ISyE 6739–The Analysis of Variance (Two-Way ANOVA) (Chapter 13)

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Two-Way ANOVA (Randomized Complete Block Designs)

Example: Three different washing solutions are being compared to study their effectiveness in retarding bacteria growth in 5-gallon milk containers. The analysis is done in a laboratory, and only three trials can be run on any day. Because days could represent a potential source of variability, the experimenter decides to use a randomized block design. Observations are taken for four days, and the data are shown here. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Days			
Solution	1	2	3	4
1	13	22	18	39
2	16	24	17	44
3	5	4	1	22

Question of interest: Do these solutions have the same efficacy?

Factor of interest: Types of solutions

Block Factor: Day

Block factor is a nuisance factor that affect the homogeneity of the study. For example, the lab conditions could be different in days 1 to 4.

$$\begin{cases} H_0 : \mu_1 = \mu_2 = \mu_3 \\ H_a : \text{at least one mean differs from others} \end{cases} \Longleftrightarrow \begin{array}{l} H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0 \\ H_1: \tau_i \neq 0 \quad \text{for at least one } i \end{array}$$

Two-Way ANOVA (Blocking)

Table 13-10 A Randomized Complete Block Design with a Treatments and b Blocks

Treatments	Blocks 1	2	$\cdots$	b	Totals	Averages
1	y_{11}	y_{12}	$\cdots$	y_{1b}	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	y_{21}	y_{22}	$\cdots$	y_{2b}	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{ab}	$y_{a.}$	$\bar{y}_{a\cdot}$
Totals	$y_{\cdot 1}$	$y_{\cdot 2}$	$\cdots$	$y_{\cdot b}$	$y_{\cdot\cdot}$	
Averages	$\bar{y}_{\cdot 1}$	$\bar{y}_{\cdot 2}$	$\cdots$	$\bar{y}_{\cdot b}$		$\bar{y}_{\cdot\cdot}$

$i = 1,2,\cdots,a \longrightarrow$ Number of treatments

$j = 1,2,\cdots,b \longrightarrow$ Number of blocks

y_{ij} Observation from treatment i and block j

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

Effect of factor

Effect of block

Variation Decomposition

$$\begin{cases} H_0 : \mu_1 = \mu_2 = \cdots = \mu_a \\ H_a : \text{at least one mean differs from others} \end{cases}$$

ANOVA partitions the total variability into three parts

Total Variations = Between-Treatment Variations + Between-Block Variations + Within-group Variations

SST = $SS_{\text{treatments}} + SS_{\text{Blocks}} + SS_{\text{Error}}$

The **sum of squares identity for the randomized complete block design** is

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{\cdot\cdot})^2 = b\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 + a\sum_{j=1}^{b}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{\cdot j} - \bar{y}_{i\cdot} + \bar{y}_{\cdot\cdot})^2 \quad (13\text{-}27)$$

or symbolically

$$SS_T = SS_{\text{Treatments}} + SS_{\text{Blocks}} + SS_E$$

Sum of Squares Calculations

SST = $SS_{treatments} + SS_{Blocks} + SS_{Error}$

The computing formulas for the sums of squares in the analysis of variance for a randomized complete block design are

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab} \tag{13-29}$$

$$SS_{\text{Treatments}} = \frac{1}{b} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{ab} \tag{13-30}$$

$$SS_{\text{Blocks}} = \frac{1}{a} \sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{ab} \tag{13-31}$$

and

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}} \tag{13-32}$$

Mean Squares and F-Statistic

$$\begin{cases} H_0 : \mu_1 = \mu_2 = \cdots = \mu_a \\ H_a : \text{at least one mean differs from others} \end{cases}$$

Mean Squares are

$$MS_{\text{Treatments}} = \frac{SS_{\text{Treatments}}}{a-1}$$

$$MS_{\text{Blocks}} = \frac{SS_{\text{Blocks}}}{b-1}$$

$$MS_E = \frac{SS_E}{(a-1)(b-1)}$$

The appropriate test statistic is

$$F_0 = \frac{MS_{\text{Treatments}}}{MS_{\text{Error}}}$$

We would reject H_0 if $F_0 > F_{\alpha, a-1, (a-1)(b-1)}$

Two-Way ANOVA Table

$$\begin{cases} H_0 : \mu_1 = \mu_2 = \cdots = \mu_a \\ H_a : \text{at least one mean differs from others} \end{cases}$$

Table 13-11 ANOVA for a Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a-1$	$\dfrac{SS_{\text{Treatments}}}{a-1}$	$\dfrac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b-1$	$\dfrac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$(a-1)(b-1)$	$\dfrac{SS_E}{(a-1)(b-1)}$	
Total	SS_T	$ab-1$		

We would reject H_0 if $F_0 > F_{\alpha, a-1, (a-1)(b-1)}$

ANOVA

Example: Three different washing solutions are being compared to study their effectiveness in retarding bacteria growth in 5-gallon milk containers. The analysis is done in a laboratory, and only three trials can be run on any day. Because days could represent a potential source of variability, the experimenter decides to use a randomized block design. Observations are taken for four days, and the data are shown here. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Days			
Solution	1	2	3	4
1	13	22	18	39
2	16	24	17	44
3	5	4	1	22

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab}$$

$$SS_{\text{Treatments}} = \frac{1}{b} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{ab}$$

$$SS_{\text{Blocks}} = \frac{1}{a} \sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{ab}$$

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}}$$

Example

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a-1$	$\frac{SS_{\text{Treatments}}}{a-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b-1$	$\frac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$(a-1)(b-1)$	$\frac{SS_E}{(a-1)(b-1)}$	
Total	SS_T	$ab-1$		

ANOVA

EXAMPLE 13-5 Fabric Strength

An experiment was performed to determine the effect of four different chemicals on the strength of a fabric. These chemicals are used as part of the permanent press finishing process. Five fabric samples were selected, and a RCBD was run by testing each chemical type once in random order on each fabric sample. The data are shown in Table 13-12. We will test for differences in means using an ANOVA with $\alpha = 0.01$.

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab}$$

$$SS_{\text{Treatments}} = \frac{1}{b}\sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{ab}$$

$$SS_{\text{Blocks}} = \frac{1}{a}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{ab}$$

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}}$$

Table 13-12 Fabric Strength Data—Randomized Complete Block Design

Chemical Type	Fabric Sample 1	2	3	4	5	Treatment Totals $y_{i.}$	Treatment Averages $\bar{y}_{i.}$
1	1.3	1.6	0.5	1.2	1.1	5.7	1.14
2	2.2	2.4	0.4	2.0	1.8	8.8	1.76
3	1.8	1.7	0.6	1.5	1.3	6.9	1.38
4	3.9	4.4	2.0	4.1	3.4	17.8	3.56
Block totals $y_{.j}$	9.2	10.1	3.5	8.8	7.6	39.2($y_{..}$)	
Block averages $\bar{y}_{.j}$	2.30	2.53	0.88	2.20	1.90		1.96($\bar{y}_{..}$)

Example

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$\frac{SS_{\text{Treatments}}}{a - 1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b - 1$	$\frac{SS_{\text{Blocks}}}{b - 1}$	
Error	SS_E (by subtraction)	$(a - 1)(b - 1)$	$\frac{SS_E}{(a - 1)(b - 1)}$	
Total	SS_T	$ab - 1$		

Assumptions in ANOVA

- Observations are normally distributed
- Observations are independent
- Variances of observations in each factor level (treatment) are equal
- Variances of observations in each block level are equal
- Effects are fixed:

$$\sum_{i=1}^{a} \tau_i = 0 \text{ and } \sum_{j=1}^{b} \beta_j = 0$$

General Two-way ANOVA

Example:

13.4.1 An Example

Aircraft primer paints are applied to aluminum surfaces by two methods—dipping and spraying. The purpose of the primer is to improve paint adhesion; some parts can be primed using either application method. A team using the DMAIC approach has identified three different primers that can be used with both application methods. Three specimens were painted with each primer using each application method, a finish paint was applied, and the adhesion force was measured. The 18 runs from this experiment were run in random order. The resulting data are shown in Table 13.1. The circled numbers in the cells are the cell totals. The objective of the experiment was to determine which combination of primer paint and application method produced the highest adhesion force. It would be desirable if at least one of the primers produced high adhesion force *regardless* of application method, as this would add some flexibility to the manufacturing process.

TABLE 13.1
Adhesion Force Data

	Application Method				
Primer Type	**Dipping**		**Spraying**		$y_{i..}$
1	4.0, 4.5, 4.3	12.8	5.4, 4.9, 5.6	15.9	28.7
2	5.6, 4.9, 5.4	15.9	5.8, 6.1, 6.3	18.2	34.1
3	3.8, 3.7, 4.0	11.5	5.5, 5.0, 5.0	15.5	27.0
$y_{.j.}$	40.2		49.6		$89.8 = y_{...}$

General Two-factor Factorial Design

		Factor B			
		1	**2**	**…**	b
	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$	…	$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
Factor A	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$	…	$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮	⋮	⋮	⋮	⋮
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$	…	$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \qquad (13.2)$$

where μ is the overall mean effect, τ_i is the effect of the ith level of factor A, β_j is the effect of the jth level of factor B, $(\tau\beta)_{ij}$ is the effect of the interaction between A and B, and ε_{ijk} is an NID$(0, \sigma^2)$ random error component. We are interested in testing the hypotheses of no significant factor A effect, no significant factor B effect, and no significant AB interaction.

Statistical Analysis: Two-way ANOVA

TABLE 13.3
The ANOVA Table for a Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{abn} \quad (13.6)$$

Main effects

$$SS_A = \sum_{i=1}^{a} \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn} \quad (13.7)$$

$$SS_B = \sum_{j=1}^{b} \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn} \quad (13.8)$$

Interaction

$$SS_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{y_{ij.}^2}{n} - \frac{y_{...}^2}{abn} - SS_A - SS_B \quad (13.9)$$

Error

$$SS_E = SS_T - SS_A - SS_B - SS_{AB} \quad (13.10)$$

Example: Primer Experiment

EXAMPLE 13.5 The Aircraft Primer Paint Problem

Use the ANOVA described above to analyze the aircraft primer paint experiment described in Section 13.4.1.

SOLUTION

The sums of squares required are

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{abn} = (4.0)^2 + (4.5)^2 + \cdots + (5.0)^2 - \frac{(89.8)^2}{18} = 10.72$$

$$SS_{\text{primers}} = \sum_{i=1}^{a} \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn} = \frac{(28.7)^2 + (34.1)^2 + (27.0)^2}{6} - \frac{(89.8)^2}{18} = 4.58$$

$$SS_{\text{methods}} = \sum_{j=1}^{b} \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn} = \frac{(40.2)^2 + (49.6)^2}{9} - \frac{(89.8)^2}{18} = 4.91$$

$$SS_{\text{interaction}} = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{y_{ij.}^2}{n} - \frac{y_{...}^2}{abn} - SS_{\text{primers}} - SS_{\text{methods}} = \frac{(12.8)^2 + (15.9)^2 + (11.5)^2 + (15.9)^2 + (18.2)^2 + (15.5)^2}{3} - \frac{(89.8)^2}{18} - 4.58 - 4.91 = 0.24$$

and

$$SS_E = SS_T - SS_{\text{primers}} - SS_{\text{methods}} - SS_{\text{interaction}} = 10.72 - 4.58 - 4.91 - 0.24 = 0.99$$

The P-values in this table were obtained from a calculator (they can also be found using the Probability Distribution function in the Calc menu in Minitab).

The ANOVA is summarized in Table 13.4. Note that the P-values for both main effects are very small, indicating that

TABLE 13.1
Adhesion Force Data

	Application Method				
Primer Type	Dipping		Spraying		$y_{i..}$
1	4.0, 4.5, 4.3	12.8	5.4, 4.9, 5.6	15.9	28.7
2	5.6, 4.9, 5.4	15.9	5.8, 6.1, 6.3	18.2	34.1
3	3.8, 3.7, 4.0	11.5	5.5, 5.0, 5.0	15.5	27.0
$y_{.j.}$	40.2		49.6		$89.8 = y_{...}$

Example: Primer Experiment

EXAMPLE 13.5 The Aircraft Primer Paint Problem

Use the ANOVA described above to analyze the aircraft primer paint experiment described in Section 13.4.1.

SOLUTION

The sums of squares required are

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{abn}$$
$$= (4.0)^2 + (4.5)^2 + \cdots + (5.0)^2 - \frac{(89.8)^2}{18} = 10.72$$

$$SS_{\text{primers}} = \sum_{i=1}^{a} \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}$$
$$= \frac{(28.7)^2 + (34.1)^2 + (27.0)^2}{6} - \frac{(89.8)^2}{18} = 4.58$$

$$SS_{\text{methods}} = \sum_{j=1}^{b} \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn}$$
$$= \frac{(40.2)^2 + (49.6)^2}{9} - \frac{(89.8)^2}{18} = 4.91$$

$$SS_{\text{interaction}} = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{y_{ij.}^2}{n} - \frac{y_{...}^2}{abn} - SS_{\text{primers}} - SS_{\text{methods}}$$
$$= \frac{(12.8)^2 + (15.9)^2 + (11.5)^2 + (15.9)^2 + (18.2)^2 + (15.5)^2}{3}$$
$$- \frac{(89.8)^2}{18} - 4.58 - 4.91 = 0.24$$

and

$$SS_E = SS_T - SS_{\text{primers}} - SS_{\text{methods}} - SS_{\text{interaction}}$$
$$= 10.72 - 4.58 - 4.91 - 0.24 = 0.99$$

The *P*-values in this table were obtained from a calculator (they can also be found using the Probability Distribution function in the Calc menu in Minitab).

The ANOVA is summarized in Table 13.4. Note that the *P*-values for both main effects are very small, indicating that *(continued)*

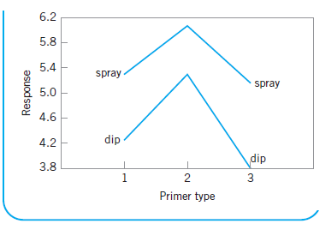
$F_{0.05,1,12} = 4.75$

$F_{0.05,2,12} = 3.89$

TABLE 13.4
ANOVA for Example 13.5

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-value
Primer types	4.58	2	2.290	27.93	1.93×10^{-4}
Application methods	4.91	1	4.910	59.88	5.28×10^{-6}
Interaction	0.24	2	0.120	1.46	0.269
Error	0.99	12	0.082		
Total	10.72	17			

Example: Primer Experiment



spraying is a superior application method and that primer type 2 is most effective. Therefore, if we wish to operate the process so as to attain maximum adhesion force, we should use primer type 2 and spray all parts.

FIGURE 13.12 Graph of average adhesion force versus primer types for Example 13.5.