

Advanced Engineering Mathematics

Mathematical Techniques for Engineering

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Dirac's Delta Function

Definition

Consider a function $f_k(t - a) = \begin{cases} \frac{1}{k} & , \quad a \leq t \leq a + k \\ 0 & , \quad \text{otherwise} \end{cases}$,

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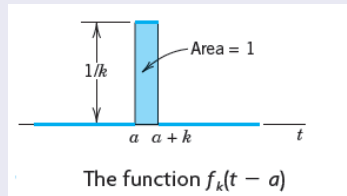
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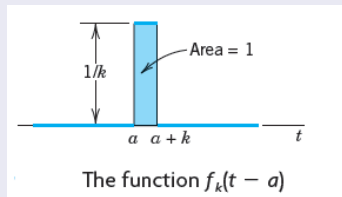


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Then Dirac's delta function (aka, unit impulse function) is defined as

$$\delta(t - a) = \lim_{k \rightarrow 0} f_k(t - a).$$

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$$\begin{aligned} L\left(\lim_{k \rightarrow 0} f_k(t-a)\right) &= \lim_{k \rightarrow 0} e^{-as} \frac{1 - e^{-ks}}{ks} \\ &= e^{-as} \lim_{k \rightarrow 0} \frac{\frac{d}{dk}(1 - e^{-ks})}{\frac{d}{dk}(ks)} \quad (\text{l'Hopital}) \end{aligned}$$

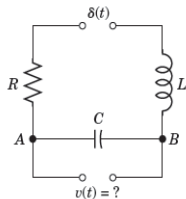
$$= e^{-as} \left[\frac{se^{-ks}}{s} \right]_{k=0} = e^{-as}$$

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Find output voltage response of the four-terminal RLC circuit given in figure if

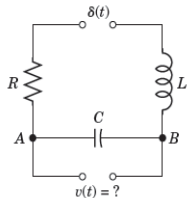
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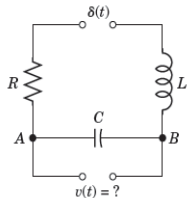
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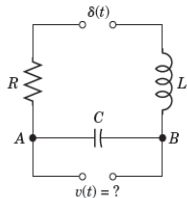
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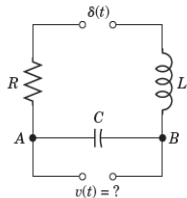
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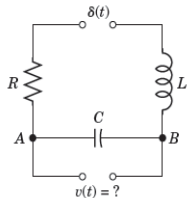
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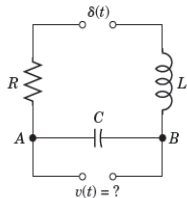
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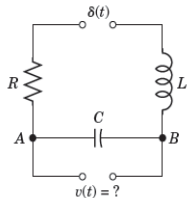
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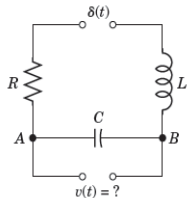
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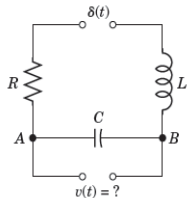
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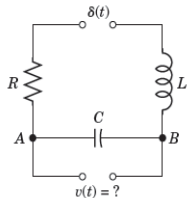
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Using Partial Fractions

$$Y = \left(\frac{1}{(s+2)} - \frac{1}{(s+3)} \right) e^{-\frac{\pi}{2}s} - \left(\frac{-2}{5} \frac{1}{(s+2)} + \frac{1}{10} \frac{s+1}{s^2+1} + \frac{3}{10} \frac{1}{(s+3)} \right) e^{-s\pi}$$

$$y(t) = L^{-1} \left(\frac{e^{-\frac{\pi}{2}s}}{s+2} \right) - L^{-1} \left(\frac{e^{-\frac{\pi}{2}s}}{s+3} \right) + \frac{2}{5} L^{-1} \left(\frac{1}{s+2} e^{-s\pi} \right) - \frac{1}{10} L^{-1} \left(\frac{s}{1+s^2} e^{-s\pi} \right) - \frac{1}{10} L^{-1} \left(\frac{1}{1+s^2} e^{-s\pi} \right) - \frac{3}{10} L^{-1} \left(\frac{1}{s+3} e^{-s\pi} \right)$$

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$$y(t) = e^{\pi-2t} u\left(t - \frac{1}{2}\pi\right) - e^{\frac{3}{2}\pi-3t} u\left(t - \frac{1}{2}\pi\right) + \frac{2}{5} e^{2\pi-2t} u(t - \pi) - \frac{1}{10} (-\cos t) u(t - \pi) - \frac{1}{10} (-\sin t) u(t - \pi) - \frac{3}{10} e^{3\pi-3t} u(t - \pi)$$

$$y(t) = \frac{1}{10} u(t - \pi) (-3e^{-3t+3\pi} + \sin(t) + \cos(t) + 4e^{-2t+2\pi}) + u(t - \pi/2) (-e^{-3t+3/2\pi} + e^{-2t+\pi}) \quad \blacksquare$$

Question: (10ed-6.4-12)

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 5y(t) = 25t - 100\delta(t - \pi), \quad y(0) = -2, \quad y'(0) = 5$$

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$$y(t) = \begin{cases} 5t - 2 & , \quad t < \pi \\ 5t - 2 - 50e^{\pi-t}\sin(2t) & , \quad t > \pi \end{cases}$$



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In words one says: product of transforms is not the transform of product, **instead** product of transforms is the transform of convolution.

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$$\begin{aligned} L^{-1} \left(\frac{1}{s^2 + \omega^2} \right) &= \int_0^t \frac{\sin(\omega \tau)}{\omega} \frac{\sin(\omega(t - \tau))}{\omega} d\tau \\ &= \frac{1}{\omega^2} \int_0^t \frac{1}{2} [-\cos(\omega t) + \cos(2\omega \tau - \omega t)] d\tau \\ &= \frac{1}{2\omega^2} \left[-\tau \cos \omega t + \frac{\sin(\omega \tau)}{\omega} \right]_{\tau=0}^t \\ &= \frac{1}{2\omega^2} \left[-t \cos \omega t + \frac{\sin(\omega t)}{\omega} \right] \end{aligned}$$

Properties of Convolution:

$$f \star g = g \star f$$

$$f \star (g + h) = f \star g + f \star h$$

$$f \star (g \star h) = (f \star g) \star h$$

$$f \star 0 = 0 \star f = 0$$

$$f \star 1 \neq f \quad (\text{why?})$$

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Substituting $u = a\tau \ln(e) - b(\tau-t) \ln(e)$ implies at

$\tau = 0$, $u = bt$ and $\tau = t$, $u = at$ and

$$du = \frac{d}{d\tau} (a\tau \ln(e) - b(\tau-t) \ln(e)) = a - b \Rightarrow \frac{du}{a-b} = d\tau$$

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$$= \int_{\ln(e)bt}^{at \ln(e)} \frac{e^u}{(a-b) \ln(e)} du = \frac{e^{at \ln(e)} - e^{\ln(e)bt}}{(a-b) \ln(e)} = \frac{e^{at} - e^{bt}}{(a-b)} \quad \blacksquare$$

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$$Y = \frac{1}{s^2 - 1}$$

$$y(t) = L^{-1}\left(\frac{1}{s^2 - 1}\right)$$

$$y(t) = \sinh t$$



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$$Y = -\frac{2s^2 - 1}{s - s^3}$$

$$Y = \frac{1}{2(s-1)} + \frac{1}{2(s+1)} + \frac{1}{s}$$

$$y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} + 1$$

$$y(t) = 1 + \cosh t \quad \blacksquare$$

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$$\begin{aligned} &= \left[\frac{1}{2} u(t-a-\tau) (e^{2\tau} + e^{2t-2a}) \right]_{\tau=0}^t \\ &= \left[\frac{1}{2} u(t-a-\tau) (e^{2\tau} + e^{2t-2a}) \right]_{\tau=t} - \\ &\quad \left[\frac{1}{2} u(t-a-\tau) (e^{2\tau} + e^{2t-2a}) \right]_{\tau=0} \end{aligned}$$

$$= \frac{1}{2} u(-a) (e^{2t-2a} + e^{2t}) - \frac{1}{2} u(t-a) (e^{2t-2a} + 1) \quad \blacksquare$$

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$$Y = \frac{1}{s^2 + 5s + 4} \left(\frac{2}{s+2} + y'(0) + sy(0) + 5y(0) \right) =$$
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$$= 2e^{-2t} \star \left(\frac{1}{3} e^{-t} - \frac{1}{3} e^{-4t} \right) = \frac{2}{3} e^{-2t} \star (e^{-t} - e^{-4t})$$

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