Row echelon form and information from it:

At the end of the Gauss elimination, the form of the coefficient matrix, the augmented matrix and the system itself are called row echelon form.

Let the row reduced form of the augmented matrix $[A|b]$ is $[R|f]$ i.e

$$[R|f] = \left[\begin{array}{cccc|c} r_{11} & r_{12} & \cdots & r_{1n} & f_1 \\ 0 & r_{22} & & r_{2n} & f_2 \\ 0 & 0 & \cdots & r_{3n} & \vdots \\ \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & & r_{tn} & f_t \\ 0 & 0 & \cdots & 0 & f_{t+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & & 0 & f_m \end{array}\right]_{m\times n}$$

Imp
<u>Note:</u> The no of non-zero rows in the row reduced coefficient matrix $[R|f]$ is called Rank of A.

Here $r < m$ and $r_{11} \neq 0$ where r is rank of A.

Three Possible cases of System :-

(i) No Solution : If $r < m$ and atleast one of the no $f_{t+1}, f_{t+2}, \ldots, f_m$ is non-zero, then the system has no solution.

<u>e.g</u>

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

Here $r = 2 < 3$ and $f_3 = 12 \neq 0$, it has no sol^n

(b) Unique Solution :- If $r = n$

i.e rank of A is equal to no of unknowns then the system has unique soln.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & \frac{95}{3} & \frac{190}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here, $r = n = 3$, it has unique soln.

(c) Infinitely many solution :-

If $r < m$ and all the numbers

$f_{r+1}, f_{r+2}, \ldots, f_m$ are zero, the there are infinitely many solutions.

e.g

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here $r = 2 < 3 = m$ and $f_3 = 0$, So the system has infinitely many solutions.

Note:

(a) If Rank $(A) \neq$ Rank $[A|b] \rightarrow$ No solution

(b) If Rank $(A) =$ Rank $[A|b] = n \rightarrow$ Unique solution

(c) If Rank $(A) =$ Rank $[A|b] < n$

$\rightarrow$ Infinitely many solution.

7.4: Rank of matrix, Vector space, Basis, Dimension, Linear independence and dependence

Rank of a Matrix: Let A be the given matrix (of any order). The rank of matrix A, denoted as $r(A)$ or rank (A), is the no of non-zero rows in the matrix A.

Procedure for finding rank of a matrix:

1. Consider the given matrix A.
2. Apply the elementary row operations (same as in Gauss-elimination method)
3. Calculate the no of non-zero rows of row equivalent matrix of A
4. The no will be rank of A

Note! If we are applying elementary row operations, then the no of non-zero rows is called row rank of the matrix.

Note! If we are applying elementary column operations, then the no of non-zero columns is called column rank of the matrix.

Note: Row rank = Column rank = Rank of matrix.

Note: Matrix A and its transpose A^T have the same rank.

Ex) Determine the rank of $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

Sol Applying elementary row operations to the matrix A, we have,

$R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - 7R_1$

$$\approx \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{2}R_2$

$$\approx \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$ This is in row echlon form

and has two non-zero rows.

Hence rank $A = 2$ #

Note! The rank of a matrix is said to be r, if it has atleast one non-zero minor of order r and every minor of A of order higher than r is zero.

Note! Let A be a square matrix of order n, If $\det(A) \neq 0$, then rank of A is n.

Vector Space :-

Definition (Field): A field of scalars consists of a set F whose elements are called scalars, together with two algebraic operations, addition "+" and multiplication "." for combining every pair of scalars $\alpha, \beta \in F$ to give new scalars $\alpha+\beta \in F$ and $\alpha.\beta \in F$.

Q:

$$10x + 4y - 2z = -4$$
$$-3w - 17x + y + 2z = 2$$
$$w + x + y = 6$$
$$8w - 34x + 16y - 10z = 4$$

Sol

$$\tilde{A} = \left[\begin{array}{cccc|c} 0 & 10 & 4 & -2 & -4 \\ -3 & -17 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -34 & 16 & -10 & 4 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cccc|c} -3 & -17 & 1 & 2 & 2 \\ 0 & 10 & 4 & -2 & -4 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -34 & 16 & -10 & 4 \end{array}\right]$$

$R_3 \to R_3 + (\frac{1}{3}R_1)$ and $R_4 \to R_4 + (\frac{8}{3}) R_1$

$$\left[\begin{array}{ccccc} -3 & -17 & 1 & 2 & 2 \\ 0 & 10 & 4 & -2 & -4 \\ 0 & -\frac{14}{3} & \frac{4}{3} & \frac{2}{3} & \frac{20}{3} \\ 0 & -\frac{238}{3} & \frac{56}{3} & -\frac{14}{3} & \frac{28}{3} \end{array}\right]$$

$R_3 \to R_3 + \frac{14}{30} R_2$

and $R_4 \to R_4 + \frac{238}{30} R_3$ $\longrightarrow$ Proceeding in similar fashion and doing back substitution

we get; $\{ w = 4, \; x = 0, \; y = 2, \; z = 6 \}$

$4 + \frac{2 \times 8}{3}$

$\frac{12 + 16}{3}$

$= \frac{28}{3}$

$6 + \frac{2}{3}$

$-34 - \frac{17 \times 8}{3}$

$\frac{-102 - 136}{3}$

$= \frac{-238}{3}$

$16 + \frac{8}{3}$

$\frac{56}{3}$

$-10 + \frac{16}{3}$

$\frac{-30 + 16}{3}$

$= -\frac{14}{3}$

$-\frac{14}{3} + 10 \times \frac{3}{14}$

$-\frac{14}{3} + 10x = 0$

$\frac{14}{30}$

Vector Space :-

Definition (Field): A field of scalars consists of a set F whose elements are called scalars, together with two algebraic operations, addition "+" and multiplication "." for combining every pair of scalars $\alpha, \beta \in F$ to give new scalars $\alpha + \beta \in F$ and $\alpha . \beta \in F$.

Note :- Every non zero element has multiplicative inverse in itself.

Defination (Vector Space) :- A vector space over a field of scalars F consists of a set V whose elements are called vectors together with two algebraic operations, '+' (addition of vectors) and '.' (multiplication by scalars).

The operations '+' and '.' are required to satisfy the following axioms.

1. <u>Vector addition</u> :

(a) For $u, v \in V$, $u+v \in V$ (Closure property)

(b) For, $u, v, w \in V$, $(u+v)+w = u+(v+w)$.
(Associativity)

(c) $\exists$ an unique $\bar{0} \in V$ s.t
for any $v \in V$, $\bar{0}+v = v+\bar{0} = v$.
(Identity)

(d) $\forall\, v \in V$, $\exists\, -v \in V$ s.t
$v+(-v) = (-v)+v = \bar{0}$ (Inverse)

(e) For all $u, v \in V$
$u+v = v+u$. (Commutative Property)

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2. Scalar multiplication :

(i) $\forall\ \alpha \in F$ and $\forall\ v \in V$ then $\alpha . v \in V$.

(ii) $\forall\ \alpha, \beta \in F$ and any vector $v \in V$ then

$$\alpha.(\beta.v) = (\alpha.\beta)v = \beta.(\alpha.v)$$

(iii) $1.v = v \quad \forall\ v \in V$ where 1 is multiplicative identity in field

Note: If we define vector space over a field of real numbers, then it is called real vector space.

If we define vector space over field of complex no then it is called complex vector space.

Definition (Subspace) :- Let V be the vector space over the scalars field F. Any non-empty subset W of V is called subspace of V if W satisfies all axioms of vector space V with same operations as in V.

Subspace test: Let V be the vector space over the scalar field F. Any non-empty subset W of V is called subspace of V if following conditions hold

i) for any $u, v \in W$, $u+v \in W$.

ii) $\alpha \in F$ and $v \in W$ then $\alpha . v \in W$.

Ex: Show that all vectors in $\mathbb{R}^3$ with $v_2 - v_1 + 4v_3 = 0$ is a subspace of vector space $\mathbb{R}^3$ over field $\mathbb{R}$.

Soln Let $u, v \in \mathbb{R}^3$ and α be any scalar

We have $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$

Consider

$$u+v = (x_1, x_2, x_3) + (y_1, y_2, y_3)$$
$$= (x_1+y_1, x_2+y_2, x_3+y_3) \quad \text{s.t}$$

$$(x_2+y_2) - (x_1+y_1) + 4(x_3+y_3)$$
$$= (x_2 - x_1 + 4x_3) + (y_2 - y_1 + 4y_3)$$
$$= 0 + 0$$
$$= 0$$

$\Rightarrow \quad u + v \in \mathbb{R}^3$

(ii) $\alpha v = \alpha(y_1, y_2, y_3) = (\alpha y_1, \alpha y_2, \alpha y_3)$ s.t

$\alpha y_2 - \alpha y_1 + 4\alpha y_3 = \alpha(y_2 - y_1 + 4y_3)$

$= \alpha . 0 = 0$

$\Rightarrow \alpha . v \in \mathbb{R}^3$

Hence all vectors in $\mathbb{R}^3$ with $v_2 - v_1 + 4v_3 = 0$ is a subspace of vector space $\mathbb{R}^3$ over $\mathbb{R}$.

Linear Combination: Given any set of n vectors $v_1, v_2, \ldots, v_n$. A linear combination of these vectors can be expressed as $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$, where $c_1, c_2, \ldots, c_n$ are any scalars.

Linear Independence: The set of n vectors $v_1, v_2, \ldots, v_n$ are called linearly independent if they satisfy the eqn

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \quad \text{s.t}$$

all $c_i = 0$ where $c_1, c_2, \ldots c_n$ are any scalars.

Linear dependence: The set of n vectors $v_1, v_2, \ldots, v_n$ are called linearly dependent if they satisfy the equation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ s.t atleast one $c_i \neq 0$, where $c_1, c_2, \ldots, c_n$ are any scalars. That means, any vector can be expressed as linear combination of other vectors.

Ex: Check whether the following set of vectors are linearly independent or not.

$\{(1,2,3,4), (2,3,4,5), (3,4,5,6), (4,5,6,7)\}$

Sol^n Consider the linear combination

$$c_1(1,2,3,4) + c_2(2,3,4,5) + c_3(3,4,5,6) + c_4(4,5,6,7) = 0$$

This can be re-written as

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying Gauss elimination method,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 4R_1$

$$\simeq \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 3R_2$

$$\simeq \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the system has infinitely sol^n. So not all c_1, c_2, c_3, c_4 are zero. Hence, these vectors are linearly dependent.

Result: Consider p vectors that each have n components. Then these vectors are linearly independent if the matrix formed, with these vectors as row vectors, has rank p.

However, these vectors are linearly dependent if that matrix has rank less than p.

Span: The <u>set of all linear combinations</u> of given vectors $v_1, v_2, v_3, \dots, v_n$ is called the span of these vectors. Span is a set having the form

$$\{ c_1 v_1 + c_2 v_2 + \dots + c_n v_n \mid c_1, c_2, \dots c_n \text{ are scalars} \}$$

Basis: Any largest possible set of independent vectors in V form a basis for V.

That means, if we add one or more vectors to that set, the set will be linearly dependent.

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Dimension $\div$ The no of vectors in a basis of vector space is called dimension of vector space. It is denoted as $\dim V$.

Ex $\dim \mathbb{R}^3 = 3$

$\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\dim \mathbb{R}^2 = \{(1,0), (0,1)\} = 2$

$\dim \mathbb{R}^n = n$

Other defination of Basis $\div$ A set of vectors is a basis for a vector space V if

(i) The vectors in the set are linearly independent (L.I)

(ii) Any vector in V can be expressed as a linear combination of the vectors in the set.

Note: Basis for vector space is not unique.

Row Space :- The row space of a matrix A is the set of all linear combinations of all of its row vectors.

Theorem: The row space and column space of a matrix A have the same dimension equal to rank A.

Null space :- For a given matrix A, the solution set of the homogeneous system $Ax = 0$ is a vector space, called the null space of A and its dimension is called nullity of A.

Note:- For a given matrix A,

$$\boxed{\text{rank}(A) + \text{nullity}(A) = \text{Number of columns of } A}$$

Q: All vectors in $\mathbb{R}^3$ with $v_2 - v_1 + 4v_3 = 0$ is a subspace of vector space $\mathbb{R}^3$. (Already proved)

Find a basis and dimension of the subspace?

Sol^n

$$\text{Basis} = \{(v_1, v_2, v_3) \mid v_2 - v_1 + 4v_3 = 0\}$$

$$= \{(v_1, v_2, v_3) \mid v_2 = v_1 - 4v_3\}$$

$$= \{(v_1, v_1 - 4v_3, v_3) \mid v_1, v_3 \in \mathbb{R}\}$$

$$= \{v_1(1, 1, 0) + v_3(0, -4, 1) \mid v_1, v_3 \in \mathbb{R}\}$$

$$= [(1, 1, 0), (0, -4, 1)]$$

Dimension $= 2$

Q1 find rank, row space, column space and null space.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Soln $R_2 \to R_2 - R_1$
$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row space $= \{ [1, 1, 1, 1], [0, 1, 2, 3] \}$

Now for Column Space :

$$A^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$

$$\simeq \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 3 & -3 \end{bmatrix}$$

$R_4 \to R_4 - 3R_2$; $R_3 \to R_3 - 2R_2$

$$\simeq \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Column space $= \{ [1, 1, 4]^T, [0, 1, -1]^T \}$

$\therefore$ rank (A) = dim (row space) = dim (column space)

$= 2$

Now for Null space

$$N(A) = \{ \bar{x} : \bar{x} \in \mathbb{R}^4 \mid A\bar{x} = 0 \}$$

i.e $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

i.e $x_1 + x_2 + x_3 + x_4 = 0$

$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

$4x_1 + 3x_2 + 2x_3 + x_4 = 0$

Use elementary row operation

$$\simeq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ie $\quad x_1 + x_2 + x_3 + x_4 = 0$

$$x_2 + 2x_3 + 3x_4 = 0$$

$$0 = 0$$

Take $\boxed{x_3 = \alpha}$

$\boxed{x_4 = \beta}$

$$\boxed{x_2 = -2\alpha - 3\beta}$$

$$x_1 = -\alpha - \beta - (-2\alpha - 3\beta)$$

$$\boxed{x_1 = \alpha + 2\beta}$$

Thus Null space

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta \\ -2\alpha - 3\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$= \left\{ [1, -2, 1, 0]^T, [2, -3, 0, 1]^T \right\}$$

$\rightarrow$ Basis for null space

and dimension of null space $= 2$

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rank (A) + nullity of (A) = # Column of A

$$\boxed{2 + 2 = 4}$$

7.8 Inverse of a Matrix.

Gauss - Jordan Elimination :-

We will discuss the Gauss - Jordan elimination method to find the inverse of a matrix.

This method is computationally efficient than adjoint and determinant method.

Row Echelon form :- Any matrix is said to be in the row echelon form if it satisfies the following properties

(i) The first non - zero entry in each row is "1", called as leading or pivotal element of that row. All the entries below the pivotal or leading entries are "0".

(ii) For any two successive rows the leading entry in the lower row occurs further right to the leading entry in the upper row.

(iii) All zero rows appear below the non-zero rows.

(Ex) Find row echelon form for the matrix

$$A = \begin{bmatrix} -4 & 1 & -6 \\ 1 & 2 & -5 \\ 6 & 3 & 4 \end{bmatrix}.$$

Note: (To avoid confusion, both row echolon form and Gauss elimination method is performed side by side)

Row echelon form

$$A = \begin{bmatrix} -4 & 1 & -6 \\ 1 & 2 & -5 \\ 6 & 3 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$: $\simeq \begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & 4 \end{bmatrix}$

$R_2 \to R_2 + 4R_1$

$R_3 \to R_3 - 6R_1$

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{bmatrix}$$

$R_2 \to \frac{1}{9}R_2$ $\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -26/9 \\ 0 & -9 & 26 \end{bmatrix}$

Gauss elimination

$$A = \begin{bmatrix} -4 & 1 & 6 \\ 1 & 2 & -5 \\ 6 & 3 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ $\simeq \begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & 6 \\ 6 & 3 & 4 \end{bmatrix}$

$R_2 \to R_2 + 4R_1$

$R_3 \to R_3 - 6R_1$

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Downarrow$ G.E.

$R_3 \to R_3 + 9R_2$

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{bmatrix}$$

Already

~~Note: One should not Gauss elimination~~

(Ex) Consider the following matrices

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For any real no "$*$", the matrices defined above are in RE (row echelon) form

(Ex) Identity matrix

$I_{1\times 1} = (1)$, $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdots$

$I_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & & & 1 \end{bmatrix} \rightarrow$

Note: The procedure to get row reduced echelon matrix ~~for~~ from a given matrix is called as Gauss - Jordan elimination.

Note: For any matrix A of order $m \times n$, the rank of matrix A is atmost minimum of m and n i.e

$\text{rank}(A) \leq \min(m, n)$.

For example, if A is matrix of order 3×4 then $\text{rank}(A) \leq 3$ i.e possible rank of A ~~can~~ can be 0 or 1 or 2 or 3.

Note: For any square matrix A of order $n \times n$, the $\text{rank}(A) \leq n$.

Q: find rank(A)

where $A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\simeq \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

$R_2 \to \frac{1}{4} R_2$

$R_3 \to \frac{1}{5} R_3$

$$\simeq \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3/5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\simeq \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/10 \end{bmatrix}$$

$\frac{3}{5} - \frac{1}{2}$

$\frac{6-5}{10}$

$R_3 \to 10 R_3$

$$\simeq \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \text{G.J.E.}$$

$\rho(A) = \text{rank}(A) = r(A) = 3$ #

Invertible Matrices : A square matrix A of order n is called invertible if there exists a square matrix B of order n such that $AB = BA = I$, where I is an identity matrix of order n.

Note: B is called as the inverse of A

i.e $A^{-1} = B$

Ex: Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$,

This matrix is invertible since there exist a matrix $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ s.t

$$AB = BA = I$$

Note! Unlike the real no system, where every non zero element has multiplicative inverse, not every nonzero matrix has an inverse.

e.g $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\nexists$ $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

~~s.t AB ≠ [illegible] ≠ I~~

$$AB = BA = I$$

i.e $AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}+b_{21} & b_{12}+b_{22} \\ 0 & \underline{\underline{0}} \end{pmatrix}$

$\neq \begin{pmatrix} 1 & 0 \\ 0 & \underline{\underline{1}} \end{pmatrix}$

Note: A square matrix for which inverse does not exist is called as a singular matrix.

Note: Let A and B be the two invertible matrices of same order. Then

(i) $(A^{-1})^{-1} = A$ (ii) $(A^T)^{-1} = (A^{-1})^T$ (iii) $(AB)^{-1} = B^{-1}A^{-1}$, where A^T is the transpose of A.

Note: If A is a matrix of order n s.t

$rank(A) < n$

$\Rightarrow A^{-1}$ does not exist.

$\Rightarrow det(A) = 0$

$\Rightarrow$ A is singular matrix.

Note: If A is a square matrix of order n s.t

$rank(A) = n$

$\Rightarrow A^{-1}$ exist

$\Rightarrow$ A is non-singular matrix.

Gauss - Jordan Elimination method to find the Inverse:

It is observe that ~~to~~ find the inverse of a matrix by adjoint method is tedious work specially when $n > 3$. So to ~~...~~ avoid that difficulty level we will use Gauss - Jordan elimination method to find the inverse of a matrix. This method is consisting two phase. ~~...~~ In first phase, "0" are introduced below the pivotal or leading element '1'. After phase one, one can justify the rank of the matrix and check the existence of inverse. If inverse exist, we use second phase in which '0' are introduced above the pivotal element '1'. At the end of phase 2, the inverse of the given matrix can be determined.

(ex): Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$$ using Gauss - Jordan elimination method

Soln Consider augmented matrix

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\simeq \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right)$$

$R_2 \to -1 \cdot R_2$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

$R_3 \to -1 R_3$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$\because \rho(A) = 3 \Rightarrow A^{-1}$ exist

Phase - II

$R_2 \rightarrow R_2 \rightarrow R_3$

$R_1 \rightarrow R_1 - 2R_3$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\sim (I \mid A^{-1})$$

In final form A is reduced to I and I is reduced to A^{-1}.

Thus $A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$

Ex: Find the inverse of the matrix

$A = \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}$ using Gauss - Jordan elimination method.

Solⁿ Consider the augmented matrix,

$$(A \mid I) = \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{array}\right]$$

$R_2 \to \frac{1}{2}R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 4R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

By the end of phase I, A is reduced in row echelon form and no of non-zero row in row echelon form is 2.

i.e rank $(A) = 2$

$\Rightarrow$ A is not invertible

$\Rightarrow$ A^{-1} does not exist #

Chapter 8.1

Eigen values, Eigen vectors, and Eigen Basis & Inverse of a matrix by Cayley Hamilton theorem:

Eigen values: Let $A = [a_{ij}]$ be an $n \times n$ matrix and λ be a scalar.

Then $\det(A - \lambda I) = |A - \lambda I| = 0$ is called charecteristic equation of A.

A root of the characteristic equation $|A - \lambda I| = 0$ is called an eigen value of matrix A.

Here $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$

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Theorem: An $n \times n$ matrix A has atleast one eigen value and atmost n numerically different eigenvalues.

e.g $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$[A - I\lambda] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$$

$\Rightarrow \lambda = 1, 1$

$\Rightarrow$ Eigen value is 1.

(Ex): Find the eigen values of the matrix $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$.

Sol^n Consider the characteristic eq^n $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(5-\lambda) - 12 = 0$

$\Rightarrow 5 - \lambda - 5\lambda + \lambda^2 - 12 = 0$

$\Rightarrow \lambda^2 - 6\lambda - 7 = 0$

$\Rightarrow \lambda^2 - 7\lambda + \lambda - 7 = 0$

$\Rightarrow \lambda(\lambda - 7) + 1(\lambda - 7) = 0$

$\Rightarrow (\lambda - 7)(\lambda + 1) = 0$

$\Rightarrow \lambda = 7, -1$

Hence the eigen values of A are $-1, 7$.

ex: Find the eigen values of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$$

Soln Characteristic eqn $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 2-\lambda & -1 \\ 3 & 2 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[-(2-\lambda)(2+\lambda) + 2\right] - (-1)\left[-(2+\lambda) + 3\right] = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2 - 4 + 2\right] + 1\left[1 - \lambda\right] = 0$

$\Rightarrow (1-\lambda)\left[(\lambda^2 - 2) + 1\right] = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2 - 1\right] = 0$

$\Rightarrow (1-\lambda)(\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \lambda = 1, 1, -1$

Hence the eign values are $1, 1, -1$ Ans

Eigen Vector : Let A be an $n \times n$ matrix.

A non-zero vector X which is a solution of the ~~vect~~ equation $AX = \lambda X$, is called an eigen vector or characteristic vector of A. Here λ is scalar.

Ex : Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Sol

Soln characteristic eqn of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(5+\lambda)\cdot(-1)(2+\lambda) - 4 = 0$$

$$\Rightarrow (5+\lambda)(2+\lambda) - 4 = 0$$

$$\Rightarrow 10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow \lambda^2 + 6\lambda + \lambda + 6 = 0$$

$$\Rightarrow \lambda(\lambda+6) + 1(\lambda+6) = 0$$

$$\Rightarrow (\lambda+6)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -1, -6$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector corresponding

to eigen value $\lambda = -1$

then $AX = \lambda X$

$\Rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow -5x_1 + 2x_2 = -x_1 \Rightarrow 4x_1 + 2x_2 = 0$

$2x_1 - 2x_2 = -x_2 \Rightarrow 2x_1 - x_2 = 0$

Let $x_1 = t$ (where $t \neq 0$)

$\Rightarrow x_2 = 2t$

So, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Hence the eigen vector corresponding to $\lambda = -1$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Similarly, let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigen vector corresponding to $\lambda = -6$ then

$\Rightarrow AX = \lambda X$

$\Rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -6 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow -5x_1 + 2x_2 = -6x_1 \Rightarrow x_1 + 2x_2 = 0$

$2x_1 + 2x_2 = -6x_2 \Rightarrow 2x_1 + 4x_2 = 0$

Let $x_1 = t\ (\neq 0)$ then $x_2 = -t/2$

~~So the eigen vect~~

So, $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -t/2 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$$

Hence the eigen vector corresponding to $\lambda = -6$ is

$$\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}.$$

Chapter 20.3

Gauss Jacobi iteration methods to solve linear system of equations –

Q: $27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110$

Sol^n

$|27| > |6| + |-1|$

$|15| > |6| + |2|$

$|54| > |1| + |1|$

So we can use Jacobi method for given system of equations

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

Take $x_0 = 0,\ y_0 = 0,\ z_0 = 0$

First iteration:

$$x^{(1)} = \frac{85}{27} = 3.148$$

$$y^{(1)} = \frac{72}{15} = 4.8$$

$$z^{(1)} = \frac{110}{54} = 2.037$$

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Second Iteration:

$$x^{(2)} = \frac{1}{27}[85 - 6(4.8) + 2.037] = 2.157$$

$$y^{(2)} = \frac{1}{15}[72 - 6(3.148) - 2(2.037)] = 3.269$$

$$z^{(2)} = \frac{1}{54}[110 - 3.148 - 4.8] = 1.89$$

Third iteration:

$$x^{(3)} = \frac{1}{27}[85 - 6(3.269) + 1.89] = 2.492$$

$$y^{(3)} = \frac{1}{15}[72 - 6(2.157) - 2(1.89)] = 3.685$$

$$z^{(3)} = \frac{1}{54}[110 - 2.157 - 3.269] = 1.937$$

Fourth iteration:

$$x^{(4)} = \frac{1}{27}[85 - 6(3.685) + (1.937)] = 2.401$$

$$y^{(4)} = \frac{1}{15}[72 - 6(2.492) - 2(1.937)] = 3.545$$

$$z^{(4)} = \frac{1}{54}[110 - 2.492 - 3.685] = 1.925$$

Fifth iteration:

$$x^{(5)} = \frac{1}{27}[85 - 6(3.595) + 1.925] = 2.432$$

$$y^{(5)} = \frac{1}{15}[72 - 6(2.401) - 2(1.925)] = 3.583$$

$$z^{(5)} = \frac{1}{54}[110 - 2.401 - 3.545] = 1.927$$

Sixth iteration:

$$x^{(6)} = 2.423$$

$$y^{(6)} = 3.57$$

$$z^{(6)} = 1.926$$

Seventh iteration:

$$x^{(7)} = 2.426$$

$$y^{(7)} = 3.574$$

$$z^{(7)} = 1.926$$

Since Sixth iteration and seventh iteration gives the same value so we can stop our iteration:

So $x = 2.42$

$y = 3.57$

$z = 1.92$

Gauss - Seidal iteration method :-

Q: Solve the system of eqn by using Gauss - Seidal iteration method.

$$2x_1 - x_2 + 0x_3 = 7 \qquad |2| > |-1| + |0|$$

$$-x_1 + 2x_2 - x_3 = 1 \qquad |2| > |-1| + |-1|$$

$$0x_1 - x_2 + 2x_3 = 1 \qquad |2| > |0| + |-1|$$

$$x_1 = \frac{1}{2}(7 + x_2)$$

$$x_2 = \frac{1}{2}(1 + x_1 + x_3)$$

$$x_3 = \frac{1}{2}(1 + x_2)$$

Initial approximation. $x_1^{(0)} = 0;\ x_2^{(0)} = 0;\ x_3^{(0)} = 0$

First iteration:

$$x_1^{(1)} = \frac{1}{2}(7 + 0) = 3.5$$

$$x_2^{(1)} = \frac{1}{2}(1 + x_1^{(1)} + x_3^{(0)})$$

$$= \frac{1}{2}(1 + 3.5 + 0) = 2.25$$

$$x_3^{(1)} = \frac{1}{2}(1+2.25) = 1.625$$

Second iteration :

$$x_1^{(2)} = \frac{1}{2}\left(7 + x_2^{(1)}\right) = \frac{1}{2}(7+2.25)$$

$$= 4.625$$

$$x_2^{(2)} = \frac{1}{2}\left(1 + x_1^{(2)} + x_3^{(1)}\right)$$

$$= \frac{1}{2}(1+4.625+1.625) = 3.625$$

$$x_3^{(2)} = \frac{1}{2}\left(1 + x_2^{(2)}\right) = \frac{1}{2}(1+3.625)$$

$$= 2.3125$$

Third iteration :

$$x_1^{(3)} = \frac{1}{2}\left(7 + x_2^{(2)}\right) = \frac{1}{2}(7+3.625) = 5.3125$$

$$x_2^{(3)} = \frac{1}{2}\left(1 + x_1^{(3)} + x_3^{(2)}\right)$$

$$= \frac{1}{2}(1+5.3125+2.3125) = 4.3125$$

$$x_3^{(3)} = \frac{1}{2}\left(1 + x_2^{(3)}\right) = \frac{1}{2}(1+4.3125) = 2.65625$$

4th iteration :

$$x_1^{(4)} = 5.65625$$

$$x_2^{(4)} = 4.65625$$

$$x_3^{(4)} = 2.828125$$

$$x_1^{(5)} = 5.8281$$

$$x_2^{(5)} = 4.8281$$

$$x_3^{(5)} = 2.9106$$

VI^{th} Iterat

$x_1^{(6)} = 5.9140$; $x_2^{(6)} = 4.9140$, $x_3^{(6)} = 2.9570$

$x_1^{(7)} = 5.9570$; $x_2^{(7)} = 4.9570$; $x_3^{(7)} = 2.9705$

$x_1 \simeq 6$

$x_2 \simeq 5$

$x_3 \simeq 3$

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Chapter 8.3 Symmetric, Skew-Symmetric and Orthogonal Matrices :

Symmetric Matrix :- A real square matrix $A = [a_{ij}]$ is said to be symmetric

if $A = A^t$ where A^t is transpose of A, i.e. $a_{ij} = a_{ji}$

ex: (i) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 7 \\ 2 & 2 & 4 \\ 7 & 4 & 3 \end{bmatrix}$

Skew-Symmetric Matrix :- ~~Skew-symmetric Matrix~~ A square matrix is said to be skew-symmetric if $A = -A^t$

e.g. $\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

Orthogonal Matrix :- A square matrix A is said to ~~square~~ orthogonal if $AA^T = I$.

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(e.g)

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

$\frac{4}{9} + \frac{2}{9} + \frac{2}{9}$
$\frac{4}{3} + \frac{2}{3} + \frac{4}{3}$
$-\frac{4}{3} + \frac{2}{3} + \frac{2}{3}$

$$A A^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem: Eigenvalues of symmetric and Skew-symmetric Matrices :-

(a) The eigenvalues of a real symmetric matrix are real.

(b) The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

e.g

$\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \to$ eigenvalues are 2, 8.

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \to$ eigenvalues are $-i, i$.

Proof: (a) Let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$; $a, b, d \in \mathbb{R}$

$\because$ A is symmetric $\Rightarrow A = A^t$.

Consider $AX = \lambda X$ where $X \neq 0$ vector and λ is any scalar

$\Rightarrow AX - \lambda X = 0$

$\Rightarrow (AX - \lambda I X) = 0$

$\Rightarrow (A - \lambda I) X = 0$

$\because X \neq 0 \Rightarrow A - \lambda I$ should be singular

$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} a-\lambda & b \\ b & d-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - (a+d)\lambda + (ad - b^2) = 0$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-b^2)}}{2}$$

$$= \frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2}$$

$\Rightarrow \lambda \in \mathbb{R}$.

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Theorem: Orthonormality of Column and row vectors.

A real square matrix is orthogonal iff its column vectors $a_1, a_2, \dots, a_n$ (also its row vectors) form an orthonormal system i.e

$$a_j^T \cdot a_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

Theo: Determinant of orthogonal matrix is ± 1.

Proof:

$\because AA^t = I$

$|AA^t| = |I|$

$\Rightarrow |A| \cdot |A^t| = 1$

$\Rightarrow |A| \cdot |A| = 1$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| = \pm 1$

$\Rightarrow$ Determinant of orthogonal matrix is ± 1.

Note: The eigenvalues of an orthogonal matrix A are real or complex conjugates in pairs and have absolute value 1.

e.g Orthogonal matrix

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

has eigenvalues

$-1, \dfrac{5 \pm i\sqrt{11}}{6}$

Chapter 8.4 Similarity of Matrices, Diagonalization of Matrices.

Eigenbases ÷

Theo. (Basis of Eigenvectors). If an $n \times n$ matrix A has n distinct eigenvalues then A has a basis of eigenvectors $x_1, x_2 \dots x_n$ for $\mathbb{R}^n$.

Theo: A symmetric matrix has an orthonormal basis of eigenvectors for $\mathbb{R}^n$.

Orthonormal Basis :-

(i) Orthogonal i.e $\bar{a}\cdot\bar{b} = 0$

(ii) Unit vector i.e $\|a\| = 1$

$\|b\| = 1$

$\bar{a}.\bar{b} = 0$

$\|a\| = 1$

Ex : Consider the symmetric matrix

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

The characteristic eqn of matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)^2 - 9 = 0$$

$$\Rightarrow 25 + \lambda^2 - 10\lambda - 9 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 16 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda - 2\lambda + 16 = 0$$

$$\Rightarrow \lambda(\lambda - 8) - 2(\lambda - 8) = 0$$

$$\Rightarrow (\lambda - 8)(\lambda - 2) = 0$$

$\Rightarrow \lambda = 8, 2$ are eigen values of matrix A.

For eigen vector corresponding to eigenvalue $\lambda = 2$

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow 5x_1 + 3x_2 = 2x_1$$

$$3x_1 + 5x_2 = 2x_2$$

$3x_1 + 3x_2 = 0$

$3x_1 + 3x_2 = 0$

$\Rightarrow x_1 = -x_2$

Take $x_2 = t$

$\Rightarrow x_1 = -t$

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ~~is eigen vector~~

Thus $X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is eigen vector

and ~~orthogon~~ orthonormal basis is

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\left(\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2} \right.$

Similarly for $\lambda = 8$, the corresponding eigen vector is soln of

$AX = 8X$

$\Rightarrow \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow 5x_1 + 3x_2 = 8x_1$

$3x_1 + 5x_2 = 8x_2$

$\Rightarrow -3x_1 + 3x_2 = 0$

$3x_1 - 3x_2 = 0$

$\Rightarrow x_1 = x_2$

take $x_2 = t \Rightarrow x_1 = t$

$X = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigenvector corresponding to eigen value $\lambda = 8$.

So the orthonormal basis is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Similar Matrices: An $n \times n$ matrix B is called similar to an $n \times n$ matrix A if $\exists$ a non-singular matrix $P_{n \times n}$ s.t

$$\boxed{B = P^{-1} A P}$$

Theorem: If B is similar to A then B and A have same set of eigenvalues.
If X is an eigenvector of A, then $Y = P^{-1} X$ is an eigenvector of B corresponding to the same eigenvalue

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Q: Verify this for A and $B = P^{-1}AP$.

If Y is an eigenvector for B then $X = PY$ is eigen vector of A where

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \text{ and } P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

Sol

For eigenvalues of A

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-3-\lambda) - 16 = 0$$

$$\Rightarrow \lambda^2 - 9 - 16 = 0$$

$$\Rightarrow \lambda = \pm 5$$

$$B = P^{-1}AP$$

$$= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$$

Now eigenvalues of B

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} -25-\lambda & 12 \\ -50 & 25-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad -(25+\lambda)(25-\lambda) + 600 = 0$

$\Rightarrow \quad \lambda^2 - 625 + 600 = 0$

$\Rightarrow \lambda^2 - 25 = 0$

$\Rightarrow \lambda = \pm 5$

So the matrix A and B have same set of eigenvalues.

Now eigenvectors for A

For $\lambda = -5$, the corresponding eigenvector is

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$ $3x_1 + 4x_2 = -5x_1$

$4x_1 - 3x_2 = -5x_2$

$\Rightarrow$ $8x_1 + 4x_2 = 0$

$\Rightarrow$ $4x_2 + 2x_2 = 0$

$\Rightarrow$ $8x_1 = -4x_2$

$\Rightarrow$ $2x_1 = -x_2$

$\Rightarrow$ Take $x_2 = 1$ then $x_1 = -1/2$

$\therefore$ $X = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

for $\lambda = 5$, the corresponding eigenvector is

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad 3x_1 + 4x_2 = 5x_1$$
$$4x_1 - 3x_2 = 5x_2$$

$$\Rightarrow \quad -2x_1 + 4x_2 = 0$$
$$4x_1 + 2x_2 = 0$$

$$\Rightarrow \quad 2x_1 = 4x_2$$

$$\Rightarrow \quad x_1 = 2x_2$$

If $x_2 = 1$ then $x_1 = 2$

$\therefore$ eigenvector X is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Next we have to find the eignvectors of B.

For $\lambda = -5$

$$\begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -5 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \quad -25y_1 + 12y_2 = -5y_1$$
$$-50y_1 + 25y_2 = -5y_2$$

$$\Rightarrow \quad -20y_1 + 12y_2 = 0$$
$$-50y_1 + 30y_2 = 0$$

$$\Rightarrow \quad 20y_1 = 12y_2$$

$$\Rightarrow \quad 5y_1 = 3y_2$$

Take $y_2 = 1 \Rightarrow y_1 = 3/5$

Thus eigen vector is $\begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$

Similarly for $\lambda = 5$, the corresponding eigenvector is

$$\begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 5 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\Rightarrow$ $-25y_1 + 12y_2 = 5y_1$

$-50y_1 + 25y_2 = 5y_2$

$\Rightarrow$ $-30y_1 + 12y_2 = 0$

$-50y_1 + 20y_2 = 0$

$\Rightarrow$ $30y_1 = 12y_2$

$\Rightarrow$ $5y_1 = 2y_2$

Take $y_2 = 1$; $y_1 = 2/5$

$\therefore$ eigen vector is $\begin{bmatrix} 2/5 \\ 1 \end{bmatrix}$

Check $X = PY$

$$X_1 = PY_1$$

$$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \times \frac{3}{5} + 2 \times 1 \\ 3 \times \frac{3}{5} - 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-12}{5} + 2 \\ \frac{9}{5} - 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \end{bmatrix}$$

$$\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \frac{1}{4} \cdot 4 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \frac{1}{4} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \frac{5}{4} X = PY_1$$

Next; $X_2 = PY_2$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2/5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \times \frac{2}{5} + 2 \\ 3 \times \frac{2}{5} - 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1/5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow 5\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\Rightarrow \boxed{5X = PY} \quad \#$$

Diagonalization of a Matrix: Diagonalization of a matrix A is the process of reducing the matrix A into diagonal form D. If a matrix A is related to D by a similarity transformation s.t $D = P^{-1}AP$, then the matrix A is reduced to the diagonal matrix D through modal matrix P.

Diagonal Matrix:- A matrix D is said to be diagonal matrix if it is given as

$$\begin{bmatrix} a & & 0 \\ & b & \\ 0 & & c \end{bmatrix}.$$

Q: Diagonalize the matrix $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$.

Soln The characteristic eqn of the matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)^2 - 16 = 0$$

$$\Rightarrow \lambda^2 + 4 - 4\lambda - 16 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 12 = 0$$

$\lambda^2 - 6\lambda + 2\lambda - 12 = 0$

$\Rightarrow \lambda(\lambda-6) + 2(\lambda-6) = 0$

$\Rightarrow (\lambda-6)(\lambda+2) = 0$

$\Rightarrow \lambda = -2, 6$

Eigen vector corresponding to eigen value $\lambda = -2$

$$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow \quad 2x_1 + 4x_2 = -2x_1$

$\quad 4x_1 + 2x_2 = -2x_2$

$\Rightarrow \quad 4x_1 + 4x_2 = 0$

$\quad 4x_1 + 4x_2 = 0$

$\Rightarrow \quad x_1 = -x_2$

Take $x_2 = 1 \Rightarrow x_1 = -1$

$\therefore \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is eigenvector corresponding to $\lambda = -2$.

Now eigenvector corresponding to $\lambda = 6$

$$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 6 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow \quad 2x_1 + 4x_2 = 6x_1$

$\quad 4x_1 + 2x_2 = 6x_2$

$\Rightarrow \quad -4x_1 + 4x_2 = 0$

$\quad 4x_1 - 4x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigenvector corresponding to $\lambda = 6$

$\therefore$ P= modal matrix is given as

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Now $$P^{-1}AP = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}\times 2 + \frac{1}{2}\times 4 & -1/2\times 4 + \frac{1}{2}\times 2 \\ \frac{1}{2}\times 2 + \frac{1}{2}\times 4 & 1/2\times 4 + \frac{1}{2}\times 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times(-1) - 1\times 1 & 1\times 1 - 1\times 1 \\ 3\times(-1) + 3\times 1 & 3\times 1 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix} = D$$

$$= \begin{bmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Section 8.5 Complex Matrices and Forms:

Notations: → $A = [a_{jk}]$ where $a_{jk} = \alpha + i\beta$

where α, β are real no.

→ $\bar{A} = [\bar{a}_{jk}]$ is conjugate matrix of A

where $\bar{a}_{jk} = \overline{\alpha + i\beta} = \alpha - i\beta$

→ $A^T = [a_{jk}]^T = [a_{kj}]$

→ $A^\theta = \bar{A}^T = [\bar{a}_{kj}]$

Definition:

(1) Hermitian Matrix: A square matrix A is said to be Hermitian matrix if

$$A = \bar{A}^T = A^\theta$$

e.g. $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ is Hermitian matrix.

Verify $A = A^\theta = \bar{A}^T$

$$\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} = A$$

(2) Skew-Hermitian : A square matrix A is said to be skew-Hermitian if $A = -\bar{A}^T = -A^\theta$

e.g. $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$

$\bar{A}^T = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix}$

$-\bar{A}^T = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} = A$

(3) Unitary Matrices :- A square matrix A is said to be unitary matrix if
$A \cdot \bar{A}^T = A \cdot A^\theta = I.$

eg $A = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$

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$$\bar{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}i \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} -\frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$$

Now; $$A.\bar{A}^T = \begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i}{2}\cdot\left(-\frac{i}{2}\right) + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} & \frac{i}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{i}{2} \\ \frac{\sqrt{3}}{2}\left(-\frac{i}{2}\right) + \frac{i}{2}\cdot\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{i}{2}\cdot\left(-\frac{i}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} + \frac{3}{4} & 0 \\ 0 & \frac{3}{4} + \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \#$$

$$= I$$

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Theorem: (a) The eigen values of a Hermitian matrix are real.

(b) The eigenvalues of a skew-Hermitian matrix are purely imaginary or zero.

(c) The eigenvalues of a unitary matrix have absolute value 1.

imaginary

i ...

Proof:

Note: If λ is eigenvalues of A then $f(\lambda)$ is eigenvalue of $f(A)$.

(a) Let λ (e.g.) be eigenvalue of A and A is Hermitian

$\Rightarrow \quad A = A^{\theta} = \bar{A}^{T}$

$\Rightarrow \lambda = \lambda^{\theta}$

$\Rightarrow \lambda = \bar{\lambda}$ $\quad$ ($\because$ A and A^T has same eigenvalues)

$\Rightarrow a+ib = \overline{a+ib}$; where $a, b \in \mathbb{R}$

$\Rightarrow \cancel{a} + ib = \cancel{a} - ib$

$\Rightarrow \quad 2ib = 0$

$\Rightarrow \quad b = 0$

$\lambda = a$ is real

$\Rightarrow$ eigenvalues of Hermitian matrix (Symmetric matrix) is real.

(b) $\because$ A is skew-symmetric

and $\lambda = a+ib$ $(a, b \in \mathbb{R})$ is eigenvalue of A

$\because$ $A = -A^{\theta}$

$\Rightarrow \lambda = -\lambda^{\theta}$

$\Rightarrow a+ib = -(\overline{a+ib})$

$\Rightarrow a+ib = -(a-ib)$

$\Rightarrow a+\cancel{ib} = -a+\cancel{ib}$

$\Rightarrow 2a = 0$

$\Rightarrow a = 0$

$\therefore \lambda = ib$ $\Rightarrow$ eigenvalue of skew-symmetric matrix is purely imaginary and when $b=0$ then eigen value will be 0.

(c) $\because$ A is unitary matrix

$\Rightarrow A \cdot A^{\theta} = I$

Let λ be eigenvalue of A then

$\Rightarrow \lambda \cdot \lambda^{\theta} = 1$

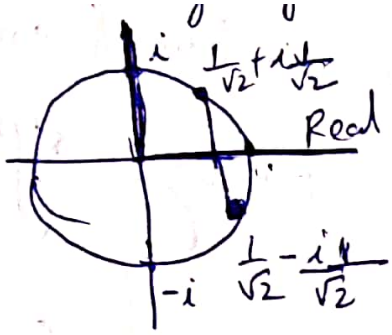
$\Rightarrow \lambda \cdot \bar{\lambda} = 1$ $\quad$ ($\because z \cdot \bar{z} = |z|^2$ modulus)

$\Rightarrow |\lambda|^2 = 1$

$\Rightarrow |\lambda| = 1$

$\Rightarrow$ eigenvalues of unitary matrix has modulus 1 #

Q: Find the eigenvalues of previous matrices?
and verify the properties of ~~Unitary System~~ eigenvalue.



Sol^n Eigenvalues of Hermiatian matrix are 9 and 2

" " Skew-Hermitian are $4i$ and $-2i$

" " Unitary matrices are $\frac{\sqrt{3}+i}{2}$ and $\left(\frac{-\sqrt{3}+i}{2}\right)$.

Unitary System: A unitary system is a set of complex vectors satisfying the relationship

$$\bar{q}_j^T q_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

e.g $q_1 = [\frac{1}{2} \quad \frac{\sqrt{3}}{2}]$

$q_2 = [\frac{\sqrt{3}}{2} \quad \frac{1}{2}]$

These forms unitary system.

Theorem: A complex square matrix is unitary if and only if its column vectors (and also its row vectors) form a unitary system.

e.g
$$A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

For this matrix, column vector and row vector each forms the unitary system.

Theo: Determinant of unitary matrix has absolute value 1.

Proof: If A is unitary matrix then

$$A \cdot A^{\theta} = I \quad (A^{\theta} = \bar{A}^{T})$$

$$\Rightarrow |AA^{\theta}| = |I|$$

$$\Rightarrow |A| \cdot |A^{\theta}| = 1$$

$\Rightarrow |A| \cdot |\overline{A^T}| = 1$

$\Rightarrow |A| \cdot \overline{|A|} = 1$

$\Rightarrow \big||A|\big|^2 = 1 \qquad \left(\because z\bar{z} = |z|^2\right)$

modulus

$\Rightarrow |\det A| = 1$

modulus

e.g $A = \begin{bmatrix} 0.8i & 0.6 \\ 0.6 & 0.8i \end{bmatrix}$ unitary system

$\det A = -1$

$|\det A| = |-1| = 1$

Theorem: A Hermitian, Skew-Hermitian, or unitary matrix has a basis of eigen vectors for $\mathbb{C}^n$.

Hermitian and Skew-Hermitian Form: —

A form in the components $x_1, x_2, \dots, x_n$ of a vector x, is given as

$$\bar{x}^T A x = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \bar{x}_j x_k$$

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$$= a_{11}\bar{x}_1 x_1 + \cdots + a_{1n}\bar{x}_1 x_n$$
$$+ a_{21}\bar{x}_2 x_1 + \cdots + a_{2n}\bar{x}_2 x_n$$
$$+ \cdots + \quad + \cdots$$
$$\vdots \qquad \vdots$$
$$+ a_{n1}\bar{x}_n x_1 + \cdots + a_{nn}\bar{x}_n x_n.$$

Here A is coefficient matrix. The form may be complex and it is called a Hermitian or Skew-Hermitian if A is Hermitian or Skew-Hermitian respectively

e.g.

$ax^2 + 2bxy + cy^2$ is symmetric form

$$\because \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} ax+by & bx+cy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= ax^2 + bxy + bxy + cy^2$$

$$= ax^2 + 2bxy + cy^2$$

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Q: Classify the matrix then find eigen value and eigen vectors?

$$A = \begin{bmatrix} 0 & 2+2i & 0 \\ 2-2i & 0 & 2+2i \\ 0 & 2-2i & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 0 & 2-2i & 0 \\ 2+2i & 0 & 2-2i \\ 0 & 2+2i & 0 \end{bmatrix}$$

$$A^{\theta} = \bar{A}^{T} = \begin{bmatrix} 0 & 2+2i & 0 \\ 2-2i & 0 & 2+2i \\ 0 & 2-2i & 0 \end{bmatrix}$$

$\because A = A^{\theta} \Rightarrow A$ is Hermitian matrix

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 2+2i & 0 \\ 2-2i & -\lambda & 2+2i \\ 0 & 2-2i & -\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda[\lambda^2 - (4+4)] - (2+2i)[-2\lambda + 2\lambda i] = 0$

$\Rightarrow \lambda(-\lambda^2 + 8) - \lambda(2+2i)(2i-2) = 0$

$\Rightarrow \lambda = 0$, or

$-\lambda^2 + 8 - (-4-4) = 0$

$\Rightarrow -\lambda^2 + 8 + 8 = 0$

$\Rightarrow \lambda = \pm 4$

Thus $\lambda = 0, \pm 4$

Now eigenvector corresponding to $\lambda = 0$ is soln of $AX = \lambda \cancel{X}$

$\Rightarrow AX = 0$

$$\Rightarrow \begin{bmatrix} 0 & 2+2i & 0 \\ 2-2i & 0 & 2+2i \\ 0 & 2-2i & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (2+2i)x_2 = 0$$

$$(2-2i)x_1 + (2+2i)x_3 = 0$$

$$(2-2i)x_2 = 0$$

$\Rightarrow \boxed{x_2 = 0}$, Now take $x_3 = 1$

then $x_1 = \dfrac{-2-2i}{(2-2i)}$

$$x_1 = -\frac{(2+2i)}{(2-2i)} \times \frac{(2+2i)}{(2+2i)}$$

$$= -\frac{(\cancel{4} - \cancel{4} + 8i)}{4+4}$$

$$= -i$$

Thus eign vectors ~~are~~ is

$$\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$$

Similarly eigenvector corresponding to $\lambda = -4$ is

$$\begin{bmatrix} i \\ -1-i \\ 1 \end{bmatrix}$$

and eigenvectors corresponding to $\lambda = 4$ is

$$\begin{bmatrix} i \\ 1+i \\ 1 \end{bmatrix}.$$

Linear Transformation (L.T) :- Let $U(F)$ and $V(F)$ are two vector spaces. A mapping $f : U \to V$ is called linear transformation of U into V if

$$\left.\begin{array}{ll} \text{(i)} & f(x+y) = f(x) + f(y) \\ \text{(ii)} & f(ax) = a f(x) \end{array}\right\} \; ; x, y \in U$$

$a \in F$

$f(x), f(y) \in V$

<u>Note:</u> $f(ax+by) = a f(x) + b f(y)$.

<u>Q:</u> Check that $\boxed{\cancel{T: V_3(\mathbb{R})}}$ $T : V_3(\mathbb{R}) \to V_2(\mathbb{R})$ by

$T(x_1, x_2, x_3) = (x_1 - x_2 , x_1 + x_3)$ is LT

<u>Sol^n</u> Let $(x_1, x_2, x_3) = x \in V_3$

and $(y_1, y_2, y_3) = y \in V_3$

Consider;

$$\begin{aligned} T(x+y) &= T[(x_1, x_2, x_3) + (y_1, y_2, y_3)] \\ &= T[x_1+y_1, x_2+y_2, x_3+y_3] \\ &= (x_1+y_1 - x_2 - y_2, x_1+y_1 + x_3 + y_3) \\ &= (x_1 - x_2 + y_1 - y_2, x_1 + x_3 + y_1 + y_3) \\ &= (x_1 - x_2, x_1 + x_3) + (y_1 - y_2, y_1 + y_3) \\ &= T(x) + T(y) \end{aligned}$$

Now

$$T(ax) = T(a(x_1, x_2, x_3))$$

$$= T(ax_1, ax_2, ax_3)$$

$$= (ax_1 - ax_2, ax_1 + ax_3)$$

$$= a(x_1 - x_2, x_1 + x_3)$$

$$= a\,T(x)$$

Thus T is L.T. #

Q. For what value of K the system has no sol^n: $3x+2y=11$; $6x+ky=21$

Sol^n

$$\begin{bmatrix} 3 & 2 \\ 6 & K \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 21 \end{bmatrix}$$

Consider

$$[A|b] = \left[\begin{array}{cc|c} 3 & 2 & 11 \\ 6 & K & 21 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$ $\qquad \frac{3}{a_2} = \frac{2}{4} \neq \frac{c_1}{c_2}$

$$\left[\begin{array}{cc|c} 3 & 2 & 11 \\ 0 & K-4 & -1 \end{array}\right]$$

Condition for no solution is

$\text{Rank}(A) \neq \text{Rank}(A|b)$

When $K = 4$ then

$\rho(A) = 1$ but $\rho(A|b) = 2$

i.e not equal

so for $K = 4$ the system has no sol^n.

If $K \neq 4$ then the system has unique solution.

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Q.2 For what value of k will the vector $(1,-2,k)$ in $\mathbb{R}^3$ be a linear combination of vectors $(3,0,-2)$ and $(2,-1,-5)$?

Sol^n

$k - (-2/3)$

Consider
$$\begin{bmatrix} 3 & 0 & -2 \\ 2 & -1 & -5 \\ 1 & -2 & k \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -2 \\ 2 & -1 & -5 \\ 0 & 0 & k+6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & k \\ 3 & 0 & -2 \\ 2 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & k \\ 0 & 6 & -2-3k \\ 0 & 3 & -5-2k \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & k \\ 0 & 6 & -2-3k \\ 0 & 0 & -5-2k-\frac{1}{2}(-2-3k) \end{bmatrix}$$

So when

$$-5-2k-\frac{1}{2}(-2-3k)=0$$

$$\Rightarrow -5-2k = \frac{1}{2}(-2-3k)$$

$$\Rightarrow -10-4k = -2-3k$$

$$\Rightarrow -8 = k$$

$$\Rightarrow \boxed{k=-8}$$

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Q1 Determine the dimension of the vector space $V = \{ (v_1, v_2, v_3) \in \mathbb{R}^3 : 2v_1 + 2v_2 + v_3 = 0$

$3v_1 + 3v_2 - 2v_3 = 0$

$v_1 + v_2 - 3v_3 = 0 \}$

Soln

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~i.e~~ $2v_1 + 2v_2 + v_3 = 0 \quad -(1)$

$3v_1 + 3v_2 - 2v_3 = 0 \quad -(2)$

$v_1 + v_2 - 3v_3 = 0 \quad (3)$

$\Rightarrow v_1 + v_2 = 3v_3 \quad -(4)$

by (4) and (1)

$2(3v_3) + v_3 = 0$

Consider $A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix}$

$\overset{R_3 \leftrightarrow R_1}{\sim} \begin{bmatrix} 1 & 1 & -3 \\ 3 & 3 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

$\underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 - 3R_1}{\sim}} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

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$$\begin{pmatrix} 1 & 1 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$$\Rightarrow \quad v_1 + v_2 - 3v_3 = 0$$

$$7 v_2 = 0$$

$$0 \cdot v_3 = 0$$

$$\Rightarrow \quad v_3 \to \text{arbitiary}$$

Take $v_3 = t$

$$v_2 = 0$$

$$v_1 = 3t$$

So the sol^n

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Thus Basis $= \{(1, 0, 3)\}$

and dim $= 1$

Note: Any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Proof: Let $A = P + Q$ —(1) where P is symmetric and Q is skew-symmetric.

Take Transpose of eqⁿ (1)

$$A^T = (P+Q)^T$$

$$= P^T + Q^T$$

$$A^T = P - Q \quad \text{—(2)} \quad \left(\because P \text{ is symmetric and } Q \text{ is skew symmetric}\right)$$

From (1) & (2)

$$P = \frac{(A + A^T)}{2} \; ; \; Q = \frac{(A - A^T)}{2}$$

Note: Any square complex matrix can be expressed as sum of Hermitian and Skew-Hermitian matrix.

Proof: $A = P + Q$ —(1) where P is Hermitian and Q is Skew Hermitian

Take conjugate transpose both side

$$A^\theta = (P+Q)^\theta$$

$$= P^\theta + Q^\theta$$

$$A^\theta = P - Q \quad \text{—(2)}$$

$\Rightarrow$ $P = \dfrac{A + A^{\theta}}{2}$

$Q = \dfrac{A - A^{\theta}}{2}$

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Q: Express the matrix

$$A = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix}$$ as sum of symmetric and skew-symmetric matrix.

Soln

$A = P + Q$

where $P = \dfrac{A + A^T}{2}$; $Q = \dfrac{A - A^T}{2}$

$$A^T = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.9 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.7 & 0.15 & 0.05 \\ 0.15 & 0.9 & 0.1 \\ 0.05 & 0.1 & 0.8 \end{bmatrix} \rightarrow \text{symmetric}$$

$$Q = \begin{bmatrix} 0 & -0.05 & -0.05 \\ 0.05 & 0 & 0.1 \\ 0.05 & -0.1 & 0 \end{bmatrix} \rightarrow \text{skew-symmetric}$$

Q: Find the dimension and basis of the vector space of $\mathbb{R}^3$ with $4v_1 + v_3 = 0$

$3v_2 = v_3$

Soln

$$4v_1 + v_3 = 0$$
$$3v_2 - v_3 = 0$$

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4v_1 + v_3 = 0 \Rightarrow 4v_1 = -v_3$

$3v_2 - v_3 = 0$

$3v_2 = v_3$

Take $v_3 = t \Rightarrow v_2 = t/3$

$v_1 = -\frac{t}{4}$

$$\therefore \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} -1/4 \\ 1/3 \\ 1 \end{bmatrix}$$

Basis $= \{ (-1/4, 1/3, 1) \}$

Dimension $= 1$

Note! Algebraic Multiplicity and Geometric multiplicity ÷ Repetition of a particular eigen value is called algebraic multiplicity for that eigenvalue.

Corresponding to a particular eigenvalue, how many independent eigen vector obtained is called geometric multiplicity of that eigen value

Note! Algebraic multiplicity ≤ Geometric multiplicity

Q; Find the eigen value and eigen vectors of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also find algebraic and geometric multiplicity of its eigen value

Soln

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 2, 2, 1$$

Algebraic multiplicity of $\lambda = 2$ is 2 (two times repetition)

Algebraic multiplicity of $\lambda = 1$ is 1

For eigen vector $X \neq 0$ is soln of

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$\Rightarrow$ $2x_1 = 2x_1 \Rightarrow x_1 = x_1 \rightarrow$ arbitrary

$2x_2 = 2x_2 \Rightarrow x_2 = x_2 \rightarrow$ arbitrary

$x_3 = 2x_3 \Rightarrow \boxed{x_3 = 0}$

Take $x_1 = t$; $x_2 = r$

So eigen vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ r \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus eigen vector corresponding to $\lambda = 2$ is 2 as we are getting 2 l.i eigen vectors $\Rightarrow$ So Geometric multiplicity is 2 for $\lambda = 2$

Now eigen vector corresponding to $\lambda = 1$ is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow$ $2x_1 = x_1 \Rightarrow x_1 = 0$

$2x_2 = x_2 \Rightarrow x_2 = 0$

$x_3 = x_3 \rightarrow$ arbitrary say t

So $X = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ So geometric multiplicity is 1 for $\lambda = 1$

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Q: Find an ordinary diff eqn for the given basis x^3, x^{-2}?

Soln General soln is

$$y = c_1 x^3 + c_2 x^{-2}$$

$$m_1 = 3 \; ; \; m_2 = -2$$

So Auxilary eqn is

$$(m-3)(m+2) = 0$$

$$\Rightarrow m^2 + 2m - 3m - 6 = 0$$

$$\Rightarrow m^2 - m - 6 = 0$$

$$\Rightarrow m^2 + (0-1)m - 6 = 0$$

$$\Rightarrow m^2 + (a-1)m + b = 0$$ corresponding to

$(a-1) = $
$a = 0$

Euler-Cauchy eqn

$$x^2 y'' + axy' + by = 0$$

$$\Rightarrow x^2 y'' + 0 - 6y = 0$$

$$\Rightarrow \boxed{x^2 y'' - 6y = 0}$$

Q: Which of the following is not subspace of $\mathbb{R}^3$

(a) W = All vectors in $\mathbb{R}^3$ with $v_1 + v_2 + v_3 = 0$

$$x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ; \quad y = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$x + y = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in W$$

Take $\alpha = -2$

$$\alpha \cdot x = -2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \in W$$

$\Rightarrow$ W is subspace

(b) W = All vectors in $\mathbb{R}^3$ whose first component is positive

$$x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ; \quad y = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$x + y = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \in W$$

$\alpha = -2$

$$\alpha \cdot x = -2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \notin W \Rightarrow \text{W is not subspace}$$

(C) W = All vectors in $\mathbb{R}^3$ having the relation $v_1 = v_2 = v_3$

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ; \quad y = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} ; \quad x + y = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \in W$$

$$\alpha = -2 ; \quad \alpha \cdot x = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} \in W \Rightarrow \text{W is subspace}$$

(d) W = all vectors in $\mathbb{R}^3$ whose 3rd componen is zero.

$$x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ; \quad y = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} ; \quad x + y = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \in W$$

$$\alpha = 2 ; \quad \alpha \cdot x = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \in W \Rightarrow \text{W is subspace}$$

Q: Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be a l.T s.t

$$T\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}4\\4\\0\\-2\end{bmatrix} \text{ and } T\begin{bmatrix}4\\0\\5\end{bmatrix} = \begin{bmatrix}4\\5\\-1\\5\end{bmatrix}. \text{ Find } T\begin{bmatrix}-7\\3\\-9\end{bmatrix}$$

$\because T(x+y) = T(x) + T(y)$

$T(\alpha x) = \alpha T(x)$

$$T\begin{bmatrix}-7\\3\\-9\end{bmatrix} = T\begin{bmatrix}1 - 2\times 4\\3 - 2\times 0\\1 - 2\times 5\end{bmatrix}$$

$$= T\begin{bmatrix}1\\3\\1\end{bmatrix} - 2T\begin{bmatrix}4\\0\\5\end{bmatrix}$$

$$= \begin{bmatrix}4\\4\\0\\-2\end{bmatrix} - 2\begin{bmatrix}4\\5\\-1\\5\end{bmatrix}$$

$$= \begin{bmatrix}4\\4\\0\\2\end{bmatrix} - \begin{bmatrix}8\\10\\-2\\10\end{bmatrix} = \begin{bmatrix}-4\\-6\\2\\-8\end{bmatrix} \text{ Ans}$$