

Advanced Engineering Mathematics

Mathematical Techniques for Engineering

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Theorem

(Laplace Transform of the Derivative $f^{(n)}$ of Any Order)

$$L\left(f^{(n)}\right) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

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Similarly let $g(t) = \sin \omega t$. Then $g(0) = 0$, $g'(t) = \omega \cos \omega t$, which gives

$$\begin{aligned} sL(g) - g(0) &= L(g') = \omega L(\cos \omega t) \\ \text{Hence } L(\sin \omega t) &= \frac{\omega}{s} L(\cos \omega t) = \frac{\omega}{s^2 + \omega^2} \quad \blacksquare \end{aligned}$$

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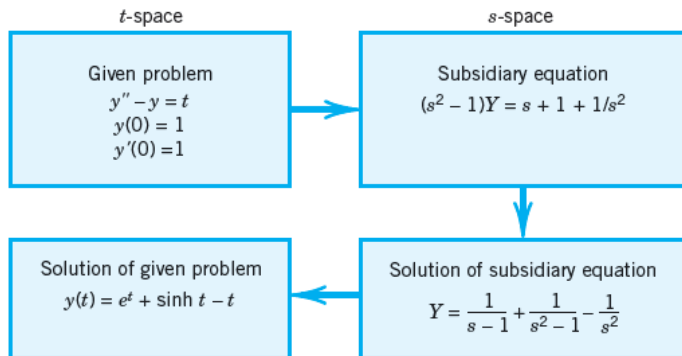
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$$y(t) = e^{-1.5t} + te^{-1.5t} + 32t - 16t^2 + 4t^3$$

$$y(t) = 4t^3 - 16t^2 + 32t + (t+1)e^{-1.5t} \quad \blacksquare$$

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Find solution of the IVP:

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Isolating $\tilde{Y} = \frac{1}{s^2+1} \left(\frac{\pi}{2} \frac{1}{s} + 2 \frac{1}{s^2} + y'(0) + sy(0) \right)$

Using ICs $\tilde{Y} = \frac{1}{s^2+1} \left(\frac{\pi}{2} \frac{1}{s} + 2 \frac{1}{s^2} + (2 - \sqrt{2}) + \frac{\pi}{2} s \right)$

$$= \frac{1}{2} \frac{s^3\pi - 2\sqrt{2}s^2 + s\pi + 4s^2 + 4}{s^2(s^2+1)}$$

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Example (10e-6.2-14)

The IVP $y'' + 2y' + 5y = 50t - 100$; $y(2) = -4$, $y'(2) = 14$
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The IVP $y'' + 2y' + 5y = 50t - 100$; $y(2) = -4$, $y'(2) = 14$ is a shifted data problem. So put $\tilde{t} = t - 2$ and obtain

$$\frac{d^2}{d\tilde{t}^2} y(\tilde{t}) + 2 \frac{d}{d\tilde{t}} y(\tilde{t}) + 5 y(\tilde{t}) = 50\tilde{t} \quad ; \quad y(0) = -4, y'(0) = 14$$

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$$\tilde{Y} = \frac{1}{s^2 + 2s + 5} \left(50 \frac{1}{s^2} + 6 - 4s \right)$$

$$\tilde{Y} = -2 \frac{2s^3 - 3s^2 - 25}{s^2(s^2 + 2s + 5)}$$

By partial fractions(DIY) $\tilde{Y} = -\frac{4}{s} + \frac{4}{(s+1)^2+4} + \frac{10}{s^2}$

$$y(\tilde{t}) = -4 + 2e^{-\tilde{t}} \sin(2\tilde{t}) + 10\tilde{t}$$

Replacing back

$$\tilde{t} = t - 2 \quad y(t) = -24 + 2e^{-t+2} \sin(2t - 4) + 10t \quad \blacksquare$$

Q (10e-6.2-15)

For the IVP $y'' + 3y' - 4y = 6e^{2t-3}$; $y(1.5) = 4, y'(1.5) = 5$
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$$s^2\tilde{Y} - y'(0) - sy(0) + 3s\tilde{Y} - 3y(0) - 4\tilde{Y} = \frac{6}{s-2} \quad \text{where}$$

$$\tilde{Y} = L(y(\tilde{t}))$$

$$\tilde{Y} = \frac{1}{s^2+3s-4} \left(\frac{6}{s-2} + y'(0) + sy(0) + 3y(0) \right)$$

Using ICs $\tilde{Y} = \frac{1}{s^2+3s-4} \left(\frac{6}{s-2} + 17 + 4s \right)$

$$\tilde{Y} = \frac{4s-7}{(s-1)(s-2)}$$

By partial fractions $\tilde{Y} = \frac{3}{s-1} + \frac{1}{s-2}$

$$y(\tilde{t}) = 3e^{\tilde{t}} + e^{2\tilde{t}}$$

Replacing back $\tilde{t} = t - 1.5$, $y(t) = 3e^{(t-1.5)} + e^{2(t-1.5)}$ ■