ISyE 6739 Homework 2

solutions

1. If X is normally distributed with mean μ and standard deviation four, and given that the probability that X is less than 32 is 0.0228, find the value of μ.

   $X \sim N(\mu, 4^2)$

   $$0.0228 = \mathrm{P}\{X < 32\} = \mathrm{P}\left\{\frac{X-\mu}{4} < \frac{32-\mu}{4}\right\}$$

   $\frac{X-\mu}{4} \sim N(0,1)$

   $$\Rightarrow \frac{32-\mu}{4} = -2$$
   $$\Rightarrow \mu = 40.$$

2. X and Y are independent and $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ Find:

   (a) $\mathrm{P}\left\{\left(\frac{X-\mu_X}{\sigma_X}\right)^2 + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2 > 2\right\}$

   $\left(\frac{X-\mu_X}{\sigma_X}\right)^2 + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2 \sim \chi^2(2)$

   $$\mathrm{P}\left\{\left(\frac{X-\mu_X}{\sigma_X}\right)^2 + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2 > 2\right\} = 0.368.$$

   (b) $\mathrm{P}\left\{\frac{\left(\frac{Y-\mu_Y}{\sigma_Y}\right)}{\sqrt{\left(\frac{X-\mu_X}{\sigma_X}\right)^2}} > \frac{\sqrt{3}}{3}\right\}$

   $\frac{\left(\frac{Y-\mu_Y}{\sigma_Y}\right)}{\sqrt{\left(\frac{X-\mu_X}{\sigma_X}\right)^2}} \sim t(1)$

   $$\mathrm{P}\left\{\frac{\left(\frac{Y-\mu_Y}{\sigma_Y}\right)}{\sqrt{\left(\frac{X-\mu_X}{\sigma_X}\right)^2}} > \frac{\sqrt{3}}{3}\right\} = 0.333.$$

3. (7-10) Suppose that random variable X has the continuous uniform distribution

   $$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

   Suppose that a random sample of $n = 12$ observations is selected from this distribution. What is the approximate probability distribution of $\bar{X} - 6$? Find the mean and variance of this quantity.

   We know that $\mathrm{E}[X] = \frac{1}{2}$, $\mathrm{Var}(X) = \frac{1}{12}$. Than $\mathrm{E}[\bar{X}] = \frac{1}{2}$, $\mathrm{Var}(\bar{X}) = \frac{1}{12^2}$.
   By CLT

   $$\frac{\bar{X} - 1/2}{\sqrt{1/12^2}} = 12\bar{X} - 6 \sim N(0,1),$$

$$\Rightarrow \bar{X} - 6 = \frac{1}{12}\left(12\bar{X} - 6\right) - \frac{11}{2} \sim N\left(-\frac{11}{2}, \frac{1}{144}\right),$$

$$\mathrm{E}[\bar{X} - 6] = -\frac{11}{2}, \quad \mathrm{Var}(\bar{X} - 6) = \frac{1}{144}.$$

4. (3-109) Because all airline passengers do not show up for their reserved seat, an airline sells 125 tickets for a flight that holds only 120 passengers. The probability that a passenger does not show up is 0.10, and the passengers behave independently.

   (a) What is the probability that every passenger who shows up can take the flight?

   Let S denote the number of passengers who show up. $S \sim Binom(125, 0.9)$.
   $\mathrm{P}\{S \leq 120\} = 1 - \mathrm{P}\{S > 120\} = 1 - \sum_{i=121}^{1} 25\binom{125}{i}0.9^i 0.1^{125-i} = 0.989$

   (b) What is the probability that the flight departs with empty seats?

   $\mathrm{P}\{S < 120\} = 1 - \mathrm{P}\{S \geq 120\} = 1 - \sum_{i=120}^{1} 25\binom{125}{i}0.9^i 0.1^{125-i} = 0.996.$

5. (3-129) A trading company uses eight computers to trade on the New York Stock Exchange (NYSE). The probability of a computer failing in a day is 0.005, and the computers fail independently. Computers are repaired in the evening, and each day is an independent trial. Note that if a computer fails on day k then the number of days until it fails is also k.

   (a) What is the probability that all eight computers fail in a day?

   Let N denote the number of computers that fail in a day. $N \sim Binom(8, 0.005)$.

$$\mathrm{P}\{N = 8\} = \binom{8}{8}0.005^8 = 3.906 \cdot 10^{-19}.$$

   (b) What is the mean number of days until a specific computer fails? What is the mean number of days until it fails three times?

   Let d_j denote the number of days until a specific computer fails j times, $j = 1, 2, \ldots$.
   $d_1 \sim Geom(p = 0.005)$.

$$\mathrm{E}[d_1] = \sum_{k=1}^{\infty} k \cdot p(1-p)^{k-1} = \frac{1}{p} = \frac{1}{0.005} = 200.$$

   $d_3 \sim NB(3, 0.005)$.

$$\mathrm{E}[d_3] = \frac{3}{0.005} = 600.$$

   (c) What is the mean number of days until all eight computers fail on the same day?

   Let D denote the number of days until all eight computers fail.
   $D \sim Geom(p^8 = 0.005^8)$

$$\mathrm{E}[D] = \frac{1}{p^8} = 2.56 \cdot 10^{18}.$$

6. A random sample of size $n_1 = 49$ is selected from a distribution with a mean of 75 and a standard deviation of 7. A second random sample of size $n_2 = 36$ is taken from another distribution with mean 70 and standard deviation 12. Let $\bar{X}_1$ and $\bar{X}_2$ be the two sample means. Find:

   (a) The probability distribution of $\bar{X}_1 - \bar{X}_2$. Explain.

   By CLT

$$\frac{\bar{X}_1 - 75}{7/7} \sim N(0,1), \quad \frac{\bar{X}_2 - 70}{12/6} \sim N(0,1)$$

$$\Rightarrow \bar{X}_1 \sim N(75, 1), \quad \bar{X}_2 \sim N(70, 4)$$

$$\Rightarrow \bar{X}_1 \bar{X}_2 \sim N(75 - 70, 1 + 4) = N(5, 5)$$

(b) The probability that $3.5 \leq \bar{X}_1 - \bar{X}_2 \leq 5.5$.

Let $Z \sim N(0,1)$

$$\mathrm{P}\left\{3.5 \leq \bar{X}_1 - \bar{X}_2 \leq 5.5\right\} = \mathrm{P}\left\{\frac{3.5-5}{\sqrt{5}} \leq Z \leq \frac{5.5-5}{\sqrt{5}}\right\} =$$

$$= \Phi\left(\frac{0.5}{\sqrt{5}}\right) - \Phi\left(\frac{-1.5}{\sqrt{5}}\right) = \Phi\left(0.2236\right) - \Phi\left(-0.6708\right) = 0.587 - 0.251 = 0.336.$$

(c) Suppose that n_1 and n_2 are unknown but it is given that $n_1 + n_2 = 150$. Find the possible values of n_1 and n_2 if $\mathrm{P}\{\bar{X}_1 - \bar{X}_2 < 6.3554\} = 0.7967$.

7. Let $Z_1, Z_2, \ldots, Z_{101}$ be a random sample from a standard normal distribution. Find:

(a)

$$\mathrm{P}\{Z_1^2 + Z_{10}^2 + Z_{100}^2 > 7.815\}$$

$Z_1^2 + Z_{10}^2 + Z_{100}^2 \sim \chi^2(3)$

$$\mathrm{P}\{Z_1^2 + Z_{10}^2 + Z_{100}^2 > 7.815\} = 0.05$$

(b)

$$\mathrm{P}\left\{\frac{Z_1 + Z_{101}}{\sqrt{Z_2^2 + Z_4^2 + \ldots + Z_{100}^2}} < 0.2598\right\}$$

$\frac{(Z_1+Z_{101})/\sqrt{2}}{\sqrt{(Z_2^2+Z_4^2+\ldots+Z_{100}^2)/50}} \sim t(50)$, which can be approximated with a standard normal distribution (because df>30).

$$\mathrm{P}\left\{\frac{Z_1 + Z_{101}}{\sqrt{Z_2^2 + Z_4^2 + \ldots + Z_{100}^2}} < 0.2598\right\} = \mathrm{P}\left\{\frac{(Z_1 + Z_{101})/\sqrt{2}}{\sqrt{(Z_2^2 + Z_4^2 + \ldots + Z_{100}^2)/50}} < 0.2598\frac{\sqrt{50}}{\sqrt{2}}\right\} =$$

$$= \mathrm{P}\left\{\frac{(Z_1 + Z_{101})/\sqrt{2}}{\sqrt{(Z_2^2 + Z_4^2 + \ldots + Z_{100}^2)/50}} < 1.299\right\} = 0.9$$

(c)

$$\mathrm{P}\left\{\frac{Z_1^2 + Z_2^2}{Z_3^2 + Z_4^2 + Z_5^2} < 10.693\right\}$$

$\frac{(Z_1^2+Z_2^2)/2}{(Z_3^2+Z_4^2+Z_5^2)/3} \sim F(2,3)$

$$\mathrm{P}\left\{\frac{Z_1^2 + Z_2^2}{Z_3^2 + Z_4^2 + Z_5^2} < 10.693\right\} = \mathrm{P}\left\{\frac{(Z_1^2 + Z_2^2)/2}{(Z_3^2 + Z_4^2 + Z_5^2)/3} < 10.693\frac{3}{2}\right\} =$$

$$= \mathrm{P}\left\{\frac{(Z_1^2 + Z_2^2)/2}{(Z_3^2 + Z_4^2 + Z_5^2)/3} < 16.04\right\} = 0.025$$

8. (a) $X_1, X_2, X_3 \sim NID(\mu, \sigma^2 = 3)$, what is $\mathrm{P}\{S^2 > 3\}$?

$\frac{2S^2}{3} \sim \chi^2(2)$

$$\mathrm{P}\{S^2 > 3\} = \mathrm{P}\left\{\frac{2S^2}{3} > 3\frac{2}{3}\right\} = \mathrm{P}\left\{\frac{2S^2}{3} > 2\right\} = 0.368.$$

(b) $X_1, X_2, \ldots, X_9 \sim NID(\mu = 100, \sigma^2)$ and $S^2 = 9$, what is $\mathrm{P}\{\bar{X} < 101.86\}$?

$\frac{\bar{X}-100}{\sqrt{9/9}} \sim t(8)$

$$\mathrm{P}\{\bar{X} < 101.86\} = \mathrm{P}\left\{\frac{\bar{X} - 100}{\sqrt{9/9}} < \frac{101.86100}{1}\right\} = \mathrm{P}\left\{\bar{X} - 100 < 1.86\right\} = 0.95.$$

(c) $X_1, X_2, \ldots, X_{100} \sim NID(\mu = 0, \sigma^2)$ find α such that $\mathrm{P}\{\alpha\bar{X} > S\} = 0.05$.

Suppose $\alpha > 0$. Than

$$\mathrm{P}\{\alpha\bar{X} > S\} = \mathrm{P}\left\{\frac{\bar{X}}{S/\sqrt{100}} > \frac{10}{\alpha}\right\} = 0.05.$$

$\frac{\bar{X}}{S/\sqrt{100}} \sim t(99)$, which can be approximated with a standard normal distribution (because df>30).

$$\Rightarrow \frac{10}{\alpha} = 1.64485 \Rightarrow \alpha = 6.07958.$$

(d) $X_1, X_2, X_3 \sim NID(\mu_X, \sigma_X^2 = 8)$ and $Y_1, Y_2, Y_3 \sim NID(\mu_Y, \sigma_Y^2 = 4)$ what is $\mathrm{P}\left\{\sum_{i=1}^{3}(X_i - \bar{X})^2 > \sum_{i=1}^{3}(Y_i - \bar{Y})^2\right\}$?

$\frac{(\sum_{i=1}^{3}(X_i-\bar{X})^2)/(2\cdot 8)}{(\sum_{i=1}^{3}(Y_i-\bar{Y})^2)/(2\cdot 4)} \sim F(2,2)$

$$\mathrm{P}\left\{\sum_{i=1}^{3}(X_i - \bar{X})^2 > \sum_{i=1}^{3}(Y_i - \bar{Y})^2\right\} = \mathrm{P}\left\{\frac{(\sum_{i=1}^{3}(X_i - \bar{X})^2)/(2\cdot 8)}{(\sum_{i=1}^{3}(Y_i - \bar{Y})^2)/(2\cdot 4)} > \frac{4}{8}\right\} =$$

$$= \mathrm{P}\left\{\frac{(\sum_{i=1}^{3}(X_i - \bar{X})^2)/(2\cdot 8)}{(\sum_{i=1}^{3}(Y_i - \bar{Y})^2)/(2\cdot 4)} > \frac{1}{2}\right\} = 0.667.$$

9. **extra credit**
$X_1, X_2, X_3 \sim NID(\mu = 0, \sigma^2)$ what is $\mathrm{P}\left\{\frac{X_1}{|X_2 - X_3|} > 1\right\}$?

$|X_2 - X_3| = \sqrt{(X_2 - X_3)^2}$

$$\mathrm{P}\left\{\frac{X_1}{|X_2 - X_3|} > 1\right\} = \mathrm{P}\left\{\frac{X_1}{\sqrt{(X_2 - X_3)^2}} > 1\right\} = \mathrm{P}\left\{\frac{X_1/\sigma}{\sqrt{(X_2 - X_3)^2/(2\sigma^2)}} > \sqrt{2}\right\}$$

$\frac{X_1/\sigma}{\sqrt{(X_2-X_3)^2/(2\sigma^2)}} \sim t(1)$

$$\Rightarrow \mathrm{P}\left\{\frac{X_1}{|X_2 - X_3|} > 1\right\} = 0.196.$$