Classroom notes of Vector Differential Calculus

based on Chapter 10 of

Advanced Engineering Mathematics, E. Kreyszig, 10th Edition

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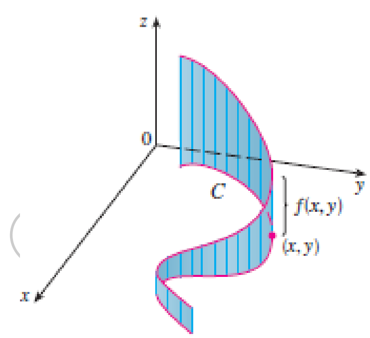
Contents

1 Line Integrals

Definition 1 *A line integral of a vector function* $\mathbf{F}(\mathbf{r})$ *over a curve* $C : \mathbf{r}(t)$ *is defined as* $I = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$.



Question: Find the value of line integral when $\mathbf{F}(\mathbf{r}) = [z, x, y]$ and $C = [\cos t, \sin t, 3t]\,;\ 0 \leq t \leq 2\pi$ is a helix.

Solution: Note that $\mathbf{F}(\mathbf{r})$ is not in terms of parameter t, hence from C we substitute $x = \cos t, y = \sin t$ and $z = 3t$ into it and get $\mathbf{F}(\mathbf{r}) = [3t, \cos t, \sin t]$. Then

$$\begin{aligned} I &= \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} \left([3t, \cos t, \sin t] \cdot \frac{d}{dt} [\cos t, \sin t, 3t] \right) dt \\ &= \int_0^{2\pi} ([3t, \cos t, \sin t] \cdot [-\sin t, \cos t, 3])\, dt \\ &= \int_0^{2\pi} \left(-3t \sin t + \cos^2 t + 3 \sin t \right) dt = 7\pi = 21.99 \end{aligned}$$

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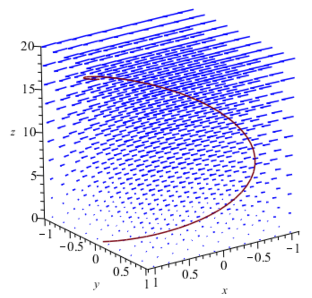
Sketch of the vector field and the curve is given below:



Vector field F(r) =[z,x,y] and curve r=[cost,sint,3 t]

Remark 2 1. *We have followig natural properties of a line integral:* $\int_C k\mathbf{F}\cdot d\mathbf{r} = k\int_C \mathbf{F}\cdot d\mathbf{r}$, $\int_C (\mathbf{F}+\mathbf{G})\cdot d\mathbf{r} = \int_{\mathbf{C}} \mathbf{F}\cdot d\mathbf{r} + \int_{\mathbf{C}} \mathbf{G}\cdot d\mathbf{r}$ *and* $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ *where* C_1 *and* C_2 *are subdividing curves of* C *with same orientation as that of* C.

2. *At times, without taking dot product we may obtain a line integral whose value is a vector rather than a scalar, as follows:*

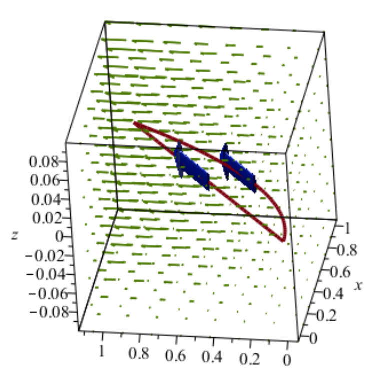
$$\int_C \mathbf{F}(\mathbf{r})\,dt = \int_a^b \mathbf{F}(\mathbf{r}(t))\,dt = \int_a^b [\mathbf{F}_1(\mathbf{r}(t)),\ \mathbf{F}_2(\mathbf{r}(t)),\ \mathbf{F}_3(\mathbf{r}(t))]\,dt$$

For the special case when $\mathbf{F}_1 = f$ *and* $\mathbf{F}_2 = \mathbf{F}_3 = 0$, *we have*

$$\int_C f(\mathbf{r})\,dt = \int_a^b f(\mathbf{r}(t))\,dt$$

3. *In general, a line integral's value would change by the change of curve* C *e.g.*

$$\int_{C:[t,t,0];0\le t\le 1} [0,xy,0]\,d\mathbf{r} = \int_0^1 \left([0,t^2,0]\cdot[1,1,0]\right)dt = \int_0^1 t^2 dt = \frac{1}{3}$$
$$\int_{C:[t,t^2,0];0\le t\le 1} [0,xy,0]\,d\mathbf{r} = \int_0^1 \left([0,t^3,0]\cdot[1,2t,0]\right)dt = \int_0^1 2t^4 dt = \frac{2}{5}$$



Vector field=[0,xy,0], Line integral along two curves [t,t,0] and [t,t^2,0] yield two different values i.e. 1/3 and 2/5, respectively

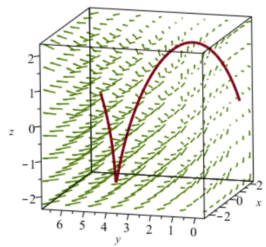
Question: (10ed-10.1-6) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} = [x-y, y-z, z-x]$ and $\mathbf{r} = [2\cos t, t, 2\sin t]$ from $(2,0,0)$ to $(2,2\pi,0)$. Show the details.

Solution: We first find variation of t : By putting $\mathbf{r} = [2\cos t, t, 2\sin t] = (2,0,0) \Rightarrow t = 0$ and $2\cos t = 2 \Rightarrow t = 2\pi$. Validity of these values of t may be checked by putting final value of t i.e. 2π and getting the terminal point, already given to us i.e. $(2,2\pi,0)$. Next we re-write $\mathbf{F}$ in terms of t by using $x = 2\cos t, y = t$ and $z = 2\sin t$ into

$\mathbf{F}=[2\cos t-t, t-2\sin t, 2\sin t-2\cos t]$. Hence

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}\left([2\cos t-t, t-2\sin t, 2\sin t-2\cos t]\cdot \frac{d}{dt}[2\cos t, t, 2\sin t]\right)dt \\
&= \int_0^{2\pi}\left([2\cos t-t, t-2\sin t, 2\sin t-2\cos t]\cdot[-2\sin t, 1, 2\cos t]\right)dt \\
&= \int_0^{2\pi}\left(t-2\sin t+2t\sin t-4\cos^2 t-4\cos t\sin t+4\sin t\cos t\right)dt \\
&= \int_0^{2\pi}\left(t-2\sin t-2\cos 2t+2t\sin t-2\right)dt \\
&= 2\pi(\pi-4)
\end{aligned}$$

Sketch:



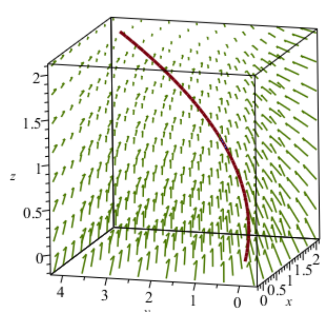
VectorField=[x-y, y-z, z-x], C=[2cos(t), t, 2sin(t)]; 0 <= t <= 2Pi ■

Question: (10ed-10.1-11) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F}=[e^{-x}, e^{-y}, e^{-z}]$ and $\mathbf{r}=[t, t^2, t]$ from $(0,0,0)$ to $(2,4,2)$. Show the details.

Solution: Initial and terminal points indicate variation of t as $0\leq t\leq 2$. Next rewriting $\mathbf{F}$ in terms of t only, by using $x=t, y=t^2$ and $z=t \Rightarrow \mathbf{F}=\left[e^{-t}, e^{-t^2}, e^{-t}\right]$. Hence

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2}\left(\left[e^{-t}, e^{-t^2}, e^{-t}\right]\cdot \frac{d}{dt}[t, t^2, t]\right)dt \\
&= \int_0^{2}\left(\left[e^{-t}, e^{-t^2}, e^{-t}\right]\cdot[1, 2t, 1]\right)dt \\
&= \int_0^{2}\left(2e^{-t}+2te^{-t^2}\right)dt \\
&= 3-\frac{2}{e^2}-\frac{1}{e^4}
\end{aligned}$$

Sketch



Vector field=[exp(-x), exp(-y), exp(-z)], C=[t, t^2, t]; 0 ≤ t ≤ 2 ■

Question: (10ed-10.1-15) Without taking dot product, evaluate the line integral with $\mathbf{F}=\left[y^2, z^2, x^2\right]$ and $C:[3\cos t, 3\sin t, 2t]$; $0\leq t\leq 4\pi$?

Solution: From the curve we have $x = 3\cos t, y = 3\sin t$ and $z = 2t$. By using the form

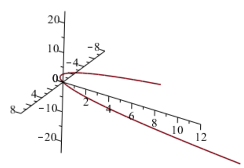
$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r})\,dt &= \int_a^b \mathbf{F}(\mathbf{r}(t))\,dt = \int_a^b \left[\mathbf{F}_1(\mathbf{r}(t)),\ \mathbf{F}_2(\mathbf{r}(t)),\ \mathbf{F}_3(\mathbf{r}(t))\right] dt \\
&= \int_0^{4\pi} \left[(3\sin t)^2, (2t)^2, (3\cos t)^2\right] dt \\
&= \left[\int_0^{4\pi} (3\sin t)^2\,dt,\ \int_0^{4\pi} (2t)^2\,dt,\ \int_0^{4\pi} (3\cos t)^2\,dt\right] \\
&= \left[18\pi, \frac{256}{3}\pi^3, 18\pi\right] \quad \blacksquare
\end{aligned}$$

Question: (10ed-10.1-19) Without taking dot product, evaluate the line integral with $f = xyz$ and $C : \left[4t, 3t^2, 12t\right];\ -2 \le t \le 2$?

Solution: Using the special case of non-dot-product line integral $\int_C f(\mathbf{r})\,dt = \int_a^b f(\mathbf{r}(t))\,dt$ we have

$$\begin{aligned}
\int_C f(\mathbf{r})\,dt &= \int_a^b f(\mathbf{r}(t))\,dt \\
&= \int_{-2}^{2} f\left(4t, 3t^2, 12t\right) dt \\
&= \int_{-2}^{2} (4t)\left(3t^2\right)(12t)\,dt \\
&= 144 \int_{-2}^{2} t^4 dt = \frac{64}{5}
\end{aligned}$$

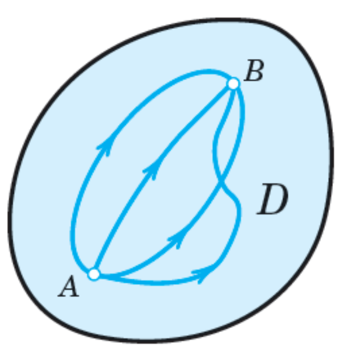
Curve C is sketched as



$\blacksquare$

2 Path Independence of Line Integrals

Definition 3 *The line integral $I = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is said to be path independent in a domain D if for every pair of endpoints A, B in domain D the integral I has same value for all paths in D that begin with A and end at B.*



Path Independence

Remark 4 *Path independence of a line integral in a domain D holds if and only if*

1. $\mathbf{F} = \operatorname{grad} f$, *and in such case we have* $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$, *Or*

2. *Integration around any closed curve C in D always gives* 0, *Or*

3. $\operatorname{curl}\mathbf{F}=\mathbf{0}$, *provided D is simply connected i.e. every closed curve in D can be continuously shurnk to any point in D without leaving D.*

Question: Evaluate the integral $I=\int\left(3x^2dx+2yzdy+y^2dz\right)$ from $A\left(0,1,2\right)$ to $B\left(1,-1,7\right)$ by showing that $\mathbf{F}$ has a potential.

Solution: If $\mathbf{F}$ has a potential f, we should have

$$f_x=3x^2,\ \ f_y=2yz,\ \ f_z=y^2$$

By integration $\int f_xdx=f=x^3+g\left(y,z\right).$ This further implies $f_y=g_y=2yz\Rightarrow g=y^2z+h\left(z\right).$ This implies $f=x^3+y^2z+h\left(z\right).$ Again $f_z=y^2+h_z=y^2\Rightarrow h_z=0$ i.e. $h\left(z\right)$ is a constant. Suppose it is 0, then $f=x^3+y^2z+0=x^3+y^2z$. Thus $\mathbf{F}$ is gradient of a potential, namely, the function f. Hence the line integral is evaluated as

$$\begin{aligned}\int_{(0,1,2)}^{(1,-1,7)}\left(3x^2dx+2yzdy+y^2dz\right) &= f\left(B\right)-f\left(A\right)\\ &= f\left(1,-1,7\right)-f\left(0,1,2\right)\\ &= 8-2=6\quad\blacksquare\end{aligned}$$

Definition 5 *In the integral $\int_C\mathbf{F}\cdot d\mathbf{r}$, the term $\mathbf{F}\cdot d\mathbf{r}=F_1dx+F_2dy+F_3dz$ is said to be a **differential form**. It is further to be said exact iff there is a function $f\left(x,y,z\right)$ in domain D such that $F_1=\frac{\partial f}{\partial x}, F_2=\frac{\partial f}{\partial y}$ and $F_3=\frac{\partial f}{\partial z}$.*

Theorem 6 *For a line integral $\int_C\mathbf{F}\cdot d\mathbf{r}$, we have following:*

1. *Differential form of I is exact iff I is path independent.*

2. *If differential form is exact then* $\operatorname{curl}\mathbf{F}=\mathbf{0}$

3. *If $\operatorname{curl}\mathbf{F}=\mathbf{0}$ in D and D is simply connected, then differential form of I is exact and hence, I is path independent.*

   Pictorially, above results are summed up as follows:

Differential form of I is exact	$\leftrightarrow$	*I is path independent*
$\downarrow$	$\nwarrow$	
$\operatorname{curl}\mathbf{F}=\mathbf{0}$	$+$	*D is simply connected*

$\blacksquare$

Remark 7 *Note that* $\operatorname{curl}\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=0\iff\frac{\partial F_3}{\partial x}=\frac{\partial F_1}{\partial z},\frac{\partial F_3}{\partial y}=\frac{\partial F_2}{\partial z}$ *and* $\frac{\partial F_2}{\partial x}=\frac{\partial F_1}{\partial y}$. *(The mnemonic for remembering is* $21,31,32$*)*

Question: If $I=\int_C\left(2xyz^2dx+\left(x^2z^2+z\cos\left(yz\right)\right)dy+\left(2x^2yz+y\cos\left(yz\right)\right)dz\right)$ is path independent, then find its value from $A\left(0,0,1\right)$ to $B\left(1,\frac{\pi}{4},2\right).$

Solution: The question requires us to first establish the path independence of I. For this we systematically find potential function $f\left(x,y,z\right)$ as follows: we have

$$f_x=2xyz^2,\quad f_y=x^2z^2+z\cos\left(yz\right),\quad f_z=2x^2yz+y\cos\left(yz\right)$$

$\int f_xdx=\int 2xyz^2dx=x^2yz^2\Rightarrow f=x^2yz^2+g\left(y,z\right)$

$f_y=g_y=x^2z^2+g_y=x^2z^2+z\cos\left(yz\right)\Rightarrow\int g_ydy=\int z\cos\left(yz\right)dy=\sin yz+h\left(z\right)\Rightarrow f=x^2yz^2+\sin yz+h\left(z\right)$

$f_z=\frac{d}{dz}\left(x^2yz^2+\sin yz\right)=2x^2yz+y\cos yz+h_z=2x^2yz+y\cos\left(yz\right)\Rightarrow h_z=0$ i.e. $h\left(z\right)$ is constant, say 0. Hence we have $f=x^2yz^2+\sin yz$. As $\mathbf{F}$ is the gradient of $f=x^2yz^2+\sin yz$ hence the line integral may be evaluated as

$$\begin{aligned}&\int_{(0,0,1)}^{\left(1,\frac{\pi}{4},2\right)}\left(2xyz^2dx+\left(x^2z^2+z\cos\left(yz\right)\right)dy+\left(2x^2yz+y\cos\left(yz\right)\right)dz\right)\\ =\ & f\left(B\right)-f\left(A\right)=\left[x^2yz^2+\sin yz\right]_{x=1,y=\frac{\pi}{4},z=2}-\left[x^2yz^2+\sin yz\right]_{x=0,y=0,z=1}\\ =\ & \left(\pi+1\right)-0=\pi+1\quad\blacksquare\end{aligned}$$

Question: (10ed-10.2-4) Show that the form under the integral $\int_{(4,0)}^{(6,1)}e^{4y}\left(2x\ dx+4x^2\ dy\right)$ is exact in the plane and evaluate the integral. Show the details of your work?

Solution: Here we have $F_1 = 2xe^{4y}, F_2 = 4e^{4y}x^2$ and no third component. Hence for exactness of the differential form we see $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(2xe^{4y}\right) = 8xe^{4y} = \frac{\partial}{\partial x}\left(4e^{4y}x^2\right) = \frac{\partial F_2}{\partial x}$, hence the form is exact and the integral is path independent. So we may evaluate it by using its potential only i.e. f. For finding f we set $f_x = 2xe^{4y}$ and $f_y = 4e^{4y}x^2$. Now
$\int f_x dx = \int \left(2xe^{4y}\right) dx = x^2e^{4y} + g(y) \Rightarrow f = x^2e^{4y} + g(y)$ and $f_y = 4x^2e^{4y} + g_y = 4e^{4y}x^2 \Rightarrow g(y) = 0$ (say). Potential is $f = x^2e^{4y}$. Integral is computed as

$$\int_{(4,0)}^{(6,1)} e^{4y}\left(2x\ dx + 4x^2\ dy\right) = f(B) - f(A) = \left[x^2e^{4y}\right]_{x=6,y=1} - \left[x^2e^{4y}\right]_{x=4,y=0} = 1949.5 \quad \blacksquare$$

Question: (10ed-10.2-9) Show that the form under the integral $\int_{(0,1,0)}^{(1,0,1)}\left(e^x\cosh y\ dx + \left(e^x\sinh y + e^z\cosh y\right)\ dy + e^z\sinh y\ dz\right)$ is exact in the space and evaluate the integral. Show the details of your work?

Solution: Here we have $F_1 = e^x\cosh y, F_2 = e^x\sinh y + e^z\cosh y$ and $F_3 = e^z\sinh y$. We have to show $\frac{\partial F_3}{\partial x} = \frac{\partial}{\partial x}\left(e^z\sinh y\right) = 0 = \frac{\partial}{\partial z}\left(e^x\cosh y\right) = \frac{\partial F_1}{\partial z}$
$\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}\left(e^z\sinh y\right) = e^z\cosh y = \frac{\partial}{\partial z}\left(e^x\sinh y + e^z\cosh y\right) = \frac{\partial F_2}{\partial z}$ and
$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(e^x\sinh y + e^z\cosh y\right) = e^x\sinh y = \frac{\partial}{\partial y}\left(e^x\cosh y\right) = \frac{\partial F_1}{\partial y}$. Hence the integral is path independent. For finding the potential we set $f_x = e^x\cosh y, f_y = e^x\sinh y + e^z\cosh y$ and $f_z = e^z\sinh y$. A mere close examination of the terms indicates $f(x,y,z) = e^x\cosh y + e^z\sinh y$. The integral now may be evaluated as

$$\begin{aligned}
&\int_{(0,1,0)}^{(1,0,1)}\left(e^x\cosh y\ dx + \left(e^x\sinh y + e^z\cosh y\right)\ dy + e^z\sinh y\ dz\right)\\
=\ & f(B) - f(A) = \left[e^x\cosh y + e^z\sinh y\right]_{x=1,y=0,z=1} - \left[e^x\cosh y + e^z\sinh y\right]_{x=1,y=0,z=1}\\
=\ & e - (\cosh 1 + \sinh 1) = -0.0001 = -\frac{1}{10000} \quad \blacksquare
\end{aligned}$$

Question: (10ed-10.2-13) Check path independence of $2e^{x^2}\left(x\cos 2y\ dx - \sin 2y\ dy\right)$ and if independent, integrate from $(0,0,0)$ to (a,b,c)?

Solution: We have $F_1 = 2e^{x^2}x\cos 2y$, $F_2 = -2e^{x^2}\sin 2y$ and $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(2e^{x^2}x\cos 2y\right) = -4x(\sin 2y)e^{x^2} = \frac{\partial}{\partial x}\left(-2e^{x^2}\sin 2y\right) = \frac{\partial F_2}{\partial x} \Rightarrow$ Exact i.e. path independence holds. So we find potential as $f(x,y) = (\cos 2y)e^{x^2}$ by guess work!Integral is given as

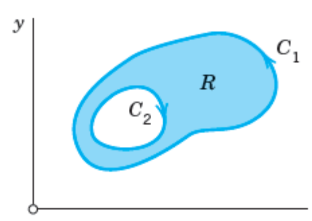
$$\begin{aligned}
\int_{(0,0,0)}^{(a,b,c)}\left(2e^{x^2}x\cos 2y\ dx - 2e^{x^2}\sin 2y\ dy\right) &= f(a,b,c) - f(0,0,0)\\
&= \left[(\cos 2y)e^{x^2}\right]_{x=a,y=b,z=c} - \left[(\cos 2y)e^{x^2}\right]_{x=0,y=0,z=0}\\
&= e^{a^2}\cos(2b) - 1 \quad \blacksquare
\end{aligned}$$

3 Green's Theorem in the Plane

Theorem 8 ***(Green's Theorem in the Plane)*** *Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R. Then*

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C \left(F_1dx + F_2dy\right)$$

Remark 9
1. *In Green's Theorem, we integrate along the entire boundary C of R in such sense that R is on the left as we advance in the direction of integration.*



2. *Green's Theorem facilitates a transformation between double integrals and line integrals. Thus it is a great computation saver.*

3. *Two vectorial forms of Green's Theorem are obtained as*

$$\iint\limits_R (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\; dxdy = \oint_C \mathbf{F}\cdot d\mathbf{r}$$

and

$$\iint\limits_R \operatorname{div}\mathbf{F} dxdy = \oint_C \mathbf{F}\cdot\mathbf{n}\; ds$$

where $\mathbf{k}$ *is the unit vector in* z*-direction and* $\mathbf{n}$ *is outer unit normal vector of* C*.*

Question: Verify Green's Theorem for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C the circle $x^2 + y^2 = 1$?

Solution: From the theorem we have $\iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \iint\limits_R \left((2y+2) - (2y-7)\right) dxdy = 9\iint\limits_R dxdy = 9\left(\pi\,(1)^2\right) = 9\pi$, using the fact that $\iint\limits_R dxdy$ is the area of a unit disk whose boundary is C.

For evaluating the RHS of theorem, we need to orient C counterclockwise i.e. $\mathbf{r}(t) = [\cos t, \sin t]$. Then $\mathbf{r}'(t) = [-\sin t, \cos t]$ and on C, $F_1 = y^2 - 7y = \sin^2 t - 7\sin t$, $F_2 = 2xy + 2x = 2\cos t\sin t + 2\cos t$. Hence

$$\begin{aligned}
\oint_C (F_1 dx + F_2 dy) &= \oint_C (F_1 x' + F_2 y')\, dt \\
&= \int_0^{2\pi} \left[\left(\sin^2 t - 7\sin t\right)(-\sin t) + (2\cos t\sin t + 2\cos t)(\cos t)\right] dt \\
&= \int_0^{2\pi} \left[7\sin^2 t - \sin^3 t + 2\cos^2 t\sin t + 2\cos^2 t\right] dt \\
&= 7\int_0^{2\pi} \sin^2 t dt - \int_0^{2\pi} \sin^3 t dt + 2\int_0^{2\pi} \cos^2 t\sin t dt + 2\int_0^{2\pi} \cos^2 t dt \\
&= 7\pi - 0 + 0 + 2\pi \\
&= 9\pi \quad \blacksquare
\end{aligned}$$

Definition 10 *For a continuous function* $w(x,y)$ *with continuous first and second partial derivatives,* $\frac{\partial w}{\partial n} = (\operatorname{grad} w)\cdot\mathbf{n}$ *is called the normal derivative of* $w(x,y)$*.*

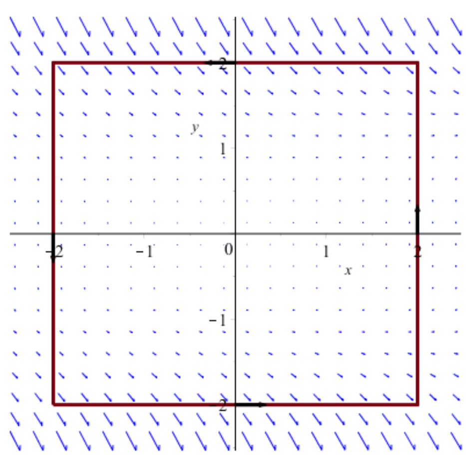
Note that $(\operatorname{grad} w)\cdot\mathbf{n} = \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right]\cdot\left[\frac{dy}{ds}, -\frac{dx}{ds}\right] = \frac{\partial w}{\partial x}\frac{dy}{ds} - \frac{\partial w}{\partial y}\frac{dx}{ds} = F_2\frac{dy}{ds} + F_1\frac{dx}{ds} = F_1 dx + F_2 dy$*. Hence now Green's Theorem may be re-stated:*

$$\iint\limits_R \nabla^2 w\; dxdy = \oint_C \frac{\partial w}{\partial n} ds$$

Question: (10ed-10.4-2) Evaluate $\oint_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's Theorem, where $\mathbf{F} = \left[6y^2, 2x - 2y^4\right]$ and R : the square with vertices $\pm(2,2), \pm(2,-2)$.

Solution:

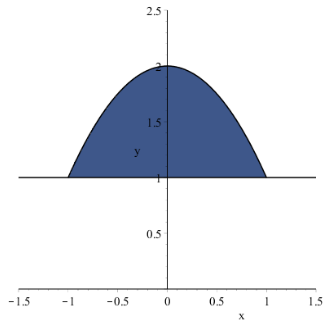
$$\begin{aligned}
\oint_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} &= \oint\limits_C \left[6y^2, 2x - 2y^4\right]\cdot d\mathbf{r} = \int_{-2}^{2}\int_{-2}^{2} \left(\frac{\partial}{\partial x}\left(2x - 2y^4\right) - \frac{\partial}{\partial y}\left(6y^2\right)\right) dxdy \\
&= \int_{-2}^{2}\int_{-2}^{2} (2 - 12y)\, dxdy = \int_{-2}^{2} (8 - 48y)\, dy = 32
\end{aligned}$$



$\blacksquare$

Question: (10ed-10.4-7) Evaluate $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's Theorem, where $\mathbf{F} = \text{grad}\left(x^3 \cos^2(xy)\right)$ and $R : 1 \le y \le 2 - x^2$?

Solution: We have the region of integration as



$$
\begin{aligned}
\mathbf{F} &= \text{grad}\left(x^3 \cos^2(xy)\right) = \left[3x^2 \cos^2 xy - 2x^3 y \cos xy \sin xy, -2x^4 \cos xy \sin xy\right] \\
F_1 &= 3x^2 \cos^2 xy - 2x^3 y \cos xy \sin xy, \qquad F_2 = -2x^4 \cos xy \sin xy
\end{aligned}
$$

$\oint_C \text{grad}\left(x^3 \cos^2(xy)\right) \cdot d\mathbf{r} = \oint_C (F_1 dx + F_2 dy) = \iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$ by Green's Theorem.

$= \iint\limits_R \left(\frac{\partial}{\partial x}\left(-2x^4 \cos xy \sin xy\right) - \frac{\partial}{\partial y}\left(3x^2 \cos^2 xy - 2x^3 y \cos xy \sin xy\right)\right) dxdy$

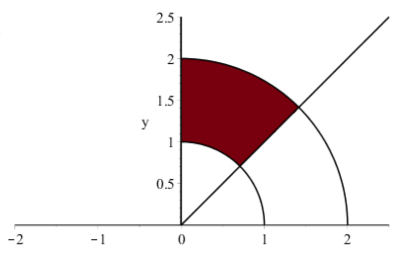
$= \iint\limits_R \left(\left(-4x^3 \sin 2xy - 2x^4 y \cos 2xy\right) - \left(-4x^3 \sin 2xy - 2x^4 y \cos 2xy\right)\right) dxdy$

$= \iint\limits_R (0)\, dxdy = 0$

Alternately as $\mathbf{F}$ is gradient of the potential $f(x, y) = x^3 \cos^2(xy)$ and $\frac{\partial F_1}{\partial y} = -4x^3 \sin 2xy - 2x^4 y \cos 2xy = \frac{\partial F_2}{\partial x}$ Hence the differential form under this integral is exact. ■

Question: (10ed-10.4-10) Evaluate $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's Theorem, where $\mathbf{F} = \left[x^2 y^2, -\frac{x}{y^2}\right]$ and $R : 1 \le x^2 + y^2 \le 4,\ x \ge 0,\ y \ge x$. Sketch R.

Solution: R is sketched as:

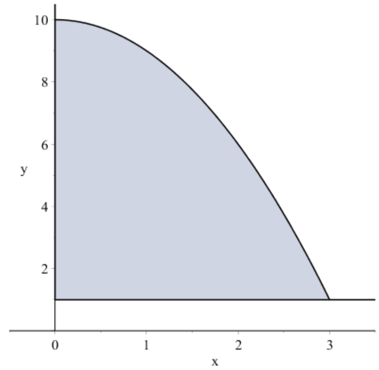


This is a portion of a circular ring (annulus) bounded by the circles of radii 1 and 2 centered at the origin, in the first quadrant bounded by $y = x$ and the y-axis. The integrand is $\frac{-1}{y^2} - 2x^2 y$. We use polar coordinates, obtaining

$$
\begin{aligned}
& \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_1^2 \left(\frac{-1}{r^2 \sin^2 \theta} - 2r^3 \cos^2 \theta \sin \theta\right) r\ drd\theta \\
= & \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[-\frac{1}{40 \sin^2 \theta}\left(40 \ln r + r^5 \sin 3\theta - r^5 \sin 5\theta + 2r^5 \sin \theta\right)\right]_1^2 d\theta \\
= & \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(-\frac{1}{40 \sin^2 \theta}\left(31 \sin 3\theta - 31 \sin 5\theta + 40 \ln 2 + 62 \sin \theta\right)\right) d\theta \\
= & -\ln 2 - \frac{31}{30}\sqrt{2} = -2.155 \qquad ■
\end{aligned}
$$

Question: (10ed-10.4-15)

Solution: The region is given as



Question: (10ed-10.4-16) Using the form $\iint\limits_R \nabla^2 w \, dxdy = \oint_C \frac{\partial w}{\partial n} ds$, of Green's Theorem, find the value of $\oint_C \frac{\partial w}{\partial n} ds$ taken counterclockwise over the boundary $C : x^2 + y^2 = 4$ of the region R where $w = x^2 + y^2$. Confirm the answer by direct integration.

Solution: $\nabla^2 w = \nabla^2 \left(x^2 + y^2\right) = 4$, Answer: 8π

Confirmation: $\mathbf{r} = [2\cos s, 2\sin s]$, $\mathbf{r}' = \left[\frac{d}{ds}(2\cos s), \frac{d}{ds}(2\sin s)\right] = [-2\sin s, 2\cos s]$. Outer normal vector $\mathbf{n} = [2\cos s, 2\sin s]$

$\operatorname{grad} w = \operatorname{grad}\left(x^2 + y^2\right) = [2x, 2y] = [4\cos s, 4\sin s]$

$\operatorname{grad}(w) \cdot \mathbf{n} = [4\cos s, 4\sin s] \cdot [-2\sin s, 2\cos s] = 8\sin s \cos s - 8\cos s \sin s = 0$

$16\int_0^{2\pi} \sin(s)\cos(s)\, ds \qquad ??$