Advanced Engineering Mathematics

Mathematical Techniques for Engineering

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Differentiation and Integration of LT

Theorem

If $F(s)$ *is the Laplace transform of* $f(t)$, *then*

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt$$

from which, also note that

$$L(tf(t)) = -F'(s) \quad \text{and} \quad L^{-1}(F'(s)) = -tf(t)$$

Corollary

By mathematical induction we also have

$$L(t^n f(t)) = (-1)^n F^{(n)}(s)$$

Example

$L^{-1}\left(\ln\left(1+\frac{\omega^2}{s^2}\right)\right)=?$

We observe that $\ln\left(1+\frac{\omega^2}{s^2}\right)=\ln\left(\frac{s^2+\omega^2}{s^2}\right)=\ln\left(s^2+\omega^2\right)-\ln s^2$. This indicates that a derivative of $\ln$ may bring terms like $\frac{s}{s^2+\omega^2}$ or $\frac{1}{s}$. Hence we set $F(s)=\ln\left(\frac{s^2+\omega^2}{s^2}\right)$ and proceed for an application of differentiation of LT theorem i.e.

$$
\begin{aligned}
F'(s) &= \frac{d}{ds}\ln\left(s^2+\omega^2\right)-\frac{d}{ds}\ln s^2 = 2\frac{s}{s^2+\omega^2}-\frac{2}{s} \\
L^{-1}\left(F'(s)\right) &= L^{-1}\left(2\frac{s}{s^2+\omega^2}-\frac{2}{s}\right)=2\cos(\omega t)-2=2\,(\\
\text{Since}\quad L^{-1}\left(F'(s)\right) &= -tf(t) \\
&\Rightarrow -tf(t)=2\left(\cos(\omega t)-1\right) \\
f(t) &= \frac{2}{t}\left(1-\cos(\omega t)\right) \qquad \blacksquare
\end{aligned}
$$

Example

(10e-6.6-2) $L(3t\sinh(4t)) = ?$

$$
\begin{aligned}
\text{We know } L(tf(t)) &= -F'(s) \\
L(3t\sinh(4t)) &= 3L(t\sinh(4t)) \\
&= 3\left(-\frac{d}{ds}L(\sinh(4t))\right) \\
&= -3\left(\frac{d}{ds}\left(\frac{4}{s^2-16}\right)\right) \\
&= 24\frac{s}{(s^2-16)^2} \qquad \blacksquare
\end{aligned}
$$

Example

For finding $L\left(t^n e^{kt}\right)$, we note

$$L\left(e^{kt}\right) = \frac{1}{s-k} \quad , \quad (= F(s))$$

by corollary $L\left(t^n f(t)\right) = (-1)^n F^{(n)}(s)$, so we take n differentiation:

$$\frac{d}{ds}\left(\frac{1}{s-k}\right) = \frac{-1}{(s-k)^2}, \quad \frac{d^2}{ds^2}\left(\frac{1}{s-k}\right) = \frac{2}{(s-k)^3}$$

$$\frac{d^3}{ds^3}\left(\frac{1}{s-k}\right) = \frac{-6}{(s-k)^4}, \quad \text{hence}$$

$$\frac{d^n}{ds^n}\left(\frac{1}{s-k}\right) = \frac{(-1)^n n!}{(s-k)^{n+1}}$$

$$L\left(t^n e^{kt}\right) = \frac{(-1)^n n!}{(s-k)^{n+1}} \qquad \blacksquare$$

Example

(10e-6.7-6)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} y_1(t) &= 5\, y_1(t) + y_2(t) \\
\frac{\mathrm{d}}{\mathrm{d}t} y_2(t) &= y_1(t) + 5\, y_2(t) \\
y_1(0) &= 1, y_2(0) = -3
\end{aligned}
$$

Taking Laplace Transforms

$$
\begin{aligned}
s\; Y_1 - y_1(0) &= 5\, Y_1 + Y_2 \\
s Y_2 - y_2(0) &= Y_1 + 5\, Y_2
\end{aligned}
$$

Using ICs

$$
\begin{aligned}
s\; Y_1 - 1 &= 5\, Y_1 + Y_2 \\
s\; Y_2 + 3 &= Y_1 + 5\, Y_2
\end{aligned}
$$

Example

(10e-6.7-12)

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2} y_1(t) &= -2\, y_1(t) + 2\, y_2(t) \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2} y_2(t) &= 2\, y_1(t) - 5\, y_2(t) \\
y_1(0) &= 1, y_2(0) = 3, y_1'(0) = 0, y_2'(0) = 0
\end{aligned}$$

Taking Laplace Transforms

$$\begin{aligned}
s^2\, Y_1 - y_1'(0) - s y_1(0) &= -2\, Y_1 + 2\, Y_2 \\
s^2\, Y_2 - y_2'(0) - s y_2(0) &= 2\, Y_1 - 5\, Y_2
\end{aligned}$$

Using ICs

$$\begin{aligned}
s^2\, Y_1 - s &= -2\, Y_1 + 2\, Y_2 \\
s^2\, Y_2 - 3\, s &= 2\, Y_1 - 5\, Y_2
\end{aligned}$$