Advanced Engineering Mathematics

Mathematical Techniques for Engineering

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Pierre Simon Marquis de Laplace (1749-1827)

Laplace, great French mathematician, was a professor in Paris. He developed the foundation of potential theory and made important contributions to celestial mechanics, astronomy in general, special functions and probablity theory. Napoleon Bonaparte was his student for a year.

Definition

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$$F(s) = \int_0^{\infty} k(s,t) f(t)\, dt$$

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Solution

$$\begin{aligned} L\left(e^{at}\right) &= \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt \\ &= \left[\frac{1}{a-s} e^{-(s-a)t}\right]_0^\infty \quad \textit{by using} \int e^{ax}\, dx = \frac{1}{a} e^{ax} \\ \therefore \quad & L\left(e^{at}\right) = \frac{1}{s-a} \quad \blacksquare \end{aligned}$$

Example (9ed-6.1-4)

For $f(t) = \sin^2(4t)$, we have

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\therefore \quad & \sin\left(\frac{x}{2}\right) = \sqrt{\frac{1-\cos(x)}{2}} \\
= \quad & \left[-\frac{1}{128se^{st} + 2s^3e^{st}}\left(8s\sin 8t - s^2\cos 8t + s^2 + 64\right)\right]_0^\infty
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= \; & 0 - \left(-\frac{64}{2s^3 + 128s}\right) \\
= \; & \frac{32}{s\left(s^2+64\right)} \quad \blacksquare
\end{aligned}$$

(9ed-6.1-7)

For e^{3a-2bt}, we have

$$\int \left(e^{3a-2bt} e^{-st} \right) dt$$

$$= \int e^{3a-2bt-st} dt = -\frac{e^{3a-2bt-st}}{2b+s} \therefore \int e^{ax} dx = \frac{1}{a} e^{ax}$$

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$$\therefore \quad L\left(e^{3a-2bt} \right) = \int_0^\infty \left(e^{3a-2bt} e^{-st} \right) dt = \left[-\frac{e^{3a-2bt-st}}{2b+s} \right]_0^\infty$$

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= &\left[-\frac{e^{3a-2bt-st}}{2b+s}\right]_{t=\infty} - \left[-\frac{e^{3a-2bt-st}}{2b+s}\right]_{t=0} \\
= &\ 0 - \left(-\frac{e^{3a}}{2b+s}\right) \\
= &\ \frac{e^{3a}}{2b+s} \quad \blacksquare
\end{aligned}$$

(9ed-6.1-11)

We find $L(\sin t \cos t)$

$$
\begin{aligned}
& L(\sin t \cos t) \\
= \; & \int_0^{\infty} (\sin t \cos t)\, e^{-st}\, dt \\
= \; & \left[-\frac{1}{8e^{st} + 2s^2 e^{st}} (2\cos 2t + s \sin 2t) \right]_0^{\infty} \\
= \; & 0 - \left(-\frac{2}{2s^2 + 8} \right) \\
= \; & \frac{1}{s^2 + 8} \quad \blacksquare
\end{aligned}
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$$= \left[\left(-\frac{1}{s^5}e^{-st}\left(s^4t^4 + 4s^3t^3 + 12s^2t^2 + 24st + 24\right)\right) - 6\left(-\frac{1}{s^3}e^{-st}\left(s^2t^2 + 2st + 2\right)\right) + \left(-\frac{9}{s}e^{-st}\right)\right]_0^\infty$$

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$$= 0 - \left(-\frac{1}{s^5}\left(9s^4 - 12s^2 + 24\right)\right) = \frac{9}{s} - \frac{12}{s^3} + \frac{24}{s^5} \quad \blacksquare$$

Table of Important Laplace Transforms

$f(t)$	$L(f)$	$f(t)$	$L(f)$
1	$\frac{1}{s}$	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
t^2	$\frac{2!}{s^3}$	$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$t^a, a \geq 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$