GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Pre-test

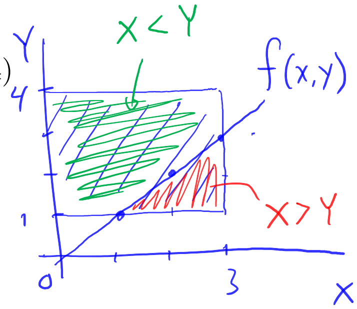
Due: January 17, 2018 Course: ECE 6254

Name: Solutions
Last, First

- Open book/internet.
- No time limit.
- The test is worth 100 points. There are ten questions, and each one is worth 10 points. In multi-part questions, each part will be weighted equally.
- All work should be performed on the test itself. If more space is needed, use the backs of the pages.
- This test will be conducted under the rules and guidelines of the Georgia Tech Honor Code and no cheating will be tolerated (i.e., no discussing the test with other students).
- Make sure to look at the question titles, they will sometimes provide valuable hints.
- Please contact Prof. Anderson (in person or via email) directly with any questions if you are unclear what a question is asking.

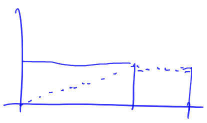
Problem 1: Random variables. Suppose that two independent random variables X and Y are distributed according to

$$X \sim \text{Uniform}(0,3) \quad Y \sim \text{Uniform}(1,4)$$



What is the probability that $X < Y$.

$$f(x,y) = \int_1^4 \int_0^Y u_{(0,3)}(x)\, u_{(1,4)}(y)\, dx\, dy$$



$$= \int_1^4 \int_0^Y \tfrac{1}{3} I_{(0,3)}(x)\, dx \cdot \tfrac{1}{3} I_{(1,4)}(y)\, dy$$

$$= \frac{1}{9}\left(\int_1^3 y\, dy + \int_3^4 3\, dy \right)$$

$$= \frac{1}{9}\left(\frac{1}{2} y^2 \Big|_1^3 + 3 \right) = \frac{1}{9}\left(\frac{9}{2} - \frac{1}{2} + 3 \right) = \boxed{\frac{7}{9}}$$

or the green area is $1 - \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{1}{9}$

$$= \boxed{\frac{7}{9}}$$

Problem 2: Independence: Suppose that the probability that A occurs is 0.4 and the probability that **both** A **and** B occur is 0.25. If A and B are independent events, what is the probability that **neither** A **nor** B occur?

$P(A) = 0.4$

$P(A \cap B) = 0.25$

Independence implies $= P(A \cap B) = P(A)P(B)$

$$\Rightarrow P(B) = \frac{0.25}{0.4} = 0.625$$

$$P(\sim A \cap \sim B) = P(\sim A)P(\sim B) = (1-0.4)\cdot(1-0.625)$$

$$= \boxed{0.225}$$

Problem 3: Conditional probability density functions and derived distributions: Suppose that X and Y have joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2x-y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) What are the marginal probability density functions for X and Y?

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^{\infty} 2e^{-2x} e^{-y}\,dy$$

$$= 2e^{-2x}\left(-e^{-y}\Big|_0^{\infty}\right)$$

$$\boxed{f_X(x) = 2e^{-2x} \quad \text{for } x \geq 0}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= e^{-y}\int_0^{\infty} 2e^{-2x}\,dx$$

$$= e^{-y}\left(-e^{-2x}\Big|_0^{\infty}\right)$$

$$\boxed{f_Y(y) = e^{-y} \quad \text{for } y \geq 0}$$

(b) What is the conditional probability density function $f_{X|Y}(x|y)$?

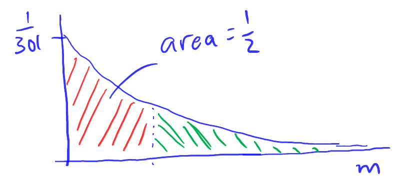
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

the variables are independent and so knowing y doesn't tell anything about x

Problem 4: The median and the cumulative distribution function: Let M be the number of miles your electric car can drive without running out of electricity, and suppose that M has probability density function given by

$$f_M(m) = \begin{cases} \frac{e^{-m/301}}{301} & m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the median range for your car? That is, how far can you drive before there is a 50% chance that your battery runs out?



$P(\text{you will lose charge before } m_0 \text{ miles}) =$

$$\underbrace{\int_0^{m_0} \frac{e^{-m/301}}{301}\, dm}_{} = \frac{1}{2}$$

$$= 1 - e^{-m_0/301} = \frac{1}{2} \Rightarrow e^{-m_0/301} = \frac{1}{2}$$

$$m_0 = -301 \cdot \ln\left(\frac{1}{2}\right) \approx \boxed{208.6}$$

Problem 5: Bayes rule and normal random variables: Let X be the score of a randomly selected student in this class on an exam. Let Y denote the mean score of the class. If we knew Y, suppose that a reasonable model for X would be that X is normal with mean Y and variance 20, that is,[1]

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi 20}} e^{-(x-y)^2/40}$$

Suppose that I think that the mean for the exam is probably between 70 and 85, *i.e.*,

$$f_Y(y) = \begin{cases} \frac{1}{15} & 70 \le y \le 85 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that I randomly select a quiz, grade it, and it turns out to receive a 100. Using Bayes rule, how should I update my distribution for Y? That is, what is $f_{Y|X}(y|X = 100)$? (Simplify as much as possible, but you may leave your answer in terms of the standard normal cumulative distribution function Φ if you wish.)

by Bayes' rule

$$f_{Y|X}(y|X=100) = \frac{f_{X|Y}(100|y)\, f_Y(y)}{\int f_{X|Y}(100|u)\, f_Y(u)\, du} \longleftarrow \text{normalization}$$

$$= \frac{\frac{1}{\sqrt{2\pi 20}} e^{-(100-y)^2/40} \cdot \cancel{\frac{1}{15}}}{\int_{70}^{85} \frac{1}{\sqrt{2\pi 20}} e^{-(100-y)^2/40} \cdot \cancel{\frac{1}{15}}} \quad \text{for } 70 \le y \le 85$$

$$= \frac{1}{\sqrt{2\pi 20}} e^{-(100-y)^2/40} \left(\frac{1}{\Phi\left(\frac{85-100}{\sqrt{20}}\right) - \Phi\left(\frac{70-100}{\sqrt{20}}\right)} \right) \quad \text{for } 70 \le y \le 85$$

$$\boxed{= \frac{1}{\sqrt{2\pi 20}} e^{-(y-100)^2/40} \left(\frac{1}{\Phi\left(\frac{30}{\sqrt{20}}\right) - \Phi\left(\frac{15}{\sqrt{20}}\right)} \right) \quad \text{for } 70 \le y \le 85}$$

[1]Note that this pdf technically allows some probability of scores above 100 and below 0. You should just ignore this for now.

Problem 6: Joint probability density functions: The *correlation coefficient* $\rho(X, Y)$ between a pair of random variables X and Y is given by

$$\rho(X, Y) = \frac{E\Big[\,(X - E[X]) \cdot (Y - E[Y])\,\Big]}{\sigma_X \sigma_Y}.$$

Suppose that X and Y have joint pdf given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{3}{4}x^2(1-y) & 0 \le x \le 2, 0 \le y \le 1 \\ 0 & \text{else.} \end{cases}$$

for any x, y. What is ρ in this case?

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$ where $f_X(x) = \frac{3}{4}x^2 \quad 0 \le x \le 2$

and $f_Y(y) = (1-y) \quad 0 \le y \le 1$

so X & Y are independent.

Thus, $E\Big[(X - E[X])(Y - E[Y])\Big] = E\big[X - E[X]\big]\, E\big[Y - E[Y]\big]$

$= 0 \cdot 0$

so $\boxed{\rho = 0}$

Problem 7: Linear Algebra: Pythagoras?

(a) Under what conditions on $\boldsymbol{x}$ and $\boldsymbol{y}$ is it true that

$$\|\boldsymbol{x}+\boldsymbol{y}\|_2^2 = \|\boldsymbol{x}\|_2^2 + \|\boldsymbol{y}\|_2^2 \ ?$$

$$\|x+y\|_2^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle$$

$$= \|x\|_2^2 + \|y\|_2^2 + 2\langle x,y\rangle$$

for real x & y

so this holds if $2\langle x,y\rangle = 0$

$\Rightarrow$ x and y are orthogonal

(b) Under what conditions on $\boldsymbol{x}$ and $\boldsymbol{y}$ is it true that

$$\|\boldsymbol{x}+\boldsymbol{y}\|_2 = \|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2 \ ?$$ square both sides

from above, $\|x+y\|_2^2 = \|x\|_2^2 + 2\langle x,y\rangle + \|y\|_2^2$

also $(\|x\|_2 + \|y\|_2)^2 = \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2$

they are equal when $\boxed{\langle x,y\rangle = \|x\|_2\|y\|_2}$

i.e. when x & y are colinear

Problem 8: Singular value decomposition.: Let

$$\boldsymbol{A} = \begin{bmatrix} -2 & 2 & 2 & -2 & 0 \\ 2 & -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) What is rank($\boldsymbol{A}$)?

rank $(A) = 2$

(b) Using Python or MATLAB (or whatever) find the singular value decomposition of $\boldsymbol{A}$. That is, find matrices $\boldsymbol{U}, \boldsymbol{\Sigma}, \boldsymbol{V}$ such that

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}$$

and $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} = \mathbf{I}$, $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V} = \mathbf{I}$, and Σ has non-negative entries along its diagonal and is zero elsewhere.

$$U = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5.657 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/2 & 0 & -.866 & 0 & 0 \\ -1/2 & 0 & -.289 & .577 & .577 \\ -1/2 & 0 & -.289 & .211 & -.789 \\ 1/2 & 0 & .289 & .789 & -.211 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(c) Describe, in words, the column space (or range) of $\boldsymbol{A}$:

$$\text{Range}(\boldsymbol{A}) = \{\boldsymbol{v} \in \mathbb{R}^5 \ : \ \boldsymbol{v} = \boldsymbol{A}\boldsymbol{x} \text{ for some } \boldsymbol{x}\}.$$

The column space of A is given by the span of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and is of the form $\begin{bmatrix} a \\ -a \\ b \end{bmatrix}$

(d) Describe, in words, the row space of $\boldsymbol{A}$ (this is the column space of $\boldsymbol{A}^{\mathrm{T}}$):

$$\text{Range}(\boldsymbol{A}^{\mathrm{T}}) = \{\boldsymbol{v} \in \mathbb{R}^5 \ : \ \boldsymbol{v} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} \text{ for some } \boldsymbol{x}\}.$$

The row space of A is given by the span of the vectors $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and has the form $\begin{bmatrix} -a \\ a \\ a \\ -a \\ b \end{bmatrix}$

Problem 9: Eigenvalues and eigenvectors. Suppose that two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, but different sets of eigenvalues. Matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with respective eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Matrix $\mathbf{B}$ has eigenvalues $\gamma_1, \ldots, \gamma_n$ with respective eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(a) What are the eigenvalues and eigenvectors of $\mathbf{A} + \mathbf{B}$? *support your answer*

$$(A+B)v_i = Av_i + Bv_i = \lambda_i v_i + \gamma_i v_i = (\lambda_i + \gamma_i) v_i$$

the eigenvalues are $(\lambda_1 + \gamma_1), \ldots, (\lambda_n + \gamma_n)$

the eigenvectors are $v_1, \ldots, v_n$

(b) What are the eigenvalues and eigenvectors of $\mathbf{AB}$? *support your answer*

$$AB v_i = A \gamma_i v_i = \gamma_i A v_i = \gamma_i \lambda_i v_i$$

The eigenvalues are $\gamma_1 \lambda_1, \ldots, \gamma_n \lambda_n$

The eigenvectors are $v_1, \ldots, v_n$

(c) What are the eigenvalues and eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$? *support your answer*

$$A^{-1} B v_i = \gamma_i A^{-1} v_i = \frac{\gamma_i}{\lambda_i} v_i$$

note, if $A v_i = \lambda_i v_i$ and A^{-1} exists then

$A^{-1} A v_i = \lambda_i A^{-1} v_i$

so $A^{-1} v_i = \frac{1}{\lambda_i} v_i$

The eigenvalues are $\frac{\gamma_1}{\lambda_1}, \ldots, \frac{\gamma_n}{\lambda_n}$

The eigenvectors are $v_1, \ldots, v_n$

Problem 10: Orthogonal projections: Let

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 4 \\ -2 \\ -6 \\ -7 \end{bmatrix} \qquad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix}.$$

Find a decomposition of $\mathbf{x}$ into $\mathbf{x} = \mathbf{x}^* + \mathbf{x}_e$ where $\mathbf{x}^*$ is in the span of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, i.e., where $\mathbf{x}^* = \alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 + \alpha_3 \mathbf{p}_3$ for some suitable choice of $\alpha_1, \alpha_2, \alpha_3$. Make sure to give both $\mathbf{x}^*$ and $\mathbf{x}_e$, and show your work/describe your method, even if you use a computer to help with the calculations.

let $P = [p_1 \; p_2 \; p_3]$

we want to find $\underline{\alpha}$ s.t. $x = \underbrace{P\alpha}_{x^*} + x_e$

There are multiple approaches, one easy way is to use the Moore-Penrose pseudoinverse

$$\alpha = (P^T P)^{-1} P^T x$$

$$\alpha = \begin{pmatrix} 1.1639 \\ -0.1361 \\ 0.0917 \end{pmatrix} \qquad P\alpha = x^* = \begin{pmatrix} 0.8946 \\ 2.9665 \\ 4.1247 \\ 5.6996 \end{pmatrix}$$

$$x_e = x - x^* = \begin{pmatrix} 0.1054 \\ -0.9665 \\ -1.1247 \\ 1.3004 \end{pmatrix}$$

Additional workspace:

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