

# Advanced Engineering Mathematics

## Mathematical Techniques for Engineering

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## Usage of Laplace Transform

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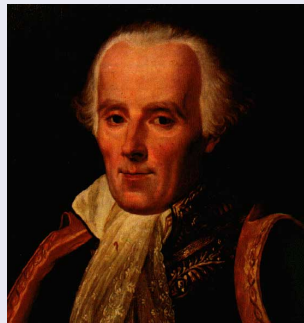
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## Pierre Simon Marquis de Laplace (1749-1827)

Laplace, great French mathematician, was a professor in Paris. He developed the foundation of potential theory and made important contributions to celestial mechanics, astronomy in general, special functions and probability theory. Napoleon Bonaparte was his student for a year.



## Definition

A function  $F(s)$  given as

$$F(s) = \int_0^{\infty} k(s, t) f(t) dt$$

is said to be **integral transform** of  $f(t)$  with **kernel**  $k(s, t)$ .

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The given function  $f(t)$  in a Laplace Transform expression is called inverse transform of  $F(s)$  and is denoted by  $L^{-1}(F)$ , more explicitly  $f(t) = L^{-1}(F)$ .

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### Solution

$$\begin{aligned} L(e^{at}) &= \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{(a-s)t} dt \\ &= \left[ \frac{1}{a-s} e^{-(s-a)t} \right]_0^{\infty} \quad \text{by using } \int e^{ax} dx = \frac{1}{a} e^{ax} \\ \therefore L(e^{at}) &= \frac{1}{s-a} \quad \blacksquare \end{aligned}$$

# Example (9ed-6.1-4)

For  $f(t) = \sin^2(4t)$ , we have

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 &= \left[ -\frac{1}{128se^{st} + 2s^3e^{st}} (8s \sin 8t - s^2 \cos 8t + s^2 + 64) \right]_0^{\infty} \\
 &= 0 - \left( -\frac{64}{2s^3 + 128s} \right) \\
 &= \frac{32}{s(s^2 + 64)} \quad \blacksquare
 \end{aligned}$$

## (9ed-6.1-7)

For  $e^{3a-2bt}$ , we have

$$\begin{aligned} & \int \left( e^{3a-2bt} e^{-st} \right) dt \\ &= \int e^{3a-2bt-st} dt = -\frac{e^{3a-2bt-st}}{2b+s} \therefore \int e^{ax} dx = \frac{1}{a} e^{ax} \end{aligned}$$



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## (9ed-6.1-11)

We find  $L(\sin t \cos t)$

$$\begin{aligned} & L(\sin t \cos t) \\ &= \int_0^{\infty} (\sin t \cos t) e^{-st} dt \\ &= \left[ -\frac{1}{8e^{st} + 2s^2 e^{st}} (2 \cos 2t + s \sin 2t) \right]_0^{\infty} \\ &= 0 - \left( -\frac{2}{2s^2 + 8} \right) \\ &= \frac{1}{s^2 + 8} \quad \blacksquare \end{aligned}$$

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$$= \left[ \left( -\frac{1}{s^5} e^{-st} (s^4 t^4 + 4s^3 t^3 + 12s^2 t^2 + 24st + 24) \right) - 6 \left( -\frac{1}{s^3} e^{-st} (s^2 t^2 + 2st + 2) \right) + \left( -\frac{9}{s} e^{-st} \right) \right]_0^{\infty}$$



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$$= 0 - \left( -\frac{1}{s^5} (9s^4 - 12s^2 + 24) \right) = \frac{9}{s} - \frac{12}{s^3} + \frac{24}{s^5} \quad \blacksquare$$

# Table of Important Laplace Transforms

$f(t)$	$L(f)$	$f(t)$	$L(f)$
1	$\frac{1}{s}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$t$	$\frac{1}{s^2}$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$t^2$	$\frac{2!}{s^3}$	$\cosh at$	$\frac{s}{s^2 - a^2}$
$t^n$	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2 - a^2}$
$t^a, a \geq 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$e^{at}$	$\frac{1}{s-a}$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$