Classroom notes of Vector Differential Calculus

from Chapter 9 of

Advanced Engineering Mathematics, E. Kreyszig, 10th Edition

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Contents

1 Vectors in 2-Space and 3-Space

Definition 1 *We have following definitions:*

1. *A vector is a quantity that has both magnitude and direction.*
2. $\mathbf{a} = \mathbf{b} \iff$ *they have same length and same direction.*
3. *For* $\mathbf{a}$ *with initial point* $P(x_1, y_1, z_1)$ *and terminal point* $Q(x_2, y_2, z_2)$, *components of* $\mathbf{a}$ *are given as*
$$a_1 = x_2 - x_1, \quad a_2 = y_2 - y_1, \quad a_3 = z_2 - z_1.$$
Moreover $\mathbf{a}$ *is written as* $\mathbf{a} = [a_1, a_2, a_3]$.
4. *Length* $|\mathbf{a}|$ *is given as* $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
5. *A vector of length* 1 *is called a unit vector. Unit vector in the direction of a given vector* $\mathbf{a}$ *may be computed as* $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$.

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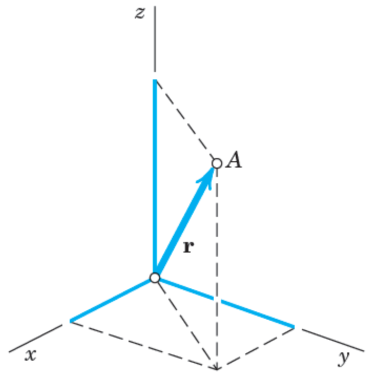
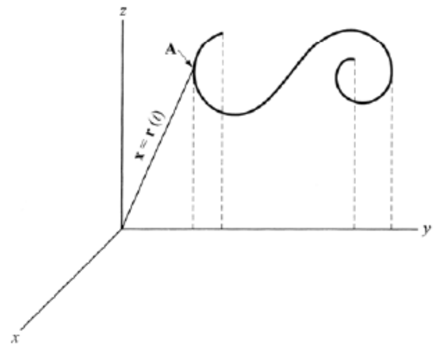
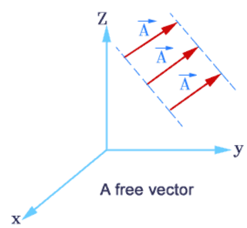
6. *Position vector* $\mathbf{r}$ *of a point* $A\,(x, y, z)$ *is the vector with the origin* $(0, 0, 0)$ *as initial point and* $A\,(x, y, z)$ *as the terminal point.*



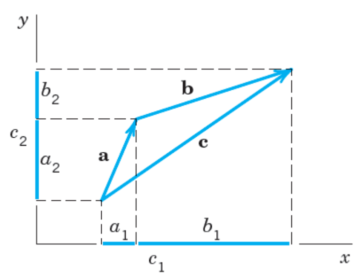


7. *A vector that can be displaced parallel to itself and applied at any point is known as a free vector.*

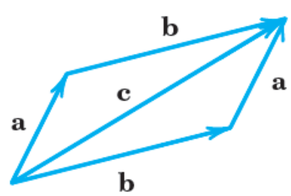


8. *For* $\mathbf{a} = [a_1, a_2, a_3]$ *and* $\mathbf{b} = [b_1, b_2, b_3]$ *and* c *any real number:*

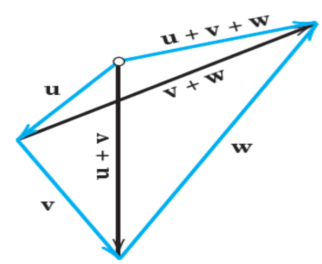
- *Vector addition is defined as* $\mathbf{a} + \mathbf{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$ *, and it has the properties:* $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, $\quad (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, $\quad \mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$, $\quad \mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.



$\mathbf{a} + \mathbf{b}$

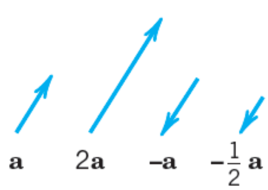


$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

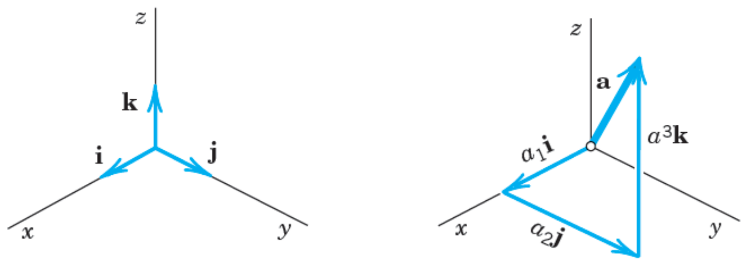


$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

- *Scalar multiplication is defined as* $c\mathbf{a} = [ca_1, ca_2, ca_3]$ *, and it has the properties:* $c\,(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$, $\quad (c + k)\,\mathbf{a} = (c\mathbf{a} + k\mathbf{a})$, $\quad c\,(k\mathbf{a}) = (ck)\,\mathbf{a}$, $\quad 1\mathbf{a} = \mathbf{a}$.



9. $\mathbf{a} = [a_1, a_2, a_3]$ *is also written as* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ *where* $\mathbf{i} = [1, 0, 0]$, $\mathbf{j} = [0, 1, 0]$ *and* $\mathbf{k} = [0, 0, 1]$.

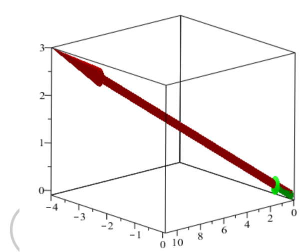


The unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *and* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Example 2 *(10ed-9.1-3) Find components of the vector with initial point* $P(-3.5, 4.0, -1.5)$ *and terminal point* $Q(7.5, 0, 1.5)$. *Find* $|\mathbf{v}|$. *Find the unit vector* $\mathbf{u}$ *in the direction of* $\mathbf{v}$. *Sketch* $\mathbf{v}$.
Solution: *We have*

$$\begin{aligned} \mathbf{v} &= [7.5 - (-3.5)\ ,\ 0 - 4.0\ ,\ 1.5 - (-1.5)] \\ &= [11.0, -4.0, 3.0] \\ |\mathbf{v}| &= \sqrt{(11.0)^2 + (-4.0)^2 + (3.0)^2} = \sqrt{146.0} = 12.083 \\ \mathbf{u} &= \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{12.083}[11.0, -4.0, 3.0] = [0.91, -0.33, 0.24] \end{aligned}$$

The sketch is given as



$\mathbf{v}$ *and* $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ *(green)*

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Example 3 *(10ed-9.1-9) Find the terminal point* Q *of the vector* $\mathbf{v}$ *with components* $3, 1, -3$ *and initial point* $P(3, -1, -1)$. *Find* $|\mathbf{v}|$.
Solution: *Let* Q *be the point* $Q(x, y, z)$, *then*

$$\begin{aligned} \overline{PQ} &= [x - 3, y + 1, z + 1] = \mathbf{v} = [3, 1, -3] \\ &\Rightarrow \left\{ \begin{array}{c} x - 3 = 3 \\ y + 1 = 1 \\ z + 1 = -3 \end{array} \right. \quad \textit{give} \quad \begin{array}{c} x = 0 \\ y = 0 \\ z = -4 \end{array} \\ \textit{Hence } Q &= (x, y, z) = (0, 0, -4) \\ |\mathbf{v}| &= \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19} \quad \blacksquare \end{aligned}$$

Example 4 *(10ed-9.1-16) For* $\mathbf{a} = [2, 3, 0]$ *and* $\mathbf{c} = [-1, 5, 3] = -\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$, *find* $\frac{6}{2}\mathbf{a} - 2\mathbf{c}$ *and* $6\left(\frac{1}{2}\mathbf{a} - \frac{1}{3}\mathbf{c}\right)$?
Solution:

$$\begin{aligned} \frac{6}{2}\mathbf{a} - 2\mathbf{c} &= \frac{6}{2}[2, 3, 0] - 2[-1, 5, 3] = [6, 9, 0] - [-2, 10, 6] = [8, -1, -6] = 8\mathbf{i} - \mathbf{j} - 6\mathbf{k} \\ 6\left(\frac{1}{2}\mathbf{a} - \frac{1}{3}\mathbf{c}\right) &= 6\left(\frac{1}{2}[2, 3, 0] - \frac{1}{3}[-1, 5, 3]\right) = 6\left(\left[1, \frac{3}{2}, 0\right] - \left[-\frac{1}{3}, \frac{5}{3}, 1\right]\right) \\ &= 6\left[\frac{4}{3}, -\frac{1}{6}, -1\right] = [8, -1, -6] = 8\mathbf{i} - \mathbf{j} - 6\mathbf{k} \quad \blacksquare \end{aligned}$$

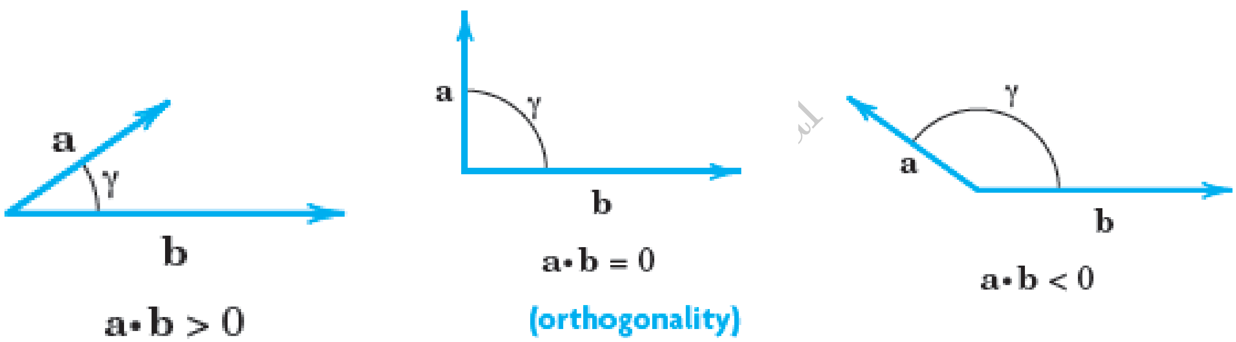
Example 5 *(10ed-9.1-24) Find the resultant in terms of components and its magnitude for* $\mathbf{p} = [-1, 2, -3]$, $\mathbf{q} = [1, 1, 1]$ *and* $\mathbf{u} = [1, -2, 2]$?
Solution: *Since resultant of force vectors is the algebraic sum of vectors, we have*

$$\begin{aligned} \mathbf{r} &= \mathbf{p} + \mathbf{q} + \mathbf{u} \\ &= [-1, 2, -3] + [1, 1, 1] + [1, -2, 2] \\ &= [-1 + 1 + 1, 2 + 1 - 2, -3 + 1 + 2] \\ &= [1, 1, 0] = \mathbf{i} + \mathbf{j} \\ |\mathbf{r}| &= \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2} \quad \blacksquare \end{aligned}$$

2 Inner Product (Dot Product)

Definition 6 *The inner product (aka dot product) of vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *with* γ $(0 \le \gamma \le \pi)$ *being the angle inbetween, is defined as* $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \gamma$. *In components form we have* $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. *Also note that* $|\mathbf{a}| = \sqrt{a \cdot a}$ *and using this one also writes*

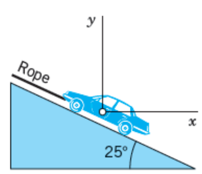
$$\cos \gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}}}.$$



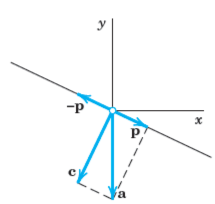
Angle between vectors and value of inner product

Remark 7 *Dot product possesses following properties:*
Linearity: $(q_1\mathbf{a} + q_2\mathbf{b}) \cdot \mathbf{c} = q_1\mathbf{a} \cdot \mathbf{c} + q_2\mathbf{b} \cdot \mathbf{c}$, *Symmetry:* $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, *Positive-definiteness:* $\mathbf{a} \cdot \mathbf{a} \ge 0$ *and* $\mathbf{a} \cdot \mathbf{a} = 0 \iff \mathbf{a} = \mathbf{0}$, *Distributivity:* $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$, *Cauchy-Schwarz inequality:* $|\mathbf{a} \cdot \mathbf{b}| \le |\mathbf{a}| |\mathbf{b}|$, *Triangle inequality:* $|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$, *Parallelogram equality:* $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2\left(|\mathbf{a}|^2 + |\mathbf{b}|^2\right)$.

Question: What force in the rope in figure will hold a car of 5000 lb in equilibrium if the ramp makes an angle of 25° with the horizontal?



Solution: Introducing coordinates as shown



the weight is $\mathbf{a} = [0, -5000]$. We have to represent $\mathbf{a}$ as sum of two forces i.e. the force exerted on the ramp by car and force dragging back the car due to slope of ramp, symbolically $\mathbf{a} = \mathbf{c} + \mathbf{p}$. A vector in direction of rope is $\mathbf{b} = [-1, \tan 25°] = [-1, 0.46631]$, thus $|\mathbf{b}| = \sqrt{(-1)^2 + (0.46631)^2} = 1.1034$.

Dirction of the balancing force has to be a unit vector opposite to that of the rope, i.e.

$$\widehat{\mathbf{u}} = -\frac{\mathbf{b}}{|\mathbf{b}|} = -\frac{1}{1.1034}[-1, 0.46631] = [0.90629, -0.42261]$$

So the required force will be in direction of opposite to $\mathbf{p}$ and, secondly, it should be such that its addition to $\mathbf{c}$ should give resultant as $\mathbf{a}$. Second condition implies, we have to find the component of $\mathbf{a}$ in direction of $\mathbf{p}$ i.e.

$$|\mathbf{p}| = \mathbf{a}\cdot\widehat{\mathbf{u}} = [0, -5000] \cdot [0.906\,29, -0.422\,61] = 2113.1 \; lb \quad \blacksquare$$

Remark 8 *For the plane $Ax + By + Cz + D = 0$, the vector $\mathbf{n} = [A, B, C]$ is normal to the given plane.*

Question: (10ed-9.2-5) Find $|\mathbf{a}+\mathbf{c}|^2 + |\mathbf{a}-\mathbf{c}|^2 - 2\left(|\mathbf{a}|^2 + |\mathbf{c}|^2\right)$ where $\mathbf{a} = [1,3,5]\,, \mathbf{b} = [4,0,8]\,, \mathbf{c} = [2,9,1]\,.$

Solution:

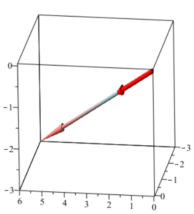
$$\begin{aligned}
& |\mathbf{a}+\mathbf{c}|^2 + |\mathbf{a}-\mathbf{c}|^2 - 2\left(|\mathbf{a}|^2 + |\mathbf{c}|^2\right) \\
= \; & |[1,3,5]|^2 + |[2,9,1]|^2 - 2\left(|[1,3,5]|^2 + |[2,9,1]|^2\right) \\
= \; & |[1,3,5]|^2 + |[2,9,1]|^2 - 2\left(|[1,3,5]|^2 + |[2,9,1]|^2\right) \\
= \; & \left(\sqrt{35}\right)^2 + \left(\sqrt{86}\right)^2 - 2\left(\left(\sqrt{35}\right)^2 + \left(\sqrt{86}\right)^2\right) \\
= \; & -121 \quad \blacksquare
\end{aligned}$$

Question: (10ed-9.2-20) Find the work done by a force $\mathbf{p} = [6, -3, -3]$ acting on a body if the body is displaced along the straight segment $\overline{AB}$ from $A : (1,5,2)$ and $B : (3,4,1)$. Sketch $\overline{AB}$ and $\mathbf{p}$. Show the details.

Solution: We have $\overline{AB} = [3-1, 4-5, 1-2] = [2,-1,-1]$ and work w is given as

$$w = \overline{AB} \cdot \mathbf{p} = [2,-1,-1] \cdot [6,-3,-3] = 18$$

Here are the sketch of both vectors $\overline{AB} = [2,-1,-1]$ and $\mathbf{p} = [6,-3,-3]$



$\blacksquare$

Question: (10ed-9.2-30) Find the distance of the point $A\,(1,0,2)$ from the plane $P : 3x + y + z = 9$. Make a sketch.

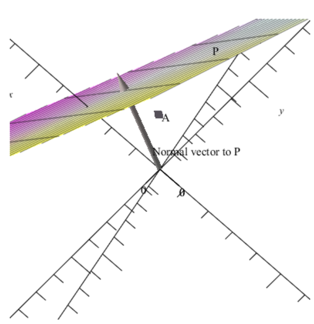
Solution: The unit vector normal to the plane P is given as

$$\mathbf{n} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{1}{\sqrt{11}}\left(3\mathbf{i} + \mathbf{j} + \mathbf{k}\right)$$

A point Q on plane is its x-intercept hence, $[3x + y + z - 9]_{x=x, y=0, z=0} \Rightarrow x = 3$ gives $Q : (3,0,0)$ and the vector $QA = [1-3, 0-0, 2-0] = [-2, 0, 2]$

Projection on plane's unit normal vector is the distance and is given as $QA \cdot \frac{\mathbf{n}}{|\mathbf{n}|} = [-2,0,2] \cdot \frac{1}{\sqrt{11}}[3,1,1] = \frac{1}{\sqrt{11}}(-6+0+2) = \frac{4}{\sqrt{11}}$

Sketch is given as



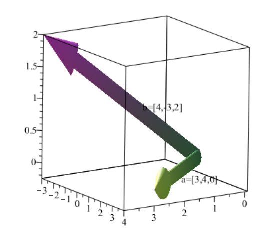
Question: (10ed-9.2-32) For what c are $P : 3x + z = 5$ and $Q : 8x - y + cz = 9$ orthogonal?
Solution: Normal vectors to the given planes are

$$\text{for } P : [3, 0, 1] \qquad \text{and for } Q : [8, -1, c]$$

Both these planes will be orthogonal iff dot product of their normal vectors is zero, i.e.

$$\begin{aligned} [3, 0, 1] \cdot [8, -1, c] &= 0 \\ 24 + c &= 0 \Rightarrow c = -24 \qquad \blacksquare \end{aligned}$$

Question: (10ed-9.2-37) Find the component of $\mathbf{a} = [3, 4, 0]$ in the direction of $\mathbf{b} = [4, -3, 2]$. Make a sketch.
Solution: The mere direction of $\mathbf{b}$ is given by its unit vector i.e. $\frac{\mathbf{b}}{|\mathbf{b}|} = \left[\frac{4}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right]$. hence the component of $\mathbf{a}$ in direction of $\mathbf{b}$ is $\mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = [3, 4, 0] \cdot \left[\frac{4}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right] = 0$. Sketch:



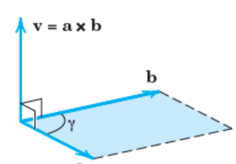
3 Vector Product (aka Cross Product, Outer Product)

Definition 9 *The **vector product** $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is the vector defined as*

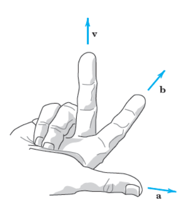
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Remark 10 *We have:*

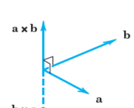
1. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \gamma$, *where γ is the angle between $\mathbf{a}$ and $\mathbf{b}$.*
2. *Magnitude (length) of $\mathbf{a} \times \mathbf{b}$ is exactly equal to the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.*



3. *The direction of $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, such that $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ (precisely in this written order) forms a right-handed triple.*



4. *If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\gamma = 0°, 180°$ then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.*

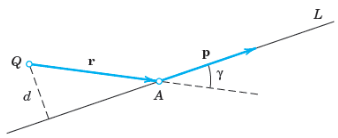
5. *Cross product is anticommutative i.e.* $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. 

Specifically we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$ *and* $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

6. *For every scalar l,*

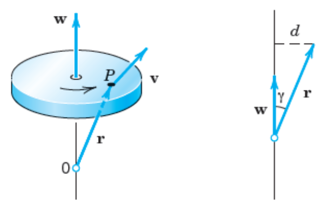
$$\begin{aligned}(l\mathbf{a})\times\mathbf{b} &= l\,(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(l\mathbf{b})\\ \mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= (\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})\\ (\mathbf{a}+\mathbf{b})\times\mathbf{c} &= (\mathbf{a}\times\mathbf{c})+(\mathbf{a}\times\mathbf{b})\\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\times\mathbf{b})\times\mathbf{c}\end{aligned}$$

7. *The moment $\mathbf{m}$ of a force $\mathbf{p}$ about a point Q is defined as $\mathbf{m}=\mathbf{r}\times\mathbf{p}$, where $\mathbf{r}$ is the vector from Q to any point A on the line of action of $\mathbf{p}$ (say) L.*



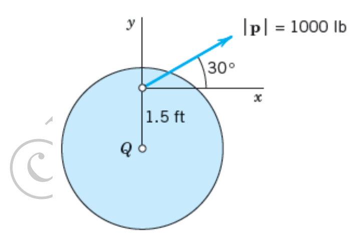
Furthermore, $|\mathbf{m}|=m=|\mathbf{p}|\,d$, where d is perpendicular distance between Q and L.

8. *The velocity vector $\mathbf{v}$ of a point P on rotating body B is given as $\mathbf{v}=\mathbf{w}\times\mathbf{r}$ (see figure for description)*



Moreover $|\mathbf{v}|=|\mathbf{w}\times\mathbf{r}|=\omega d$, where ω is the angular speed and $|\mathbf{w}|=\omega$.

Question: Find the moment of the force $\mathbf{p}$ about the centre Q of the wheel given in figure below:



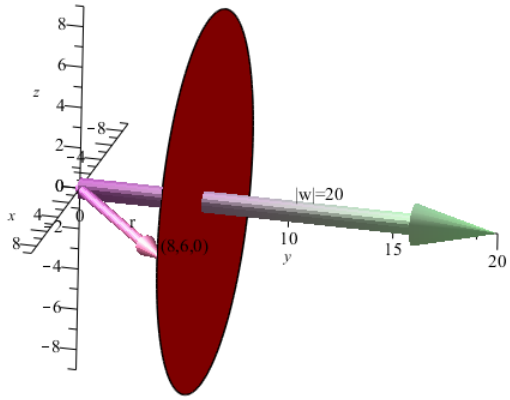
Solution: Introducing coordinates, we have $\mathbf{p}=[1000\cos 30°, 1000\sin 30°, 0]=[866,500,0]$ and $\mathbf{r}=[0,1.5,0]$. Hence the moment is given as

$$\mathbf{m}=\mathbf{r}\times\mathbf{p}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1.5 & 0\\ 866 & 500 & 0\end{vmatrix}=[0,0,-1299]=-1299\mathbf{k} \quad \blacksquare$$

Question: (10ed-9.3-7) A wheel is rotating about the y-axis with angular speed $\omega=20\,\text{sec}^{-1}$. The rotation appears clockwise if one looks from the origin in the positive y-direction. Find the velocity and speed at the point $[8,6,0]$. Make a sketch.

Solution: As the wheel is on positive y-axis and we also know that magnitude of the vector $\mathbf{w}$ equals the angular velocity ω, hence we have $\mathbf{w}=[0,20,0]$. Position vector of point $(8,6,0)$ is given as $\mathbf{r}=[8,6,0]$. Hence $\mathbf{v}=|\mathbf{w}\times\mathbf{r}|=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 20 & 0\\ 8 & 6 & 0\end{vmatrix}=$

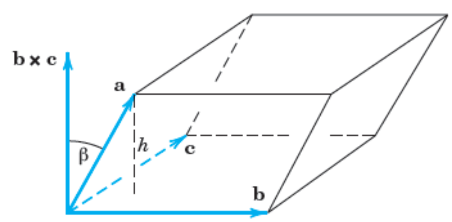
$|0\mathbf{i}+0\mathbf{j}-160\mathbf{k}| = 160$. Sketch is given as



Definition 11 *Scalar triple product (aka Mixed Product) of three vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *is defined and denoted as* $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Remark 12 *We have the properties of scalar triple product as*

1. $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$
2. $|(\mathbf{a}\ \mathbf{b}\ \mathbf{c})|$ *is the volume of parallelpiped with edges* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$.



Geometric interpretation of a scalar triple product

3. $|(\mathbf{a}\ \mathbf{b}\ \mathbf{c})| = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = |\mathbf{a}|\,|\mathbf{b}\times\mathbf{c}|\,|\cos\gamma|$

Question: (10ed-9.3-13) With respect to right-handed Cartesian coordinates, showing the details find $\mathbf{c}\times(\mathbf{a}+\mathbf{b})$ and $\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$, where $\mathbf{a} = [1,-2,0]$, $\mathbf{b} = [-2,3,0]$, $\mathbf{c} = [2,-4,-1]$.

Solution:

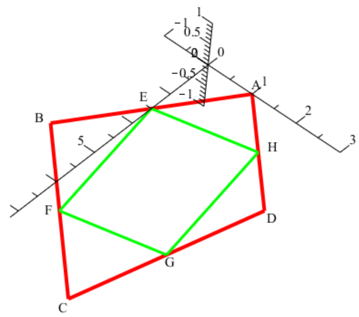
$$\begin{aligned}
\mathbf{c}\times(\mathbf{a}+\mathbf{b}) &= [2,-4,-1]\times([1,-2,0]+[-2,3,0]) \\
&= [2,-4,-1]\times[-1,1,0] \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -1 \\ -1 & 1 & 0 \end{vmatrix} = (0+1)\,\mathbf{i}-(0-1)\,\mathbf{j}+(2-4)\,\mathbf{k} = \mathbf{i}+\mathbf{j}-2\mathbf{k} \\
\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c} &= [1,-2,0]\times[2,-4,-1]+[-2,3,0]\times[2,-4,-1] \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 2 & -4 & -1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ 2 & -4 & -1 \end{vmatrix} \\
&= (2\mathbf{i}+\mathbf{j}) + (-3\mathbf{i}-2\mathbf{j}+2\mathbf{k}) \\
&= -\mathbf{i}-\mathbf{j}+2\mathbf{k} = -(\mathbf{i}+\mathbf{j}-2\mathbf{k}) \quad \blacksquare
\end{aligned}$$

Question: (10ed-9.3-28) Find the area of quadrangle Q whose vertices are the midpoints of of the sides of quadrangle P with vertices $A(2,1,0)$, $B(5,-1,0)$, $C(8,2,0)$ and $D(4,3,0)$. Verify that Q is a parallelogram.

Solution: Let E, F, G and H be mid points of the segments AB, BC, CD and AD, respectively. Then

$$E = midpoint\,(A, B) = \left(\frac{2+5}{2}, \frac{1-1}{2}, \frac{0+0}{2}\right) = \left(\frac{7}{2}, 0, 0\right)$$

Similarly $F\left(\frac{13}{2}, \frac{1}{2}, 0\right)$, $G\left(6, \frac{5}{2}, 0\right)$ and $H\,(3, 2, 0)$. Sketch is given as

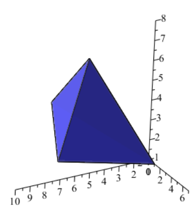


Hence area of Q is given as

$$\begin{aligned}
\left|\overline{EF} \times \overline{FG}\right| &= \left|\left[\frac{13}{2} - \frac{7}{2}, \frac{1}{2} - 0, 0 - 0\right] \times \left[6 - \frac{13}{2}, \frac{5}{2} - \frac{1}{2}, 0 - 0\right]\right| \\
&= \left|\left[3, \frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, 2, 0\right]\right| \\
&= \left|\left[0, 0, , \frac{25}{4}\right]\right| = \frac{25}{4} \quad \blacksquare
\end{aligned}$$

Question: (10ed-9.3-33) Find the volume of the tetrahedron whose vertices are $(1, 1, 1), (5, -7, 3), (7, 4, 8)$ and $(10, 7, 4)$.
Solution: $\mathbf{a} = (5, -7, 3) - (1, 1, 1) = \begin{bmatrix} 4 & -8 & 2 \end{bmatrix}$
$\mathbf{b} = (7, 4, 8) - (1, 1, 1) = \begin{bmatrix} 6 & 3 & 7 \end{bmatrix}$
$\mathbf{c} = (10, 7, 4) - (1, 1, 1) = \begin{bmatrix} 9 & 6 & 3 \end{bmatrix}$

volume of tetrahedron$=\frac{1}{6}$vol of parallelopiped$=\frac{1}{6}(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = \frac{1}{6}\begin{vmatrix} 4 & -8 & 2 \\ 6 & 3 & 7 \\ 9 & 6 & 3 \end{vmatrix} = \frac{1}{6}|-474| = 79$



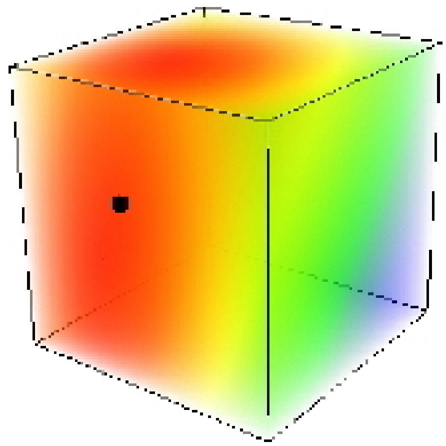
$\blacksquare$

4 Vector and Scalar Functions and their Fields

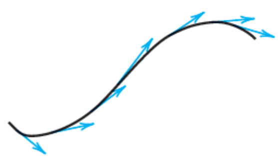
Definition 13 *For any point $P\,(x, y, z)$,*

- *the vector function $\mathbf{v}$ is defined as $\mathbf{v} = \mathbf{v}\,(P) = [v_1\,(x, y, z),\ v_2\,(x, y, z),\ v_3\,(x, y, z)]$,*
- *the scalar function f is defined as $f\,(P) = f\,(x, y, z)$.*

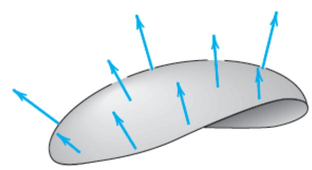
Remark 14 *Field is a region in which every point has a defined value or vector attached to it through a scalar or vector function. The field is accordingly named as scalar field or vector field. Examples of scalar fields are: Temperature field of a* *body and pressure field of air in Earth's atmosphere.*



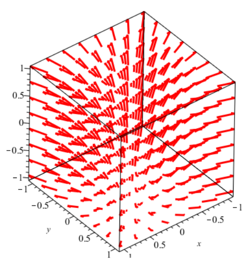
Examples of vector field are: gravitational field, electromagnetic field, flow field around an aircraft, field of tangent vectors of a curve and field of normal vectors of a surface.

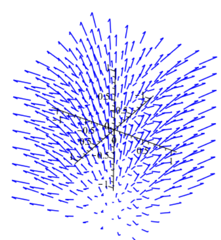


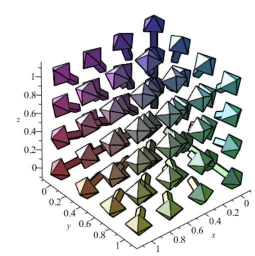
Field of tangents



Field of normals

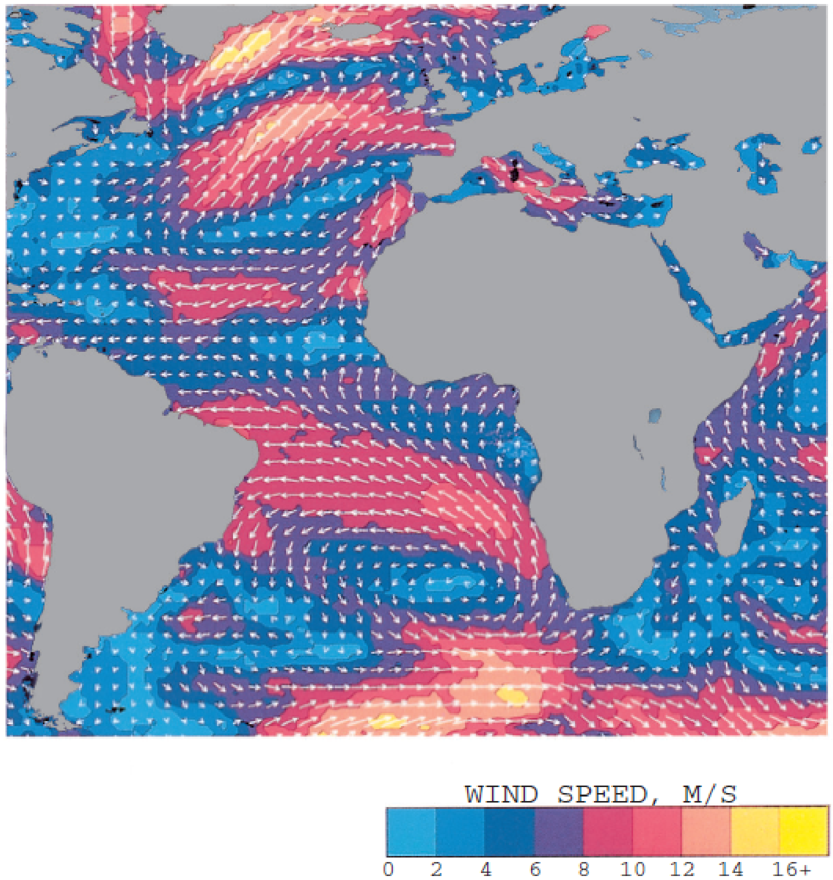






Remark 15 *NASA's Seasat used radar to take 350,000 wind measurements over the world's oceans. In the figure below, the arrows show wind direction; their length and the color contouring indicate speed: hence a vector field! Notice the heavy storm*

south of Greenland.



4.1 Vector Calculus

Definition 16 *The derivative of a vector function* $\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)]$ *is given as* $\mathbf{v}'(t) = [v_1'(t), v_2'(t), v_3'(t)]$.

Question: (10ed-9.4-5) Let the temperature T in a body be independent of z so that it is given by a scalar function $T = T(x,t)$. Identify the isotherms $T(x,y) = const$ for $T(x,y) = \frac{y}{x^2+y^2}$. Also sketch some of them.

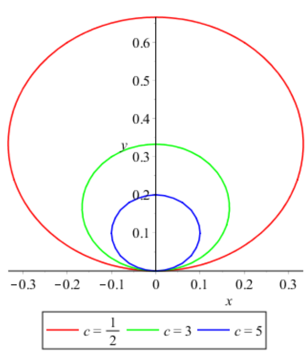
Solution: For isotherms put $T(x,y) = \frac{y}{x^2+y^2} = c$, where c is a constant. In such questions we try to unearth an equation of a familiar curve. Here denominator indicates something near to a circle, perhaps. So we proceed as follows:

$$\frac{y}{x^2+y^2} = c \Rightarrow \frac{x^2+y^2}{y} = \frac{1}{c} \Rightarrow x^2+y^2 = \frac{y}{c} \Rightarrow x^2+y^2-\frac{y}{c} = 0$$

$$\text{Completeing square } x^2+y^2-\frac{y}{c}+\left(\frac{1}{2c}\right)^2-\left(\frac{1}{2c}\right)^2 = 0$$

$$x^2+\left(y-\frac{1}{2c}\right)^2 = \frac{1}{4c^2}$$

Hence the isotherms for this scalar field are circles with centre $\left(0, \frac{1}{2c}\right)$ and radius $\frac{1}{2c}$. Isotherms for $c = \frac{1}{2}, 3$ and 5 are sketched below:



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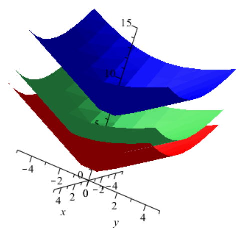
Question: (10ed-9.4-12) What kind of surfaces are the level surfaces $f(x,y,z) = const$, if $f(x,y,z) = z - \sqrt{x^2+y^2}$?

Solution: For finding the level surfaces $f(x,y,z) = z - \sqrt{x^2+y^2} = c$ where c is a constant.

$$z - \sqrt{x^2+y^2} = c \quad \Rightarrow z - c = \sqrt{x^2+y^2}$$

is a typical equation of one branch of hyperbola with origin at $(0,0,c)$. For $c = \frac{1}{2}, 2$ and 6 we have the sketches as



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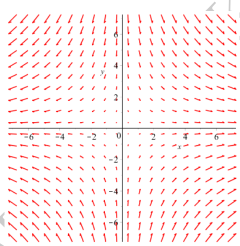
Question: (10ed-9.4-15) Sketch the vector field given by $\mathbf{v} = \mathbf{i} - \mathbf{j}$.
Solution: Consider some arbitrary points $(1,2)$, $(7,4)$ and $(2,6)$. Then

$$\begin{aligned}
\text{at point } (1,2), \quad \mathbf{v} &= \mathbf{i} - 2\mathbf{j} \quad \text{and} \quad |\mathbf{v}| = \sqrt{5} = 2.2 \\
\text{at point } (7,4), \quad \mathbf{v} &= 7\mathbf{i} - 4\mathbf{j} \quad \text{and} \quad |\mathbf{v}| = \sqrt{65} = 8.1 \\
\text{at point } (2,6), \quad \mathbf{v} &= 2\mathbf{i} - 6\mathbf{j} \quad \text{and} \quad |\mathbf{v}| = \sqrt{40} = 6.3
\end{aligned}$$

Doing the same for enough points and then on each point draw an arrow in the direction of $\mathbf{v}$ with length $|\mathbf{v}|$ we get



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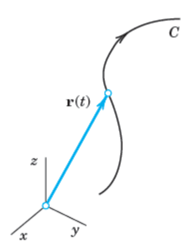
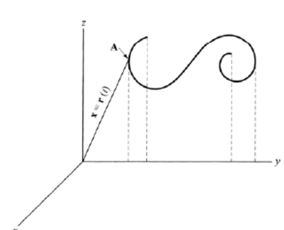
Question: (10ed-9.4-24) Find the partial derivative of $\mathbf{v}_1 = [e^x \cos y, e^x \sin y]$ and $\mathbf{v}_2 = [\cos x \cosh y, -\sin x \sinh y]$.
Solution:
$\frac{\partial \mathbf{v}_1}{\partial x} = \left[\frac{\partial}{\partial x}(e^x \cos y), \frac{\partial}{\partial x}(e^x \sin y)\right] = [e^x \cos y, e^x \sin y]$
$\frac{\partial \mathbf{v}_1}{\partial y} = \left[\frac{\partial}{\partial y}(e^x \cos y), \frac{\partial}{\partial y}(e^x \sin y)\right] = [-e^x \sin y, e^x \cos y]$
For $\mathbf{v}_2$,
$\frac{\partial \mathbf{v}_2}{\partial x} = \left[\frac{\partial}{\partial x}(\cos x \cosh y), \frac{\partial}{\partial x}(-\sin x \sinh y)\right] = [-\sin x \cosh y, -\cos x \sinh y]$
$\frac{\partial \mathbf{v}_2}{\partial y} = \left[\frac{\partial}{\partial y}(\cos x \cosh y), \frac{\partial}{\partial y}(-\sin x \sinh y)\right] = [\cos x \sinh y, -\sin x \cosh y]$ $\blacksquare$

5 Curves and Arc Length

Notation 17 *A parametric representation of a space curve is given as* $\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.





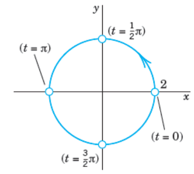
Such representation has two distinct advantages:

1. *the coordinates x, y, z play an equal role, i.e. all three are dependent variables,*

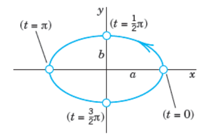
2. *this representation induces an orientation, i.e. a beginning and an end equivalently a sense of direction, of the curve.*

Example 18 *Following are few examples of space curves with their parametric representations:*

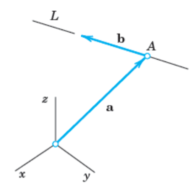
$$\mathbf{r}(t) = [a\cos t, b\sin t, 0] = a\cos(t)\,\mathbf{i} + b\sin(t)\,\mathbf{j};\ \ a = b$$



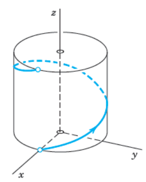
$$\mathbf{r}(t) = [a\cos t, b\sin t, 0] = a\cos(t)\,\mathbf{i} + b\sin(t)\,\mathbf{j};\ \ a \neq b$$



$$\mathbf{r}(t) = \mathbf{a} + \mathbf{b}t = [a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t]$$



$$\mathbf{r}(t) = [a\cos t, a\sin t, ct] = a\cos(t)\,\mathbf{i} + a\sin(t)\,\mathbf{j} + ct\mathbf{k}$$



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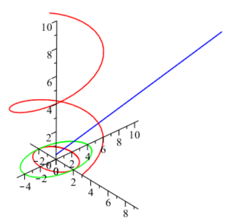
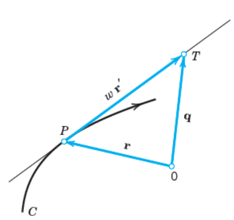
On a combined plot these are shown as:



Definition 19 *We have:*

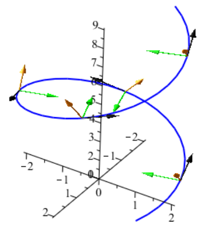
1. *The tangent vector to the curve* $\mathbf{r}(t)$ *at point* P *is* $\mathbf{r}'(t)$.
2. *Unit tangent is computed as* $\mathbf{u} = \frac{1}{|\mathbf{r}'|}\mathbf{r}'$. *Both* $\mathbf{r}'$ *and* $\mathbf{u}$ *are in the direction of increasing* t *(a blessing of parametric notation!).*
3. *Vector equation of the tangent line passing through point* P *on curve* $\mathbf{r}(t)$ *is given as* $\mathbf{q}(w) = \mathbf{r} + w\mathbf{r}'$



4. *Length of the curve* $\mathbf{r}(t)$ *from* $t = a$ *to an arbitrary point in* t *is given as* $s(t) = \int_a^t \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\,d\tau$, *where* $\mathbf{r}' = \frac{d\mathbf{r}}{d\tau}$.
5. *If a curve is representing the path of a moving body, as usually is the case in Mechanics, then velocity and acceleration of the body are given as* $\mathbf{v}(t) = \mathbf{r}'(t)$ *and* $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.
6. *Acceleration vector has its tangential and normal components i.e.* $\mathbf{a} = \mathbf{a}_{\tan} + \mathbf{a}_{norm}$, *which are obtained as* $\mathbf{a}_{\tan} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$ *and* $\mathbf{a}_{norm} = \mathbf{a} - \mathbf{a}_{\tan}$.

7. *At any given point on a space curve we have three defining unit vectors:*

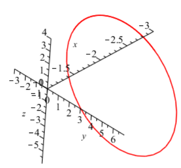
   (a) *a **unit tangent**,*
   (b) *a **unit normal** which is perpendicular to unit tangent but lies in the same plane as that of tangent, and*
   (c) *a **unit binormal**, which is perpendicular to both i.e. unit tangent and unit normal vectors.*



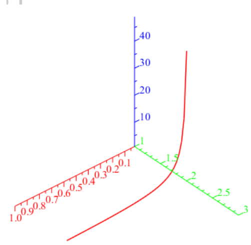
Question: (10ed-9.5-4,10) Sketch the curves $[-2, 2+5\cos t, -1+5\sin t]$ and $\left[t, 2, \frac{1}{t}\right]$?
Solution: We compute the table as

t	x-coord$=-2$	y-coord$=2+5\cos t$	z-coord$=-1+5\sin t$
0	-2	7	-1
2	-2	-0.008	3. 5
3	-2	-2.9	-0.29
5	-2	3. 4	-5.8

Carefully ploting yields:



Similarly for $\left[t, 2, \frac{1}{t}\right]$, we have



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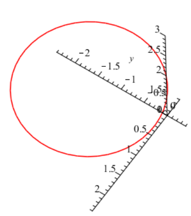
Question: (10ed-9.5-11) Find parametric representation of a circle in the plane $z = 2$ with centre $(1, -1)$ and passing through origin.
Solution: Equation of a circle with centre $(1, -1)$ is given as $(x-1)^2 + (y+1)^2 = r^2$. As $(0, 0)$ is on the circle so we have $(0-1)^2 + (0+1)^2 = r^2 \Rightarrow r = \sqrt{2}$

Parametric equation of circle with centre (h, k) and radius r is given as $[h + r\cos t, k + r\sin t]$. Hence the required parametric equation is $\left[1 + \sqrt{2}\cos t, -1 + \sqrt{2}\sin t\right]$. Sketch:



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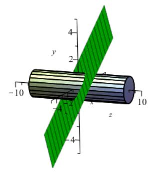
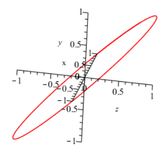
Question: (10ed-9.5-16) Find parametric representation of the intersection of the circular cylinder of radius 1 about z-axis and the plane $z = y$.

Solution: Equation of the cylinder is $x^2 + y^2 = 1$ and it calls for puting $x = \cos t$ and $y = \sin t$. This gives the equation of plane as $z = \sin t$. Hence the parametric equation of the circle of intersection is given as

$$[\cos t, \sin t, \sin t]$$

Sketches are given as





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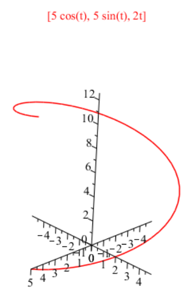
Question: (10ed-9.5-18) Helix: $x^2 + y^2 = 25$, $z = 2\arctan\left(\frac{y}{x}\right)$. Write its parametric equation.

Solution: Put $x = 5\cos t$ and $y = 5\sin t \Rightarrow z = 2\arctan\left(\frac{5\sin t}{5\cos t}\right) = 2\arctan(\tan t) = 2t$ Hence the parametric equation of the helix is given as

$$[5\cos t, 5\sin t, 2t]$$

Sketch:



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Question: (10ed-9.5-27) Given a curve $C : \mathbf{r}(t) = \left[t, \frac{4}{t}, 0\right]$, find tangent vector $\mathbf{r}'(t)$, a unit tangent vector $\mathbf{u}'(t)$ and tangent of C at $P(4, 1, 0)$.

Solution: Tangent vector: $\mathbf{r}'(t) = \frac{d}{dt}\left(\left[t, \frac{4}{t}, 0\right]\right) = \left[1, -\frac{4}{t^2}, 0\right]$

Unit tangent vector: $\mathbf{u}' = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{1}{\left|\left[1,-\frac{4}{t^2},0\right]\right|}\left[1, -\frac{4}{t^2}, 0\right] = \frac{1}{\sqrt{\frac{16}{t^4}+1}}\left[1, -\frac{4}{t^2}, 0\right] = \left[\frac{1}{\sqrt{\frac{16}{t^4}+1}}, -\frac{4}{t^2\sqrt{\frac{16}{t^4}+1}}, 0\right]$

Tangent line from $P : q(w) = [4 + w,]$

Question: (10ed-9.5-30) Find the length and sketch the curve given by $\mathbf{r}(t) = [4\cos t, 4\sin t, 5t]$ from $(4, 0, 0)$ to $(4, 0, 10\pi)$?

Solution: We have the formula $s(t) = \int_a^t \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, d\tau$.

$\mathbf{r}'(t) = \left[\frac{d}{dt}(4\cos t), \frac{d}{dt}(4\sin t), \frac{d}{dt}(5t)\right] = [-4\sin t, 4\cos t, 5] \quad \Rightarrow \quad \mathbf{r}' \cdot \mathbf{r}' = 16\cos^2 t + 16\sin^2 t + 25$

As the point $(4, 0, 0)$ is on the curve, hence for some t, $\mathbf{r}(t) = (4\cos t, 4\sin t, 5t) = (4, 0, 0) \Rightarrow 5t = 0 \Rightarrow t = 0$.

Also the second point $(4, 0, 10\pi)$ is on the curve, hence for some t, $\mathbf{r}(t) = (4\cos t, 4\sin t, 5t) = (4, 0, 10\pi) \Rightarrow 5t = 10\pi \Rightarrow t = 2\pi$.

Hence puting values in the formula

$$s = \int_0^{2\pi} \sqrt{16\cos^2 t + 16\sin^2 t + 25}\, d\tau = 2\sqrt{41}\pi \quad \blacksquare$$

Question: (10ed-9.5-35) Find speed, velocity and tangential and normal acceleration for the parabola $\mathbf{r}(t) = \left[t, 4t^2, 0\right]$?

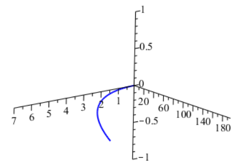
Solution: $\mathbf{v} = \mathbf{r}'(t) = [1, 8t, 0]$

$\mathbf{a} = \mathbf{v}' = \mathbf{r}''(t) = [0, 8, 0]$

$\mathbf{a}_{\tan} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{[0,8,0]\cdot[1,8t,0]}{[1,8t,0]\cdot[1,8t,0]}[1, 8t, 0] = \frac{64t}{64t^2+1}[1, 8t, 0]$

$\mathbf{a}_{norm} = \mathbf{a} - \mathbf{a}_{\tan} = [0, 8, 0] - \frac{64t}{64t^2+1}[1, 8t, 0] = \left[\frac{-64t}{64t^2+1}, 8 - \frac{512t^2}{64t^2+1}, 0\right]$

Sketch:



[t, 4*t^2, 0] ■

Question: (10ed-9.5-46) A satellite in a circular orbit 450 miles above Earth's surface and completes 1 revolution in 100 min. Find the acceleration of gravity at the orbit from these data and from the radius of Earth (3960 *miles*).

Solution: $R = 3960 + 450 = 4410$ mi.

$2\pi R = 100\,|\mathbf{v}|$ and $\mathbf{v} = 277.1$ mi/min

$g = |\mathbf{a}| = \omega^2 R = \frac{|\mathbf{v}|^2}{R} = 17.41 \text{ mi/min}^2 = 25.53\ ft/\sec^2 = 7.78\ m/\sec^2$ ■

6 Gradient of a Scalar Field

Definition 20 *Gradient of a scalar function $f(x,y,z)$ is denoted as* grad f *or* ∇f *(read as* ***nabla*** *f) and is defined as*

$$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

We also write the ***differential operator*** *as* $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$.

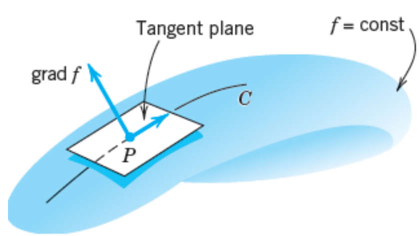
Remark 21 *Major usages of gradient are as follows:*

1. *Rate of change of $f(x,y,z)$ in any direction in space, technically called Directional Derivative.*

   Definition 22 *The directional derivative $D_{\mathbf{a}} f$ of a scalar function $f(x,y,z)$ at a point P in the direction of a vector* $\mathbf{a}$ *is given as* $D_{\mathbf{a}} f = \frac{1}{|\mathbf{a}|}\mathbf{a} \cdot \operatorname{grad} f$.

2. $\operatorname{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$ *points in the direction of maximum increase of $f(x,y,z)$.*

3. *For a curve $C = [x(t), y(t), z(t)]$ liying on a level surface $f(x,y,z) = c = const$,* grad f *is normal vector of S at P.*



4. *Obtainig vector field from a scalar field: as the gradiant of the scalar function. A vector field obtained in this manner is relatively easily studied using $f(x,y,z)$ only. For such vector fields, $f(x,y,z)$ is said to be its* ***potential****. Furthermore such vector field is said to be* ***conservative*** *if no energy is lost or gained in displacing a body from one point to another and then back.*

Definition 23 *The potential $f(x,y,z)$ of a conservative vector field satisfies the Laplace's equation, given as*

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

It is universally agreed that Laplace equation is The Most Important partial differential equation in today's Physics and its numerous applications.

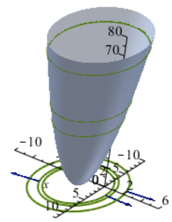
Question: (10ed-9.7-4) Find grad f where $f=(x-2)^2+(2y+4)^2$. Graph some level curves $f=const$ Indicate ∇f by arrows at some points of these curves.

Solution: grad $\left((x-2)^2+(2y+4)^2\right)=\left[\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right]\left((x-2)^2+(2y+4)^2\right)$

$=\left[\frac{\partial}{\partial x}\left((x-2)^2+(2y+4)^2\right),\frac{\partial}{\partial y}\left((x-2)^2+(2y+4)^2\right),\frac{\partial}{\partial z}\left((x-2)^2+(2y+4)^2\right)\right]=[2x-4,8y+16,0]$

We choose points $(-1,2),(2,1)$ and $(5,-5)$ for computation of gradiant vectors. Sketch is given below:

Gradient Vectors



For the function $f(x,y)=(x-2)^2+(2y+4)^2$, level curves, their projections to the xy-plane, and gradient vectors at the point(s) $[(-1,2),(2,1),(5,-5)]$. ■

Question: (10ed-9.7-10) Prove that $\nabla^2(fg)=g\nabla^2 f+2\nabla f\cdot\nabla g+f\nabla^2 g$

Proof:

$\nabla^2(fg)=\frac{\partial^2}{\partial x^2}(fg)+\frac{\partial^2}{\partial y^2}(fg)+\frac{\partial^2}{\partial z^2}(fg)$

Consider $\frac{\partial^2}{\partial x^2}(fg)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(fg)\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}g+f\frac{\partial g}{\partial x}\right)=\left(\frac{\partial^2 f}{\partial x^2}g+\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}\right)$

similar computation for $\frac{\partial^2}{\partial y^2}(fg)$ and $\frac{\partial^2}{\partial z^2}(fg)$ yield

$\nabla^2(fg)=\frac{\partial^2}{\partial x^2}(fg)+\frac{\partial^2}{\partial y^2}(fg)+\frac{\partial^2}{\partial z^2}(fg)$

$=\left(\frac{\partial^2 f}{\partial x^2}g+\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial^2 f}{\partial y^2}g+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right)+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}\right)+\left(\frac{\partial^2 f}{\partial z^2}g+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}\right)$

$=\left(\frac{\partial^2 f}{\partial x^2}g+\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right)+\left(\frac{\partial^2 f}{\partial y^2}g+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right)+\left(\frac{\partial^2 f}{\partial z^2}g+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}\right)$

$=\left(\frac{\partial^2 f}{\partial x^2}g+\frac{\partial^2 f}{\partial y^2}g+\frac{\partial^2 f}{\partial z^2}g\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)+\left(f\frac{\partial^2 g}{\partial x^2}+f\frac{\partial^2 g}{\partial y^2}+f\frac{\partial^2 g}{\partial z^2}\right)$

$=g\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)+2\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)+f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)$

$=g\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)+2\left[\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right]\cdot\left[\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z}\right]+f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)$

$=g\nabla^2 f+2\nabla f\cdot\nabla g+f\nabla^2 g$ ■

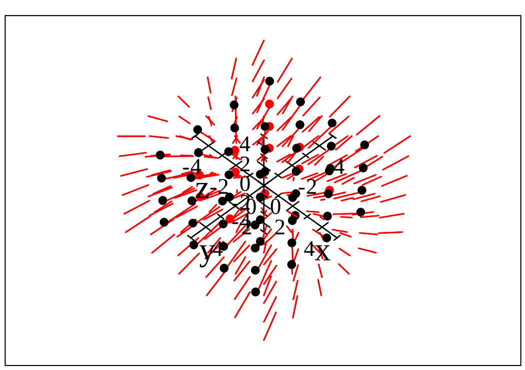
Question: (10ed-9.7-14) The force in an electric field given by $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$ has the direction of the gradient. Find ∇f and its value at $P(12,0,16)$?

Solution: $\nabla f=\nabla\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)=\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)=\frac{-1}{(x^2+y^2+z^2)^{\frac{3}{2}}}[x,y,z]$

$\nabla f|_P=\left(\frac{-1}{(x^2+y^2+z^2)^{\frac{3}{2}}}[x,y,z]\right)_{x=12,y=0,z=16}=\frac{1}{500}\left[\frac{-3}{4},0,-1\right]=\frac{-3}{2000}\mathbf{i}-\frac{\mathbf{1}}{\mathbf{500}}\mathbf{k}$ ■

Question: (10ed-9.7-16) For what points $P(x.y.z)$ does ∇f with $f=25x^2+9y^2+16z^2$ have the direction from P to the origin?

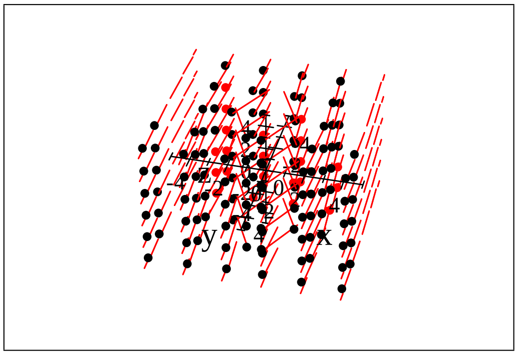
Solution: $\nabla\left(25x^2+9y^2+16z^2\right)=[50x,18y,32z]$. Presence of integer multiples of x,y and z only in the gradient indicates that all points on any of the three axes would have direction from P to origin. Sketch of gradient vector field is given as

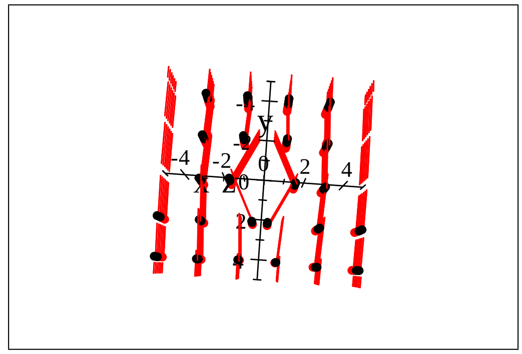


■

Question: (10ed-9.7-20) Given the velocity potential $f = x\left(1 + \left(x^2 + y^2\right)^{-1}\right)$, find the velocity $\mathbf{v} = \nabla f$ of the field and its value $\mathbf{v}(P)$ at P. Sketch $\mathbf{v}(P)$ and the curve $f = const$ passing through $P(1,1)$?

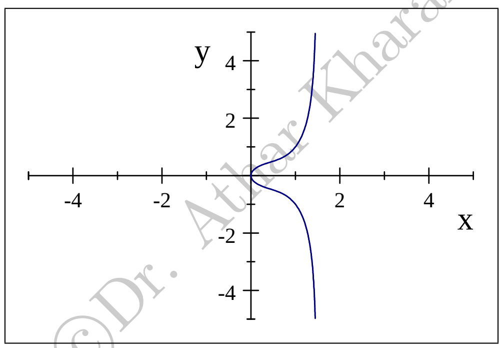
Solution: $\mathbf{v} = \nabla f = \nabla\left(x\left(1 + \left(x^2 + y^2\right)^{-1}\right)\right) = \left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + 1, -2x\frac{y}{(x^2+y^2)^2}, 0\right]$ and this velocity field is sketched below (two different views) as follows:





$\mathbf{v}(P) = \left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + 1, -2x\frac{y}{(x^2+y^2)^2}, 0\right]_{x=1,y=1} = \left[1, -\frac{1}{2}, 0\right]$

Equation of curve is of the form $x\left(1 + \left(x^2 + y^2\right)^{-1}\right) = c$, since the curve has to pass through $P(1,1)$, hence to find c put $x = 1, y = 1$ in f i.e. $x\left(1 + \left(x^2 + y^2\right)^{-1}\right)|_{(1,1)} = \frac{3}{2} = c$. Equation of curve passing through $P(1,1)$ is $x\left(1 + \left(x^2 + y^2\right)^{-1}\right) = \frac{3}{2}$. Sketch of the required curve is given as:



■

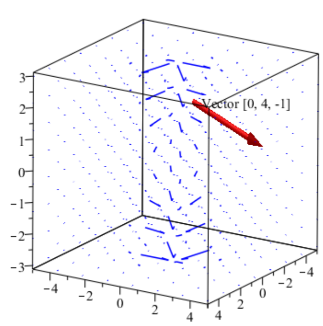
Question: (10ed-9.7-25) Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature $T = \frac{z}{(x^2+y^2)}$. Find this direction in general and at the given point $P(0,1,2)$. Sketch that direction at P as an arrow.

Solution: As ∇f points in the direction of maximum increase hence

Direction of maximum decrease$= -\nabla T = -\nabla\left(\frac{z}{(x^2+y^2)}\right) = \left[\frac{2xz}{(x^2+y^2)^2}, \frac{2yz}{(x^2+y^2)^2}, -\frac{1}{x^2+y^2}\right]$

Direction of maximum decrease at $P = \left[\frac{2xz}{(x^2+y^2)^2}, \frac{2yz}{(x^2+y^2)^2}, -\frac{1}{x^2+y^2}\right]_{x=0,y=1,z=2} = [0, 4, -1]$

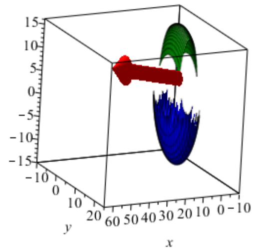
Sketch of general direction of the field and at point P are given as



■

Question: (10ed-9.7-33) Find the normal vector of the surface $6x^2 + 2y^2 + z^2 = 225$ at the point $P(5,5,5)$?

Solution: Since on a surface $f(x,y,z) = const$ normal vector is given by ∇f, hence we compute $\nabla\left(6x^2+2y^2+z^2\right) = [12x, 4y, 2z]$ and the normal at the given point$= [12x, 4y, 2z]_{x=5,y=5,z=5} = [60, 20, 10]$. We sketch the surface and the normal as



■

Question: (10ed-9.7-39) Find directional derivative of $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ at $P(3,0,4)$ in the direction of $\mathbf{a} = [1,1,1]$. Also sketch it.

Solution: We have $D_{\mathbf{a}}f = \frac{1}{|\mathbf{a}|}\mathbf{a}\cdot \operatorname{grad} f$. So we compute

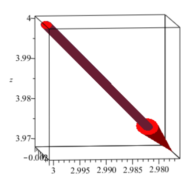
$\operatorname{grad} f = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \left[-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$

$D_{\mathbf{a}}f = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]\cdot\left[-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$

$D_{\mathbf{a}}f|_{(3,0,4)} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]\cdot\left[-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]_{(3,0,4)}$

$= \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]\cdot\left[-\frac{3}{125}, 0, -\frac{4}{125}\right] = -\frac{7}{375}\sqrt{3} = -.032$

Hence the required vector is $\left[-\frac{3}{125}, 0, -\frac{4}{125}\right]$ at the point $P(3,0,4)$ and is sktched as:



■

Question: Find a potential $f = \operatorname{grad} f$ for the given $\mathbf{v}(x,y,z) = \left[ye^x, e^x, z^2\right]$.

Solution: By examining $\mathbf{v} = \left[ye^x, e^x, z^2\right]$, we conclude that $f(x,y,z) = ye^x + \frac{1}{3}z^3$, because

$$\begin{aligned}\operatorname{grad} f &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) f(x,y,z)\\ &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(ye^x+\frac{1}{3}z^3\right)\\ &= \left[ye^x, e^x, z^2\right] \quad \blacksquare\end{aligned}$$

Question: Find a unit normal vector $\mathbf{n}$ of the cone of revolution $z^2 = 4\left(x^2+y^2\right)$ at the point $P:(1,0,2)$

Solution: The cone is the level surface $f = 0$ of $f(x,y,z) = 4\left(x^2+y^2\right) - z^2$. Thus

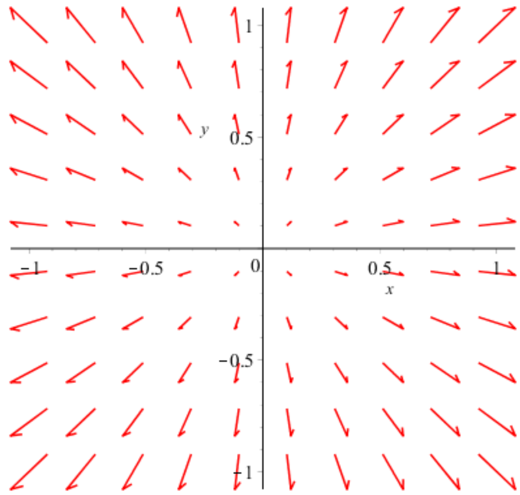
$$\begin{aligned}\nabla f &= [8x, 8y, -2z], \quad \nabla f(P) = [8, 0, -4]\\ \mathbf{n} &= \frac{1}{|\nabla f(P)|}\nabla f(P) = \left[\frac{2}{\sqrt{5}}, 0, \frac{-1}{\sqrt{5}}\right]\end{aligned}$$

$\mathbf{n}$ points downward since it has a negative z-component. The other unit normal vector of the cone at P is $-\mathbf{n}$. ■

7 Divergence of a Vector Field

From a scalar field we can obtain a vector field by the gradient. Conversely, from a vector field we can obtain a scalar field by the divergence or another vector field by the curl.

Definition 24 *For a differentiable vector function* $\mathbf{v}(x,y,z) = [v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)]$, *the divergence is denoted and defined as* $\operatorname{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$.



Visualization of Divergence

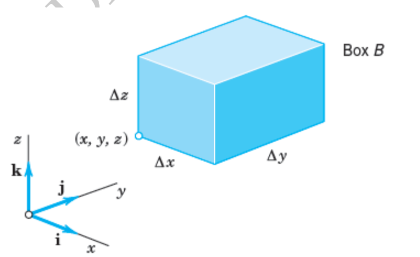
Remark 25 1. $\operatorname{div}\mathbf{v}$ *is also denoted as* $\nabla\cdot\mathbf{v}$ *because* $\operatorname{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \left[\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right]\cdot[v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}] = \nabla\cdot\mathbf{v}$

2. *If* $\mathbf{v} = \operatorname{grad}(f)$ *and* f *is twice differentiable scalar function* $f(x,y,z)$, *we have* $\operatorname{div}\mathbf{v} = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$.

We now intend to present a physical interpretation of the notion of divergence. For this we need following definitions:

Definition 26 ***Flux*** *is the total loss of mass leaving an object per unit of time.* ***Compressible fluid*** *is the fluid whose density* ρ *(mass per unit volume) depends upon coordinates* x, y, z *(and possibly on time* t*). Examples are gases and vapors. Water is an incompressible fluid. If density* ρ *is independent of time* t*, the flow is said to be* ***steady.***

Question: Give a physical interpretation of divergence? OR Derive equation of continuity for fluids?
Derivation: Consider motion of a compressible fluid in region R with no source or sink in R. Consider flow of the fluid through the box B with volume $\Delta V = \Delta x\Delta y\Delta z$ (here Δ is denoting a small quantity and not the Laplacian).



Physical interpretation of the divergence

Let $\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be the velocity vector of the motion. We set

$$\mathbf{u} = \rho\mathbf{v} = [u_1, u_2, u_3] = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

Consider the flow through the xz face whose area is $\Delta x\Delta z$. Since the vectors $v_1\mathbf{i}$ and $v_3\mathbf{k}$ are parallel to xz face, the components v_1 and v_3 contribute nothing to this flow.

Hence the mass of fluid entering through xz face during a short time interval Δt is

$$(\rho v_2)\,\Delta x\Delta z\Delta t = (u_2)_y\,\Delta x\Delta z\Delta t$$

and the mass leaving from opposite face is $(u_2)_{y+\Delta y}\,\Delta x\Delta z\Delta t$. Hence the difference

$$\Delta u_2\Delta x\Delta y\Delta z = \frac{\Delta u_2}{\Delta y}\Delta y\Delta t \quad \text{where} \quad \Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y$$

is the approximate loss of mass. Other two faces also give similar expressions and the total loss of mass in B during the time interval Δt is approximately

$$\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right)\Delta V\Delta t \tag{1}$$

where $\Delta u_1 = (u_1)_{x+\Delta x} - (u_1)_x$ and $\Delta u_3 = (u_3)_{z+\Delta z} - (u_3)_z$.

This loss of mass in B is caused by the time rate of change of the density and is thus equals to

$$-\frac{\partial \rho}{\partial t}\Delta V \Delta t \tag{2}$$

Equating (1) and (2) and letting small changes approach to zero we get

$$\begin{aligned} \operatorname{div} \mathbf{u} &= \operatorname{div}(\rho \mathbf{v}) = -\frac{\partial \rho}{\partial t} \\ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0 \end{aligned}$$

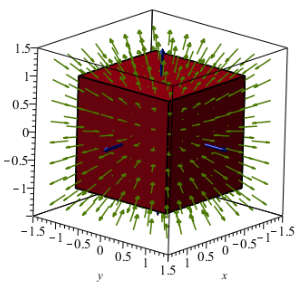
is the '**equation of continuity** of a compressible fluid flow' also called 'condition for the coservation of mass'.

If the flow is steady then $\frac{\partial \rho}{\partial t} = 0$ and the equation becomes $\operatorname{div}(\rho \mathbf{v}) = 0$ and if the density is constant i.e. the fluid is incompressible then $\operatorname{div} \mathbf{v} = 0$. ■

Question: (10ed-9.8-5) Find $\operatorname{div} \mathbf{v}$ at $P(-1,3,-2)$ where $\mathbf{v} = [x^2yz, xy^2z, xyz^2]$?

Solution: $\operatorname{div} \mathbf{v} = \operatorname{div}\left([x^2yz, xy^2z, xyz^2]\right) = 6xyz \Rightarrow \operatorname{div} \mathbf{v}|_{P(-1,3,-2)} = 6(-1 \times 3 \times -2) = 36$

The vector field $\mathbf{v} = [x^2yz, xy^2z, xyz^2]$ around a box is sketched below:



The vector field arrows, the surface through which the field passes, and vectors normal to the surface. ■

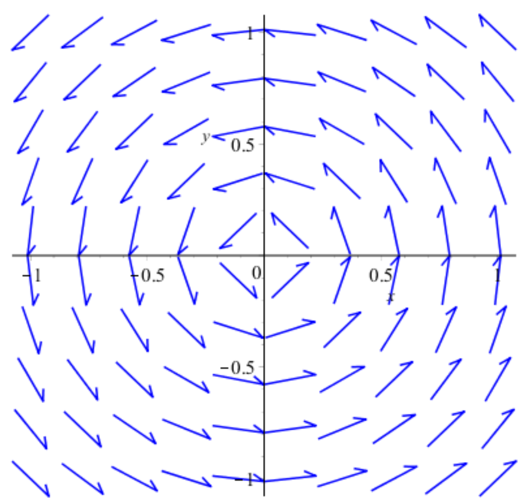
Question: (10ed-9.8-17) Find $\nabla^2 f$ by the formula $\nabla^2 f = \operatorname{div}(\operatorname{grad} f)$, where $f = \ln(x^2 + y^2)$?

Solution: $\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \operatorname{div}\left(\operatorname{grad}\left(\ln\left(x^2+y^2\right)\right)\right) = \operatorname{div}\left(\left[\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}, 0\right]\right) = \frac{4}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} - \frac{4x^2}{(x^2+y^2)^2} = 0$ ■

8 Curl of a Vector Field

Definition 27 *Curl of a vector function* $\mathbf{v}(x,y,z) = [v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)]$ *is defined as*

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$



Visualization of Curl

Remark 28 *We have:*

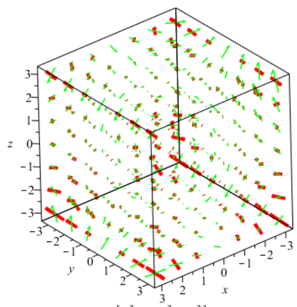
1. ***Gradient fields are irrotational.*** *That is, if a continuously differentiable vector function is the gradient of a scalar function, then its curl is the zero vector,*

$$\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$$

2. *Furthermore, the divergence of the curl of a twice continuously differentiable vector function* $\mathbf{v}$ *is zero,*

$$\operatorname{div}(\operatorname{curl} f) = 0.$$

Question: (10ed-9.9-5) Find $\operatorname{curl}\mathbf{v}$ for $\mathbf{v} = xyz\left[x^2, y^2, z^2\right]$?
Solution: $\operatorname{curl}\mathbf{v} = \operatorname{curl}\left(xyz\left[x^2, y^2, z^2\right]\right)$
$= \operatorname{curl}\left(\left[x^3yz, xy^3z, xyz^3\right]\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3yz & xy^3z & xyz^3 \end{vmatrix} = \left(xz^3 - xy^3\right)\mathbf{i} + \left(x^3y - yz^3\right)\mathbf{j} + \left(y^3z - x^3z\right)\mathbf{k}$
The vector field and its curl are sketched below:



Vector field $[x^3yz, xy^3, xyz^3]$ and its Curl ■

Question: (10ed-9.9-10) Let $\mathbf{v} = [\sec x, \csc x, 0]$ be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles).
Solution: $\operatorname{curl}([\sec x, \csc x, 0]) = -\frac{\cos x}{\sin^2 x}\mathbf{k} \neq \mathbf{0} \Rightarrow$ Not irrotational.
$\operatorname{div}([\sec x, \csc x, 0]) = \frac{1}{\cos^2 x}\sin x \neq 0 \Rightarrow$ Compressible. ■
Question: (10ed-9.9-11) Let $\mathbf{v} = [y, -2x, 0]$ be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles).
Solution: $\operatorname{curl}(\mathbf{v}) = \operatorname{curl}([y, -2x, 0]) = -3\mathbf{k} \neq \mathbf{0} \Rightarrow$ Not irrotational.
$\operatorname{div}([y, -2x, 0]) = 0 \Rightarrow$ incompressible. ■