Advanced Engineering Mathematics

Mathematical Techniques for Engineering

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Definition: Heaviside Unit Step Function

The unit step function or Heaviside function $u(t-a)$ is defined as:

$$u(t-a) = \begin{cases} 0 & if \quad t < a \\ 1 & if \quad t > a \end{cases} \qquad ; \quad a \geq 0$$

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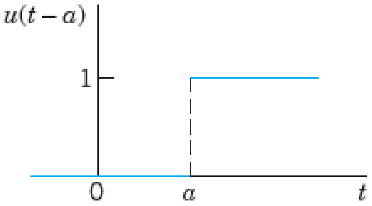
Its graph is given as



Heaviside Unit Step Function: Explained

Let $f(t) = 0$ for all negative t. Then $f(t-a)\,u(t-a)$ with $a > 0$ is $f(t)$ shifted (translated) to the right by the amount a.

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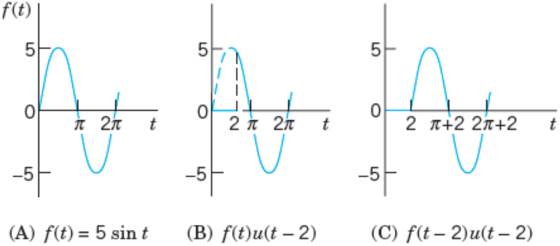
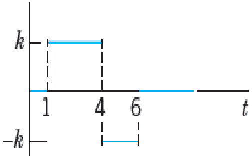
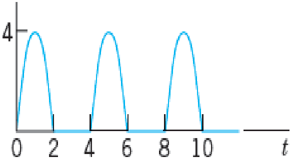


Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.

Heaviside Unit Step Function: Explained



(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$



(B) $4 \sin\left(\frac{1}{2}\pi t\right)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Use of many unit step functions.

Laplace Transform of Unit Step Function

LT of Heaviside unit step function may be found as

$$\begin{aligned} L\left(u\left(t-a\right)\right) &= \int_0^\infty e^{-st} u\left(t-a\right) dt \\ &= \int_0^\infty e^{-st} \left(1\right) dt \qquad \text{since } a \geq 0 \\ &= \left[\frac{e^{-st}}{s}\right]_{t=a}^{\infty} \\ &= \frac{e^{-as}}{s} \qquad \blacksquare \end{aligned}$$

Second Shifting Theorem; t-Shifting

Theorem: (**Second Shifting Theorem; t-Shifting**) If $f(t)$ has the transform $F(s)$, then the 'shifted function'

$$\widetilde{f}(t) = f(t-a)\,u(t-a)$$

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Remark: Practically the t-shifting theorem amounts to this: if we know $F(s)$, we can obtain the transform of $f(t-a)\,u(t-a)$ by just multiplying $F(s)$ by e^{-as}.

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Remark: Practically the t-shifting theorem amounts to this: if we know $F(s)$, we can obtain the transform of $f(t-a)\,u(t-a)$ by just multiplying $F(s)$ by e^{-as}.
If the conversion of $f(t)$ to $f(t-a)$ is difficult, we may use following form as well:

$$L(f(t)\,u(t-a)) = e^{-as}L(f(t+a))$$

Question: Find LT of the piecewise function

$$f(t) = \begin{cases} 2 & if \quad 0 < t < 1 \\ \frac{t^2}{2} & if \quad 1 < t < \frac{\pi}{2} \\ \cos t & if \quad t > \frac{\pi}{2} \end{cases}$$

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Applying LT on both sides

$$L(f(t)) = 2L(u(t)) - 2L(u(t-1)) + \tfrac{1}{2}L\left(t^2 u(t-1)\right) - \tfrac{1}{2}L\left(t^2 u\left(t - \tfrac{\pi}{2}\right)\right) + L\left(\cos(t)\, u\left(t - \tfrac{\pi}{2}\right)\right)$$

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Writing each term in $f(t)$ in the form $f(t-a)$, so that the LT of the form $f(t-a)\, u(t-a)$, of t-shifting theorem, may be applied.

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for $\frac{1}{2}t^2u\left(t-\frac{\pi}{2}\right)$, $\frac{1}{2}t^2 = \frac{1}{2}\left(t-\frac{\pi}{2}\right)^2 + \frac{1}{2}\pi\left(t-\frac{\pi}{2}\right) + \frac{1}{8}\pi^2$

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and for $\cos\left(t\right)u\left(t-\frac{\pi}{2}\right),\quad \cos\left(t\right)=-\sin\left(t-\frac{1}{2}\pi\right)$

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$$L\left(f\left(t\right)\right) =$$
$$\frac{2}{s} - \frac{2}{s}e^{-s} + \frac{1}{2}L\left(t^2\right)e^{-s} - \frac{1}{2}L\left(t^2\right)e^{-\frac{\pi}{2}s} + L\left(\cos\left(t\right)\right)e^{-\frac{\pi}{2}s}$$

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$$L\left(f\left(t\right)\right)=\frac{2}{s}-\frac{2}{s}e^{-s}+\frac{1}{2}\frac{\left(s^2+2s+2\right)}{s^3}e^{-s}-\frac{1}{8}\left(\frac{\pi^2}{s}+\frac{4\pi}{s^2}+\frac{8}{s^3}\right)e^{-\frac{\pi}{2}s}-\frac{1}{s^2+1}e^{-\frac{\pi}{2}s} \quad \blacksquare$$

Question: (10ed-6.3-10)

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$$L(f(t)) = L(\sinh(t)\ \ u(t)) - \cosh(2)\ \ L(\sinh(t)\ \ u(t-2)) + \sinh(2)\ \ L(\cosh(t)\ \ u(t-2))$$

$$= \frac{1}{s^2-1} - \cosh(2)\left(\frac{1}{s^2-1}\right)e^{-2s} + \sinh(2)\left(\frac{s}{s^2-1}\right)e^{-2s}$$

$$= \frac{1}{s^2-1} - \left(\frac{\sinh(2)s+\cosh(2)}{s^2-1}\right)e^{-2s} \qquad \blacksquare$$

Question: (10ed-6.3-14)

$$
\begin{aligned}
L^{-1}\left(\frac{4\left(e^{-2s}-2e^{-5s}\right)}{s}\right) &= L^{-1}\left(\frac{1}{s}\left(4e^{-2s}-8e^{-5s}\right)\right) \\
&= 4L^{-1}\left(\frac{1}{s}e^{-2s}\right)-8L^{-1}\left(\frac{1}{s}e^{-5s}\right)
\end{aligned}
$$

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Though not required, but this would be beneficial if student transforms above $f(t)$ into a piecewise representation.

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f\left(t\right) &= 4\ u\left(t-2\right)-8\ u\left(t-5\right) \qquad \blacksquare
\end{aligned}$$

Though not required, but this would be beneficial if student transforms above $f\left(t\right)$ into a piecewise representation. For this $f\left(t\right)$ it is given as

$$f\left(t\right)=4\ u\left(t-2\right)-8\ u\left(t-5\right)=\begin{cases} 4 & if \quad 2<t<5 \\ -4 & if \quad t>5 \end{cases}$$

Question: (DIY, hints are given) Writing the Heaviside form of

$$f(t) = \begin{cases} \frac{1}{2}t & if \quad 0 < t < \frac{1}{2} \\ 3t - 2 & if \quad \frac{1}{2} < t < \frac{\pi}{2} \\ e^t & if \quad t > \frac{\pi}{2} \end{cases}$$

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$$f\left(t\right)=\underbrace{\frac{1}{2}t\ u\left(t\right)+\frac{5}{2}t\ u\left(t-\frac{1}{2}\right)}_{First\ \ piece}$$

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$$f(t) = \underbrace{\frac{1}{2}t\ u(t) + \frac{5}{2}t\ u\left(t - \frac{1}{2}\right)}_{First\ \ piece} -$$

$$\underbrace{3t\ u\left(t - \frac{\pi}{2}\right) + 2\,u\left(t - \frac{\pi}{2}\right) - 2\,u\left(t - \frac{1}{2}\right)}_{Second\ \ piece}$$

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gives

$$f(t) = \underbrace{\frac{1}{2}t\ u(t) + \frac{5}{2}t\ u\left(t - \frac{1}{2}\right)}_{First\ \ piece} -$$

$$\underbrace{3t\ u\left(t - \frac{\pi}{2}\right) + 2\,u\left(t - \frac{\pi}{2}\right) - 2\,u\left(t - \frac{1}{2}\right)}_{Second\ \ piece} + \underbrace{e^t\ u\left(t - \frac{\pi}{2}\right)}_{Third\ \ piece}$$

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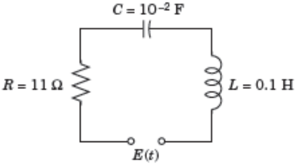
Applying t-shifting theorem to obtain the LT as

$L(f(t)) = \frac{1}{2}\frac{1}{s^2} + \frac{5}{4}\frac{(s+2)}{s^2}e^{\frac{-1}{2}s} - \frac{3}{2}\left(\frac{2}{s^2} + \frac{\pi}{s}\right)e^{-\frac{\pi}{2}s} + \frac{2}{s}e^{-\frac{\pi}{2}s} -$
$\frac{2}{s}e^{-\frac{1}{2}s} + \frac{e^{-\frac{\pi}{2}(s-1)}}{s-1}$ ∎

Question: Find the response (the current) of the RLC-circuit given in figure, where $E(t)$ is sinuosoidal,



acting for a short time interval only, say $E(t) = 100 \sin(400t)$ if $0 < t < 2\pi$ and $E(t) = 0$ if $t > 2\pi$.

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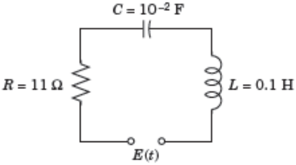

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Solution: The forcing function

$E(t) = \begin{cases} 100\sin(400t) & \text{if} \quad 0 < t < 2\pi \\ 0 & \text{if} \quad t > 2\pi \end{cases}$ may be written in terms of unit-step function as

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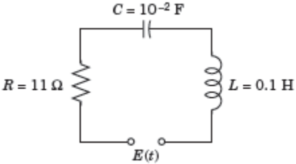

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$$E(t) = -100\sin(400\,t)\;u(t - 2\,\pi) + 100\sin(400\,t)\;u(t).$$

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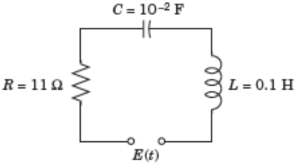

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Model for current $i(t)$ is the integro-differential equation

$0.1\frac{d}{dt}i(t) + 11\,i(t) + 100\int_0^t i(\tau)\,d\tau = -100\sin(400t)\; u(t-2\pi) + 100\sin(400t)\; u(t) \quad ; \; i(0) = i'(0) = 0$

$$0.1sY - 0.1i\left(0\right) + 11Y + 100\tfrac{1}{s}Y = \tfrac{40000}{\left(s^2+160000\right)}\left(1 - e^{-2\pi s}\right)$$

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$$Y = -\frac{\left(-i(0)s^2+400000\ \ e^{-2\pi\, s}-160000\ \ i(0)-400000\right)s}{\left(s^2+160000\right)\left(s^2+110\, s+1000\right)}$$

$$Y = -\frac{\left(-400000+400000\, e^{-2\pi\, s}\right)s}{\left(s^2+160000\right)\left(s^2+110\, s+1000\right)} = \frac{400000\ \ s-400000\, se^{-2\pi\, s}}{\left(s^2+160000\right)\left(s^2+110\, s+1000\right)}$$

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By partial fractions of first term, $Y =$

$$\left(\frac{400}{153(s+100)} - \frac{4000}{14\,409(s+10)} - \frac{\frac{63\,600}{27\,217}s-\frac{7040\,000}{27\,217}}{s^2+160\,000}\right) - \left(\frac{400000\,s\ \ e^{-2\pi\,s}}{(s^2+160000)(s^2+110\,s+1000)}\right)$$

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takinging Inverse LT

$$i(t) = \left(\frac{400}{153}e^{-100t} - \frac{4000}{14\,409}e^{-10t} - \left(\frac{63\,600}{27\,217}\cos 400t - \frac{17\,600}{27\,217}\sin 400t\right)\right)$$
$$-u(t-2\pi)$$
$$\left(\frac{400}{153}e^{200\pi-100t} - \frac{4000}{14\,409}e^{20\pi-10t} - \frac{63\,600}{27\,217}\cos 400t + \frac{17\,600}{27\,217}\sin 400t\right) \quad \blacksquare$$

Question: (10ed-6.3-23)

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = f(t) = \begin{cases} 3\sin(t) - \cos(t) & \text{if} \quad 0 < t < 2\pi \\ 3\sin(2t) - \cos(2t) & \text{if} \quad t > 2\pi \end{cases} \quad \text{where}$$

$$y(0) = 1, \; y'(0) = 0$$

Solution:

Question: (10ed-6.3-23)

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Solution: Transforming the forcing function into Heaviside form:

$$f(t) =$$
$$-3\sin(t)\,u(t-2\pi) + 3\sin(t)\,u(t) + \cos(t)\,u(t-2\pi)$$

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Taking LT on both sides of the given DE

$s^2Y - y'(0) - sy(0) + sY - y(0) - 2Y =$

$\frac{1}{s^2+1}\left(3 - s + 3\frac{e^{-2s\pi}(s+2)(s-1)}{s^2+4}\right)$

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$$f(t)=$$
$$-3\sin(t)\,u(t-2\pi)+3\sin(t)\,u(t)+\cos(t)\,u(t-2\pi)-\cos(t)\,u(t)+3u(t-2\pi)\sin(2t)-u(t-2\pi)\cos(2t)$$

Taking LT on both sides of the given DE

$$s^2Y-y'(0)-sy(0)+sY-y(0)-2Y=$$

$$\frac{1}{s^2+1}\left(3-s+3\frac{e^{-2s\pi}(s+2)(s-1)}{s^2+4}\right)$$

$$Y=$$
$$\frac{1}{s^2+s-2}\left(\frac{1}{s^2+1}\left(3-s+3\frac{e^{-2s\pi}(s+2)(s-1)}{s^2+4}\right)+y'(0)+sy(0)+y(0)\right)$$

$$Y = \frac{1}{s^2+s-2}\left(\frac{1}{s^2+1}\left(3 - s + 3\frac{e^{-2s\pi}(s+2)(s-1)}{s^2+4}\right) + 1 + s\right)$$

$$Y = \frac{s^4-s^3+3e^{-2s\pi}s+6s^2-3e^{-2s\pi}-4s+8}{(s-1)(s^2+1)(s^2+4)}$$

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$$L\left(y\left(t\right)\right) = Y = \frac{s^4-s^3+6s^2+\left(3e^{-2s\pi}-4\right)s-3e^{-2s\pi}+8}{(s-1)(s^2+1)(s^2+4)}$$

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$$y(t) = e^t - \sin(t) + \frac{1}{2}(2\sin(t) - \sin(2t)) \ \ u(t-2\pi)$$

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$$y(t) = \Bigg\{$$

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$$y(t) = \begin{cases} e^t - \sin(t) & \text{if} \quad 0 < t < 2\pi \\ e^t - \frac{1}{2}\sin(2t) & \text{if} \quad t > 2\pi \end{cases} \qquad \blacksquare$$

Question: (10e-6.3-27) Put $\widetilde{t} = t - 1$ in $E(t) = \begin{cases} 8t^2 & 0 < t < 5 \\ 0 & t > 5 \end{cases}$ and writing it in Heaviside terms:

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$E\left(\widetilde{t}\right)=8\,u\left(\widetilde{t}+1\right)\widetilde{t}^{2}-8\,u\left(\widetilde{t}-4\right)\widetilde{t}^{2}+16\,u\left(\widetilde{t}+1\right)\widetilde{t}-16\,u\left(\widetilde{t}-4\right)\widetilde{t}+8\,u\left(\widetilde{t}+1\right)-8\,u\left(\widetilde{t}-4\right)$

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$$E(\widetilde{t}) = 8\,u(\widetilde{t}+1)\,\widetilde{t}^2 - 8\,u(\widetilde{t}-4)\,\widetilde{t}^2 + 16\,u(\widetilde{t}+1)\,\widetilde{t} - 16\,u(\widetilde{t}-4)\,\widetilde{t} + 8\,u(\widetilde{t}+1) - 8\,u(\widetilde{t}-4)$$

$$\frac{d^2}{d\widetilde{t}^2}y(\widetilde{t}) + 4\,y(\widetilde{t}) = E(\widetilde{t}) \quad ; \quad y(0) = 1 + \cos(2), \quad y'(0) = 4 - 2\sin(2)$$

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$$\frac{d^2}{d\widetilde{t}^2} y(\widetilde{t}) + 4\,y(\widetilde{t}) = E(\widetilde{t}) \quad ; \quad y(0) = 1 + \cos(2)\,, \quad y'(0) = 4 - 2\sin(2)$$

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$$s^2\,\widetilde{Y} - y'(0) - sy(0) + 4\,\widetilde{Y} = 16\,\frac{1}{s^2} + 8\,\frac{1}{s} + 8\,\frac{-e^{-4\,s}\left(25\,s^2+10\,s+2\right)+2}{s^3}$$

$$s^2\,\widetilde{Y} - 4 + 2\,\sin(2) - s\,(1+\cos(2)) + 4\,\widetilde{Y} =$$
$$16\,\frac{1}{s^2} + 8\,\frac{1}{s} + 8\,\frac{-e^{-4\,s}\left(25\,s^2+10\,s+2\right)+2}{s^3}$$ using ICs

$$\widetilde{Y} =$$
$$\frac{1}{s^2+4}\left(\frac{16}{s^2} + \frac{8}{s} + 8\,\frac{-e^{-4\,s}\left(25\,s^2+10\,s+2\right)+2}{s^3} + 4 - 2\sin(2) + s\,(1+\cos(2))\right)$$

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$$\widetilde{Y} =$$
$$-\frac{-s^4\cos(2)+2\,\sin(2)s^3-s^4+200\,e^{-4\,s}s^2-4\,s^3+80\,e^{-4\,s}s-8\,s^2+16\,e^{-4\,s}-16\,s-16}{s^3\left(s^2+4\right)}$$

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$$y\left(\widetilde{t}\right) =$$
$$1 + \cos\left(2\right)\cos\left(2\,\widetilde{t}\right) - \sin\left(2\right)\sin\left(2\,\widetilde{t}\right) + 2\,\widetilde{t}^2 + 4\,\widetilde{t} - u\left(\widetilde{t} - 4\right)$$
$$\left(100\,\left(\sin\left(\widetilde{t} - 4\right)\right)^2 + 2\,\widetilde{t}^2 + \cos\left(2\,\widetilde{t} - 8\right) - 10\,\sin\left(2\,\widetilde{t} - 8\right) + 4\,\widetilde{t} - 49\right)$$

$$y\left(\tilde{t}\right) =$$
$$\begin{cases} 1+\cos(2)\cos\left(2\,\tilde{t}\right)-\sin(2)\sin\left(2\,\tilde{t}\right)+2\,\tilde{t}^2+4\,\tilde{t} & \tilde{t}<4 \\ \left.\begin{array}{l} -50+\cos(2)\cos\left(2\,\tilde{t}\right)-\sin(2)\sin\left(2\,\tilde{t}\right)+ \\ \quad 100\,\left(\cos\left(\tilde{t}-4\right)\right)^2-\cos\left(2\,\tilde{t}-8\right)+10\,\sin\left(2\,\tilde{t}-8\right) \end{array}\right\} & \tilde{t}>4 \end{cases}$$

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putting back $\widetilde{t}=t-1,\ y(t) =$

$$\begin{cases} \left.\begin{array}{l} -3+\cos(2)\cos(2\,t-2)-\sin(2)\sin(2\,t-2)+ \\ 2\,(t-1)^2+4\,t \end{array}\right\} & t<5 \\ \left.\begin{array}{l} -50+\cos(2)\cos(2\,t-2)-\sin(2)\sin(2\,t-2)+ \\ \quad 100\,(\cos(t-5))^2-\cos(2\,t-10)+10\sin(2\,t-10) \end{array}\right\} & t>5 \end{cases}$$

$$y(t) =$$
$$\begin{cases} \cos(2t)+2t^2-1 & \textit{if} \quad t<5 \\ \cos(2t)+49\cos(2t-10)+10\sin(2t-10) & \textit{if} \quad t>5 \end{cases} \qquad \blacksquare$$

Question: (10ed-6.3-30) $0.5\frac{d}{dt}i(t)+10i(t)=E(t)=$

$$\begin{cases} 200t & , \quad 0<t<2 \\ 0 & , \quad t>2 \end{cases} \quad ; \quad i(0)=0$$

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$$Y = -400\frac{2e^{-2s}s + e^{-2s} - 1}{s^2(s+20)}$$

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$$0.5sY+10Y=200\left(1-e^{-2s}(2s+1)\right)s^{-2}$$

$$Y=-400\frac{2e^{-2s}s+e^{-2s}-1}{s^2(s+20)}$$

$$i(t)=-1+20t\ u(2-t)+e^{-20t}+$$
$$2\ u(t-2)\left(e^{(20-10t)}\sinh(10t-20)+20\ e^{(-20t+40)}\right)$$

$$i(t)=$$
$$\begin{cases}-1+e^{-20t}+20t & , \quad t<2\\ -1+e^{-20t}+2\ e^{(20-10t)}\sinh(10t-20)+40\ e^{(-20t+40)} & , \quad t>2\end{cases}$$

Question: (10ed-6.3-40)

$$\frac{d}{dt}i(t) + 2i(t) + 10\int_0^t i(\tau)\,d\tau = E(t) =$$

$$\begin{cases} 255\sin(t) & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases} \quad ; \quad i(0) = 0$$

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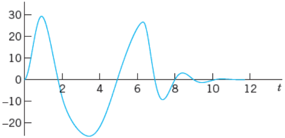

$$\begin{cases} 255\sin(t) & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases} \quad ; \quad i(0) = 0$$

$$E(t) = 255\sin(t)\ u(t) - 255\sin(t)\ u(-2\pi + t)$$

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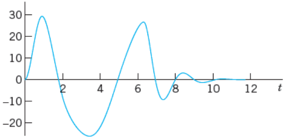

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$$255\sin(t)\ u(t) - 255\sin(t)\ u(-2\pi + t)$$

$$s\ Y - i(0) + 2Y + 10\frac{Y}{s} = 255\frac{1-e^{-2s\pi}}{s^2+1}$$

$$Y = 1\left(255\frac{1-e^{-2s\pi}}{s^2+1} + i(0)\right)\left(s + 2 + 10s^{-1}\right)^{-1}$$

$$Y = 255\frac{1-e^{-2s\pi}}{(s^2+1)\left(s+2+\frac{10}{s}\right)}$$

$$Y = -255\frac{\left(-1+e^{-2s\pi}\right)s}{(s^2+1)(s^2+2s+10)}$$

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$$i\,(t) = 3\;u\,(2\pi - t)\,(9\cos(t) + 2\sin(t)) +$$

$$\left(e^{2\pi-t}\;u\,(-2\pi+t) - e^{-t}\right)(27\cos(3t) + 11\sin(3t))$$

$$i\,(t) =$$

$$\begin{cases} 27\cos(t) + 6\sin(t) - (27\cos(3t) + 11\sin(3t))\,e^{-t} & t < 2\pi \\ (27\cos(3t) + 11\sin(3t))\left(-e^{-t} + e^{(2\pi-t)}\right) & t > 2\pi \end{cases}$$

$$Y = 255\frac{1-e^{-2s\pi}}{(s^2+1)\left(s+2+\frac{10}{s}\right)}$$

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On plotting the current $i\,(t)$ we have