ISyE 6739 – Group Activity 12
solutions

1. (a)

$$P\{\text{we reject } H_0 \mid H_0 \text{ is true}\} = P\{Z_0 > Z_\alpha \mid \mu = \mu_0\} = P\left\{\frac{\bar{X} - \mu_0}{\sigma\sqrt{n}} > Z_\alpha \middle| \mu = \mu_0\right\} = \alpha$$

which is the probability of Type I error or significance level. The power of the test is the probability to reject the null hypothesis when it is false.

$$P\{\text{reject } H_0 | H_0 \text{ is false}\} = 1 - \beta,$$

where β is the probability of Type II error.

(b) A confidence of the test is the probability of failing to reject the null hypothesis when it is true.

$$P\{\text{fail reject } H_0 | H_0 \text{ is true}\} = 1 - \alpha$$

where α is the significance level.

(c)

$$\beta(\mu_1) = P\{\text{we accept } H_0 | H_0 \text{ is false}\} = P\{Z_0 \le Z_\alpha | \mu = \mu_1\} = P\left\{\frac{\bar{X} - \mu_0}{\sigma\sqrt{n}} < Z_\alpha | \mu = \mu_1\right\} = \beta$$

which is the probability of Type II error.

2.

$$\bar{X} = 10.5, \quad \mu_0 = 10, \quad \sigma = 1$$

$$\beta(\mu_1) = \Phi\left\{Z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right\}.$$

(a) $\mu_1 = 11$, $n = 9$ then

$$\beta(11) = \Phi\left(Z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right) = \Phi\left(1.64 - \frac{11 - 10}{1/3}\right) = \Phi(-1.36) = 0.087$$

$\Rightarrow$ the probability of Type II error is 0.087 and the power of the est is 0.913.

(b) $\mu_1 = 11$, $n = 16$ then

$$\beta(11) = \Phi\left(Z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right) = \Phi\left(1.64 - \frac{11 - 10}{1/4}\right) = \Phi(-2.36) = 0.009$$

$\Rightarrow$ the probability of Type II error is 0.009 and the power of the test is 0.991.

(c) $\mu_1 = 12$, $n = 9$ then

$$\beta(12) = \Phi\left(Z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right) = \Phi\left(1.64 - \frac{12 - 10}{1/3}\right) = \Phi(-4.36) = 6.503 \cdot 10^{-6}.$$

$\Rightarrow$ the probability of Type II error is $6.503 \cdot 10^{-6}$ and the power of the test is 0.9999935.

(d) When the sample size increases the probability of Type II error decreases; when the difference between μ_0 and the actual mean increases the probability of Type II error decreases.

3. (a) $\mu_1 = 11$, $\beta = 0.1$:

$$n = \frac{\left(Z_\alpha + Z_\beta\right)^2 \sigma^2}{(\mu_1 - \mu_0)^2} = \frac{(1.64 + 1.282)^2}{1} = 8.54 \approx 9.$$

(b) $\mu_1 = 11$, $\beta = 0.1$:

$$n = \frac{\left(Z_{\alpha/2} + Z_\beta\right)^2 \sigma^2}{(\mu_1 - \mu_0)^2} = \frac{(1.96 + 1.282)^2}{1} = 10.51 \approx 11.$$

We need larger sample size for two-sided test than for one-sided in order to achieve the same power.