ISyE 6739 – Statistical Methods

Point Estimation – Methods (Ch. 7)

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Methods of Point Estimation

- Method of Moments (MoM)
- Method of Maximum Likelihood
- Method of Least Square Error
- Bayesian Method
- …

Methods of Moments

Population and samples moments

Let $X_1, X_2, \ldots, X_n$ be a random sample from the probability distribution $f(x)$, where $f(x)$ can be a discrete probability mass function or a continuous probability density function. The kth **population moment** (or **distribution moment**) is $E(X^k)$, $k = 1, 2, \ldots$. The corresponding kth **sample moment** is $(1/n)\sum_{i=1}^{n} X_i^k$, $k = 1, 2, \ldots$.

Population moments

$$\mu_k' = \begin{cases} \int_x x^k f(x)dx & \textbf{If } \boldsymbol{x} \textbf{ is continuous} \\ \sum_x x^k f(x) & \textbf{If } \boldsymbol{x} \textbf{ is discrete} \end{cases}$$

Sample moments

$$m_k' = \frac{\sum_{i=1}^{n} X_i^k}{n}$$

Methods of Moments

Let $X_1, X_2, \ldots, X_n$ be a random sample from either a probability mass function or probability density function with m unknown parameters $\theta_1, \theta_2, \ldots, \theta_m$. The **moment estimators** $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_m$ are found by equating the first m population moments to the first m sample moments and solving the resulting equations for the unknown parameters.

***m* equations for *m* parameters**

$$\begin{cases} m_1' = \mu_1' \\ m_2' = \mu_2' \\ \vdots \\ m_m' = \mu_m' \end{cases}$$

Example

What is the point estimator of λ in the exponential distribution?

What is the point estimator of p in the Bernoulli distribution?

Example

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with parameters μ and σ^2. For the normal distribution $E(X) = \mu$ and $E(X^2) = \mu^2 + \sigma^2$. Equating $E(X)$ to $\overline{X}$ and $E(X^2)$ to $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ gives

$$\mu = \overline{X}, \qquad \mu^2 + \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Solving these equations gives the moment estimators

$$\hat{\mu} = \overline{X}, \qquad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2}{n} = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n}$$

Notice that the moment estimator of σ^2 is not an unbiased estimator.

Method of Maximum Likelihood

Suppose that X is a random variable with probability distribution $f(x;\theta)$, where θ is a single unknown parameter. Let $x_1, x_2, \ldots, x_n$ be the observed values in a random sample of size n. Then the **likelihood function** of the sample is

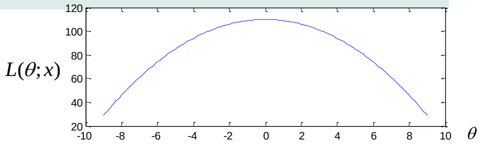
$$L(\theta) = f(x_1;\theta)\cdot f(x_2;\theta)\cdot \cdots \cdot f(x_n;\theta) \tag{7-9}$$

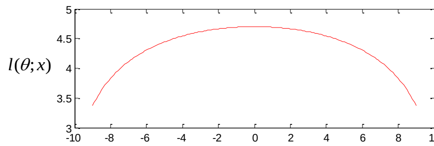
Note that the likelihood function is now a function of only the unknown parameter θ. The **maximum likelihood estimator (MLE)** of θ is the value of θ that maximizes the likelihood function $L(\theta)$.

$$L(\theta;x)=\prod_{i=1}^{n} f(x_i;\theta)=f(x_1;\theta)\ldots f(x_n;\theta)$$

$$l(\theta;x)=\sum_{i=}^{n}\log\left[f(x_i;\theta)\right]$$

$$\hat{\Theta}(x)=\arg\max_{\theta} L(\theta;x)=\arg\max_{\theta} l(\theta;x)$$





Example

Let X be a Bernoulli random variable. The probability mass function is

$$f(x;p)=\begin{cases} p^x(1-p)^{1-x}, & x=0,1 \\ 0, & \text{otherwise} \end{cases}$$

where p is the parameter to be estimated. The likelihood function of a random sample of size n is

Example

Let X be normally distributed with mean μ and variance σ^2, where both μ and σ^2 are unknown. The likelihood function for a random sample of size n is

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_i-\mu)^2/(2\sigma^2)} = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-(1/2\sigma^2)\sum_{i=1}^{n}(x_i-\mu)^2}$$

Exponential MLE

Let X be a exponential random variable with parameter λ.
The likelihood function of a random sample of size n is:

MLE Properties

Under very general and not restrictive conditions, when the sample size n is large and if $\hat{\Theta}$ is the maximum likelihood estimator of the parameter θ,

(1) $\hat{\Theta}$ is an approximately unbiased estimator for θ $[E(\hat{\Theta}) \simeq \theta]$,

(2) the variance of $\hat{\Theta}$ is nearly as small as the variance that could be obtained with any other estimator, and

(3) $\hat{\Theta}$ has an approximate normal distribution.

Example:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Invariance Property

Let $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k$ be the maximum likelihood estimators of the parameters $\theta_1, \theta_2, \ldots, \theta_k$. Then the maximum likelihood estimator of any function $h(\theta_1, \theta_2, \ldots, \theta_k)$ of these parameters is the same function $h(\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k)$ of the estimators $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k$.

Example:

In the normal distribution case, the maximum likelihood estimators of μ and σ^2 were $\hat{\mu} = \overline{X}$ and $\hat{\sigma}^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$. To obtain the maximum likelihood estimator of the function $h(\mu, \sigma^2) = \sqrt{\sigma^2} = \sigma$, substitute the estimators $\hat{\mu}$ and $\hat{\sigma}^2$ into the function h, which yields

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2\right]^{1/2}$$

Thus, the maximum likelihood estimator of the standard deviation σ is *not* the sample standard deviation S.

Complications in Using MLE

- It is not always easy to maximize the likelihood function because the equation(s) obtained from $dL(\theta)/d\theta = 0$ may be difficult to solve.

- It may not always be possible to use calculus methods directly to determine the maximum of $L(\theta)$.

Example: Uniform Distribution MLE

Let *X* be uniformly distributed on the interval 0 to a.

$$f(x) = 1/a \text{ for } 0 \le x \le a$$

$$L(a) = \prod_{i=1}^{n} \frac{1}{a} = \frac{1}{a^n} = a^{-n} \text{ for } 0 \le x_i \le a$$

$$\frac{dL(a)}{da} = \frac{-n}{a^{n+1}} = -na^{-(n+1)}$$

$$a = \max(x_i)$$

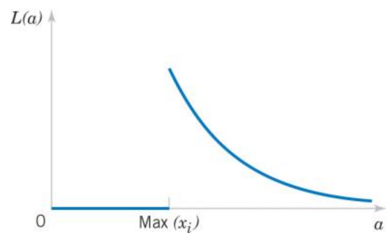


Figure 7-8 The likelihood function for this uniform distribution

Calculus methods don't work here because $L(a)$ is maximized at the discontinuity.
Clearly, *a* cannot be smaller than $\max(x_i)$, thus the MLE is $\max(x_i)$.