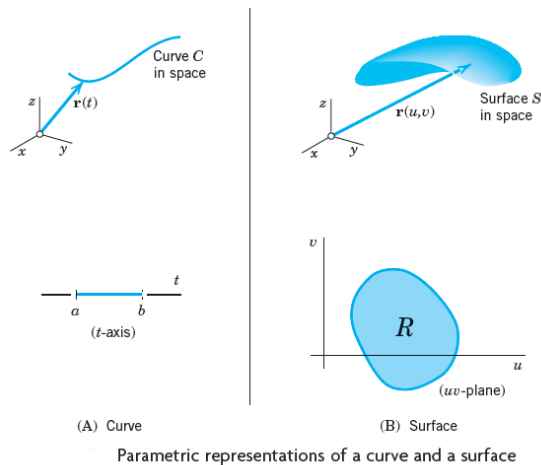
$\text{grad } w = \text{grad } (x^2 + y^2) = [2x, 2y] = [4 \cos s, 4 \sin s]$

$\text{grad } (w) \cdot \mathbf{n} = [4 \cos s, 4 \sin s] \cdot [-2 \sin s, 2 \cos s] = 8 \sin s \cos s - 8 \cos s \sin s = 0$

$16 \int_0^{2\pi} \sin(s) \cos(s) ds \quad ??$

4 Surfaces for Surface Integrals

With line integrals, we integrate over curves in space, with surface integrals we integrate over surfaces in space. Each curve in space is represented by a parametric equation. This suggests that we should also find parametric representations for the surfaces in space. As surfaces are two dimensional objects, we need two parameters which we call u and v .

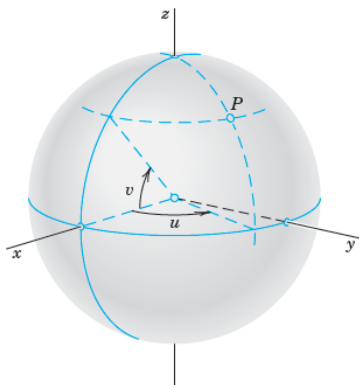
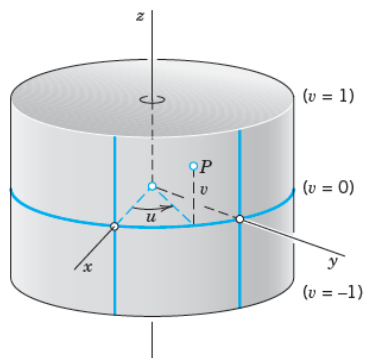


Thus **parametric representation** of a surface S in space is of the form

$$\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)] = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies in some region R of the uv -plane.

Example 11 *Following are the parametric representations of a cylinder, sphere and cone respectively:*



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