ISyE 6739 – Group Activity 8
solutions

1.
$$n = 9, \quad \bar{X} = 10.5, \quad \sigma^2 = 4.$$

(a) Lower bound of 95% CI:
$$L = \bar{X} - Z_{0.025}\frac{\sigma}{\sqrt{n}} = 10.5 - 1.96\frac{2}{3} = 9.193,$$
Upper bound of 95% CI:
$$U = \bar{X} + Z_{0.025}\frac{\sigma}{\sqrt{n}} = 10.5 + 1.96\frac{2}{3} = 11.807.$$

(b)
$$|\mu - \bar{X}| \leq Z_{0.025}\frac{\sigma}{\sqrt{n}} = 1.96\frac{2}{3} = 1.307.$$

(c)
$$W = \bar{X} + Z_{0.025}\frac{\sigma}{\sqrt{n}} - (\bar{X} - Z_{0.025}\frac{\sigma}{\sqrt{n}}) = 2 \cdot Z_{0.025}\frac{\sigma}{\sqrt{n}} = 2 \cdot 1.96\frac{2}{3} = 2.613.$$

(d) $E = 1$ then
$$n_E = \left(Z_{0.025}\frac{\sigma}{E}\right)^2 = \left(\frac{1.96 \cdot 2}{1}\right)^2 = 15.36 \approx 16.$$

(e)
$$W_{new} = 2 \cdot Z_{0.025}\frac{\sigma}{\sqrt{n_E}} = 2 \cdot 1.96\frac{2}{4} = 1.96.$$

2. We know that $\Phi^{-1}(1-\alpha) = Z_\alpha$. Then the length of the CI is following
$$L = \frac{\sigma(Z_{\alpha_1} + Z_{\alpha_2})}{\sqrt{n}} = \frac{\sigma(\Phi^{-1}(1-\alpha_1) + \Phi^{-1}(1-\alpha_2))}{\sqrt{n}}.$$
We want to solve the following optimization problem:
$$\text{minimize} \quad \Phi^{-1}(1-\alpha_1) + \Phi^{-1}(1-\alpha_2)\alpha_1 + \alpha_2 = \alpha$$
where $\alpha \in (0, 1)$ Then by KKT stationarity condition
$$-\nabla(\Phi^{-1}(1-\alpha_1) + \Phi^{-1}(1-\alpha_2)) = \lambda\nabla(\alpha_1 + \alpha_2 - \alpha),$$
for some λ.
Using $\left(\Phi^{-1}\right)' = (\Phi')^{-1}$
$$-\nabla(\Phi^{-1}(1-\alpha_1) + \Phi^{-1}(1-\alpha_2)) = \left(\frac{1}{\Phi'(Z_{\alpha_1})}, \frac{1}{\Phi'(Z_{\alpha_2})}\right),$$
$$\lambda\nabla(\alpha_1 + \alpha_2 - \alpha) = \lambda(1, 1) = (\lambda, \lambda).$$
$$\Rightarrow \left(\frac{1}{\Phi'(Z_{\alpha_1})}, \frac{1}{\Phi'(Z_{\alpha_2})}\right) = (\lambda, \lambda)$$

$$\Rightarrow \frac{1}{\Phi'(Z_{\alpha_1})} = \frac{1}{\Phi'(Z_{\alpha_2})}$$

$$\Rightarrow \Phi'(Z_{\alpha_1}) = \Phi'(Z_{\alpha_2}) \quad \Rightarrow Z_{\alpha_1} = Z_{\alpha_2}$$

the last equation is because for $x > 0$ pdf $\Phi'(x)$ is an increasing function.

$$Z_{\alpha_1} = Z_{\alpha_2} \alpha_1 + \alpha_2 = \alpha$$

$$\Rightarrow \alpha_1 = \alpha_2 = \frac{\alpha}{2}.$$

3. (a) By definition of the median $\mathrm{P}\{X_i \geq \tilde{\mu}\} = \mathrm{P}\{X_i < \tilde{\mu}\} = \frac{1}{2}$.

$\mathrm{P}\{\min\limits_i X_i < \tilde{\mu} < \max\limits_i X_i\} = 1 - \mathrm{P}\{\tilde{\mu} \leq \min\limits_i X_i \cup \tilde{\mu} \geq \max\limits_i X_i\} = 1 - \mathrm{P}\{\tilde{\mu} \leq \min\limits_i X_i\} - \mathrm{P}\{\tilde{\mu} \geq \max\limits_i X_i\} =$

$$= 1 - \mathrm{P}\{\forall i \;\; \tilde{\mu} \leq X_i\} - \mathrm{P}\{\forall i \;\; \tilde{\mu} \geq X_i\} = 1 - \prod_{i=1}^{n} \mathrm{P}\{\tilde{\mu} \leq X_i\} - \prod_{i=1}^{n} \mathrm{P}\{\tilde{\mu} \geq X_i\} =$$

$$= 1 - \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2^{n-1}}.$$

4. Let P denote the percentage of CI's that cover the true mean. Compute P:

```
n<-5
alpha<-0.05
# generate 1000 samples of 5 std normal random numbers
simdata<-rnorm(1000*n, mean=0, sd=1)
X<-matrix(simdata, nrow=1000, ncol=n)
# calculate a sample mean for each sample
X_bar<-1/n*rowSums(X)
# calculate the alpha/2-quantile for the std normal distribution
Za2<-qnorm(alpha/2, 0, 1, lower.tail = FALSE)
# lower and upper bounds for (1-alpha)*100% CI
L<-X_bar-Za2/sqrt(n)
U<-X_bar+Za2/sqrt(n)
numb<-0
# for each CI check if the true mean is in it
for (i in seq(from=1, to=1000, by=1)) {
  if (0>=L[i] & 0<=U[i]){numb<-numb+1}
}
P<-numb/1000
# print the proportion of CI's that cover the true mean
print(P)
```

Using the code above we have got that $P = 0.946 \approx 0.95$. That agrees with the interpretation of "$100(1-\alpha)\%$ confidence": in the long run, $100(1-\alpha)\%$ of all computed CI's will contain the true value of an estimated parameter.

5. (a) $2\lambda T_r \sim \chi^2_{2r}$ then

$$\mathrm{P}\{\chi^2_{1-\alpha/2,2r} \leq 2\lambda T_r \leq \chi^2_{\alpha/2,2r}\} = \mathrm{P}\{\frac{2T_r}{\chi^2_{\alpha/2,2r}} \leq \frac{1}{\lambda} \leq \frac{2T_r}{\chi^2_{1-\alpha/2,2r}}\},$$

$\Rightarrow 100(1-\alpha)\%$ confidence interval for mean μ is

$$\frac{2T_r}{\chi^2_{\alpha/2,2r}} \leq \mu \leq \frac{2T_r}{\chi^2_{1-\alpha/2,2r}}.$$

(b)

$$T_{10} = (15 + 18 + \cdots + 28 + 29) + (20 - 10)29 = 510,$$

$$\chi^2_{0.025,20} = 34.17, \quad \chi^2_{1-0.025,20} = 9.591,$$

then 95% CI for μ is

$$\frac{2 \cdot 510}{34.17} \leq \mu \leq \frac{2 \cdot 510}{9.591},$$

$$29.85 \leq \mu \leq 106.35.$$