Advanced Engineering Mathematics

Mathematical Techniques for Engineering

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Theorem

(Laplace Transform of the Derivative $f^{(n)}$ of Any Order)

$$L\left(f^{(n)}\right) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

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Similarly let $g(t) = \sin \omega t$. Then $g(0) = 0$, $g'(t) = \omega \cos \omega t$, which gives

$$\begin{aligned} sL(g) - g(0) &= L(g') = \omega L(\cos \omega t) \\ \text{Hence } L(\sin \omega t) &= \frac{\omega}{s} L(\cos \omega t) = \frac{\omega}{s^2 + \omega^2} \quad \blacksquare \end{aligned}$$

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= & \frac{5}{\pi^2}e^{2(\pi\tau)}-\frac{10}{\pi}\tau-\frac{5}{\pi^2} \quad \blacksquare
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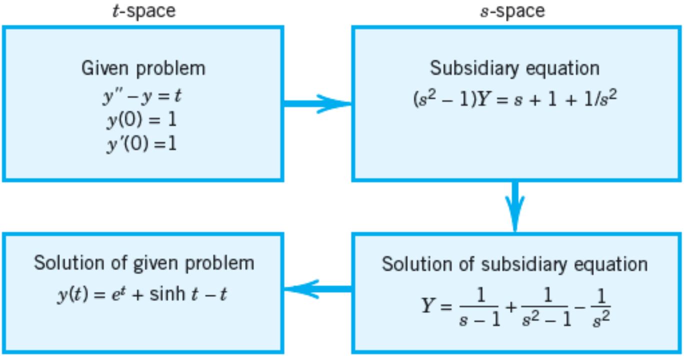

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$$y(t) = -t + \cosh(t) + 2\sinh(t) \qquad \blacksquare$$

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Isolating Laplace term $\quad L(y(t)) = \frac{1}{s^2+9}\left(\frac{10}{s+1} + y'(0) + sy(0)\right)$

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Solve the IVP $y'' + 9y = 10e^{-t}$; $y(0) = y'(0) = 0$ using Laplace Transforms.

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Example (10e-6.1-10)

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By partial fractions(DIY) $\quad Y = \frac{2}{2s+3} + \frac{4}{(2s+3)^2} + \frac{32}{s^2} - \frac{32}{s^3} + \frac{24}{s^4}$

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$$y\left(t\right) = e^{-1.5t} + te^{-1.5t} + 32t - 16t^{2} + 4t^{3}$$
$$y\left(t\right) = 4t^{3} - 16t^{2} + 32t + \left(t+1\right)e^{-1.5t} \quad \blacksquare$$

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Isolating $\quad \widetilde{Y} = \frac{1}{s^2+1}\left(\frac{\pi}{2}\frac{1}{s} + 2\frac{1}{s^2} + y'(0) + sy(0)\right)$

Using ICs $\quad \widetilde{Y} = \frac{1}{s^2+1}\left(\frac{\pi}{2}\frac{1}{s} + 2\frac{1}{s^2} + \left(2 - \sqrt{2}\right) + \frac{\pi}{2}s\right)$

$$= \frac{1}{2}\frac{s^3\pi - 2\sqrt{2}s^2 + s\pi + 4s^2 + 4}{s^2(s^2+1)}$$

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Example (10e-6.2-14)

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$s^2\,\widetilde{Y} - y'(0) - sy(0) + 2\,s\widetilde{Y} - 2y(0) + 5\widetilde{Y} = 50\frac{1}{s^2}$ where $\widetilde{Y} = L\left(y\left(\widetilde{t}\right)\right)$

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$s^2\,\widetilde{Y} - y'\left(0\right) - sy\left(0\right) + 2\,s\widetilde{Y} - 2y\left(0\right) + 5\,\widetilde{Y} = 50\frac{1}{s^2}$ where $\widetilde{Y} = L\left(y\left(\widetilde{t}\right)\right)$

$$\widetilde{Y} = \frac{1}{s^2+2\,s+5}\left(50\frac{1}{s^2} + y'\left(0\right) + sy\left(0\right) + 2\,y\left(0\right)\right)$$

$$\widetilde{Y} = \frac{1}{s^2+2\,s+5}\left(50\,\frac{1}{s^2} + 6 - 4s\right)$$

$$\widetilde{Y} = -2\,\frac{2\,s^3 - 3s^2 - 25}{s^2\left(s^2+2s+5\right)}$$

By partial fractions(DIY) $\widetilde{Y} = -\frac{4}{s} + \frac{4}{(s+1)^2+4} + \frac{10}{s^2}$

$y\left(\widetilde{t}\right) = -4 + 2e^{-\widetilde{t}}\sin\left(2\widetilde{t}\right) + 10\widetilde{t}$

Replacing back

$\widetilde{t} = t - 2 \qquad y\left(t\right) = -24 + 2\,e^{-t+2}\sin\left(2\,t - 4\right) + 10t$ ■

Q (10e-6.2-15)

For the IVP $y'' + 3y' - 4y = 6e^{2t-3} \quad ; \quad y(1.5) = 4, y'(1.5) = 5$
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$s^2\,\widetilde{Y} - y'(0) - sy(0) + 3s\widetilde{Y} - 3y(0) - 4\,\widetilde{Y} = \frac{6}{s-2}$ where
$\widetilde{Y} = L\left(y\left(\widetilde{t}\right)\right)$

$$\widetilde{Y} = \frac{1}{s^2+3s-4}\left(\frac{6}{s-2} + y'(0) + sy(0) + 3y(0)\right)$$

Using ICs $\widetilde{Y} = \frac{1}{s^2+3s-4}\left(\frac{6}{s-2} + 17 + 4s\right)$

$$\widetilde{Y} = \frac{4s-7}{(s-1)(s-2)}$$

By partial fractions $\widetilde{Y} = \frac{3}{s-1} + \frac{1}{s-2}$

$y\left(\widetilde{t}\right) = 3e^{\widetilde{t}} + e^{2\widetilde{t}}$

Replacing back $\widetilde{t} = t - 1.5,$ $y\left(t\right) = 3e^{(t-1.5)} + e^{2(t-1.5)}$ $\blacksquare$