Advanced Engineering Mathematics

Vector Differential Calculus

Dr Athar Kharal atharkharal@gmail.com

Humanities and Science Department
College of Aeronautical Engineering
National University of Sciences and Tecnology (NUST), Pakistan

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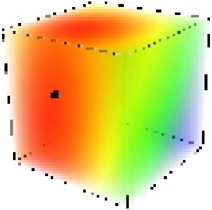

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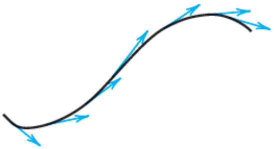

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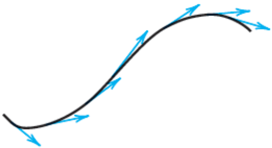

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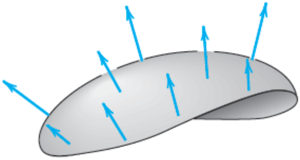

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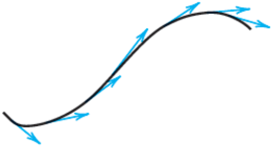

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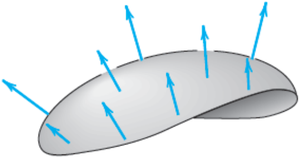

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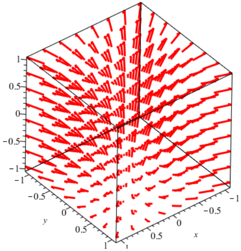

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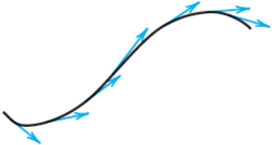

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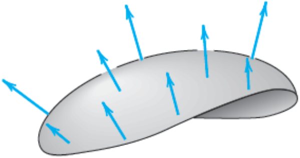

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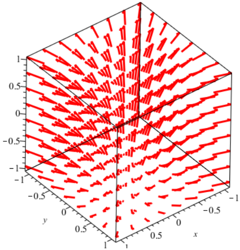

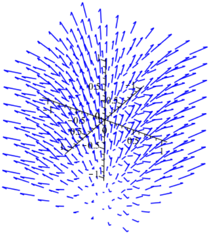

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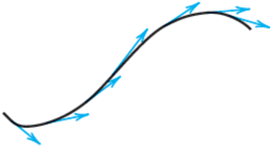

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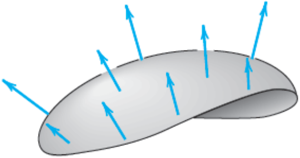

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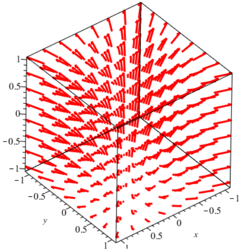
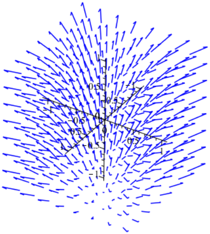
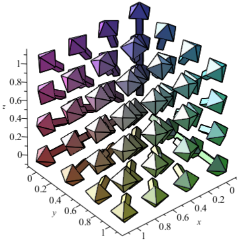
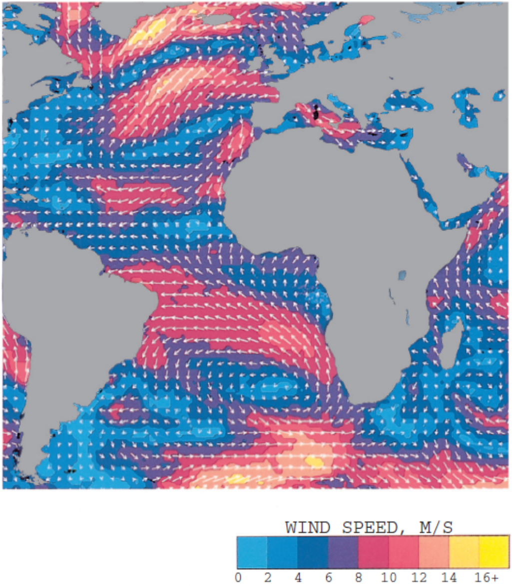
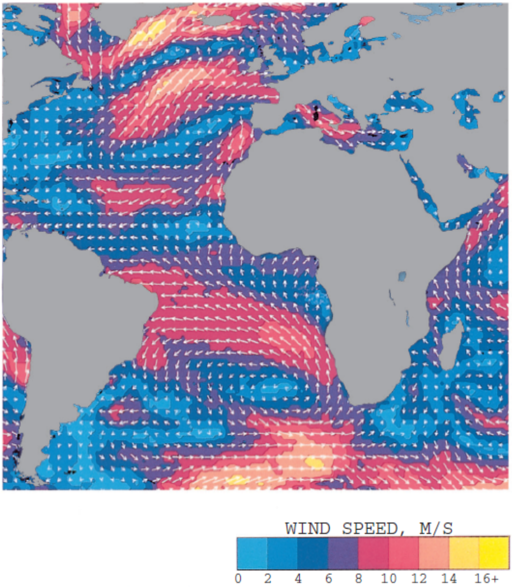


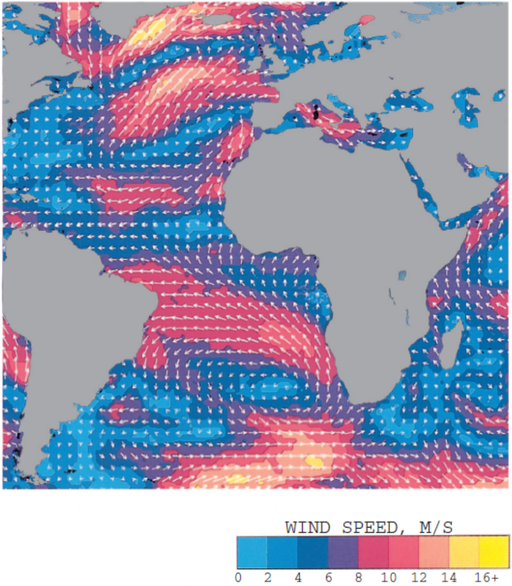





WIND SPEED, M/S
0 2 4 6 8 10 12 14 16+



NASA's Seasat used radar to take 350,000 wind measurements over the world's oceans. In the figure, the arrows show wind direction; their length and the color contouring indicate speed:



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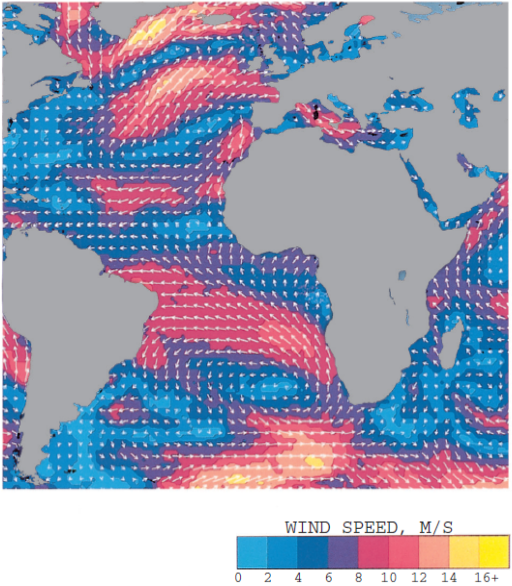

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The derivative of a vector function $\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)]$ is given as $\mathbf{v}'(t) = [v_1'(t), v_2'(t), v_3'(t)]$.

Question: (10ed-9.4-5) Let the temperature T in a body be independent of z so that it is given by a scalar function $T = T(x, t)$. Identify the isotherms $T(x, y) = const$ for $T(x, y) = \frac{y}{x^2+y^2}$. Also sketch some of them.

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Completeing square

$$x^2 + y^2 - \frac{y}{c} + \left(\frac{1}{2c}\right)^2 - \left(\frac{1}{2c}\right)^2 = 0$$

$$x^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{4c^2}$$

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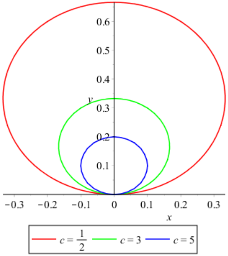
Isotherms for $c = \frac{1}{2}, 3$ and 5 are sketched below:



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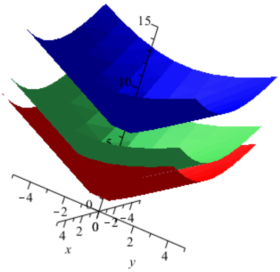

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Solution: Consider some arbitrary points $(1,2)$, $(7,4)$ and $(2,6)$.
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$$\text{at point }(1,2)\,,\ \mathbf{v}=\mathbf{i}-2\mathbf{j}\ \text{ and }\ |\mathbf{v}|=\sqrt{5}=2.2$$
$$\text{at point }(7,4)\,,\ \mathbf{v}=7\mathbf{i}-4\mathbf{j}\ \text{ and }\ |\mathbf{v}|=\sqrt{65}=8.1$$
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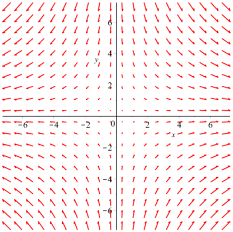
Doing the same for enough points and then on each point draw an arrow in the direction of $\mathbf{v}$ with length $|\mathbf{v}|$ we get



Question: (10ed-9.4-24) Find the partial derivative of $\mathbf{v}_1 = [e^x \cos y, e^x \sin y]$ and $\mathbf{v}_2 = [\cos x \cosh y, -\sin x \sinh y]$.

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Solution:

$$\frac{\partial \mathbf{v}_1}{\partial x} = \left[\frac{\partial}{\partial x}\left(e^x \cos y\right), \frac{\partial}{\partial x}\left(e^x \sin y\right)\right] = [e^x \cos y, e^x \sin y]$$

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\end{aligned}$$

For $\mathbf{v}_2$,

$$\begin{aligned}
\frac{\partial \mathbf{v}_2}{\partial x} &= \left[\frac{\partial}{\partial x}\left(\cos x \cosh y\right), \frac{\partial}{\partial x}\left(-\sin x \sinh y\right)\right] \\
&= [-\sin x \cosh y, -\cos x \sinh y] \\
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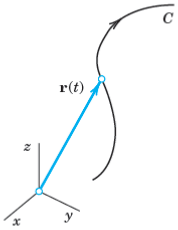
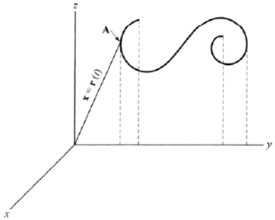
&= [\cos x \sinh y, -\sin x \cosh y] \quad \blacksquare
\end{aligned}$$

Curves and Arc Length

A parametric representation of a space curve is given as
$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$





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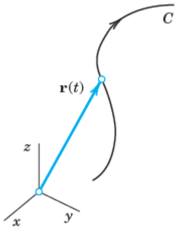

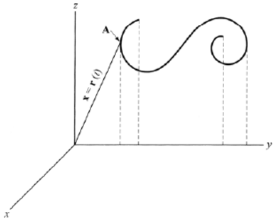

Such a representation has two distinct advantages:

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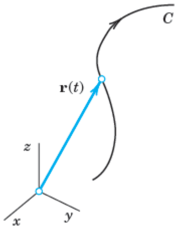

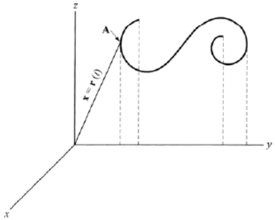

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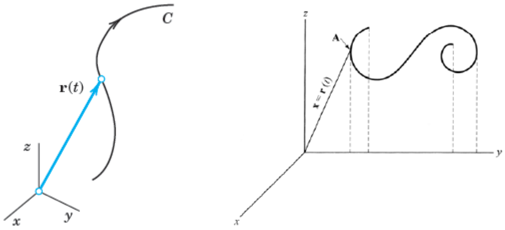

Such a representation has two distinct advantages:

1. the coordinates x, y, z play an equal role, i.e. all three are dependent variables,
2. this representation induces an orientation, i.e. a beginning and an end equivalently a sense of direction, of the curve.

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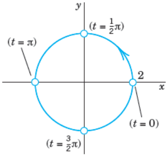
$\mathbf{r}(t) = [h + a\cos t, k + a\sin t, 0] = (h + a\cos(t))\,\mathbf{i} + (k + a\sin(t))\,\mathbf{j}$; radius$=a$, centre$=(h, k)$



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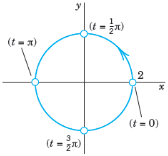

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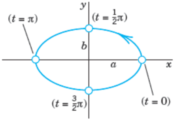

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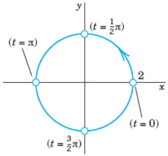

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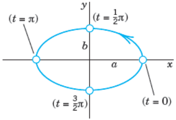

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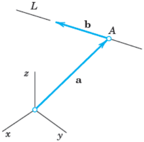

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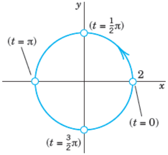

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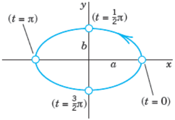

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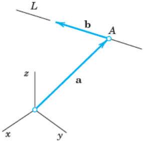
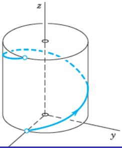
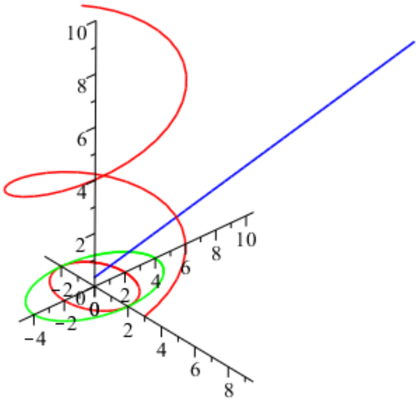


$\mathbf{r}(t) = [a\cos t, a\sin t, ct] = a\cos(t)\,\mathbf{i} + a\sin(t)\,\mathbf{j} + ct\mathbf{k}$





10
8
6
4
2
0
-2
-4
2
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6
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10

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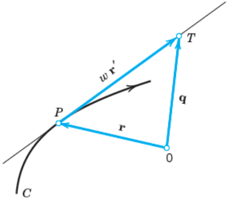

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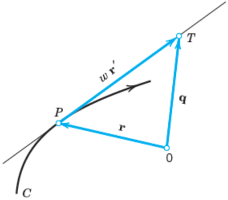

- Length of the curve $\mathbf{r}(t)$ from $t = a$ to an arbitrary point in t is given as $s(t) = \int_a^t \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, d\tau$, where $\mathbf{r}' = \frac{d\mathbf{r}}{d\tau}$.

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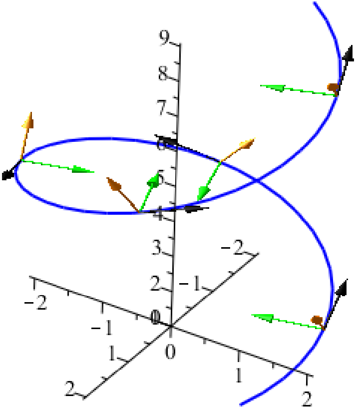
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Question: (10ed-9.5-4,10) Sketch the curves $\left[-2, 2+5\cos t, -1+5\sin t\right]$ and $\left[t, 2, \frac{1}{t}\right]$?

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Solution: We compute the table as

t	x-coord$=-2$	y-coord$=2+5\cos t$	z-coord$=-1+5\sin t$
0	-2	7	-1
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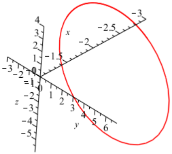
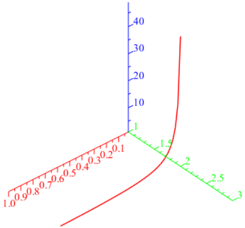
Carefully ploting yields and by a similar computation for $\left[t, 2, \frac{1}{t}\right]$, we have:





Question: (10ed-9.5-11) Find parametric representation of a circle in the plane $z = 2$ with centre $(1, -1)$ and passing through origin.

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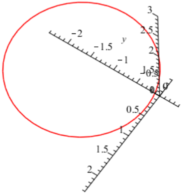
Parametric equation of circle with centre (h, k) and radius r is given as $[h + r\cos t, k + r\sin t]$. Hence the required parametric equation is $\left[1 + \sqrt{2}\cos t, -1 + \sqrt{2}\sin t\right]$. Sketch:



Question: (10ed-9.5-16) Find parametric representation of the intersection of the circular cylinder of radius 1 about z-axis and the plane $z = y$.

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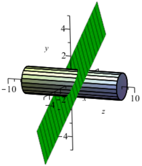
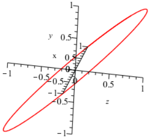
Solution: Equation of the cylinder is $x^2 + y^2 = 1$ and it calls for puting $x = \cos t$ and $y = \sin t$. This gives the equation of plane as $z = \sin t$. Hence the parametric equation of the circle of intersection is given as

$$[\cos t, \sin t, \sin t]$$

Sketches are given as





Question: (10ed-9.5-18) Helix: $x^2 + y^2 = 25, z = 2\arctan\left(\frac{y}{x}\right).$
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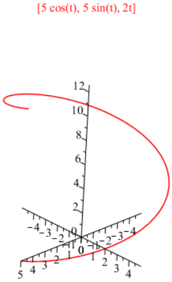
Solution: Put $x = 5\cos t$ and $y = 5\sin t \Rightarrow z = 2\arctan\left(\frac{5\sin t}{5\cos t}\right) = 2\arctan(\tan t) = 2t$ Hence the parametric equation of the helix is given as

$$[5\cos t, 5\sin t, 2t]$$

Sketch:



Question: (10ed-9.5-27) Given a curve $C : \mathbf{r}(t) = \left[t, \frac{4}{t}, 0\right]$, find tangent vector $\mathbf{r}'(t)$, a unit tangent vector $\mathbf{u}'(t)$ and tangent of C at $P(4, 1, 0)$.

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$$\frac{1}{\sqrt{\frac{16}{t^4}+1}}\left[1, -\frac{4}{t^2}, 0\right] = \left[\frac{1}{\sqrt{\frac{16}{t^4}+1}}, -\frac{4}{t^2\sqrt{\frac{16}{t^4}+1}}, 0\right]$$

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Tangent line from $P : q(w) = [4 + w,]$

Question: (10ed-9.5-30) Find the length and sketch the curve given by $\mathbf{r}(t) = [4\cos t, 4\sin t, 5t]$ from $(4, 0, 0)$ to $(4, 0, 10\pi)$?

Solution: We have the formula $s(t) = \int_a^t \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, d\tau$.

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$[-4\sin t, 4\cos t, 5] \quad \Rightarrow \quad \mathbf{r}' \cdot \mathbf{r}' = 16\cos^2 t + 16\sin^2 t + 25$

As the point $(4,0,0)$ is on the curve, hence for some t, $\mathbf{r}(t) = (4\cos t, 4\sin t, 5t) = (4,0,0) \Rightarrow 5t = 0 \Rightarrow t = 0$.

Also the second point $(4,0,10\pi)$ is on the curve, hence for some t, $\mathbf{r}(t) = (4\cos t, 4\sin t, 5t) = (4,0,10\pi) \Rightarrow 5t = 10\pi \Rightarrow t = 2\pi$.

Hence puting values in the formula

$$s = \int_0^{2\pi} \sqrt{16\cos^2 t + 16\sin^2 t + 25}\, d\tau = 2\sqrt{41}\pi \qquad \blacksquare$$

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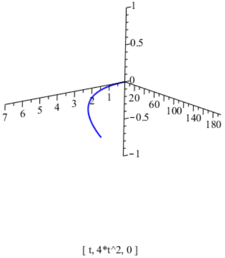
Solution: $\mathbf{v} = \mathbf{r}'(t) = [1, 8t, 0]$

$\mathbf{a} = \mathbf{v}' = \mathbf{r}''(t) = [0, 8, 0]$

$\mathbf{a}_{\tan} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{[0,8,0]\cdot[1,8t,0]}{[1,8t,0]\cdot[1,8t,0]}[1, 8t, 0] = \frac{64t}{64t^2+1}[1, 8t, 0]$

$\mathbf{a}_{norm} = \mathbf{a} - \mathbf{a}_{\tan} = [0, 8, 0] - \frac{64t}{64t^2+1}[1, 8t, 0] =$
$\left[\frac{-64t}{64t^2+1}, 8 - \frac{512t^2}{64t^2+1}, 0\right]$

Sketch:



Question: (10ed-9.5-46) A satellite in a circular orbit 450 miles above Earth's surface and completes 1 revolution in 100 min . Find the acceleration of gravity at the orbit from these data and from the radius of Earth (3960 *miles*)?

Question: (10ed-9.5-46) A satellite in a circular orbit 450 miles above Earth's surface and completes 1 revolution in 100 min . Find the acceleration of gravity at the orbit from these data and from the radius of Earth (3960 *miles*)?

Solution: $R = 3960 + 450 = 4410$ mi.
$2\pi R = 100\,|\mathbf{v}|$ and $\mathbf{v} = 277.1$ mi/min
$g = |\mathbf{a}| = \omega^2 R = \frac{|\mathbf{v}|^2}{R} = 17.41$ mi/min$^2 = 25.53\ ft/\sec^2 = 7.78$ $m/\sec^2$ ■

Definition

Gradient of a scalar function $f(x, y, z)$ is denoted as $\text{grad}\, f$ or ∇f (read as **nabla** f) and is defined as

$$\text{grad}\, f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

Gradient of a Scalar Field

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We also write the **differential operator** as $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$.

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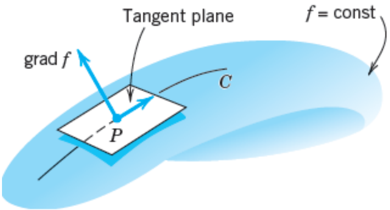

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Definition

The potential $f(x, y, z)$ of a conservative vector field satisfies the Laplace's equation, given as

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It is universally agreed that Laplace equation is The Most Important partial differential equation in today's Physics and its numerous applications.

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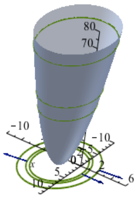
Gradient Vectors



For the function $f(x,y) = (x-2)^2 + (2y+4)^2$, level curves, their projections to the xy-plane, and gradient vectors at the point(s) $[(-1,2), (2,1), (5,-5)]$.

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$$= g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g \qquad \blacksquare$$

Question: (10ed-9.7-14) The force in an electric field given by $f(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ has the direction of the gradient. Find ∇f and its value at $P(12, 0, 16)$?

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Solution: $\nabla f = \nabla\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right) = \left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right) = \frac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}[x,y,z]$

$\nabla f|_P = \left(\frac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}[x,y,z]\right)_{x=12,y=0,z=16} = \frac{1}{500}\left[\frac{-3}{4},0,-1\right] = \frac{-3}{2000}\mathbf{i} - \frac{1}{500}\mathbf{k}$ ■

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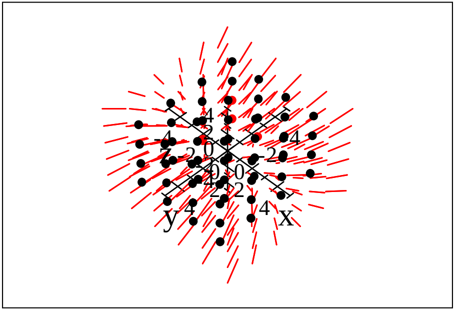
Solution: $\nabla\left(25x^2 + 9y^2 + 16z^2\right) = [50x, 18y, 32z]$. Presence of integer multiples of x, y and z only in the gradient indicates that all points on any of the three axes would have direction from P to origin. Sketch of gradient vector field is given as



■

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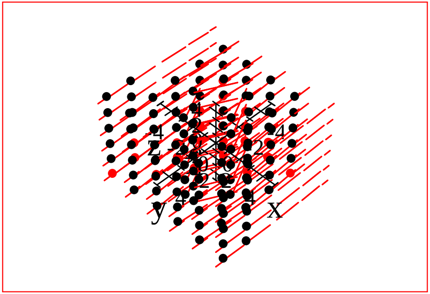
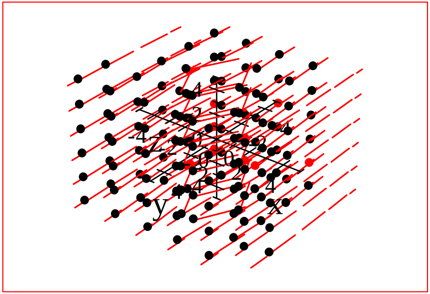
Solution: $\mathbf{v} = \nabla f = \nabla\left(x\left(1 + \left(x^2 + y^2\right)^{-1}\right)\right) =$ $\left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + 1, -2x\frac{y}{(x^2+y^2)^2}, 0\right]$ and this velocity field is sketched below (two different views) as follows:





Equation of curve is of the form $x\left(1+\left(x^2+y^2\right)^{-1}\right)=c$, since the curve has to pass through $P\,(1,1)$, hence to find c put $x=1, y=1$ in f i.e. $x\left(1+\left(x^2+y^2\right)^{-1}\right)|_{(1,1)}=\frac{3}{2}=c$.
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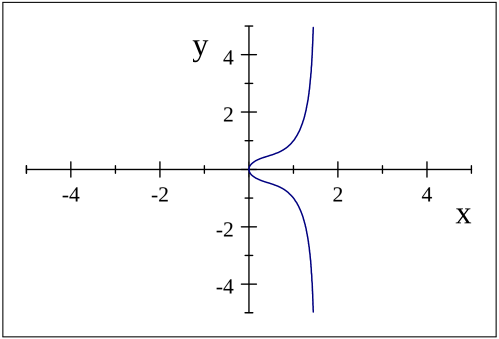
Equation of curve passing through $P\left(1,1\right)$ is $x\left(1+\left(x^2+y^2\right)^{-1}\right)=\frac{3}{2}$. Sketch of the required curve is an implicit graph and is given as:



Question: (10ed-9.7-25) Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature $T = \frac{z}{(x^2+y^2)}$. Find this direction in general and at the given point $P\,(0,1,2)$. Sketch that direction at P as an arrow.

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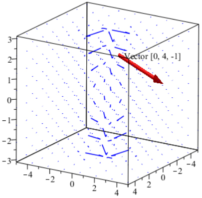
Sketch of general direction of the field and at point P are given as



Question: (10ed-9.7-33) Find the normal vector of the surface $6x^2 + 2y^2 + z^2 = 225$ at the point $P(5, 5, 5)$?

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Solution: Since on a surface $f(x, y, z) = const$ normal vector is given by ∇f, hence we compute $\nabla(6x^2 + 2y^2 + z^2) = [12x, 4y, 2z]$ and the normal at the given point$= [12x, 4y, 2z]_{x=5, y=5, z=5} = [60, 20, 10]$.

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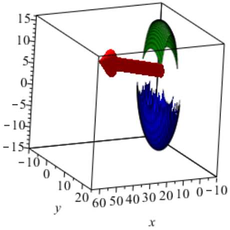

Question: (10ed-9.7-39) Find directional derivative of $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ at $P(3, 0, 4)$ in the direction of $\mathbf{a} = [1, 1, 1]$. Also sketch it.

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$$\operatorname{grad} f = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) =$$

$$\left[-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

$$D_{\mathbf{a}} f =$$

$$\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \cdot \left[-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

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$$= \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \cdot \left[-\frac{3}{125}, 0, -\frac{4}{125}\right]$$

$$= -\frac{7}{375}\sqrt{3} = -.032$$

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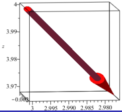
Hence the required vector is

$\left[-\frac{3}{125}, 0, -\frac{4}{125}\right]$ at the point $P(3,0,4)$



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Solution: By examining $\mathbf{v} = [ye^x, e^x, z^2]$, we conclude that $f(x, y, z) = ye^x + \frac{1}{3}z^3$, because

$$
\begin{aligned}
\operatorname{grad} f &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) f(x, y, z) \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(ye^x + \frac{1}{3}z^3\right) \\
&= [ye^x, e^x, z^2] \qquad \blacksquare
\end{aligned}
$$

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Solution: The cone is the level surface $f = 0$ of $f\left(x, y, z\right) = 4\left(x^2 + y^2\right) - z^2$. Thus

$$\begin{aligned} \nabla f &= [8x, 8y, -2z], \quad \nabla f\left(P\right) = [8, 0, -4] \\ \mathbf{n} &= \frac{1}{\left|\nabla f\left(P\right)\right|}\nabla f\left(P\right) = \left[\frac{2}{\sqrt{5}}, 0, \frac{-1}{\sqrt{5}}\right] \end{aligned}$$

n points downward since it has a negative z-component. The other unit normal vector of the cone at P is $-\mathbf{n}$. ■

Definition: Divergence of a Vector Field

For a differentiable vector function $\mathbf{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$, the divergence is denoted and defined as $\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$.

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If $\mathbf{v} = \operatorname{grad}(f)$ and f is twice differentiable scalar function $f(x, y, z)$, we have
$\operatorname{div} \mathbf{v} = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f.$

For a differentiable vector function $\mathbf{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$, the divergence is denoted and defined as $\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$.

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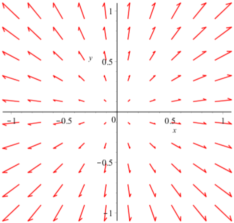

Visualization of Divergence

From a scalar field we can obtain a vector field by the gradient.

Usage of Divergence

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$$\textbf{Scalar Field} \quad \underset{\text{Div}}{\overset{\text{Grad}}{\rightleftarrows}} \quad \textbf{Vector Field}$$

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Definition

If density ρ is independent of time t, the flow is said to be **steady**.

Question: Give a physical interpretation of divergence? OR Derive equation of continuity for fluids?

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Derivation: Consider motion of a compressible fluid in region R with no source or sink in R. Consider flow of the fluid through the box B with volume $\Delta V = \Delta x \Delta y \Delta z$ (here Δ is denoting a small quantity and not the Laplacian).

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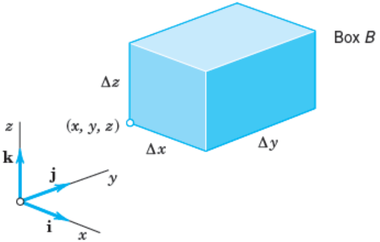

Physical interpretation of the divergence

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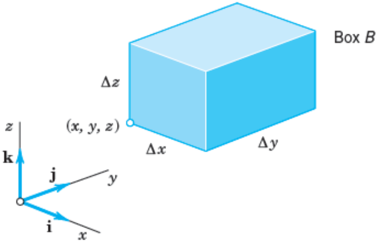


Physical interpretation of the divergence

Let $\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be the velocity vector of the motion. We set

$$\mathbf{u} = \rho\mathbf{v} = [u_1, u_2, u_3] = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

Consider the flow through the xz face whose area is $\Delta x \Delta z$. Since the vectors $v_1\mathbf{i}$ and $v_3\mathbf{k}$ are parallel to xz face, the components v_1 and v_3 contribute nothing to this flow.

Consider the flow through the xz face whose area is $\Delta x \Delta z$. Since the vectors $v_1 \mathbf{i}$ and $v_3 \mathbf{k}$ are parallel to xz face, the components v_1 and v_3 contribute nothing to this flow. Hence the mass of fluid entering through xz face during a short time interval Δt is

$$(\rho v_2) \Delta x \Delta z \Delta t = (u_2)_y \Delta x \Delta z \Delta t$$

and the mass leaving from opposite face is $(u_2)_{y+\Delta y} \Delta x \Delta z \Delta t$. Hence the difference

$$\Delta u_2 \Delta x \Delta y \Delta z = \frac{\Delta u_2}{\Delta y} \Delta y \Delta t \quad \text{where} \quad \Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y$$

is the approximate loss of mass. Other two faces also give similar expressions and the total loss of mass in B during the time interval Δt is approximately

$$\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z} \right) \Delta V \Delta t \tag{1}$$

where $\Delta u_1 = (u_1)_{x+\Delta x} - (u_1)_x$ and $\Delta u_3 = (u_3)_{z+\Delta z} - (u_3)_z$.

This loss of mass in B is caused by the time rate of change of the density and is thus equals to

$$-\frac{\partial\rho}{\partial t}\Delta V\Delta t \tag{2}$$

Equating (1) and (2) and letting small changes approach to zero we get

$$\begin{aligned} \operatorname{div}\mathbf{u} &= \operatorname{div}(\rho\mathbf{v}) = -\frac{\partial\rho}{\partial t} \\ \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) &= 0 \end{aligned}$$

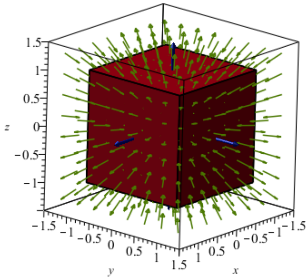
is the '**equation of continuity** of a compressible fluid flow' also called 'condition for the coservation of mass'.
If the flow is steady then $\frac{\partial\rho}{\partial t} = 0$ and the equation becomes $\operatorname{div}(\rho\mathbf{v}) = 0$ and if the density is constant i.e. the fluid is incompressible then $\operatorname{div}\mathbf{v} = 0$. ■

Question: (10ed-9.8-5) Find $\operatorname{div}\mathbf{v}$ at $P(-1,3,-2)$ where $\mathbf{v}=\left[x^2yz, xy^2z, xyz^2\right]$?

Question: (10ed-9.8-5) Find $\operatorname{div} \mathbf{v}$ at $P\,(-1, 3, -2)$ where $\mathbf{v} = \left[x^2yz, xy^2z, xyz^2\right]$?

Solution: $\operatorname{div} \mathbf{v} = \operatorname{div}\left(\left[x^2yz, xy^2z, xyz^2\right]\right) = 6xyz \Rightarrow$
$\operatorname{div} \mathbf{v}|_{P(-1,3,-2)} = 6\,(-1 \times 3 \times -2) = 36$

The vector field $\mathbf{v} = \left[x^2yz, xy^2z, xyz^2\right]$ around a box is sketched below:



The vector field arrows, the surface through which the field passes, and vectors normal to the surface.

Question: (10ed-9.8-17) Find $\nabla^2 f$ by the formula $\nabla^2 f = \mathrm{div}\,(\mathrm{grad}\, f)$, where $f = \ln\left(x^2 + y^2\right)$?

Question: (10ed-9.8-17) Find $\nabla^2 f$ by the formula $\nabla^2 f = \operatorname{div}(\operatorname{grad} f)$, where $f = \ln(x^2 + y^2)$?

Solution: $\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \operatorname{div}\left(\operatorname{grad}\left(\ln\left(x^2 + y^2\right)\right)\right) = \operatorname{div}\left(\left[\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}, 0\right]\right) = \frac{4}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} - \frac{4x^2}{(x^2+y^2)^2} = 0$ ■

Definition: Curl of a Vector Field

Curl of a vector function
$\mathbf{v}(x,y,z) = [v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)]$ is defined as

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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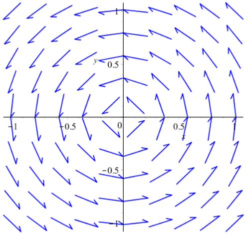

Visualization of Curl

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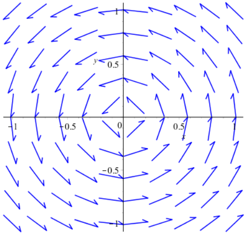
$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$



Visualization of Curl

From a scalar field we can obtain a vector field by the gradient.

Definition: Curl of a Vector Field

Curl of a vector function
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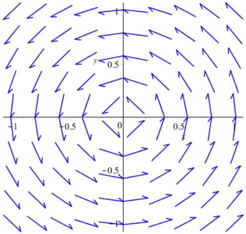

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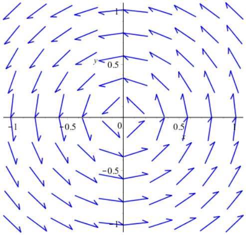

Visualization of Curl

From a scalar field we can obtain a vector field by the gradient. Conversely, from a vector field we can obtain a scalar field by the divergence or yet another vector field by the curl.

Scalar Field $\underset{\text{Div}}{\overset{\text{Grad}}{\rightleftarrows}}$ **Vector Field**

Definition: Curl of a Vector Field

Curl of a vector function
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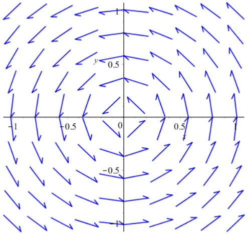


Visualization of Curl

From a scalar field we can obtain a vector field by the gradient. Conversely, from a vector field we can obtain a scalar field by the divergence or yet another vector field by the curl.

$$\textbf{Scalar Field} \quad \underset{\text{Div}}{\overset{\text{Grad}}{\rightleftarrows}} \quad \textbf{Vector Field} \quad \xrightarrow{\text{Curl}} \quad \textbf{Another Vector Field}$$

Curl (cont'd)

We also have:

1. **Gradient fields are irrotational.** That is, if a continuously differentiable vector function is the gradient of a scalar function, then its curl is the zero vector,

$$\mathrm{curl}\,(\mathrm{grad}\, f) = \mathbf{0}$$

We also have:

1. **Gradient fields are irrotational.** That is, if a continuously differentiable vector function is the gradient of a scalar function, then its curl is the zero vector,

$$\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$$

2. Divergence of the curl of a twice continuously differentiable vector function $\mathbf{v}$ is zero,

$$\operatorname{div}(\operatorname{curl} f) = 0.$$

Question: (10ed-9.9-5) Find $\mathrm{curl}\,\mathbf{v}$ for $\mathbf{v} = xyz\left[x^2, y^2, z^2\right]$?

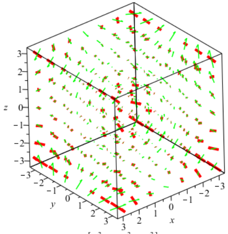
Question: (10ed-9.9-5) Find $\operatorname{curl} \mathbf{v}$ for $\mathbf{v} = xyz\left[x^2, y^2, z^2\right]$?

Solution: $\operatorname{curl} \mathbf{v} = \operatorname{curl}\left(xyz\left[x^2, y^2, z^2\right]\right)$

$$= \operatorname{curl}\left(\left[x^3yz, xy^3z, xyz^3\right]\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3yz & xy^3z & xyz^3 \end{vmatrix} =$$

$$\left(xz^3 - xy^3\right)\mathbf{i} + \left(x^3y - yz^3\right)\mathbf{j} + \left(y^3z - x^3z\right)\mathbf{k}$$

The vector field and its curl are sketched below:



Vector field $[x^3yz, xy^3, xyz^3]$ and its Curl

Question: (10ed-9.9-10) Let $\mathbf{v} = [\sec x, \csc x, 0]$ be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles).

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Solution: $\operatorname{curl}([\sec x, \csc x, 0]) = -\frac{\cos x}{\sin^2 x}\mathbf{k} \neq \mathbf{0} \Rightarrow$ Not irrotational.

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Question: (10ed-9.9-11) Let $\mathbf{v} = [y, -2x, 0]$ be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles)?

Question: (10ed-9.9-10) Let $\mathbf{v} = [\sec x, \csc x, 0]$ be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles).

Solution: $\operatorname{curl}([\sec x, \csc x, 0]) = -\frac{\cos x}{\sin^2 x}\mathbf{k} \neq \mathbf{0} \Rightarrow$ Not irrotational.

$\operatorname{div}([\sec x, \csc x, 0]) = \frac{1}{\cos^2 x} \sin x \neq 0 \Rightarrow$ Compressible. ■

Question: (10ed-9.9-11) Let $\mathbf{v} = [y, -2x, 0]$ be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles)?

Solution: $\operatorname{curl}(\mathbf{v}) = \operatorname{curl}([y, -2x, 0]) = -3\mathbf{k} \neq \mathbf{0} \Rightarrow$ Not irrotational.

$\operatorname{div}([y, -2x, 0]) = 0 \Rightarrow$ incompressible. ■