Q. 5

(a) $$P'_{m+1} = \begin{bmatrix} K_m + \delta I & 0 \\ 0 & \delta \end{bmatrix}^{-1}$$ Since this is a block diagonal matrix, inversion is easy.

$$\therefore P'_{m+1} = \begin{bmatrix} (K_m + \delta I)^{-1} & 0 \\ 0 & 1/\delta \end{bmatrix} = \begin{bmatrix} P_m & 0 \\ 0 & 1/\delta \end{bmatrix}$$

(b) $$P_m^{-1} = \begin{bmatrix} k(t_1, t_1)+\delta & k(t_2, t_1) & \cdots & k(t_m, t_1) \\ k(t_1, t_2) & k(t_2, t_2)+\delta & \cdots & k(t_m, t_2) \\ \vdots & & & \vdots \\ k(t_1, t_m) & k(t_2, t_m) & \cdots & k(t_m, t_m)+\delta \end{bmatrix}$$

$$P_{m+1}^{-1} = \begin{bmatrix} k(t_1, t_1)+\delta & k(t_2, t_1) & \cdots & k(t_m, t_1) & k(t_{m+1}, t_1) \\ k(t_1, t_2) & k(t_2, t_2)+\delta & \cdots & k(t_m, t_2) & k(t_{m+1}, t_2) \\ \vdots & \vdots & & & \vdots \\ k(t_1, t_m) & k(t_2, t_m) & & k(t_m, t_m)+\delta & k(t_{m+1}, t_m) \\ k(t_1, t_{m+1}) & k(t_2, t_{m+1}) & & k(t_m, t_{m+1}) & k(t_{m+1}, t_{m+1})+\delta \end{bmatrix}$$

Split P_{m+1}^{-1}

$$\therefore P_{m+1}^{-1} = \underbrace{\begin{bmatrix} k(t_1, t_1)+\delta & \cdots & k(t_m, t_1) & 0 \\ k(t_1, t_2) & \cdots & k(t_m, t_2) & 0 \\ \vdots & & & \\ k(t_1, t_m) & \cdots & k(t_m, t_m)+\delta & 0 \\ 0 & 0 \cdots & 0 & \delta \end{bmatrix}}_{P'^{-1}_{m+1}} + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & k(t_{m+1}, t_m) \\ 0 & 0 & \cdots & k(t_{m+1}, t_2) \\ \vdots & & 0 & \\ k(t_1, t_{m+1}) & \cdots & \cdots & k(t_{m+1}, t_{m+1}) \end{bmatrix}}_{BC^T}$$

where $B = \begin{bmatrix} k(t_{m+1}, t_1) & 0 \\ k(t_{m+1}, t_2) & 0 \\ \vdots & 0 \\ \vdots & \vdots \\ k(t_{m+1}, t_{m+1}) & 1 \end{bmatrix}$; $C^T = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ k(t_{m+1}, t_1) & k(t_{m+1}, t_2) & \cdots & 0 \end{bmatrix}$

$$\therefore C = \begin{bmatrix} 0 & k(t_{m+1}, t_1) \\ 0 & k(t_{m+1}, t_2) \\ \vdots & \vdots \\ 0 & k(t_{m+1}, t_m) \\ 1 & 0 \end{bmatrix}$$

$$\therefore BC^T = \underbrace{\begin{bmatrix} k(t_{m+1}, t_1) & 0 \\ k(t_{m+1}, t_2) & 0 \\ \vdots & \vdots \\ k(t_{m+1}, t_{m+1}) & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 1 \\ k(t_{m+1}, t_1) & k(t_{m+1}, t_2) & \cdots\ k(t_{m+1}, t_m) & 0 \end{bmatrix}}_{C^T}$$

Let $W = P'^{-1}_{m+1}$, $X^T = B$, $Y = I$ and $Z = C^T$

Applying Matrix Inversion Lemma,

$$\Rightarrow P_{m+1} = P'_{m+1} - P'_{m+1} B \left(I + C^T P'_{m+1} B \right)^{-1} C^T P'_{m+1}$$

Here, $C^T \rightarrow R \times (M+1)$

$B \rightarrow (M+1) \times R$ $\qquad$ $R = 2$ here

$P'_{m+1} \rightarrow (M+1) \times (M+1)$

$C^T P'_{m+1} B \rightarrow R \times R$

$\therefore \left(I + C^T P'_{m+1} B \right)^{-1} \rightarrow$ Solving for $R \times R$ system of equations.

Matrix inversion lemma thus allows us to get solutions by solving this $R \times R$ system, which has been reduced from $(M+1)\times(M+1)$, which we would have had to do otherwise.

Matrix vector multiplication that surround the $R \times R$ system can be done in a relatively shorter time.

(c) $P_{m+1} = P'_{m+1} - P'_{m+1} B(I + C^T P'_{m+1} B)^{-1} C^T P'_{m+1}$

$$\hat{\alpha}_m = P_m y_m \quad \text{and} \quad \hat{\alpha}_{m+1} = P_{m+1} y_{m+1}$$

$$\hat{\alpha}_{m+1} = P'_{m+1} y_{m+1} - P'_{m+1} B(I + C^T P'_{m+1} B)^{-1} C^T P'_{m+1} y_{m+1} \quad -① $$

We know from (a) $P'_{m+1} = \begin{bmatrix} P_m & 0 \\ \underline{0} & 1/\delta \end{bmatrix}$

$$P_{m+1} y_{m+1} = \begin{bmatrix} P_m & \underline{0} \\ \underline{0} & 1/\delta \end{bmatrix} \left[\frac{y_m}{y_{m+1}}\right]$$ → not division. (augmenting the column with the new y_{m+1})

where,

$\underline{y_m} = [y_1 \ y_2 \ \cdots \ y_m]^T$

$$\therefore P'_{m+1} y_{m+1} = \begin{bmatrix} P_m y_m \\ (y_{m+1})/\delta \end{bmatrix} \rightarrow (M+1)\times 1 \text{ column.} \quad = \begin{bmatrix} \hat{\alpha}_m \\ (y_{m+1})/\delta \end{bmatrix}$$

From ①,

$$\hat{\alpha}_{m+1} = (I - G) P'_{m+1} y_{m+1}, \text{ where } G = P'_{m+1} B(I + C^T P'_{m+1} B)^{-1} C^T$$

∴ on combining all this data,

$$\hat{\alpha}_{m+1} = [I - G] \begin{bmatrix} \hat{\alpha}_m \\ (y_{m+1})/\delta \end{bmatrix}$$

PSEUDO CODE:

Initialization: $P_m = (K_m + \delta I)^{-1}, \delta$

$$\hat{\alpha}_m = (K_m + \delta I)^{-1} y_m$$

for $k = m+1, m+2, \dots$ do

(t_k, y_k) streams in.

Compute $P'_k = \begin{bmatrix} P_{k-1} & \underline{0} \\ \underline{0} & 1/\delta \end{bmatrix}$

Construct B and C as:

$$B = \begin{bmatrix} k(t_{m+1}, t_1) & \cdots & 0 \\ k(t_{m+1}, t_2) & & 0 \\ \vdots & & \vdots \\ k(t_{m+1}, t_m) & \cdots & 0 \\ k(t_{m+1}, t_{m+1}) & - & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & - & k(t_{m+1}, t_1) \\ 0 & & k(t_{m+1}, t_2) \\ \vdots & & \vdots \\ 1 & & k(t_{m+1}, t_{m+1}) \end{bmatrix}$$

$(m+1) \times 2 \rightarrow R$ $\qquad$ $(m+1) \times 2$

$P_k = P'_k - G P'_k$ where $G = P'_k B (I + C^T P'_k B)^{-1} C^T$

$$\hat{\alpha}_k = [I - G] \begin{bmatrix} \hat{\alpha}_{k-1} \\ y_k/\delta \end{bmatrix}$$

end for.

(d) <u>Only q</u> latest samples are used for prediction (at any given time).

$$P_m^{-1} = \begin{bmatrix} k(t_{m-q+1}, t_{m-q+1})+\delta & k(t_{m-q+2}, t_{m-q+1}) & \cdots & k(t_m, t_{m-q+1}) \\ k(t_{m-q+1}, t_{m-q+2}) & k(t_{m-q+2}, t_{m-q+2})+\delta & \cdots & k(t_m, t_{m-q+2}) \\ \vdots & \vdots & & \vdots \\ k(t_{m-q+1}, t_m) & k(t_{m-q+2}, t_m) & \cdots & k(t_m, t_m)+\delta \end{bmatrix}$$

$$P_{m+1}^{-1} = \begin{bmatrix} k(t_{m-q+2}, t_{m-q+2})+\delta & k(t_{m-q+3}, t_{m-q+2}) & \cdots & k(t_{m+1}, t_{m-q+2}) \\ k(t_{m-q+2}, t_{m-q+3}) & k(t_{m-q+3}, t_{m-q+3})+\delta & \cdots & k(t_{m+1}, t_{m-q+3}) \\ \vdots & \vdots & & \\ k(t_{m-q+2}, t_{m+1}) & k(t_{m-q+3}, t_{m+1}) & \cdots & k(t_{m+1}, t_{m+1})+\delta \end{bmatrix}$$

Split P_{m+1}^{-1} as follows:

$$P_{m+1}^{-1} = \underbrace{\begin{bmatrix} k(t_{m-q+2}, t_{m-q+2})+\delta & \cdots & k(t_m, t_{m-q+2}) & 0 \\ k(t_{m-q+2}, t_{m-q+3}) & \cdots & k(t_m, t_{m-q+3}) & 0 \\ \vdots & & & \vdots \\ k(t_{m-q+2}, t_m) & \cdots & k(t_m, t_m)+\delta & 0 \\ 0 & \cdots & 0 & \delta \end{bmatrix}}_{P'_{m+1}} + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & k(t_{m+1}, t_{m-q+2}) \\ 0 & 0 & \cdots & 0 & \\ \vdots & & & & \\ k(t_{m-q+2}, t_{m+1}) & k(t_{m-q+3}, t_{m+1}) & \cdots & & k(t_{m+1}, t_{m+1}) \end{bmatrix}}_{BC^T}$$

split BC^T,

where $B = \begin{bmatrix} k(t_{m+1}, t_{m-q+2}) & 0 \\ k(t_{m+1}, t_{m-q+3}) & 0 \\ \vdots & \vdots \\ k(t_{m+1}, t_m) & 0 \\ k(t_{m+1}, t_{m+1}) & 1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 0 & k(t_{m-q+2}, t_{m+1}) \\ 0 & k(t_{m-q+3}, t_{m+1}) \\ \vdots & \vdots \\ & k(t_m, t_{m+1}) \\ 1 & 0 \end{bmatrix}$

Now, $$P'^{-1}_{m+1} = \begin{bmatrix} k(t_{m-q+2}, t_{m-q+2})+\delta & \cdots & k(t_m, t_{m-q+2}) \\ k(t_{m-q+2}, t_{m-q+3}) & k(t_{m-q+3}, t_{m-q+3})+\delta & k(t_m, t_{m-q+3}) \\ \vdots & & \\ k(t_{m-q+2}, t_m) & \cdots\ k(t_{m-q+3}, t_m) & \cdots\ k(t_m, t_m)+\delta \end{bmatrix}$$

$$P_m^{-1} = \left[\begin{array}{c|ccc} k(t_{m-q+1}, t_{m-q+1})+\delta & k(t_{m-q+2}, t_{m-q+1}) & \cdots & k(t_m, t_{m-q+1}) \\ \hline k(t_{m-q+1}, t_{m-q+2}) & & & \\ \vdots & & P'^{-1}_{m+1} & \\ k(t_{m-q+1}, t_m) & & & \end{array}\right]$$

Split P_m^{-1} as :

$$P_m^{-1} = \underbrace{\left[\begin{array}{c|c} k(t_{m-q+1}, t_{m-q+1})+\delta & \underline{0} \\ \hline \underline{0} & P'^{-1}_{m+1} \end{array}\right]}_{Q'} + \underbrace{\left[\begin{array}{c|ccc} 0 & k(t_{m-q+2}, t_{m-q+1}) & \cdots & k(t_m, t_{m-q+1}) \\ \hline k(t_{m-q+1}, t_{m-q+2}) & & & \\ \vdots & & \underline{0} & \\ k(t_{m-q+1}, t_m) & & & \end{array}\right]}_{FG^T}$$

Split FG^T as :

$$F = \begin{bmatrix} 1 & 0 \\ 0 & k(t_{m-q+2}, t_{m-q+1}) \\ 0 & k(t_{m-q+3}, t_{m-q+1}) \\ \vdots & \vdots \\ 0 & k(t_m, t_{m-q+1}) \end{bmatrix} \qquad G^T = \begin{bmatrix} 0 & k(t_{m-q+2}, t_{m-q+1}) & \cdots & k(t_m, t_{m-q+1}) \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\therefore P_m^{-1} = \begin{bmatrix} k(t_{m-q+1}, t_{m-q+1}) + \delta & \underline{0} \\ \underline{0} & P'^{-1}_{m+1} \end{bmatrix} + FG^T$$

$$\longrightarrow \begin{bmatrix} a & \underline{0} \\ \underline{0} & P'^{-1}_{m+1} \end{bmatrix} \text{ where } a = k(t_{m-q+1}, t_{m-q+1}) + \delta$$

$$\therefore P_m^{-1} - FG^T = \begin{bmatrix} a & \underline{0} \\ \underline{0} & P'^{-1}_{m+1} \end{bmatrix}$$

Take inverse on both sides :

$$(P_m^{-1} - FG^T)^{-1} = \begin{bmatrix} 1/a & \underline{0} \\ \underline{0} & P'_{m+1} \end{bmatrix}$$

By Matrix - Inversion Lemma:

Now, $W = P_m^{-1}$ $X^T = -F$, $Z = G^T$

$$\therefore (P_m^{-1} - FG^T)^{-1} = P_m - P_m(-F)\left(I + G^T P_m (-F)\right)^{-1} G^T P_m$$

$$= P_m + P_m F(I - G^T P_m F)^{-1} G^T P_m$$

$\therefore$ We can compute P'_{m+1} from P_m by taking the $q-1 \times q-1$ submatrix here.

& then, $P_{m+1} = (P'^{-1}_{m+1} + BC^T)^{-1}$

$\therefore$ Again, by matrix inversion lemma

$$P_{m+1} = P'_{m+1} - P'_{m+1} B(I + C^T P'_{m+1} B) C^T P'_{m+1}$$

Plug in P'_{m+1}, B and c to get update P_{m+1}.

Thus, had to do two low-rank updates, once to remove the effect of (t_{m-q+1}, y_{m-q+1}) and once to add effect of (t_{m+1}, y_{m+1}).