ISyE 6739 – Group Activity 6
solutions

1.
$$\mathrm{P}\{50^{th}\text{component fails after 200 hours}\} = \mathrm{E}\{\sum_{i=1}^{50} Y_i > 200\}.$$

$Y_1, Y_2, \ldots, Y_{50} \sim Exp(\lambda = 1/2)$, $\mathrm{E}[Y_i] = 1/\lambda$, $\mathrm{Var}(Y_i) = 1/\lambda^2$.

$$\Rightarrow \mathrm{E}[\sum_{i=1}^{50} Y_i] = \frac{50}{\lambda} = 100, \quad \mathrm{Var}(\sum_{i=1}^{50} Y_i) = \frac{50}{\lambda^2} = 200,$$

therefore, by CLT $\sum_{i=1}^{50} Y_i \sim N(100, 200)$.

$$\Rightarrow \mathrm{P}\left\{\sum_{i=1}^{50} Y_i > 200\right\} = \mathrm{P}\left\{N(100, 200) > 200\right\} =$$

$$= 1 - \mathrm{P}\left\{N(0,1) \le \frac{200-100}{\sqrt{200}}\right\} = 1 - \mathrm{P}\left\{N(0,1) \le 5\sqrt{2}\right\} = 7.69 \cdot 10^{-13}.$$

2. Note that $\left\{\frac{|X|}{|Y|} > 0.1\right\} = \left\{\frac{X}{|Y|} > 0.1 \text{ or } \frac{X}{|Y|} < -0.1\right\} = \left\{\frac{X}{\sqrt{Y^2}} > 0.1 \text{ or } \frac{X}{\sqrt{Y^2}} < -0.1\right\}$.

$$\frac{X/\sqrt{0.01}}{\sqrt{Y^2/0.04}} \sim t(1)$$

$$\Rightarrow \mathrm{P}\left\{\frac{|X|}{|Y|} > 0.1\right\} = \mathrm{P}\left\{\frac{X}{\sqrt{Y^2}} > 0.1\right\} + \mathrm{P}\left\{\frac{X}{\sqrt{Y^2}} < -0.1\right\} =$$

$$= \mathrm{P}\left\{\frac{X/\sqrt{0.01}}{\sqrt{Y^2/0.04}} > 0.1\sqrt{\frac{0.04}{0.01}}\right\} + \mathrm{P}\left\{\frac{X/\sqrt{0.01}}{\sqrt{Y^2/0.04}} < -0.1\sqrt{\frac{0.04}{0.01}}\right\} =$$

$$= \mathrm{P}\left\{t(1) > 0.2\right\} + \mathrm{P}\left\{t(1) < -0.2\right\} = 1 - F_{t(1)}(0.2) + F_{t(1)}(-0.2) = 0.874$$

Now $\bar{X} \sim N(0, 0.01/5)$ and $\bar{Y} \sim N(0, 0.04/5)$

$$\Rightarrow \frac{\bar{X}/\sqrt{0.01/5}}{\sqrt{\bar{Y}^2/(0.04/5)}} = 2\frac{\bar{X}}{\sqrt{\bar{Y}^2}} \sim t(1),$$

$$\mathrm{P}\left\{\frac{|\bar{X}|}{|\bar{Y}|} > 0.1\right\} = \mathrm{P}\left\{2\frac{\bar{X}}{\sqrt{\bar{Y}^2}} > 0.2\right\} + \mathrm{P}\left\{2\frac{\bar{X}}{\sqrt{\bar{Y}^2}} < -0.2\right\} =$$

$$= \mathrm{P}\left\{t(1) > 0.2\right\} + \mathrm{P}\left\{t(1) < -0.2\right\} = 0.874.$$

3. Let $X_1, X_2, \ldots, X_n$ be a sample from the Geometric distribution with the probability of success p.
$\mathrm{E}[X_i] = \frac{1}{p}$, $\mathrm{Var}(X_i) = \frac{1}{p}\left(\frac{1}{p} - 1\right)$.

$$\mathrm{E}[\bar{X}] = \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}n\frac{1}{p} = \frac{1}{p}.$$

$\Rightarrow \bar{X}$ is not an unbiased estimator of the parameter p.

$$\mathrm{Var}(\bar{X}) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}n\frac{1}{p}\left(\frac{1}{p} - 1\right) = \frac{1}{np}\left(\frac{1}{p} - 1\right).$$

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Uniform distribution on the interval $[\theta, 3\theta]$. $\mathrm{E}[X_i] = 2\theta$, $\mathrm{Var}(X_i) = \frac{\theta^2}{3}$.

$$\mathrm{E}[\bar{X}] = \frac{1}{n}\mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n} \cdot n \cdot 2\theta = 2\theta,$$

$\Rightarrow \hat{\theta} = \frac{\bar{X}}{2}$ is an unbiased estimator of the parameter θ.

$$\mathrm{Var}(\hat{\theta}) = \frac{1}{4n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{4n^2} n \frac{\theta^2}{3} = \frac{\theta^2}{12n}.$$

5.

$$\mathrm{E}\left[\hat{\Theta}_1\right] = \mathrm{E}\left[\frac{\sum_{i=1}^{7} X_i}{7}\right] = \frac{\sum_{i=1}^{7} \mathrm{E}[X_i]}{7} = \frac{7 \cdot \mu}{7} = \mu,$$

$$\mathrm{E}[\hat{\Theta}_2] = \mathrm{E}\left[\frac{2X_1 - X_6 + X_4}{2}\right] = \frac{2\mathrm{E}[X_1] - \mathrm{E}[X_6] + \mathrm{E}[X_4]}{2} = \frac{2\mu - \mu + \mu}{2} = \frac{2\mu}{2} = \mu.$$

$\Rightarrow$ both $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are unbiased:

$$\mathrm{bias}(\hat{\Theta}_1) = \mathrm{bias}(\hat{\Theta}_2) = 0.$$

$$\mathrm{Var}(\hat{\Theta}_1) = \mathrm{Var}\left(\frac{\sum_{i=1}^{7} X_i}{7}\right) = \frac{7\sigma^2}{49} = \frac{\sigma^2}{7},$$

$$\mathrm{Var}(\hat{\Theta}_2) = \mathrm{Var}\left(\frac{2X_1 - X_6 + X_4}{2}\right) = \frac{4\sigma^2 + \sigma^2 + \sigma^2}{4} = \frac{3\sigma^2}{2}.$$

$\Rightarrow \mathrm{Var}(\hat{\Theta}_1) < \mathrm{Var}(\hat{\Theta}_2)$ — $\hat{\Theta}_1$ is more efficient.

6. $X_{A,1}, X_{A,2}, \ldots, X_{A,20}$ and $X_{B,1}, X_{B,2}, \ldots, X_{B,5}$ are random samples from a distribution with mean μ and variance σ^2.

$$\mathrm{E}[\bar{X}_A] = \mathrm{E}[\bar{X}_B] = \mu,$$

$$\mathrm{Var}(\bar{X}_A) = \frac{\sigma^2}{20}, \quad \mathrm{Var}(\bar{X}_B) = \frac{\sigma^2}{5}.$$

(a)

$$\mathrm{E}[\hat{\mu}_1] = \frac{1}{2}(\mathrm{E}[\bar{X}_A] + \mathrm{E}[\bar{X}_B]) = \frac{1}{2}(\mu + \mu) = \mu,$$

$$\mathrm{E}[\hat{\mu}_2] = \mathrm{E}\left[\frac{4}{5}\bar{X}_A + \frac{1}{5}\bar{X}_B\right] = \frac{4}{5}\mu + \frac{1}{5}\mu = \mu.$$

$\Rightarrow$ both estimators are unbiased.

(b)

$$\mathrm{Var}(\hat{\mu}_1) = \mathrm{Var}\left(\frac{1}{2}(\bar{X}_A + \bar{X}_B)\right) = \frac{1}{4}\left(\frac{\sigma^2}{20} + \frac{\sigma^2}{5}\right) = \frac{\sigma^2}{16},$$

$$\mathrm{Var}(\hat{\mu}_2) = \mathrm{Var}\left(\frac{4}{5}\bar{X}_A + \frac{1}{5}\bar{X}_B\right) = \frac{16}{25}\frac{\sigma^2}{20} + \frac{1}{25}\frac{\sigma^2}{5} = \frac{\sigma^2}{25}.$$

$\mathrm{Var}(\hat{\mu}_1) > \mathrm{Var}(\hat{\mu}_2) \Rightarrow \hat{\mu}_2$ is more efficient.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Bernoulli distribution with the probability of success p. $\mathrm{E}[X_i] = p$, $\mathrm{Var}(X_i) = p(1-p)$, $\mathrm{E}[X_i^2] = \mathrm{Var}(X_i) + (\mathrm{E}[X_i])^2 = p(1-p) + p^2 = p$, and the pmf is $f(x) = p^x(1-p)^{1-x}$, $x = 0, 1$.

$$\mathrm{E}[\bar{X}] = \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}np = p$$

$\Rightarrow \bar{X}$ is an unbiased estimator for p.

$$\text{Var}(\bar{X}) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{p(1-p)}{n},$$

Find the Cramer-Rao lower bound (C-R):

$$\ln(f(x)) = x\ln(p) + (1-x)\ln(1-p), \quad x = 0, 1.$$

$$\frac{\partial \ln(f(x))}{\partial p} = \frac{x}{p} - \frac{1-x}{1-p}, \quad x = 0, 1.$$

$$\text{E}\left[\left(\frac{\partial \ln(f(x))}{\partial p}\right)^2\right] = \text{E}\left[\left(\frac{X}{p} - \frac{1-X}{1-p}\right)^2\right] = \text{E}\left[\frac{X^2}{p^2} - 2\frac{X}{p}\frac{(1-X)}{1-p} + \frac{(1-X)^2}{(1-p)^2}\right] =$$

$$= \text{E}\left[\frac{X^2}{p^2} - 2\frac{X - X^2}{p(1-p)} + \frac{(X^2 - 2X + 1)}{(1-p)^2}\right] = \frac{p}{p^2} - 2\frac{p-p}{p(1-p)} + \frac{p - 2p + 1}{(1-p)^2} =$$

$$= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}.$$

$$\Rightarrow \text{C-R} = \frac{1}{n\text{E}\left[\left(\frac{\partial \ln(f(x))}{\partial p}\right)^2\right]} = \frac{1}{n/(p(1-p))} = \frac{p(1-p)}{n}$$

C-R $= \text{Var}(\bar{X}) \Rightarrow \bar{X}$ is an MVUE for parameter p.

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson distribution with the rate λ.

$$\text{E}[X_i] = \lambda, \quad \text{Var}(X_i) = \lambda, \quad \text{E}[X_i^2] = \lambda + \lambda^2, \quad f(x) = \frac{\lambda^x}{x!}e^{-\lambda}, \quad x = 0, 1, 2, \ldots.$$

where $f(x)$ is a pmf.

$$\text{E}[\bar{X}] = \text{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}n\lambda = \lambda$$

$\Rightarrow \bar{X}$ is an unbiased estimator of λ.

$$\text{Var}(\bar{X}) = \frac{\lambda}{n}.$$

Find the Cramer-Rao lower bound (C-R):

$$\ln(f(x)) = \ln\left(\frac{\lambda^x}{x!}e^{-\lambda}\right) = x\ln(\lambda) - \text{x!} - \lambda,$$

$$\frac{\partial \ln(f(x))}{\partial \lambda} = \frac{x}{\lambda} - 1,$$

$$\text{E}\left[\left(\frac{\partial \ln(f(x))}{\partial \lambda}\right)^2\right] = \text{E}\left[\left(\frac{X}{\lambda} - 1\right)^2\right] = \text{E}\left[\frac{X^2}{\lambda^2} - 2\frac{X}{\lambda} + 1\right] = \frac{\lambda + \lambda^2}{\lambda^2} - 2\frac{\lambda}{\lambda} + 1 = \frac{1}{\lambda} + 1 - 2 + 1 = \frac{1}{\lambda}.$$

$$\Rightarrow \text{C-R} = \frac{1}{n\text{E}\left[\left(\frac{\partial \ln(f(x))}{\partial \lambda}\right)^2\right]} = \frac{1}{n/\lambda} = \frac{\lambda}{n}.$$

C-R $= \text{Var}(\bar{X}) \Rightarrow \bar{X}$ is an MVUE for parameter λ.

9. Let $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$.

$$\text{E}[(X_i - \mu)] = 0, \quad \text{Var}(X_i - \mu) = \sigma^2, \text{Var}\left((X_i - \mu)^2\right) = \sigma^2, \quad \text{Var}(\left(X_i - \mu\right)^4) = 3\sigma^4.$$

We know that S^2 is an unbiased estimator of σ^2 and that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

$$\text{Var}\left(\chi^2(n-1)\right) = 2(n-1) \Rightarrow \text{Var}(S^2) = \text{Var}\left(\frac{\sigma^2}{n-1} \cdot \frac{(n-1)S^2}{\sigma^2}\right) = \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{n-1}.$$

Find the Cramer-Rao lower bound (C-R):

$$\ln(f(x)) = \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) =$$

$$= -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2},$$

$$\frac{\partial \ln(f(x))}{\partial\sigma^2} = -\frac{1}{2\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4},$$

$$\mathrm{E}\left[\left(\frac{\partial \ln(f(x))}{\partial\sigma^2}\right)^2\right] = \mathrm{E}\left[\left(-\frac{1}{2\sigma^2} + \frac{(X-\mu)^2}{2\sigma^4}\right)^2\right] =$$

$$= \frac{1}{4\sigma^4}\mathrm{E}\left[\left(-1 + \frac{(X-\mu)^2}{\sigma^2}\right)^2\right] = \frac{1}{4\sigma^4}\mathrm{E}\left[1 + \frac{(X-\mu)^4}{\sigma^4} - 2\frac{(X-\mu)^2}{\sigma^2}\right] =$$

$$= \frac{1}{4\sigma^4}\left(1 + \frac{3\sigma^4}{\sigma^4} - 2\frac{\sigma^2}{\sigma^2}\right) = \frac{1}{4\sigma^4}\left(1+3-2\right) = \frac{1}{2\sigma^4}.$$

$$\Rightarrow \text{C-R} = \frac{1}{n\mathrm{E}\left[\left(\frac{\partial \ln(f(x))}{\partial\sigma^2}\right)^2\right]} = \frac{1}{n/(2\sigma^4)} = \frac{2\sigma^4}{n}$$

C-R $<$ Var$(S^2) \Rightarrow S^2$ is not an MVUE for parameter σ^2.