ISyE 6739 – Group Activity 7
solutions

Notations: $\bar{X}^k = \frac{1}{n}\sum_{i=1}^n X_i^k$.

1. Let X be distributed with the pdf $f(x) = \lambda e^{-\lambda(x-\gamma)}, \quad 0 \le \gamma \le x, \ 0 < \lambda$.

   $$\mathrm{E}[X] = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \int_{\gamma}^{+\infty} x \cdot \lambda e^{-\lambda(x-\gamma)}dx = -x \cdot e{-}\lambda(x-\gamma)\Big|_{\gamma}^{+\infty} + \int_{\gamma}^{+\infty} e^{-\lambda(x-\gamma)}dx =$$

   $$= \gamma - \frac{e^{-\lambda(x-\gamma)}}{\lambda}\Big|_{\gamma}^{+\infty} = \gamma + \frac{1}{\lambda},$$

   $$\mathrm{E}[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f(x)dx = \int_{\gamma}^{+\infty} x \cdot \lambda e^{-\lambda(x-\gamma)}dx = -x^2 \cdot e{-}\lambda(x-\gamma)\Big|_{\gamma}^{+\infty} + 2\int_{\gamma}^{+\infty} x \cdot e^{-\lambda(x-\gamma)}dx = \gamma^2 + \frac{2}{\lambda^2}.$$

   By MOM:

   $$\bar{X} = \gamma + \frac{1}{\lambda}, \quad \bar{X}^2 = \gamma^2 + \frac{2}{\lambda^2},$$

   $$\Rightarrow \hat{\lambda}_{MOM} = \frac{1}{\sqrt{\bar{X}^2 - (\bar{X})^2}}, \hat{\gamma}_{MOM} = \bar{X} + \sqrt{\bar{X}^2 - (\bar{X})^2}.$$

2. By MOM:

   $$\bar{X} = \frac{r}{\lambda}, \quad \bar{X}^2 = \frac{r}{\lambda^2} + \frac{r^2}{\lambda^2},$$

   $$\Rightarrow \hat{r}_{MOM} = \frac{(\bar{X})^2}{\bar{X}^2 - (\bar{X})^2}, \quad \hat{\lambda}_{MOM} = \frac{\bar{X}}{\bar{X}^2 - (\bar{X})^2}.$$

3.

   $$L(\theta) = \prod_{i=1}^n (\theta+1)X_i = (\theta+1)^n \left(\prod_{i=1}^n X_i\right)^{\theta},$$

   $$l(\theta) = \ln L(\theta) = n\ \ln(\theta+1) + \theta\sum_{i=1}^n \ln X_i,$$

   $$\frac{\partial l(\theta)}{\partial\theta} = \frac{n}{\theta+1} + \sum_{i=1}^n \ln X_i = 0$$

   $$\Rightarrow \hat{\theta}_{MOM} = -1 - \frac{n}{\sum_{i=1}^n X_i} = -1 - \left(\bar{X}\right)^{-1}.$$

4.

   $$L(\theta) = \prod_{i=1}^n \frac{1}{\theta^2} X_i e^{-\frac{X_i}{\theta}} = \frac{1}{\theta^{2n}}\prod_{i=1}^n X_i e^{-\frac{\sum_{i=1}^n X_i}{\theta}},$$

   $$l(\theta) = \ln L(\theta) = -2n\ \ln\theta + \sum_{i=1}^n \ln X_i - \frac{\sum_{i=1}^n X_i}{\theta},$$

   $$\frac{\partial l(\theta)}{\partial\theta} = \frac{-2n}{\theta} + \frac{\sum_{i=1}^n X_i}{\theta^2} = 0$$

   $$\Rightarrow \hat{\theta}_{MOM} = \frac{\sum_{i=1}^n X_i}{2n} = \frac{\bar{X}}{2}.$$

5.
$$f(x) = \frac{1}{a}\mathbb{I}\{0 \le x \le a\},$$
where $\mathbb{I}(A)$ is an indicator function, i.e.
$$\mathbb{I}(A) = \begin{cases} 1 & x \in A, \\ 0 & \text{otherwise.} \end{cases}$$
Then
$$L(a) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{a}\mathbb{I}\{0 \le x \le a\} = \frac{1}{a^n}\mathbb{I}\{\min_i x_i \ge 0, \max_i x_i \le a\}$$
$\Rightarrow L(a)$ achieves its maximum when $a = \max_i x_i$.
$$\Rightarrow \hat{a}_{MLE} = \max_i X_i.$$

6.
$$f(x) = \lambda e^{-\lambda(x-\theta)}\mathbb{I}\{x \ge \theta\},$$
$$L(\lambda, \theta) = \prod_{i=1}^{n} \lambda e^{-\lambda(X_i-\theta)}\mathbb{I}\{X_i \ge \theta\} = \lambda^n e^{-\lambda\sum_{i=1}^n X_i} e^{n\lambda\theta}\mathbb{I}\{\min_i X_i \ge \theta\}.$$
To maximize $L(\theta)$ θ should be as large as possible
$$\Rightarrow \hat{\theta}_{MLE} = \min_i X_i.$$
Fix θ and suppose $\theta \le X_i \ \forall i$ (otherwise $L(\lambda, \theta) = 0$)
$$l(\lambda, \theta) = \ln L(\lambda, \theta) = n \ln\lambda - \lambda\sum_{i=1}^{n} X_i + n\theta\lambda,$$
$$\frac{\partial l(\lambda, \theta)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i + n\theta = 0$$
$$\Rightarrow \hat{\lambda}_{MLE} = \frac{1}{\bar{X} - \hat{\theta}_{MLE}} = \frac{1}{\bar{X} - \min_i X_i}.$$

7. (a) By the definition of a probability density function:
$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_{-1}^{1} c(1+\theta x)dx = c\frac{(1+\theta x)^2}{2\theta}\bigg|_{-1}^{1} = c\frac{4\theta}{2\theta} = 2c$$
$\Rightarrow c = \frac{1}{2}$.

(b)
$$M_1 = \mathrm{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-1}^{1} \frac{x}{2}(1+\theta x)dx = \frac{\theta x^3}{6}\bigg|_{-1}^{1} = \frac{\theta}{3}.$$
By MOM $\bar{X} = M_1$
$$\Rightarrow \hat{\theta}_{MOM} = 3\bar{X}.$$

(c)
$$\mathrm{E}[\hat{\theta}] = \mathrm{E}\left[\frac{3}{n}\sum_{i-1}^{n} X_i\right] = \frac{3}{n} \cdot n \cdot 3\theta = \theta$$
$\Rightarrow \hat{\theta}$ is an unbiased estimator for θ.

(d)

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2}(1+\theta X_i) = \frac{1}{2^n}\prod_{i=1}^{n}(1+\theta X_i),$$

$$l(\theta) = \ln L(\theta) = -n\ \ln 2 + \sum_{i=1}^{n} \ln(1+\theta X_i),$$

The MLE can be derived from the following equation

$$\frac{\partial l(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{X_i}{1+\theta X_i} = 0.$$

8. (a) Consider $\hat{\theta} = \frac{\bar{X}^2}{2}$

$$\mathrm{E}[\hat{\theta}] = \mathrm{E}\left[\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right] = \frac{1}{2n}\cdot n \cdot 2\theta = \theta$$

$\Rightarrow \hat{\theta}$ is an unbiased estimator for θ.

(b)

$$L(\theta) = \prod_{i=1}^{n} f(X_i) = \prod_{i=1}^{n} \frac{X_i}{\theta} e^{-X_i^2/2\theta} = \frac{\prod_{i=1}^{n} X_i}{\theta^n} e^{-\sum_{i=1}^{n} X_i^2/2\theta},$$

$$l(\theta) = \ln L(\theta) = -n\ \ln\theta + \sum_{i=1}^{n} \ln X_i - \frac{1}{2\theta}\sum_{i=1}^{n} X_i^2,$$

$$\frac{\partial l(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2 = 0.$$

$$\hat{\theta}_{MLE} = \frac{\bar{X}^2}{2}$$

which is equal to the estimator in the part (a).

(c) Find the median m of the distribution. By definition

$$\mathrm{P}\{X < m\} = \frac{1}{2}.$$

$$\frac{1}{2} = \int_0^m \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}\, dx = \int_0^m e^{-\frac{x^2}{2\theta}}\, d\left(\frac{x^2}{2\theta}\right) = -e^{-\frac{x^2}{2\theta}}\Big|_0^m = 1 - e^{-\frac{m^2}{2\theta}}.$$

$$\Rightarrow m = \sqrt{-2\theta \ln\frac{1}{2}}.$$

By the invariance property of the MLE:

$$\hat{m}_{MLE} = \sqrt{-2\hat{\theta}_{MLE}\ln\frac{1}{2}} = \sqrt{-\bar{X}^2\ln\frac{1}{2}}.$$

9. (a) $\beta = 2$ then

$$f(x) = \frac{2}{\delta^2} x e^{-\frac{x^2}{\delta^2}}, \quad x > 0.$$

$$L(\delta) = \prod_{i=1}^{n} \frac{2}{\delta^2} X_i e^{-\frac{X_i^2}{\delta^2}} = \frac{2^n}{\delta^{2n}} \prod_{i=1}^{n} X_i e^{-\frac{\sum_{i=1}^{n} X_i^2}{\delta^2}},$$

$$l(\delta) = \ln L(\delta) = -2n\ \ln\frac{\delta}{\sqrt{2}} + \sum_{i=1}^{n} \ln X_i - \frac{\sum_{i=1}^{n} X_i^2}{\delta^2},$$

$$\frac{\partial l(\delta)}{\partial \delta} = -\frac{2n}{\delta} + \frac{2\sum_{i=1}^{n} X_i^2}{\delta^3} = 0.$$

$$\Rightarrow \hat{\delta}_{MLE} = \sqrt{\bar{X}^2}.$$

(b) By MOM:

$$\bar{X} = \delta\Gamma(3/2)$$

$$\Rightarrow \hat{\delta}_{MOM} = \frac{\bar{X}}{\Gamma(3/2)}.$$

$$\mathrm{E}[\hat{\delta}_{MOM}] = \mathrm{E}\left[\frac{1}{\Gamma(3/2)n}\sum_{i=1}^{n} X_i\right] = \frac{1}{\Gamma(3/2)n} \cdot n \cdot \delta\Gamma(3/2) = \delta.$$

$\Rightarrow \hat{\delta}_{MOM}$ is an unbiased estimator for δ.