proof of Gram-Schmidt orthogonalization procedure*

rspuzio†

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Note that, while we state the following as a theorem for the sake of logical completeness and to establish notation, our definition of Gram-Schmidt orthogonalization is wholly equivalent to that given in the defining entry.

Theorem. (Gram-Schmidt Orthogonalization) *Let* $\{\mathbf{u_k}\}_{k=1}^{n}$ *be a basis for an inner product space* V *with inner product* $\langle , \rangle$*. Define* $\mathbf{v_1} = \frac{\mathbf{u_1}}{||\mathbf{u_1}||}$ *and* $\{\mathbf{m_k}\}_{k=2}^{n}$ *recursively by*

$$\mathbf{m_k} = \mathbf{u_k} - \langle \mathbf{u_k}, \mathbf{v_1} \rangle \mathbf{v_1} - \langle \mathbf{u_k}, \mathbf{v_2} \rangle \mathbf{v_2} - \ldots - \langle \mathbf{u_k}, \mathbf{v_{k-1}} \rangle \mathbf{v_{k-1}}, \tag{1}$$

where $\mathbf{v_k} = \frac{\mathbf{m_k}}{||\mathbf{m_k}||}$ *for* $2 \leq k \leq n$*. Then* $\{\mathbf{v_k}\}_{k=1}^{n}$ *is an orthonormal basis for* V*.*

Proof. We proceed by induction on n. In the case $n = 1$, we suppose $\{\mathbf{u_k}\}_{k=1}^{n} = \{\mathbf{u_k}\}_{k=1}^{1} = \mathbf{u_1}$ is a basis for the inner product space V. Letting $\mathbf{v_1} = \frac{\mathbf{u_1}}{||\mathbf{u_1}||}$, it is clear that $\mathbf{v_1} \in \mathrm{Span}\left(\mathbf{u_1}\right)$, whence it follows that $\mathrm{Span}\left(\mathbf{v_1}\right) = \mathrm{Span}\left(\mathbf{u_1}\right) = V$. Thus $\mathbf{v_1}$ is an orthonormal basis for V, and the result holds for $n = 1$. Now let $n \geq 1 \in \mathbb{N}$, and suppose the result holds for arbitrary n. Let $\{\mathbf{u_k}\}_{k=1}^{n+1}$ be a basis for an inner product space V. By the inductive hypothesis we may use $\{\mathbf{u_k}\}_{k=1}^{n}$ to construct an orthonormal set of vectors $\{\mathbf{v_k}\}_{k=1}^{n}$ such that $\mathrm{Span}\left(\{\mathbf{v_k}\}_{k=1}^{n}\right) = \mathrm{Span}\left(\{\mathbf{u_k}\}_{k=1}^{n}\right)$. In accordance with the procedure outlined in the statement of the theorem, let $\mathbf{m_{n+1}}$ be defined as

$$\mathbf{u_{n+1}} - \langle \mathbf{u_{n+1}}, \mathbf{v_1} \rangle \mathbf{v_1} - \langle \mathbf{u_{n+1}}, \mathbf{v_2} \rangle \mathbf{v_2} - \ldots - \langle \mathbf{u_{n+1}}, \mathbf{v_n} \rangle \mathbf{v_n} = \mathbf{u_{n+1}} - \sum_{i=1}^{n} \langle \mathbf{u_{n+1}}, \mathbf{v_i} \rangle \mathbf{v_i}.$$

First we show that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}, \mathbf{m_{n+1}}$ are mutually orthogonal. Consider the inner product of $\mathbf{m_{n+1}}$ with $\mathbf{v_j}$ for $1 \leq j \leq n$. By construction, we have

$$\langle \mathbf{m_{n+1}}, \mathbf{v_j} \rangle = \left\langle \mathbf{u_{n+1}} - \sum_{i=1}^{n} \langle \mathbf{u_{n+1}}, \mathbf{v_i} \rangle \mathbf{v_i}, \mathbf{v_j} \right\rangle = \langle \mathbf{u_{n+1}}, \mathbf{v_j} \rangle - \left\langle \sum_{i=1}^{n} \langle \mathbf{u_{n+1}}, \mathbf{v_i} \rangle \mathbf{v_i}, \mathbf{v_j} \right\rangle.$$

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Now since $\{\mathbf{v_k}\}_{k=1}^{n}$ is an orthonormal set of vectors, whence $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \delta_{ij}$, each term in the summation on the right-hand side of the preceding equation will vanish except for the term where $i = j$. Thus by this and the preceding equation, we have

$$\langle \mathbf{m_{n+1}}, \mathbf{v_j} \rangle = \langle \mathbf{u_{n+1}}, \mathbf{v_j} \rangle - \left\langle \sum_{i=1}^{n} \langle \mathbf{u_{n+1}}, \mathbf{v_i} \rangle \mathbf{v_i}, \mathbf{v_j} \right\rangle = \langle \mathbf{u_{n+1}}, \mathbf{v_j} \rangle - \Big\langle \langle \mathbf{u_{n+1}}, \mathbf{v_j} \rangle \mathbf{v_j}, \mathbf{v_j} \Big\rangle$$
$$= \langle \mathbf{u_{n+1}}, \mathbf{v_j} \rangle - \langle \mathbf{u_{n+1}}, \mathbf{v_j} \rangle \langle \mathbf{v_j}, \mathbf{v_j} \rangle = \langle \mathbf{u_{n+1}}, \mathbf{v_j} \rangle - \langle \mathbf{u_{n+1}}, \mathbf{v_j} \rangle = 0.$$

Thus $\mathbf{m_{n+1}}$ is orthogonal to $\mathbf{v_j}$ for $1 \le j \le n$, so we may take $\mathbf{v_{n+1}} = \frac{\mathbf{m_{n+1}}}{||\mathbf{m_{n+1}}||}$ to have $\{\mathbf{v_k}\}_{k=1}^{n+1}$ an orthonormal set of vectors. Finally we show that $\{\mathbf{v_k}\}_{k=1}^{n+1}$ is a basis for V. By construction, each $\mathbf{v_k}$ is a linear combination of the vectors $\{\mathbf{u_k}\}_{k=1}^{n+1}$, so we have $n+1$ orthogonal, hence linearly independent vectors in the $n+1$ dimensional space V, from which it follows that $\{\mathbf{v_k}\}_{k=1}^{n+1}$ is a basis for V. Thus the result holds for $n+1$, and by the principle of induction, for all n. $\square$