Advanced Engineering Mathematics

Mathematical Techniques for Engineering

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Differentiation and Integration of LT

Theorem

If $F(s)$ *is the Laplace transform of* $f(t)$, *then*

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt$$

from which, also note that

$$L(tf(t)) = -F'(s) \quad \text{and} \quad L^{-1}(F'(s)) = -tf(t)$$

Corollary

By mathematical induction we also have

$$L(t^n f(t)) = (-1)^n F^{(n)}(s)$$

Example

$L^{-1}\left(\ln\left(1+\frac{\omega^2}{s^2}\right)\right) = ?$

We observe that $\ln\left(1+\frac{\omega^2}{s^2}\right) = \ln\left(\frac{s^2+\omega^2}{s^2}\right) = \ln\left(s^2+\omega^2\right) - \ln s^2$. This indicates that a derivative of $\ln$ may bring terms like $\frac{s}{s^2+\omega^2}$ or $\frac{1}{s}$. Hence we set $F(s) = \ln\left(\frac{s^2+\omega^2}{s^2}\right)$ and proceed for an application of differentiation of LT theorem i.e.

$$
\begin{aligned}
F'(s) &= \frac{d}{ds}\ln\left(s^2+\omega^2\right) - \frac{d}{ds}\ln s^2 = 2\frac{s}{s^2+\omega^2} - \frac{2}{s} \\
L^{-1}\left(F'(s)\right) &= L^{-1}\left(2\frac{s}{s^2+\omega^2} - \frac{2}{s}\right) = 2\cos(\omega t) - 2 = 2\,(\\
\text{Since} \quad L^{-1}\left(F'(s)\right) &= -tf(t) \\
&\Rightarrow -tf(t) = 2\left(\cos(\omega t) - 1\right) \\
f(t) &= \frac{2}{t}\left(1-\cos(\omega t)\right) \qquad \blacksquare
\end{aligned}
$$

Example

(10e-6.6-2) $L\left(3t\sinh\left(4t\right)\right) = ?$

$$\begin{aligned} \text{We know } L\left(tf\left(t\right)\right) &= -F'\left(s\right) \\ L\left(3t\sinh\left(4t\right)\right) &= 3L\left(t\sinh\left(4t\right)\right) \\ &= 3\left(-\frac{d}{ds}L\left(\sinh\left(4t\right)\right)\right) \\ &= -3\left(\frac{d}{ds}\left(\frac{4}{s^2-16}\right)\right) \\ &= 24\frac{s}{\left(s^2-16\right)^2} \qquad \blacksquare \end{aligned}$$

Example

For finding $L\left(t^n e^{kt}\right)$, we note

$$
\begin{aligned}
L\left(e^{kt}\right) &= \frac{1}{s-k} \quad , \quad (= F(s)) \\
\text{by corollary } L\left(t^n f(t)\right) &= (-1)^n F^{(n)}(s), \text{ so we take } n \text{ differentiation:} \\
\frac{d}{ds}\left(\frac{1}{s-k}\right) &= \frac{-1}{(s-k)^2}, \quad \frac{d^2}{ds^2}\left(\frac{1}{s-k}\right) = \frac{2}{(s-k)^3} \\
\frac{d^3}{ds^3}\left(\frac{1}{s-k}\right) &= \frac{-6}{(s-k)^4}, \quad \text{hence} \\
\frac{d^n}{ds^n}\left(\frac{1}{s-k}\right) &= \frac{(-1)^n n!}{(s-k)^{n+1}} \\
L\left(t^n e^{kt}\right) &= \frac{(-1)^n n!}{(s-k)^{n+1}} \qquad \blacksquare
\end{aligned}
$$

Example

(10e-6.7-6)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}y_1(t) &= 5\,y_1(t) + y_2(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}y_2(t) &= y_1(t) + 5\,y_2(t) \\
y_1(0) &= 1, y_2(0) = -3
\end{aligned}$$

Taking Laplace Transforms

$$\begin{aligned}
s\;Y_1 - y_1(0) &= 5\,Y_1 + Y_2 \\
sY_2 - y_2(0) &= Y_1 + 5\,Y_2
\end{aligned}$$

Using ICs

$$\begin{aligned}
s\;Y_1 - 1 &= 5\,Y_1 + Y_2 \\
s\;Y_2 + 3 &= Y_1 + 5\,Y_2
\end{aligned}$$

Example

(10e-6.7-12)

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} y_1(t) &= -2\,y_1(t) + 2\,y_2(t)\\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} y_2(t) &= 2\,y_1(t) - 5\,y_2(t)\\ y_1(0) &= 1, y_2(0) = 3, y_1'(0) = 0, y_2'(0) = 0\end{aligned}$$

Taking Laplace Transforms

$$\begin{aligned}s^2\,Y_1 - y_1'(0) - s y_1(0) &= -2\,Y_1 + 2\,Y_2\\ s^2\,Y_2 - y_2'(0) - s y_2(0) &= 2\,Y_1 - 5\,Y_2\end{aligned}$$

Using ICs

$$\begin{aligned}s^2\,Y_1 - s &= -2\,Y_1 + 2\,Y_2\\ s^2\,Y_2 - 3\,s &= 2\,Y_1 - 5\,Y_2\end{aligned}$$