Advanced Engineering Mathematics

Mathematical Techniques for Engineering

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Definition

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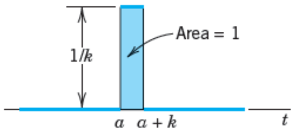

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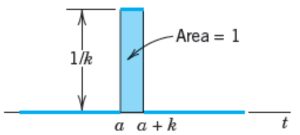

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Then Dirac's delta function (aka, unit impulse function) is defined as

$$\delta\left(t-a\right)=\lim_{k\to 0}f_k\left(t-a\right).$$

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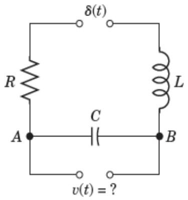
&= e^{-as}\lim_{k\to 0}\frac{\frac{d}{dk}\left(1-e^{-ks}\right)}{\frac{d}{dk}(ks)} \qquad \text{(l'Hopital)} \\
&= e^{-as}\left[\frac{se^{-ks}}{s}\right]_{k=0} = e^{-as} \\
L\left(\delta(t-a)\right) &= e^{-as} \qquad \blacksquare
\end{aligned}
$$

Question:

Find output voltage response of the four-terminal RLC circuit given in figure if $R = 20\Omega$, $L = 1H$, $C = 10^{-4}F$, the input is an impulse, current and charge are zero at time $t = 0$?



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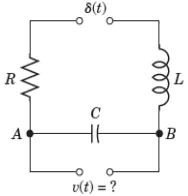

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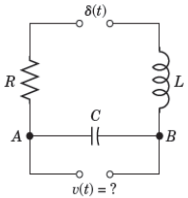

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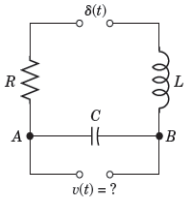

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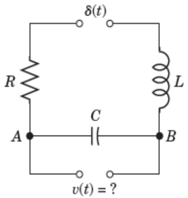

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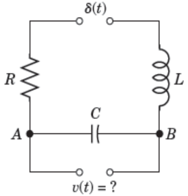

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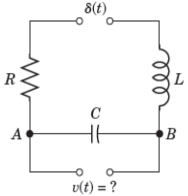

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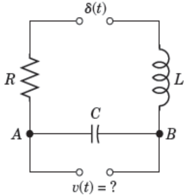

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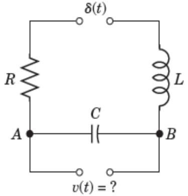

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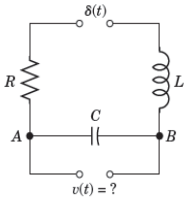

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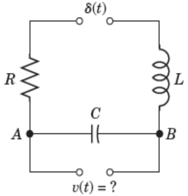

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$y'' + 5y' + 6y = \delta\left(t - \frac{1}{2}\pi\right) + u\left(t - \pi\right)\cos t \quad ; \quad y\left(0\right) = y'\left(0\right) = 0$

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Using Partial Fractions

$$Y = \left(\frac{1}{(s+2)} - \frac{1}{(s+3)}\right)e^{-\frac{\pi}{2}s} - \left(\frac{-2}{5}\,\frac{1}{(s+2)} + \frac{1}{10}\,\frac{s+1}{s^2+1} + \frac{3}{10}\,\frac{1}{(s+3)}\right)e^{-s\pi}$$

$$y(t) = L^{-1}\left(\frac{e^{-\frac{\pi}{2}s}}{s+2}\right) - L^{-1}\left(\frac{e^{-\frac{\pi}{2}s}}{s+3}\right) + \frac{2}{5}L^{-1}\left(\frac{1}{s+2}e^{-s\pi}\right) - \frac{1}{10}L^{-1}\left(\frac{s}{1+s^2}e^{-s\pi}\right) - \frac{1}{10}L^{-1}\left(\frac{1}{1+s^2}e^{-s\pi}\right) - \frac{3}{10}L^{-1}\left(\frac{1}{s+3}e^{-s\pi}\right)$$

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$$y(t) = e^{\pi-2t}u\left(t-\frac{1}{2}\pi\right) - e^{\frac{3}{2}\pi-3t}u\left(t-\frac{1}{2}\pi\right) + \frac{2}{5}e^{2\pi-2t}u(t-\pi) - \frac{1}{10}\left(-(\cos t)\,u(t-\pi)\right) - \frac{1}{10}\left(-(\sin t)\,u(t-\pi)\right) - \frac{3}{10}e^{3\pi-3t}u(t-\pi)$$

$$y(t) = \frac{1}{10}u(t-\pi)\left(-3e^{-3t+3\pi} + \sin(t) + \cos(t) + 4e^{-2t+2\pi}\right) + u(t-\pi/2)\left(-e^{-3t+3/2\pi} + e^{-2t+\pi}\right) \quad \blacksquare$$

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$$Y = \frac{1}{s^2+2s+5}\left(25\frac{1}{s^2} - 100e^{-s\pi} + 1 - 2s\right)$$

$$Y = -\frac{100e^{-s\pi}s^2 + 2s^3 - s^2 - 25}{s^2(s^2+2s+5)}$$

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$$Y = \frac{1}{s^2+2s+5}\left(25\frac{1}{s^2} - 100e^{-s\pi} + y'(0) + sy(0) + 2y(0)\right)$$

$$Y = \frac{1}{s^2+2s+5}\left(25\frac{1}{s^2} - 100e^{-s\pi} + 1 - 2s\right)$$

$$Y = -\frac{100e^{-s\pi}s^2 + 2s^3 - s^2 - 25}{s^2(s^2+2s+5)}$$

$$Y = -100\frac{1}{s^2+2s+5}e^{-s\pi} - 2\frac{s}{s^2+2s+5} + \frac{1}{s^2+2s+5} + \frac{25}{s^2(s^2+2s+5)}$$

$$y(t) = -50u(t-\pi)e^{-t+\pi}\sin(2t) + 5t - 2$$

Question: (10ed-6.4-12)

$$\frac{d^2}{dt^2}y\left(t\right)+2\,\frac{d}{dt}y\left(t\right)+5\,y\left(t\right)=$$
$$25\,t-100\,\delta\left(t-\pi\right)\qquad,\qquad y\left(0\right)=-2,\quad y'\left(0\right)=5$$

$$s^2\,Y-y'\left(0\right)-sy\left(0\right)+2s\;Y-2\,y\left(0\right)+5\,Y=25\,\frac{1}{s^2}-100\,e^{-s\pi}$$

$$Y=\frac{1}{s^2+2\,s+5}\left(25\,\frac{1}{s^2}-100\,e^{-s\pi}+y'\left(0\right)+sy\left(0\right)+2\,y\left(0\right)\right)$$

$$Y=\frac{1}{s^2+2\,s+5}\left(25\,\frac{1}{s^2}-100\,e^{-s\pi}+1-2\,s\right)$$

$$Y=-\frac{100\,e^{-s\pi}s^2+2\,s^3-s^2-25}{s^2\left(s^2+2\,s+5\right)}$$

$$Y=-100\,\frac{1}{s^2+2\,s+5}e^{-s\pi}-2\,\frac{s}{s^2+2\,s+5}+\frac{1}{s^2+2\,s+5}+\frac{25}{s^2\left(s^2+2\,s+5\right)}$$

$$y\left(t\right)=-50\,u\left(t-\pi\right)e^{-t+\pi}\sin\left(2\,t\right)+5\,t-2$$
$$y\left(t\right)=\begin{cases}5t-2 & ,\quad t<\pi\\ 5t-2-50e^{\pi-t}\sin\left(2t\right) & ,\quad t>\pi\end{cases}\qquad\blacksquare$$

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$$(f \star g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

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$$\left(f\star g\right)\left(t\right)=\int_0^t f\left(\tau\right)g\left(t-\tau\right)d\tau$$

In words one says: product of transforms is not the transform of product, instead product of transforms is the transform of convolution.

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L^{-1}\left(\frac{1}{s^2+\omega^2}\right) &= \int_0^t \frac{\sin(\omega\tau)}{\omega} \frac{\sin(\omega(t-\tau))}{\omega} d\tau
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L^{-1}\left(\frac{1}{s^2+\omega^2}\right) &= \int_0^t \frac{\sin(\omega\tau)}{\omega} \frac{\sin(\omega(t-\tau))}{\omega} d\tau \\
&= \frac{1}{\omega^2}\int_0^t \frac{1}{2}\left[-\cos(\omega t) + \cos(2\omega\tau - \omega t)\right] d\tau \\
&= \frac{1}{2\omega^2}\left[-\tau\cos\omega t + \frac{\sin(\omega\tau)}{\omega}\right]_{\tau=0}^{t} \\
&= \frac{1}{2\omega^2}\left[-t\cos\omega t + \frac{\sin(\omega t)}{\omega}\right] \qquad \blacksquare
\end{aligned}$$

Properties of Convolution:

$$
\begin{aligned}
f \star g &= g \star f \\
f \star (g + h) &= f \star g + f \star h \\
f \star (g \star h) &= (f \star g) \star h \\
f \star 0 &= 0 \star f = 0 \\
f \star 1 &\neq f \qquad \text{(why?)}
\end{aligned}
$$

Question: (10ed-6.5-6) Find convolution $e^{at} \star e^{bt}$, $(a \neq b)$?

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$$= \int_0^t e^{a\tau \ln(e)} e^{b(t-\tau)\ln(e)}\, d\tau = \int_0^t e^{a\tau \ln(e) - b(\tau - t)\ln(e)}\, d\tau$$

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Substituting $u = a\tau \ln(e) - b(\tau - t)\ln(e)$ implies at $\tau = 0,\ u = bt$ and $\tau = t,\ \ u = at$ and
$du = \frac{d}{d\tau}\left(a\tau \ln(e) - b(\tau - t)\ln(e)\right) = a - b \Rightarrow \frac{du}{a-b} = d\tau$

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$$= \int_{\ln(e)bt}^{at\ln(e)} \frac{e^u}{(a-b)\ln(e)}\, du = \frac{e^{at\ln(e)} - e^{\ln(e)bt}}{(a-b)\ln(e)} = \frac{e^{at} - e^{bt}}{(a-b)} \qquad \blacksquare$$

Question: (10ed-6.5-13)

$$y(t) + 2e^{t}\int_{0}^{t} y(\tau)\, e^{-\tau} d\tau \quad = \quad te^{t}$$

Question: (10ed-6.5-13)

$$\begin{aligned} y(t) + 2e^t \int_0^t y(\tau)\, e^{-\tau} d\tau &= te^t \\ y(t) + 2 \int_0^t y(\tau)\, e^{t-\tau} d\tau &= te^t \end{aligned}$$

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$$
\begin{aligned}
y(t) + 2e^t \int_0^t y(\tau) e^{-\tau} d\tau &= te^t \\
y(t) + 2 \int_0^t y(\tau) e^{t-\tau} d\tau &= te^t \\
y(t) + 2\left(y(t) \star e^t\right) &= te^t
\end{aligned}
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y(t) + 2\left(y(t) \star e^{t}\right) &= te^{t} \\
Y + 2\left(Y\frac{1}{s-1}\right) &= \frac{1}{(s-1)^2}
\end{aligned}
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Y + 2\left(Y \frac{1}{s-1}\right) &= \frac{1}{(s-1)^2} \\
Y &= \frac{1}{s^2-1} \\
y(t) &= L^{-1}\left(\frac{1}{s^2-1}\right) \\
y(t) &= \sinh t \qquad \blacksquare
\end{aligned}$$

Question: (10ed-6.5-14)

$$y(t) - \int_0^t y(\tau)(t-\tau)\,d\tau = 2 - \frac{1}{2}t^2$$

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y(t) - (y(t) \star t) &= 2 - \frac{1}{2}t^2 \\
Y - \left(Y \frac{1}{s^2} \right) &= \frac{2}{s} - \frac{1}{s^3}
\end{aligned}
$$

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Y - \left(Y\frac{1}{s^2}\right) &= \frac{2}{s} - \frac{1}{s^3} \\
Y &= -\frac{2s^2-1}{s-s^3} \\
Y &= \frac{1}{2(s-1)} + \frac{1}{2(s+1)} + \frac{1}{s} \\
y(t) &= \frac{1}{2}e^t + \frac{1}{2}e^{-t} + 1 \\
y(t) &= 1 + \cosh t \quad \blacksquare
\end{aligned}$$

Question: (10ed-6.5-22)

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$$\frac{e^{-as}}{s(s-2)} = \frac{1}{s-2}\frac{e^{-as}}{s} = L\left(e^{2t}\right) L\left(u\left(t-a\right)\right) = \left(e^{2t}\right) \star u\left(t-a\right)$$

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choose $f\left(t\right) = e^{2t}$ and $g\left(t\right) = u\left(t-a\right)$ in convolution integral $\left(f \star g\right)\left(t\right) = \int_0^t f\left(\tau\right)g\left(t-\tau\right)d\tau$

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$\left(e^{2t}\right) \star u\left(t-a\right) = \int_0^t e^{2\tau} u((t-\tau)-a) d\tau$

$= \left[\frac{1}{2} u\left(t-a-\tau\right)\left(e^{2\tau} + e^{2t-2a}\right)\right]_{\tau=0}^{t}$
$= \left[\frac{1}{2} u\left(t-a-\tau\right)\left(e^{2\tau} + e^{2t-2a}\right)\right]_{\tau=t} -$
$\left[\frac{1}{2} u\left(t-a-\tau\right)\left(e^{2\tau} + e^{2t-2a}\right)\right]_{\tau=0}$

$= \frac{1}{2} u\left(-a\right)\left(e^{2t-2a} + e^{2t}\right) - \frac{1}{2} u\left(t-a\right)\left(e^{2t-2a} + 1\right)$ ■

Question: (9ed-6.5-20) Using convolution theorem solve the following IVP

$$y'' + 5y' + 4y = 2e^{-2t} \quad ; \quad y(0) = 0,\ y'(0) = 0$$

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$$s^2\,Y - y'\left(0\right) - sy\left(0\right) + 5\,sY - 5\,y\left(0\right) + 4\,Y = \frac{2}{s+2}$$

$$Y = \frac{1}{s^2+5\,s+4}\left(\frac{2}{s+2} + y'\left(0\right) + sy\left(0\right) + 5\,y\left(0\right)\right) =$$
$$2\,\frac{1}{(s+2)}\left(\frac{1}{(s+4)(s+1)}\right)$$

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$$2\,\tfrac{1}{(s+2)}\left(\tfrac{1}{(s+4)(s+1)}\right)$$

$$Y = \tfrac{2}{(s+2)}\left[L\left(e^{-4t}\right)L\left(e^{-t}\right)\right] = 2L\left(e^{-2t}\right)\left[L\left(e^{-4t}\right)L\left(e^{-t}\right)\right]$$

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$$Y = \tfrac{2}{(s+2)}\left[L\left(e^{-4t}\right)L\left(e^{-t}\right)\right] = 2L\left(e^{-2t}\right)\left[L\left(e^{-4t}\right)L\left(e^{-t}\right)\right]$$
$$Y = 2e^{-2t} \star \left[e^{-4t} \star e^{-t}\right] = 2e^{-2t} \star \int_0^t e^{-4\tau} e^{-(t-\tau)}\,d\tau$$

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$$Y = \frac{2}{(s+2)}\left[L\left(e^{-4t}\right)L\left(e^{-t}\right)\right] = 2L\left(e^{-2t}\right)\left[L\left(e^{-4t}\right)L\left(e^{-t}\right)\right]$$

$$Y = 2e^{-2t} \star \left[e^{-4t} \star e^{-t}\right] = 2e^{-2t} \star \int_0^t e^{-4\tau} e^{-(t-\tau)}\,d\tau$$

$$Y = 2e^{-2t} \star \left[-\frac{1}{3}e^{-t-3\tau}\right]_{\tau=0}^{\tau=t}$$

$$= 2e^{-2t} \star \left(\frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}\right) = \frac{2}{3}e^{-2t} \star \left(e^{-t} - e^{-4t}\right)$$

$$Y = \frac{2}{3}\int_0^t e^{-2\tau}\left(e^{-(t-\tau)} - e^{-4(t-\tau)}\right)d\tau$$

$$Y = \frac{2}{3}\int_0^t e^{-2\tau}\left(e^{-(t-\tau)} - e^{-4(t-\tau)}\right)d\tau$$

$$y(t) = \frac{2}{3}\left(\frac{1}{2}e^{-t}\left(e^{-t}-1\right)^2\left(e^{-t}+2\right)\right)$$

$$y(t) = \frac{1}{3}e^{-t}\left(e^{-t}-1\right)^2\left(e^{-t}+2\right) \quad \blacksquare$$