Frames and Riesz Bases in Hilbert Space.

A. Nonorthogonal Bases in Finite Dimensions.

Definition 1 *A finite collection of elements, $\{x_i\}_{i=1}^n$ in a linear space L is* linearly independent *if any collection of scalars $\{\alpha_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n \alpha_i\, x_i = 0$ must all be zero. Otherwise, the collection $\{x_i\}_{i=1}^n$ is* linearly dependent.

*A linear space is n–*dimensional *if there exist n linearly independent elements in L but every set of $n+1$ elements is linearly dependent. If there is a set of n linearly independent elements for every n, then L is* infinite–dimensional.

A set of n linearly independent elements in an n–dimensional linear space is a basis *for that space.*

Remarks.

1. If $\{v_k\}_{k=1}^n$ is a basis for the n–dimensional linear space then every $x \in L$ can be written uniquely as $x = \sum_k \alpha_k\, v_k$. Every such linear space has a basis and any n–dimensional linear space L (over $\mathbf{R}$ or $\mathbf{C}$) is isomorphic to $\mathbf{R}^n$ (or to $\mathbf{C}^n$). Also by means of this isomorphism, any such L can be equipped with an inner product, so that we can talk about *orthogonality* of such vectors. From now on we will treat L as though it were $\mathbf{R}^n$ or $\mathbf{C}^n$.
2. Any finite-dimensional linear space L with an inner product has an orthonormal basis. This can be obtained from any basis using the Gram-Schmidt process.
3. Given two bases for L, $\{v_k\}_{k=1}^n$ and $\{w_k\}_{k=1}^n$ there is a unique $n \times n$ matrix T with the property that if $\alpha = (\alpha_1, \ldots, \alpha_n)$ contains the expansion coefficients of x in the first basis, then $T\alpha$ contains the expansion coefficients of x in the second. T is called the *change-of-basis* matrix.
4. Any basis $\{v_k\}_{k=1}^n$ for $\mathbf{R}^n$ is the image under an invertible linear transformation of an orthonormal basis.

B. Riesz Bases in Hilbert Spaces.

Definition 2 *A collection of vectors $\{x_k\}_k$ in a Hilbert space H is a* Riesz basis *for H if it is the image of an orthonormal basis for H under an invertible linear transformation. In other words, if there is an orthonormal basis $\{e_k\}$ for H and an invertible transformation T such that $Te_k = x_k$ for all k.*

Theorem 1 *Let $\{x_k\}$ be a collection of vectors in a Hilbert space H.*

(a) *If $\{x_k\}$ is a Riesz basis for H then there is a unique collection $\{y_k\}$ such that $\langle x_k, y_k\rangle = \delta_k$, that is such that $\{y_k\}$ is biorthogonal to $\{x_k\}$. In this case $\{y_k\}$ is also a Riesz basis.*

(b) *If $\{x_k\}$ is a Riesz basis for H then there are constants $0 \le A \le B$ such that for all $x \in H$, $A\,\|x\|^2 \le \sum_k |\langle x, x_k\rangle|^2 \le B \le \|x\|^2$. This inequality is called the* frame inequality.

(c) $\{x_k\}$ *is a Riesz basis for* H *if and only if there are constants* $0 \leq A \leq B$ *such that for all finite sequences* $\{\alpha_k\}$, $A \sum_k |\alpha_k|^2 \leq \left\| \sum_k \alpha_k \, x_k \right\|^2 \leq B \sum_k |\alpha_k|^2$.

(d) *If* $\{x_k\}$ *is a Riesz basis for* H *then for each* $x \in H$ *there is a unique collection of scalars* $\{\alpha_k\}$ *such that* $x = \sum_k \alpha_k \, x_k$ *and* $\sum_k |\alpha_k|^2 \leq \infty$.

Remarks.

1. Note that if $\{x_k\}$ were an orthonormal basis then (a) would be obvious (just take $y_k = x_k$) and (b) would hold with $A = B = 1$ (Plancherel's formula). In fact we have seen that if $\{x_k\}$ satisfies the frame inequality with $A = B = 1$ and if $\|x_k\| = 1$ for all k, then $\{x_k\}$ is an orthonormal basis for H.
2. A different way to characterize some of these properties is to think of two operators associated to a Riesz basis $\{x_k\}$. The first is the *analysis operator* $T \colon H \to l^2$ given by $T(x) = \{\langle x, x_k \rangle\}_k$. That $T(x) \in l^2$ (and further that T is a bounded linear operator) follows from the frame inequality. The second is the *synthesis operator* from $l^2 \to H$ given by $\{\alpha_k\} \mapsto \sum_k \alpha_k \, x_k$. Since the synthesis operator is the adjoint of T, we will just denote it by T^*.
3. In this language, $\{x_k\}$ is a Riesz basis if and only if T is a bounded linear bijection from H onto l^2. In other words there is a one to one correspondence between sequences of the form $\{\langle x, x_k \rangle\}_k$ and sequences in l^2. In other words, every l^2 sequence gets "hit" by something in H through the analysis operator. In still other words, this statement is equivalent to (c) above.

Definition 3 *Let* H *be an infinite-dimensional Hilbert space. An infinite collection* $\{x_k\}$ *of vectors in* H *is* (finnitely) linearly independent *if every finite subset of* $\{x_k\}$ *is linearly independent. It is* ω–linearly independent *if a sequence* $\{\alpha_k\}$ *such that* $\sum_k \alpha_k \, x_k$ *converges in the norm of* H *to* 0 *must be identically zero.*

Lemma 1 *If* $\{x_k\}$ *is* ω*–linearly independent then it is linearly independent. However a sequence can be finitely linearly independent without being* ω*–linearly independent.*

Remark.

If $\{x_k\}$ is an orthonormal basis, then it is ω–linearly independent. If $\{x_k\}$ is a Riesz basis, then it is ω–linearly independent. Both of these facts follow from the assertion that an orthonormal or Riesz basis has a biorthogonal sequence.

Theorem 2 *A sequence* $\{x_k\}$ *in a Hilbert space* H *is a Riesz basis for* H *if and only if* $\{x_k\}$ *satisfies the frame condition and is* ω*–linearly independent.*

C. Frames in Hilbert Spaces.

Definition 4 *A sequence $\{x_k\}$ in a Hilbert space H is a frame if there exist numbers $A, B > 0$ such that for all $x \in H$ we have*

$$A\|x\|^2 \;\leq\; \sum_k |\langle x, x_n\rangle|^2 \;\leq\; B\|x\|^2.$$

The numbers A, B are called the frame bounds. The frame is tight if $A = B$. The frame is exact if it ceases to be a frame whenever any single element is deleted from the sequence.

Remarks.

1. From the Plancherel formula we see that every orthonormal basis is a tight exact frame with $A = B = 1$. For orthonormal bases, the Plancherel formula is equivalent to the basis property, which gives a decomposition of the Hilbert space. The weakened form of the Plancherel formula satisfied by frames also gives a decomposition, although the representations need not be unique.
2. A frame is a complete set since if $x \in H$ satisfies $\langle x, x_n\rangle = 0$ for all n, then $A\|x\|^2 \leq \sum |\langle x, x_n\rangle|^2 = 0$, so $x = 0$.
3. A tight frame need not be exact and vice versa. For example let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Then

(a) $\{e_1, e_1, e_2, e_2, e_3, e_3, \cdots\}$ is a tight inexact frame with bounds $A = B = 2$, but is not an orthonormal basis, although it contains one.

(b) $\left\{e_1, e_2/2, e_3/3, \cdots\right\}$ is a complete orthogonal sequence, but not a frame.

(c) $\left\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, \cdots\right\}$ is a tight frame with bounds $A = B = 1$ but is not an exact sequence, and no nonredundant subsequence is a frame.

(d) $\{2e_1, e_2, e_3, \cdots\}$ is an exact frame with bounds $A = 1, B = 2$ and is not a tight frame.

Theorem 3 *Given a sequence $\{x_n\}$ in a Hilbert space H, the following two statements are equivalent:*

(a) *$\{x_n\}$ is a frame with bounds A, B.*

(b) *$Sx = \sum\langle x, x_n\rangle x_n$ is a bounded linear operator with $AI \leq S \leq BI$, called the* frame operator *for $\{x_n\}$.*

Corollary 1 (a) *S is invertible and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.*

(b) *$\{S^{-1}x_n\}$ is a frame with bounds B^{-1}, A^{-1}, called the* dual frame *of $\{x_n\}$.*

(c) *Every $x \in H$ can be written $x = \sum\langle x, S^{-1}x_n\rangle x_n = \sum\langle x, x_n\rangle S^{-1}x_n$.*

Theorem 4 *Given a frame $\{x_n\}$ and given $x \in H$ let $a_n = \langle x, S^{-1}x_n\rangle$, so $x = \sum a_n x_n$. If it is possible to find other scalars c_n such that $x = \sum c_n x_n$ then $\sum |c_n|^2 = \sum |a_n|^2 + \sum |a_n - c_n|^2$.*

Theorem 5 *The removal of a vector from a frame leaves either a frame or an incomplete set. In particular, if for a given m, $\langle x_m, S^{-1}x_m\rangle \neq 1$ then $\{x_n\}_{n\neq m}$ is a frame; and if $\langle x_m, S^{-1}x_m\rangle = 1$ then $\{x_n\}_{n\neq m}$ is not complete in H.*

Corollary 2 *If $\{x_n\}$ is an exact frame, then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal, i.e., $\langle x_m, S^{-1}x_n\rangle = \delta_{mn}$.*

Theorem 6 *A sequence $\{x_n\}$ in a Hilbert space H is an exact frame for H if and only if it is a Riesz basis for H.*

D. Example: Nonharmonic Fourier Series.

One illustrative way to generate examples of Riesz bases and frames is as "perturbations" of orthonormal bases. One classic result in this direction is the following

Theorem 7 *(Paley-Wiener) Let $\{e_k\}$ be an orthonormal basis for the Hilbert space H and suppose that $\{x_k\}$ is a sequence in H with the property that for some $0 \leq \lambda < 1$,*

$$\Big\|\sum_k \alpha_k(e_k - x_k)\Big\| \leq \lambda\Big(\sum_k |\alpha_k|^2\Big)^{1/2}$$

for every finite sequence $\{\alpha_k\}$. Then $\{x_k\}$ is a Riesz basis for H.

The proof of this theorem uses the following important result from operator theory.

Lemma 2 *A bounded linear operator T on a Hilbert space is invertible whenever $\|I - T\| < 1$.*

One application of the theorem of Paley and Wiener is to the problem of *nonharmonic Fourier series.* We know that the collection $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0,1]$. What are the basis properties of collections of the form $\{e^{2\pi i\lambda_n t}\}_{n\in\mathbf{Z}}$ where $\{\lambda_n\}$ is a sequence of real or complex numbers?

The following result follows directly from the above theorem of Paley and Wiener.

Theorem 8 *There is an $\epsilon > 0$ such that whenever $|\lambda_n - n| < \epsilon$ then $\{e^{2\pi i\lambda_n t}\}_{n\in\mathbf{Z}}$ is a Riesz basis for $L^2[0,1]$.*

An interesting question is: What is the largest value of ϵ for which the above theorem is valid? The answer is $\epsilon = 1/4$. This result is known as the Kadec 1/4 theorem. Implicit in this solution is the statement that the theorem fails if $\epsilon \geq 1/4$. In fact the following is true.

Theorem 9 *The collection $\{e^{\pm 2\pi i(n-1/4)t}\}_{n=1}^{\infty}$ is complete in $L^2[0,1]$.*

This implies that the collection $\{e^{2\pi i\lambda_n t}\}_{n\in\mathbf{Z}}$ where $\lambda_n = n - 1/4$ if $n > 0$, $n + 1/4$ if $n < 0$ and 0 if $n = 0$ is not a Riesz basis for $L^2[0,1]$.

In terms of frames of exponentials, the following result is due to Duffin and Schaeffer.

Theorem 10 *Suppose there are constants $L > 0$ and $\epsilon > 0$ such that (1) for all n, $|\lambda_n - n| < L$, and (2) for all $n \neq m$, $|\lambda_n - \lambda_m| \geq \epsilon$. Then the collection $\{e^{2\pi i\lambda_n t}\}_{n\in\mathbf{Z}}$ is a frame for $L^2[0,\gamma]$ for all $0 \leq \gamma < 1$.*