ISyE 6739 Homework 4

solutions

1. **(8-22)** Ishikawa et al. (*Journal of Bioscience and Bioengineering, 2012*) studied the adhesion of various biofilms to solid surfaces for possible use in environmental technologies. Adhesion assay is conducted by measuring absorbance at A_{590}. Suppose that for the bacterial strain *Acinetobacter*, five measurements gave readings of 2.69, 5.76, 2.67, 1.62 and 4.12 dyne-cm^2. Assume that the standard deviation is known to be 0.66 dyne-cm^2.

   (a) Find a 95% confidence interval for the mean adhesion.

   $$\bar{X} = 3.372$$

   $\Rightarrow$ 95%-CI:

   $$\left[\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = [2.79, 3.951]$$

   (b) If the scientists want the confidence interval to be no wider than 0.55 dyne-cm^2, how many observations should they take?

   $$2Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < 0.55$$

   $$\Rightarrow n_{new} > \left(\frac{2Z_{\alpha/2}\sigma}{0.55}\right)^2 = 22.1268$$

   Then we should take $n_{new} = 23$ observations.
   Or if we consider

2. **(8-23)** Dairy cows at large commercial farms often receive injections of bST (Bovine Somatotropin), a hormone used to spur milk production. Bauman et al. (*Journal of Dairy Science, 1989*) reported that 12 cows given bST produced an average of 28.0 kg/d of milk. Assume that the standard deviation of milk production is 2.25 kg/d.

   (a) Find a 99% confidence interval for the true mean milk production.

   $$Z_{0.01/2} = 2.58$$

   $\Rightarrow$ 99%-CI (two-sided):

   $$\left[\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = [26.327, 29.673]$$

   Lower bound:

   $$\mu > 26.49$$

   (b) If the farms want the confidence interval to be no wider than ± 1.25 kg/d, what level of confidence would they need to use?

   $$2Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < 1.25$$

   $$\Rightarrow Z_{\alpha/2} < \frac{1.25\sqrt{n}}{2\sigma} = 1.925$$

   $\alpha/2 = 0.168$ so they need to use the confidence level $1 - \alpha = 0.9457$.

3. Assume $X_1, X_2, \ldots, X_n$ is a random sample from exponential distribution with rate λ. If n is large, find a 95% approximate (general) confidence interval for λ.
$X \sim Exp(\lambda)$, $\mathrm{E}[X] = \frac{1}{\lambda}$, $\mathrm{Var}[X] = \frac{1}{\lambda^2}$. We know that the MLE for parameter λ is $\hat{\lambda} = \frac{1}{\bar{X}}$. Let $\theta = \frac{1}{\lambda}$. Then by the invariance property of MLE

$$\hat{\theta} = \frac{1}{\hat{\lambda}} = \bar{X},$$

$$\mathrm{Var}[\hat{\theta}] = \mathrm{Var}[\bar{X}] = \frac{1}{n\lambda^2}$$

We can approximate $\mathrm{Var}[\hat{\theta}]$ by setting $\lambda \approx \hat{\lambda}$:

$$\mathrm{Var}[\hat{\theta}] = \frac{(\bar{X})^2}{n}.$$

Then the confidence interval for parameter θ is

$$\hat{\theta} - Z_{\alpha/2}\sqrt{\mathrm{Var}[\hat{\theta}]} \le \theta \le \hat{\theta} + Z_{\alpha/2}\sqrt{\mathrm{Var}[\hat{\theta}]}$$

$$\bar{X} - Z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}} \le \frac{1}{\lambda} \le \bar{X} + Z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}}$$

where s is a sample standard deviation, n is a sample size. $\Rightarrow$ If $\bar{X} - Z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}} > 0$ then the confidence interval for parameter λ is

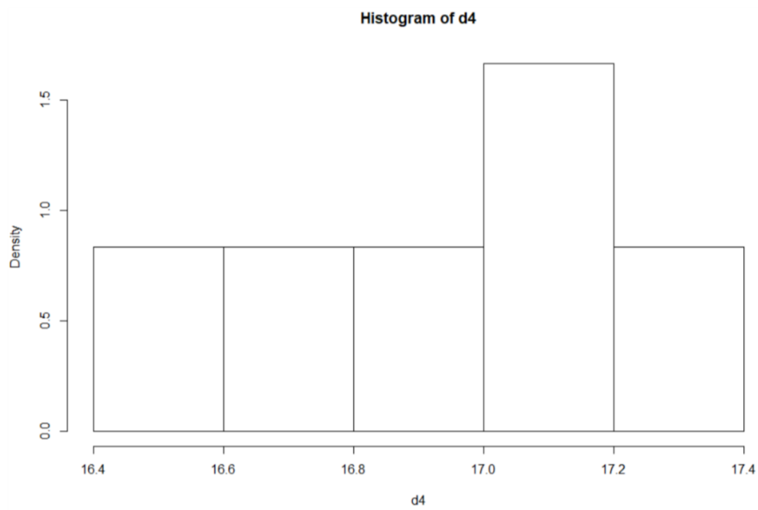
$$\frac{1}{\bar{X} + Z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}}} \le \lambda \le \frac{1}{\bar{X} - Z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}}}$$

otherwise

$$\lambda \ge \frac{1}{\bar{X} + Z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}}}$$

4. **(8-38)** A particular brand of diet margarine was analyzed to determine the level of polyunsaturated fatty acid (in percentages). A sample of six packages resulted in the following data: 16.8, 17.2, 17.4, 16.9, 16.5, 17.1.

   (a) Check the assumption that the level of polyunsaturated fatty acid is normally distributed.



   (b) Calculate a 99% confidence interval on the mean μ. Provide a practical interpretation of this interval.

$$\bar{X} = 16.98, \quad S = 0.319, \quad t_{0.01/2, 6-1} = 4.032$$

Then 99%-CI:

$$\left[\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right] = [16.458, 17.51]$$

That means in a long run 0.99% of all CI's will contain the true mean μ.

(c) Calculate a 99% lower confidence bound on the mean. Compare this bound with the lower bound of the two-sided confidence interval and discuss why they are different.

$$\mu > \bar{X} - t_{\alpha,n-1}\frac{S}{\sqrt{n}} = 16.55$$

5. **(8-57)** From the data on the pH of rain in Ingham County, Michigan: 5.47 5.37 5.38 4.63 5.37 3.74 3.71 4.96 4.64 5.11 5.65 5.39 4.16 5.62 4.57 4.64 5.48 4.57 4.57 4.51 4.86 4.56 4.61 4.32 3.98 5.70 4.15 3.98 5.65 3.10 5.04 4.62 4.51 4.34 4.16 4.64 5.12 3.71 4.64 Find a two-sided 95% confidence interval for the standard deviation of pH.

$$S = 0.6295, \quad \chi^2_{\alpha/2,n-1} = 56.9, \quad \chi^2_{1-\alpha/2,n-1} = 22.88$$

95% confidence interval for the standard deviation of pH

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right] = [0.514, 0.811]$$

6. **(8-60)** An article in *Knee Surgery, Sports Traumatology, Arthroscopy* ["Arthroscopic Meniscal Repair with an Absorbable Screw: Results and Surgical Technique" (2005, Vol. 13, pp. 273–279)] showed that only 25 out of 37 tears (67.6%) located between 3 and 6 mm from the meniscus rim were healed.

(a) Calculate a two-sided 95% confidence interval on the proportion of such tears that will heal.

$$\hat{p} = 0.676$$

$$\left[\hat{p} - Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] = [0.525, 0.827]$$

(b) Calculate a 95% lower confidence bound on the proportion of such tears that will heal.

$$p > \hat{p} - Z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.549$$

7. **(9-10)** The heat evolved in calories per gram of a cement mixture is approximately normally distributed. The mean is thought to be 100, and the standard deviation is 2. You wish to test H_0: $\mu = 100$ versus H_1: $\mu \neq 100$ with a sample of $n = 9$ specimens.

(a) If the acceptance region is defined as $98.5 \le \bar{x} \le 101.5$, find the type I error probability α.

$$Z_{\alpha/2}\frac{\sigma}{sqrtn} + \mu = 101.5$$

$$Z_{\alpha/2} = 2.25$$

$$\Rightarrow \alpha = 0.0244$$

(b) Find β for the case in which the true mean heat evolved is 103.

$$\mu_1 = 103$$

$$\beta = \Phi\left(Z_{\alpha/2} - \frac{(\mu_1 - \mu)\sqrt{n}}{\sigma}\right) - \Phi\left(-Z_{\alpha/2} - \frac{(\mu_1 - \mu)\sqrt{n}}{\sigma}\right) = 0.0122$$

(c) Find β for the case where the true mean heat evolved is 105. This value of β is smaller than the one found in part (b). Why?

$$\mu_1 = 105$$

$$\beta = \Phi\left(Z_{\alpha/2} - \frac{(\mu_1 - \mu)\sqrt{n}}{\sigma}\right) - \Phi\left(-Z_{\alpha/2} - \frac{(\mu_1 - \mu)\sqrt{n}}{\sigma}\right) = 7.6 \cdot 10^{-8}$$

The value is smaller because the probability of Type II error decreases as the mean shift increases.

8. **(9-14)** In Exercise 9-10, calculate the P-value if the observed statistic is

(a) $\bar{x} = 98$

$$\mathrm{p-value} = 2\left(1 - \Phi\left(\left|\frac{\bar{X} - 100}{\sigma/n}\right|\right)\right) = 0.0027$$

(b) $\bar{x} = 101$

$$\mathrm{p-value} = 0.1336$$

(c) $\bar{x} = 102$

$$\mathrm{p-value} = 0.0027$$

9. **(9-44)** A melting point test of $n = 10$ samples of a binder used in manufacturing a rocket propellant resulted in $x = 154.2$°F. Assume that the melting point is normally distributed with $\sigma = 1.5$°F.

(a) Test $\mathrm{H}_0 : \mu = 155$ versus $\mathrm{H}_1 : \mu \neq 155$ using $\alpha = 0.01$.

$$|Z_0| = \left|\frac{\bar{X} - 155}{1.5/n}\right| = 1.686548 < 2.576 = Z_{0.01/2}$$

We fail to reject the null.

(b) What is the P-value for this test?

$$\mathrm{p-value} = 2\left(1 - \Phi\left(\left|\frac{\bar{X} - 100}{\sigma/n}\right|\right)\right) = 0.0917$$

(c) What is the β-error if the true mean is $\mu = 150$?

$$\beta = \Phi\left(Z_{\alpha/2} - \frac{(150 - \mu)\sqrt{n}}{\sigma}\right) - \Phi\left(-Z_{\alpha/2} - \frac{(150 - \mu)\sqrt{n}}{\sigma}\right) \approx 0$$

(d) What value of n would be required if we want $\beta < 0.1$ when $\mu = 150$? Assume that $\alpha = 0.01$.

$$n_{new} = \frac{(Z_{\alpha/2} + Z_\beta)^2\sigma^2}{|150 - 155|^2} = 1.34 \approx 2$$

10. **(9-50)** Humans are known to have a mean gestation period of 280 days (from last menstruation) with a standard deviation of about 9 days. A hospital wondered whether there was any evidence that their patients were at risk for giving birth prematurely. In a random sample of 70 women, the average gestation time was 274.3 days.

(a) Is the alternative hypothesis one- or two-sided?
One-sided, because we are interested only in prematurity.

(b) Test the null hypothesis at $\alpha = 0.05$.

$$Z_0 = \frac{\bar{X} - 280}{1.5/n} = -5.299 < -1.69 = -Z_{0.05}$$

We reject the null.

(c) What is the P-value of the test statistic?

$$\mathrm{p-value} = \Phi\left(\frac{\bar{X} - 280}{\sigma/n}\right) = 5.82 \cdot 10^{-8} \approx 0$$