Problem1:

The log likelihood function of $\beta$ is

$$\begin{aligned} l(\beta|X_1,\dots,X_n,\alpha) &= log\ L(\beta|X_1,\dots,X_n,\alpha) \\ &= \sum_{i=1}^{n} log\ f(X_i|\alpha,\beta) \\ &= \sum_{i=1}^{n} log[\frac{\beta^\alpha}{\Gamma(\alpha)} X_i^{\alpha-1} e^{-\beta X_i}] \end{aligned}$$

As MLE of $\beta$ maximizes $l(\beta|X_1,\dots,X_n,\alpha)$, we use the first order and second order conditions and get

$$\begin{aligned} &\frac{\partial}{\partial\beta} l(\beta|X_1,\dots,X_n,\alpha)|_{\beta=\hat\beta} = 0 \\ &\sum_{i=1}^{n} (\frac{\alpha}{\hat\beta} - X_i) = 0 \\ &\hat\beta = \frac{1}{n\alpha}\sum_{i=1}^{n} X_i, \end{aligned}$$

and,

$$\begin{aligned} &\frac{\partial^2}{\partial\beta^2} l(\beta|X_1,\dots,X_n,\alpha)|_{\beta=\hat\beta} \\ &= \sum_{i=1}^{n} -\frac{\alpha}{\hat\beta^2} < 0. \end{aligned}$$

Thus, the MLE of $\beta$ is $\hat\beta = \frac{1}{n\alpha}\sum_{i=1}^{n} X_i$.

Problem2:

(a)The joint density of $X_1,\dots,X_n$ is

$$f(x_1,\dots,x_n|\theta) = \theta^n \prod_{i=1}^{n} x_i^{-2} I[x_1 \ge \theta,\dots,x_n \ge \theta] = \theta^n \prod_{i=1}^{n} x_i^{-2} I[\min_{1\le i\le n} x_i \ge \theta],$$

where $I(\cdot)$ is indicator function. By factorization theorem, we know $\min_{1\le i\le n} x_i$ is sufficient statistic for $\theta$.

(b)The MLE of $\theta$ maximizes $f(x_1,\dots,x_n|\theta)$, so, $\hat\beta = \min_{1\le i\le n} x_i$.

(c)Since $E[X_i] = \infty$, let us consider $E[\frac{1}{X_i}] = \frac{1}{2\theta}$. Replace $E[\frac{1}{X_i}]$ with $\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i}$, then we get a moment estimator of $\theta$: $\hat\theta = \frac{n}{2\sum_{i=1}^{n}\frac{1}{X_i}}$.

Problem3:

(b)The $p.d.f.$ of $X_i$ is $f(x|\alpha,\beta) = \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1}, 0 \le x \le \beta$. The likelihood function of $(\alpha,\beta)$ is the joint density of $X_1,\dots,X_n$. So the likelihood function is

$$
\begin{aligned}
L(\alpha,\beta|x_1,\dots,x_n) &= \prod_{i=1}^{n} \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1} I[0 \le x_i \le \beta] \\
&= [\prod_{i=1}^{n} \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1}] I[\min_{1\le i\le n} x_i \ge 0] I[\max_{1\le i\le n} x_i \le \beta].
\end{aligned}
$$

To minimize likelihood function, we should choose $\beta$ as small as possible. But as $\max_{1\le i\le n} x_i \le \beta$, $\hat{\beta} = \max_{1\le i\le n} X_i$. Take the fist order condition of log likelihood function,

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha} l(\alpha,\beta|x_1,\dots,x_n)|_{\alpha=\hat{\alpha}} = 0 \\
&\frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} log(\frac{X_i}{\hat{\beta}}) = 0 \\
&\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} log(\frac{\hat{\beta}}{X_i})}.
\end{aligned}
$$

(c)$\hat{\beta} = \max_{1\le i\le n} X_i = 25$. $\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} log(\frac{\hat{\beta}}{X_i})} = 12.59487$.

Problem5:

MLE of $\theta$ makes $f(x|\theta)$ attain its maximum for any $x$, thus

$$
\hat{\theta} = \begin{cases} 1 & x = 0 \text{ } or \text{ } 1, \\ 2 \text{ } or \text{ } 3 & x = 2, \\ 3 & x = 3 \text{ } or \text{ } 4. \end{cases}
$$

Problem6:

$L(\theta|\mathbf{x}) = \prod_i \frac{1}{2}e^{-\frac{1}{2}|x_i-\theta|} = \frac{1}{2^n}e^{-\frac{1}{2}\Sigma_i|x_i-\theta|}$, so the MLE minimizes $\sum_i |x_i - \theta| = \sum_i |x_{(i)} - \theta|$, where $x_{(1)}, \ldots, x_{(n)}$ are the order statistics. For $x_{(j)} \le \theta \le x_{(j+1)}$,

$$\sum_{i=1}^{n} |x_{(i)} - \theta| = \sum_{i=1}^{j}(\theta - x_{(i)}) + \sum_{i=j+1}^{n}(x_{(i)} - \theta) = (2j - n)\theta - \sum_{i=1}^{j} x_{(i)} + \sum_{i=j+1}^{n} x_{(i)}.$$

This is a linear function of $\theta$ that decreases for $j < n/2$ and increases for $j > n/2$. If $n$ is even, $2j - n = 0$ if $j = n/2$. So the likelihood is constant between $x_{(n/2)}$ and $x_{((n/2)+1)}$, and any value in this interval is the MLE. Usually the midpoint of this interval is taken as the MLE. If $n$ is odd, the likelihood is minimized at $\hat{\theta} = x_{((n+1)/2)}$.