ISyE 6739 – Statistical Methods
Hypothesis Tests – Two Populations
(Chapter 10)

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Inference for Differences in Means of Two Normal Distributions (known variances)

Null Hypothesis

$$H_0 : \mu_1 - \mu_2 = \Delta_0$$

Test Statistic

$$Z_0 = \frac{(\bar{X} - \bar{Y}) - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Distribution under H0

$$Z_0 \sim N(0,1)$$

Alternative Hypothesis	Rejection/Critical Region (H0 is rejected)
$H_1 : \mu_1 - \mu_2 \neq \Delta_0$	$\lvert Z_0 \rvert > Z_{\alpha/2}$
$H_1 : \mu_1 - \mu_2 > \Delta_0$	$Z_0 > Z_\alpha$
$H_1 : \mu_1 - \mu_2 < \Delta_0$	$Z_0 < -Z_\alpha$

Critical values

Hypothesis Testing Using Confidence Intervals Mean of Two Normal Populations – Known Variance

- **Collect a sample and construct a 100(1- α)% CI**

$$H_0: \mu_1 - \mu_2 = \Delta_0 \quad H_1: \mu_1 - \mu_2 \neq \Delta_0 \Rightarrow (\bar{X}-\bar{Y}) - Z_{\alpha/2} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq (\bar{X}-\bar{Y}) + Z_{\alpha/2} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$H_0: \mu_1 - \mu_2 = \Delta_0 \quad H_1: \mu_1 - \mu_2 > \Delta_0 \Rightarrow \left[(\bar{X}-\bar{Y}) - Z_{\alpha} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, +\infty\right)$$

Lower CI

$$H_0: \mu_1 - \mu_2 = \Delta_0 \quad H_1: \mu_1 - \mu_2 < \Delta_0 \Rightarrow \left(-\infty, (\bar{X}-\bar{Y}) + Z_{\alpha} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right]$$

Upper CI

- **If the Confidence Interval does NOT include Δ_0, then Reject H0**

Inference for Differences in Means of Two Normal Distributions (known variances)

Null Hypothesis	Test Statistic	Distribution under H0
$H_0: \mu_1 - \mu_2 = \Delta_0$	$Z_0 = \dfrac{(\bar{X}-\bar{Y}) - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$	$Z_0 \sim N(0,1)$

Alternative Hypothesis	Rejection/Critical Region (H0 is rejected)	Test using CI	P-values
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	$\lvert Z_0 \rvert > Z_{\alpha/2}$	$\left[(\bar{X}-\bar{Y}) - Z_{\alpha/2} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, (\bar{X}-\bar{Y}) + Z_{\alpha/2} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right]$	$2[1-\Phi(\lvert Z_0 \rvert)]$
$H_1: \mu_1 - \mu_2 > \Delta_0$	$Z_0 > Z_{\alpha}$	$\left[(\bar{X}-\bar{Y}) - Z_{\alpha} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, +\infty\right)$	$1-\Phi(Z_0)$
$H_1: \mu_1 - \mu_2 < \Delta_0$	$Z_0 < -Z_{\alpha}$	$\left(-\infty, (\bar{X}-\bar{Y}) + Z_{\alpha} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right]$	$\Phi(Z_0)$

Example

A product developer is interested in reducing the drying time of a primer paint. Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order. The two sample average drying times are $\bar{x}_1 = 121$ minutes and $\bar{x}_2 = 112$ minutes, respectively. What conclusions can the product developer draw about the effectiveness of the new ingredient, using $\alpha = 0.05$?

Choice of Sample Size

- Two-sided hypothesis test

For the two-sided alternative hypothesis with significance level α, the sample size $n_1 = n_2 = n$ required to detect a true difference in means of Δ with power at least $1 - \beta$ is

$$n \simeq \frac{(z_{\alpha/2} + z_\beta)^2(\sigma_1^2 + \sigma_2^2)}{(\Delta - \Delta_0)^2} \qquad (10\text{-}5)$$

- One-sided hypothesis test

For a one-sided alternative hypothesis with significance level α, the sample size $n_1 = n_2 = n$ required to detect a true difference in means of $\Delta(\neq\Delta_0)$ with power at least $1 - \beta$ is

$$n = \frac{(z_{\alpha} + z_\beta)^2(\sigma_1^2 + \sigma_2^2)}{(\Delta - \Delta_0)^2} \qquad (10\text{-}6)$$

Example: Choice of Sample Size

To illustrate the use of these sample size equations, consider the situation described in Example 10-1, and suppose that if the true difference in drying times is as much as 10 minutes, we want to detect this with probability at least 0.90. Under the null hypothesis, $\Delta_0 = 0$. We have a one-sided alternative hypothesis with $\Delta = 10$, $\alpha = 0.05$ (so $z_\alpha = z_{0.05} = 1.645$), and since the power is 0.9, $\beta = 0.10$ (so $z_\beta = z_{0.10} = 1.28$).

Inference Difference in Means of Two Normal Distributions (unknown variances)

Case I: Two independent random samples from two Normal distributions with unknown (but equal) variances $\sigma_1^2 = \sigma_2^2 = \sigma = ?$

Null Hypothesis

$$H_0 : \mu_1 - \mu_2 = \Delta_0$$

Test Statistic

$$t_0 = \frac{(\bar{X} - \bar{Y}) - \Delta_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 + n_2 - 2)}$$

Distribution under H0

$$t_0 \sim t(n_1 + n_2 - 2)$$

Alternative Hypothesis	Rejection Region (H0 is rejected)	Test using CI (If the Confidence Interval does NOT include Δ_0, then Reject H0)	P-values
$H_1 : \mu_1 - \mu_2 \neq \Delta_0$	$\lvert t_0 \rvert > t_{\alpha/2, n_1+n_2-2}$	$\left[(\bar{X} - \bar{Y}) - t_{\alpha/2, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, (\bar{X} - \bar{Y}) + t_{\alpha/2, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right]$	$2[1 - T_{n_1+n_2-2}(\lvert t_0 \rvert)]$
$H_1 : \mu_1 - \mu_2 > \Delta_0$	$t_0 > t_{\alpha, n_1+n_2-2}$	$\left[(\bar{X} - \bar{Y}) - t_{\alpha, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, +\infty\right)$	$1 - T_{n_1+n_2-2}(t_0)$
$H_1 : \mu_1 - \mu_2 < \Delta_0$	$t_0 < -t_{\alpha, n_1+n_2-2}$	$\left(-\infty, (\bar{X} - \bar{Y}) + t_{\alpha, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right]$	$T_{n_1+n_2-2}(t_0)$

Example

Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Specifically, catalyst 1 is currently in use, but catalyst 2 is acceptable. Since catalyst 2 is cheaper, it should be adopted, providing it does not change the process yield. A test is run in the pilot plant and results in the data shown in Table 10-1. Is there any difference between the mean yields? Use $\alpha = 0.05$, and assume equal variances.

Table 10-1 Catalyst Yield Data, Example 10-5

Observation Number	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
3	92.18	90.46
4	95.39	93.21
5	91.79	97.19
6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
	$\bar{x}_1 = 92.255$ $s_1 = 2.39$	$\bar{x}_2 = 92.733$ $s_2 = 2.98$

Inference Difference in Means of Two Normal Distributions (unknown variances)

Case II: Two independent random samples from two Normal distributions with unknown and not equal variances $\sigma_1^2 \neq \sigma_2^2$

Null Hypothesis

$$H_0: \mu_1 - \mu_2 = \Delta_0$$

Test Statistic

$$t_0 = \frac{(\bar{X} - \bar{Y}) - \Delta_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

Distribution under H0

$$t_0 \approx t(v) \qquad v = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}}$$

Alternative Hypothesis	Rejection Region (H0 is rejected)	Test using CI (If the Confidence Interval does NOT include Δ_0, then Reject H0)	P-values
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	$\lvert t_0 \rvert > t_{\alpha/2,v}$	$\left[(\bar{X} - \bar{Y}) - t_{\alpha/2,v}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}, (\bar{X} - \bar{Y}) + t_{\alpha/2,v}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}\right]$	$2[1 - T_v(\lvert t_0 \rvert)]$
$H_1: \mu_1 - \mu_2 > \Delta_0$	$t_0 > t_{\alpha,v}$	$\left[(\bar{X} - \bar{Y}) - t_{\alpha,v}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}, +\infty\right)$	$1 - T_v(t_0)$
$H_1: \mu_1 - \mu_2 < \Delta_0$	$t_0 < -t_{\alpha,v}$	$\left(-\infty, (\bar{X} - \bar{Y}) + t_{\alpha,v}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}\right]$	$T_v(t_0)$

Example

Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Specifically, catalyst 1 is currently in use, but catalyst 2 is acceptable. Since catalyst 2 is cheaper, it should be adopted, providing it does not change the process yield. A test is run in the pilot plant and results in the data shown in Table 10-1. Is there any difference between the mean yields? Use $\alpha = 0.05$, and assume $\sigma_1^2 \neq \sigma_2^2$

Table 10-1 Catalyst Yield Data, Example 10-5

Observation Number	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
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6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
	$\bar{x}_1 = 92.255$	$\bar{x}_2 = 92.733$
	$s_1 = 2.39$	$s_2 = 2.98$

Paired-t test:
Inference for Differences in Means of Two Normal Distributions (Dependent Samples)

• A special case of the two-sample *t*-tests occurs when the observations on the two populations of interest are collected in pairs.

• Each pair of observations, say (X_{1j} , X_{2j}), is taken under homogeneous conditions, but these conditions may change from one pair to another.

•Examples include

- Analyzing the differences between hardness readings on each specimen
- Analyzing the efficacy of a diet program (record body weights before and after the program)

• Observations in each pair are NOT independent.

Paired-t test: Inference for Differences in Means of Two Normal Distributions (Dependent Samples)

Calculate the difference of each pair of observations $d_i = X_i - Y_i; \quad i = 1,2,\ldots,n$

Null Hypothesis	Test Statistic	Distribution under H0
$H_0: \mu_1 - \mu_2 = \Delta_0$ $H_0: \mu_d = \Delta_0$	$t_0 = \dfrac{\bar{d} - \Delta_0}{S_d / \sqrt{n}}$	$t_0 \sim t(n-1)$

Alternative Hypothesis	Rejection/Critical Region (H0 is rejected)	Test using CI (If the Confidence Interval does NOT include Δ_0, then Reject H0)	P-values
$H_1: \mu_d \neq \Delta_0$	$\lvert t_0 \rvert > t_{\alpha/2,n-1}$	$\left[\bar{d} - t_{\alpha/2,n-1}\dfrac{S_d}{\sqrt{n}}, \bar{d} + t_{\alpha/2,n-1}\dfrac{S_d}{\sqrt{n}}\right]$	$2\left[1 - T_{n-1}(\lvert t_0 \rvert)\right]$
$H_1: \mu_d > \Delta_0$	$t_0 > t_{\alpha,n-1}$	$\left[\bar{d} - t_{\alpha,n-1}\dfrac{S_d}{\sqrt{n}}, +\infty\right)$	$\left[1 - T_{n-1}(t_0)\right]$
$H_1: \mu_d < \Delta_0$	$t_0 < -t_{\alpha,n-1}$	$\left(-\infty, \bar{d} + t_{\alpha,n-1}\dfrac{S_d}{\sqrt{n}}\right]$	$\left[T_{n-1}(t_0)\right]$

Example

An article in the *Journal of Strain Analysis* (1983, Vol. 18, No. 2) compares several methods for predicting the shear strength for steel plate girders. Data for two of these methods, the Karlsruhe and Lehigh procedures, when applied to nine specific girders, are shown in Table 10-2. We wish to determine whether there is any difference (on the average) between the two methods.

Table 10-2 Strength Predictions for Nine Steel Plate Girders (Predicted Load/Observed Load)

Girder	Karlsruhe Method	Lehigh Method	Difference d_j
S1/1	1.186	1.061	0.119
S2/1	1.151	0.992	0.159
S3/1	1.322	1.063	0.259
S4/1	1.339	1.062	0.277
S5/1	1.200	1.065	0.138
S2/1	1.402	1.178	0.224
S2/2	1.365	1.037	0.328
S2/3	1.537	1.086	0.451
S2/4	1.559	1.052	0.507

Inference on the Variance of Two Normal Populations

Null Hypothesis	Test Statistic	Distribution under H0
$H_0 : \sigma_1^2 = \sigma_2^2$	$F_0 = \dfrac{S_1^2}{S_2^2}$	$F_0 \sim F(n_1 - 1, n_2 - 1)$

Alternative Hypothesis	**Rejection/Region** **(H0 is rejected)**	**Test using CI** (If the Confidence Interval does NOT include 1, then Reject H0)
$H_1 : \sigma_1^2 \neq \sigma_2^2$	$F_0 > F_{\alpha/2, n_1-1, n_2-1}$ **or** $F_0 < F_{1-\alpha/2, n_1-1, n_2-1}$	$\dfrac{S_1^2}{S_2^2} F_{1-\alpha/2, n_2-1, n_1-1} \leq \dfrac{\sigma_1^2}{\sigma_2^2} \leq \dfrac{S_1^2}{S_2^2} F_{\alpha/2, n_2-1, n_1-1}$
$H_1 : \sigma_1^2 > \sigma_2^2$	$F_0 > F_{\alpha, n_1-1, n_2-1}$	$\left[\dfrac{S_1^2}{S_2^2} F_{1-\alpha, n_2-1, n_1-1}, +\infty\right)$
$H_1 : \sigma_1^2 < \sigma_2^2$	$F_0 < F_{1-\alpha, n_1-1, n_2-1}$	$\left(0, \dfrac{S_1^2}{S_2^2} F_{\alpha, n_2-1, n_1-1}\right]$

Example

Oxide layers on semiconductor wafers are etched in a mixture of gases to achieve the proper thickness. The variability in the thickness of these oxide layers is a critical characteristic of the wafer, and low variability is desirable for subsequent processing steps. Two different mixtures of gases are being studied to determine whether one is superior in reducing the variability of the oxide thickness. Twenty wafers are etched in each gas. The sample standard deviations of oxide thickness are $s_1 = 1.96$ angstroms and $s_2 = 2.13$ angstroms, respectively. Is there any evidence to indicate that either gas is preferable? Use $\alpha = 0.05$.

Inference on Two Population Proportions

Null Hypothesis	Test Statistic		Asymptotic Distribution under H0
$H_0: p_1 = p_2$	$Z_0 = \dfrac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}(1-\hat{p})\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$	$\hat{p}_1 = \dfrac{X_1}{n_1}$; $\hat{p}_2 = \dfrac{X_2}{n_2}$ $X_1 = \sum_{i=1}^{n_1} X_i$, $X_2 = \sum_{i=1}^{n_2} Y_i$ $\hat{p} = \dfrac{X_1 + X_2}{n_1 + n_2}$	$Z_0 \sim N(0,1)$

Alternative Hypothesis	Rejection/ Region (H0 is rejected)	Test using CI (If the Confidence Interval does NOT include 0, then Reject H0)	P-values
$H_1: p_1 \neq p_2$	$\lvert Z_0 \rvert > Z_{\alpha/2}$	$\left[(\hat{p}_1 - \hat{p}_2) - Z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, (\hat{p}_1 - \hat{p}_2) + Z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right]$	$2[1 - \Phi(\lvert Z_0 \rvert)]$
$H_1: p_1 > p_2$	$Z_0 > Z_\alpha$	$\left[(\hat{p}_1 - \hat{p}_2) - Z_{\alpha}\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, +\infty\right)$	$1 - \Phi(Z_0)$
$H_1: p_1 < p_2$	$Z_0 < -Z_\alpha$	$\left(-\infty, (\hat{p}_1 - \hat{p}_2) + Z_{\alpha}\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right]$	$\Phi(Z_0)$

Example

Extracts of St. John's Wort are widely used to treat depression. An article in the April 18, 2001 issue of the *Journal of the American Medical Association* ("Effectiveness of St. John's Wort on Major Depression: A Randomized Controlled Trial") compared the efficacy of a standard extract of St. John's Wort with a placebo in 200 outpatients diagnosed with major depression. Patients were randomly assigned to two groups; one group received the St. John's Wort, and the other received the placebo. After eight weeks, 19 of the placebo-treated patients showed improvement, whereas 27 of those treated with St. John's Wort improved. Is there any reason to believe that St. John's Wort is effective in treating major depression? Use $\alpha = 0.05$.

The eight-step hypothesis testing procedure leads to the following results:

Type II Error

If the alternative hypothesis is two sided, the β-error is

$$\beta = \Phi\left[\frac{z_{\alpha/2}\sqrt{\bar{p}\bar{q}(1/n_1 + 1/n_2)} - (p_1 - p_2)}{\sigma_{\hat{P}_1-\hat{P}_2}}\right] - \Phi\left[\frac{-z_{\alpha/2}\sqrt{\bar{p}\bar{q}(1/n_1 + 1/n_2)} - (p_1 - p_2)}{\sigma_{\hat{P}_1-\hat{P}_2}}\right] \quad (10\text{-}35)$$

If the alternative hypothesis is H_1: $p_1 > p_2$,

$$\beta = \Phi\left[\frac{z_{\alpha}\sqrt{\bar{p}\bar{q}(1/n_1 + 1/n_2)} - (p_1 - p_2)}{\sigma_{\hat{P}_1-\hat{P}_2}}\right] \quad (10\text{-}36)$$

and if the alternative hypothesis is H_1: $p_1 < p_2$,

$$\beta = 1 - \Phi\left[\frac{-z_{\alpha}\sqrt{\bar{p}\bar{q}(1/n_1 + 1/n_2)} - (p_1 - p_2)}{\sigma_{\hat{P}_1-\hat{P}_2}}\right] \quad (10\text{-}37)$$

Choice of Sample Size

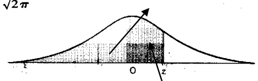
For the two-sided alternative, the common sample size is

$$n = \frac{\left[z_{\alpha/2}\sqrt{(p_1 + p_2)(q_1 + q_2)/2} + z_{\beta}\sqrt{p_1 q_1 + p_2 q_2}\right]^2}{(p_1 - p_2)^2} \quad (10\text{-}38)$$

where $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$.

Appendix II Cumulative Standard Normal Distribution

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$



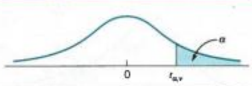
z	0.00	0.01	0.02	0.03	0.04	z
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.0
0.1	0.53983	0.54379	0.54776	0.55172	0.55567	0.1
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.2
0.3	0.61791	0.62172	0.62551	0.62930	0.63307	0.3
0.4	0.65542	0.65910	0.62276	0.66640	0.67003	0.4
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.5
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.6
0.7	0.75803	0.76115	0.76424	0.76730	0.77035	0.7
0.8	0.78814	0.79103	0.79389	0.79673	0.79954	0.8
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.9
1.0	0.84134	0.84375	0.84613	0.84849	0.85083	1.0
1.1	0.86433	0.86650	0.86864	0.87076	0.87285	1.1
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	1.2
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	1.3
1.4	0.91924	0.92073	0.92219	0.92364	0.92506	1.4
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	1.5
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	1.6
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	1.7
1.8	0.96407	0.96485	0.96562	0.96637	0.96711	1.8
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	1.9
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	2.0
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	2.1
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	2.2
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	2.3
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	2.4
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	2.5
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	2.6
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	2.7
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	2.8
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	2.9
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	3.0
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	3.1
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	3.2
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	3.3
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	3.4
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	3.5
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	3.6
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	3.7
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	3.8
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	3.9

Appendix II (*Continued*)

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

z	0.05	0.06	0.07	0.08	0.09	z
0.0	0.51994	0.52392	0.52790	0.53188	0.53586	0.0
0.1	0.55962	0.56356	0.56749	0.57142	0.57534	0.1
0.2	0.59871	0.60257	0.60642	0.61026	0.61409	0.2
0.3	0.63683	0.64058	0.64431	0.64803	0.65173	0.3
0.4	0.67364	0.67724	0.68082	0.68438	0.68793	0.4
0.5	0.70884	0.71226	0.71566	0.71904	0.72240	0.5
0.6	0.74215	0.74537	0.74857	0.75175	0.75490	0.6
0.7	0.77337	0.77637	0.77935	0.78230	0.78523	0.7
0.8	0.80234	0.80510	0.80785	0.81057	0.81327	0.8
0.9	0.82894	0.83147	0.83397	0.83646	0.83891	0.9
1.0	0.85314	0.85543	0.85769	0.85993	0.86214	1.0
1.1	0.87493	0.87697	0.87900	0.88100	0.88297	1.1
1.2	0.89435	0.89616	0.89796	0.89973	0.90147	1.2
1.3	0.91149	0.91308	0.91465	0.91621	0.91773	1.3
1.4	0.92647	0.92785	0.92922	0.93056	0.93189	1.4
1.5	0.93943	0.94062	0.94179	0.94295	0.94408	1.5
1.6	0.95053	0.95154	0.95254	0.95352	0.95448	1.6
1.7	0.95994	0.96080	0.96164	0.96246	0.96327	1.7
1.8	0.96784	0.96856	0.96926	0.96995	0.97062	1.8
1.9	0.97441	0.97500	0.97558	0.97615	0.97670	1.9
2.0	0.97982	0.98030	0.98077	0.98124	0.98169	2.0
2.1	0.98422	0.98461	0.98500	0.98537	0.98574	2.1
2.2	0.98778	0.98809	0.98840	0.98870	0.98899	2.2
2.3	0.99061	0.99086	0.99111	0.99134	0.99158	2.3
2.4	0.99286	0.99305	0.99324	0.99343	0.99361	2.4
2.5	0.99461	0.99477	0.99492	0.99506	0.99520	2.5
2.6	0.99598	0.99609	0.99621	0.99632	0.99643	2.6
2.7	0.99702	0.99711	0.99720	0.99728	0.99736	2.7
2.8	0.99781	0.99788	0.99795	0.99801	0.99807	2.8
2.9	0.99841	0.99846	0.99851	0.99856	0.99861	2.9
3.0	0.99886	0.99889	0.99893	0.99897	0.99900	3.0
3.1	0.99918	0.99921	0.99924	0.99926	0.99929	3.1
3.2	0.99942	0.99944	0.99946	0.99948	0.99950	3.2
3.3	0.99960	0.99961	0.99962	0.99964	0.99965	3.3
3.4	0.99972	0.99973	0.99974	0.99975	0.99976	3.4
3.5	0.99981	0.99981	0.99982	0.99983	0.99983	3.5
3.6	0.99987	0.99987	0.99988	0.99988	0.99989	3.6
3.7	0.99991	0.99992	0.99992	0.99992	0.99992	3.7
3.8	0.99994	0.99994	0.99995	0.99995	0.99995	3.8
3.9	0.99996	0.99996	0.99996	0.99997	0.99997	3.9

Percentage Points of the t Distribution[a]



	α									
ν	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.265	0.727	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.49	4.019	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	2.20	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.992
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646