Introduction to Statistical Learning

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Spring 2015

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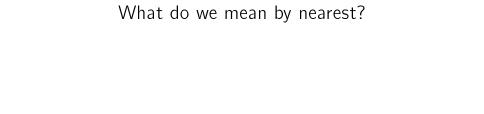
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 - Wouldn't it depend on what x are observed?



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- For Boolean, Jaccard distance:
$$d_i(p,q) = \frac{|p \cup q| - |p \cap q|}{|p| |q|}$$

Applications: Recommender systems, finding similar

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Curse of Dimensionality

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- Define neighborhood too tightly, nothing in there.
- If we expand it, nearest neighbors can be far away
- How do we solve the problem?
- One way: parametric and structural models $f(x) = \beta_0 + \beta_1 * X_1 + \beta_2 * X_2 + \dots, \beta_p * X_p$

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 - Add boundary constraint: linear before 1st knot or after last knot. (Natural spline.)

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$$-\epsilon = Y - \hat{f}(x) - E[(Y - \hat{f}(X)^2 | X = x)] = (f(x) - \hat{f}(X))^2 + Var(\epsilon)$$

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- BIC = Deviance + df * log(n)
- BIC underfits when large n.

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- Net prediction error on test set can vary a lot.

A Clarification

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- Larger *n* allows for more precise estimation of relationship

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 - Caveat: Assumes independence

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