ISyE 6739 – Group Activity 2
solution

1. $p(x)$ is a probability mass function:

$$1 = \sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} kr^x = k\frac{1}{1-r}$$

$$\Rightarrow k = 1 - r.$$

2. (a) The percentage of the calculators that will fail within the warranty period is numerically equal to the probability that the time of failure is less than 1:

$$\Pr[X < 1] = \int_{-\infty}^{1} f(x)dx = \int_{0}^{1} e^{-0.125x}dx = -e^{-0.125x}\Big|_0^1 = 1 - e^{-0.125} = 0.1175.$$

11.75% of the calculators will fail within the warranty period.

(b) If the manufacturer does not offer the warranty then:

$$\textit{Profit per sale} = \$25;$$

if the manufacturer offers the warranty then consider the profit function $p(x)$, $x \geq 0$ s.t.:

$$p(x) = \begin{cases} -25 & \text{if } x < 1, \\ 25 & \text{if } x \geq 1 \end{cases}$$

$$\Rightarrow \textit{Profit per sale} = \mathrm{E}[p(x)] = \int_{-\infty}^{+\infty} p(x)f(x)dx = \int_0^1 (-25)e^{-0.125x}dx + \int_1^{+\infty} 25e^{-0.125x}dx$$

$$= -25(1 - e^{-0.125}) + 25e^{-0.125} = -25 + 50e^{-0.125} = 19.125 = \$19.13$$

Profit per sale decreases by \$19.13 if the manufacturer offers the warranty.

3. Consider $\epsilon > 0$ small enough.

(a) $\Pr[X \leq 50] = F(50) = 1$

(b) $\Pr[X \leq 40] = F(40) = 0.75$

(c) $\Pr[40 \leq X \leq 60] = \Pr[X \leq 60] - \Pr[X < 40] = F(60) - F(40 - \epsilon) = 1 - 0.75 = 0.25$

(d) $\Pr[X < 0] = F(-\epsilon) = 0.25$

(e) $\Pr[0 \leq X < 10] = \Pr[X < 10] - \Pr[X < 0] = F(10 - \epsilon) - F(-\epsilon) = 0.25 - 0.25 = 0$

(f) $\Pr[-10 < X < 10] = \Pr[X < 10] - \Pr[X \leq -10] = F(10 - \epsilon) - F(-10) = 0.25 - 0.25 = 0.$

4.

$$\mathrm{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx = \int_{\gamma}^{+\infty} x\lambda e^{-\lambda(x-\gamma)}dx$$

$$= x(-e^{-\lambda(x-\gamma)})\Big|_{\gamma}^{+\infty} + \int_{\gamma}^{+\infty} e^{-\lambda(x-\gamma)}dx = \gamma - \frac{1}{\lambda}e^{-\lambda(x-\gamma)}\Big|_{\gamma}^{+\infty} = \gamma + \frac{1}{\lambda},$$

$$\mathrm{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{\gamma}^{+\infty} x^2\lambda e^{-\lambda(x-\gamma)}dx = x^2(-e^{-\lambda(x-\gamma)})\Big|_{\gamma}^{+\infty} + \int_{\gamma}^{+\infty} 2xe^{-\lambda(x-\gamma)}dx$$
$$= \gamma^2 + 2\frac{1}{\lambda}\mathrm{E}[X] = \gamma^2 + 2\gamma\frac{1}{\lambda} + 2\frac{1}{\lambda^2}.$$
$$\mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = \gamma^2 + 2\gamma\frac{1}{\lambda} + 2\frac{1}{\lambda^2} - (\gamma + \frac{1}{\lambda})^2 = \frac{1}{\lambda^2}.$$

5.
$$\mathrm{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b \frac{x}{b-a}dx = \frac{x^2}{2(b-a)}\Big|_a^b = \frac{a+b}{2},$$
$$\mathrm{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_a^b \frac{x^2}{b-a}dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{a^2+ab+b^2}{3},$$
$$\mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$
$$\sigma_X = \sqrt{\mathrm{Var}(X)} = \frac{|b-a|}{2\sqrt{3}}.$$

If $a = 0$ and $b = 1$:
$$\mathrm{E}[X] = \frac{1}{2},$$
$$\mathrm{Var}(X) = \frac{1}{12}.$$

6. Denote P, L and W as the perimeter, the length and the width of the sheet respectively. Then $P = 2(L + W)$.
$$\mathrm{E}[P] = \mathrm{E}[2(L+W)] = 2(\mathrm{E}[L] + \mathrm{E}[W]) = 2(10 + 15) = 50,$$
$$\mathrm{Var}(P) = \mathrm{Var}(2(L+W)) = 4(\mathrm{Var}(L) + \mathrm{Var}(W) + 2\mathrm{Corr}(L,W)) = 4(1 + 1 + 2 * 0.5) = 12,$$
$$\sigma_P = \sqrt{\mathrm{Var}(P)} = 2\sqrt{3}.$$