ISyE 2028 – Group Activity 8

Names: Group Number:

Problem 1. GM has a supplier manufacturing crankshafts. The lead time, the difference between the order time and the time of delivery, is a normal random variable with an unknown mean. A production manager in GM is interested in finding a 95% confidence interval for the mean lead time. She randomly picks 9 previous orders and compute the average lead time as 10.5 days. Based on previous experience the variance of the lead time is 4 days.

a) What is the lower and upper bound of this CI?
b) Find an error bound for the point estimation with 95% confidence.
c) What is the width of the CI?
d) Find *n* such that the point estimation error is less than 1 day.
e) Using *n* found in part d, find the new expected width of CI.

Problem 2.

8-114. Consider a two-sided confidence interval for the mean μ when σ is known:

$$\bar{x} - z_{\alpha_1}\,\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{\alpha_2}\,\sigma/\sqrt{n}$$

where $\alpha_1 + \alpha_2 = \alpha$. If $\alpha_1 = \alpha_2 = \alpha/2$, you have the usual $100(1-\alpha)\%$ confidence interval for μ. In the preceding, when $\alpha_1 \neq \alpha_2$, the interval is not symmetric about μ. The length of the interval is $L = \sigma(z_{\alpha_1} + z_{\alpha_2})/\sqrt{n}$. Prove that the length of the interval L is minimized when $\alpha_1 = \alpha_2 = \alpha/2$. *Hint:* Remember that $\Phi(z_\alpha) = 1-\alpha$, so $\Phi^{-1}(1-\alpha) = z_\alpha$, and the relationship between the derivative of a function $y = f(x)$ and the inverse $x = f^{-1}(y)$ is $(d/dy)f^{-1}(y) = 1/\left[(d/dy)f(x)\right]$.

Problem 3.

8-116. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a continuous probability distribution with median $\tilde{\mu}$

(a) Show that

$$p\{\min(X_i) < \tilde{\mu} < \max(X_i)\} = 1 - \left(\frac{1}{2}\right)^{n-1}$$

[*Hint:* The complement of the event $[\min(X_i) < \tilde{\mu} < \max(X_i)]$ is $[\max(X_i) \le \tilde{\mu}] \cup [\min(X_i) \le \tilde{\mu}]$ but $\max(X_i) \le \tilde{\mu}$ if and only if $X_i \le \tilde{\mu}$ for all i.]

(b) Write down a $100(1-\alpha)\%$ confidence interval for the median $\tilde{\mu}$ where

$$\alpha = \left(\frac{1}{2}\right)^{n-1}$$

Problem 4. Understanding the concept of confidence interval

Using R, generate 1000 samples of 5 standard normal random numbers.

- For each sample (row), find a 95% confidence interval.
- What is the percentage of intervals that cover the true mean? What is your conclusion?

Here are pieces of codes you should use:

```
simdata = rnorm((100*n), mean = 0 , sd = 1)
X = matrix(simdata, nrow = 100, ncol = n)
rowSums(X)
qnorm(alpha/2,0,1,lower.tail=FALSE) #this gives you Zα/2
```

Bonus Problem:

8-113. An electrical component has a time-to-failure (or lifetime) distribution that is exponential with parameter λ, so the mean lifetime is $\mu = 1/\lambda$. Suppose that a sample of n of these components is put on test, and let X_i be the observed lifetime of component i. The test continues only until the rth unit fails, where $r < n$. This results in a **censored** life test. Let X_1 denote the time at which the first failure occurred, X_2 denote the time at which the second failure occurred, and so on. Then the total lifetime that has been accumulated at test termination is

$$T_r = \sum_{i=1}^{r} X_i + (n-r)X_r$$

We have previously shown in Exercise 7-81 that Tr/r is an unbiased estimator for μ.

(a) It can be shown that $2\lambda T_r$ has a chi-square distribution with $2r$ degrees of freedom. Use this fact to develop a $100(1-\alpha)\%$ confidence interval for mean lifetime $\mu = 1/\lambda$.

(b) Suppose that 20 units were tested, and the test terminated after 10 failures occurred. The failure times (in hours) are 15, 18, 19, 20, 21, 21, 22, 27, 28, and 29. Find a 95% confidence interval on mean lifetime.