BYJU'S Classes

PHYSICS

WAVES

THE WAVE EQUATION

NOTES

What you already know

- Introduction to waves
- String wave
- Travelling wave
- Wave parameters

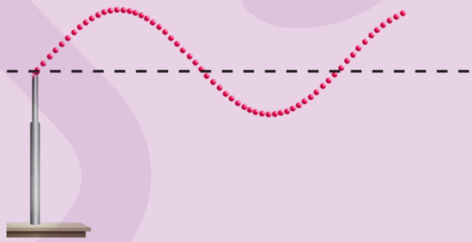
What you will learn

- Sinusoidal wave
- Wave equation
- Angular wave number
- Velocity of a wave on a string

Sinusoidal Wave

We know that any mathematical expression or formula that either is or can be manipulated as $g(x \pm vt)$ or $\left(t \pm \frac{x}{v}\right)$ must represent a travelling wave.

- If we put $x = 0$, then we shall get $f(t)$ and it represents the equation of motion of the particle at $x = 0$, which is the equation of motion of the source.
- If we put $t = 0$, then we shall get $g(x)$ and it represents the initial shape of the string.



If the generator (source) of the travelling wave executes SHM, then the produced wave resembles the sine curve. Hence, the produced wave is known as sinusoidal wave.

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Wave Equation

We know that the equation of motion of a particle at $x = 0$ also represents the equation of motion of the source, and if the source executes SHM, then it must be followed by the wave.

The equation of motion of the source executing SHM is, $y(x = 0, t) = f(t) = A \sin(\omega t + \phi)$,

where ω and ϕ is the angular frequency and epoch of the source, respectively.

Therefore, the wave equation will be,

$$y(x,t) = f\left(x = 0,\ t - \frac{x}{v}\right) = f\left(t - \frac{x}{v}\right)$$

$$\Rightarrow y(x,t) = A \sin\left[\omega\left(t - \frac{x}{v}\right) + \phi\right]$$

$$\Rightarrow y(x,t) = A \sin\left[\omega t - \left(\frac{\omega}{v}\right)x + \phi\right]$$

$$\Rightarrow y(x,t) = A \sin(\omega t - kx + \phi) \quad \left[\text{Assume } k = \frac{\omega}{v}\right]$$

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Angular wave number or propagation constant

The angular wave number or propagation constant of a wave is represented by k. Mathematically, it is defined as the ratio of angular frequency of the wave and the wave velocity.

Thus,

$$k = \frac{\omega}{v}$$

$$\Rightarrow k = \frac{2\pi f}{v} \quad [\text{Where } f \text{ is the frequency}]$$

$$\Rightarrow k = \frac{2\pi}{\left(\frac{v}{f}\right)}$$

$$\boxed{\Rightarrow k = \frac{2\pi}{\lambda}} \quad \left[\text{Since we know that } \lambda = \frac{v}{f}\right]$$

Therefore, the angular wave number or the propagation constant is, $\boxed{k = \frac{\omega}{v} = \frac{2\pi}{\lambda}}$

The equation of a wave travelling on a string is, $y = (0.10\ mm) \sin [(31.4\ m^{-1})x + (314\ s^{-1})t]$

(a) In which direction does the wave travel?

(b) Find the wave velocity and the wavelength.

(c) What is the maximum displacement and the frequency of the wave?

Solution

Given,

The equation of the wave travelling on a string is,

$$y = (0.10\ mm) \sin\left[\left(31.4\ m^{-1}\right)x + \left(314\ s^{-1}\right)t\right]$$

$$\Rightarrow y = (0.10\ mm) \sin\left[\left(10\pi\ m^{-1}\right)x + \left(100\pi\ s^{-1}\right)t\right] \quad \ldots(i)$$

(a) Equation (*i*) is of the form $y = A \sin (kx + \omega t + \phi)$. Recall that the wave equation in the form of $f\left(t + \frac{x}{v}\right)$ or $g(x + vt)$ denotes the wave travelling in the negative x-direction. Thus, the wave is travelling in the negative x-direction on the string.

(b) By comparing equation (*i*) with $y = A \sin (kx + \omega t + \phi)$, we get the following:

Amplitude of the wave, $A = 0.1\ mm$

Angular wave number, $k = 10\pi\ m^{-1}$

Angular frequency, $\omega = 100\pi\ s^{-1}$

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Therefore,

The wave velocity is, $v = \frac{\omega}{k} = \frac{100\pi}{10\pi} = 10\ ms^{-1}$

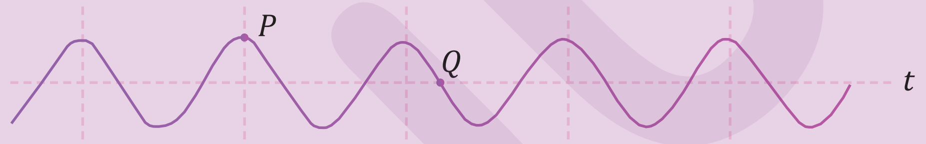
The wave length is, $\lambda = \frac{2\pi}{k} = \frac{2\pi}{10\pi} = 0.2\ m$

(c) Maximum displacement is equal to the amplitude of the wave. Thus, the maximum displacement is, $y_{max} = 0.1\ mm$

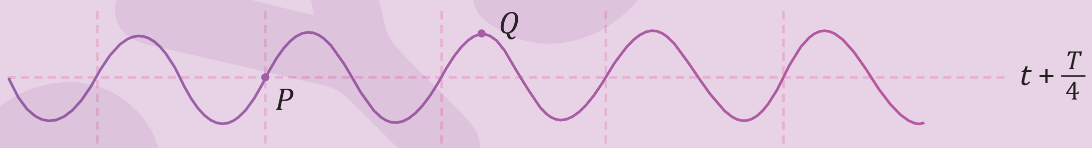
The frequency of the wave is, $f = \frac{\omega}{2\pi} = \frac{100\pi}{2\pi} = 50\ Hz$

Picturisation of Wave Equation

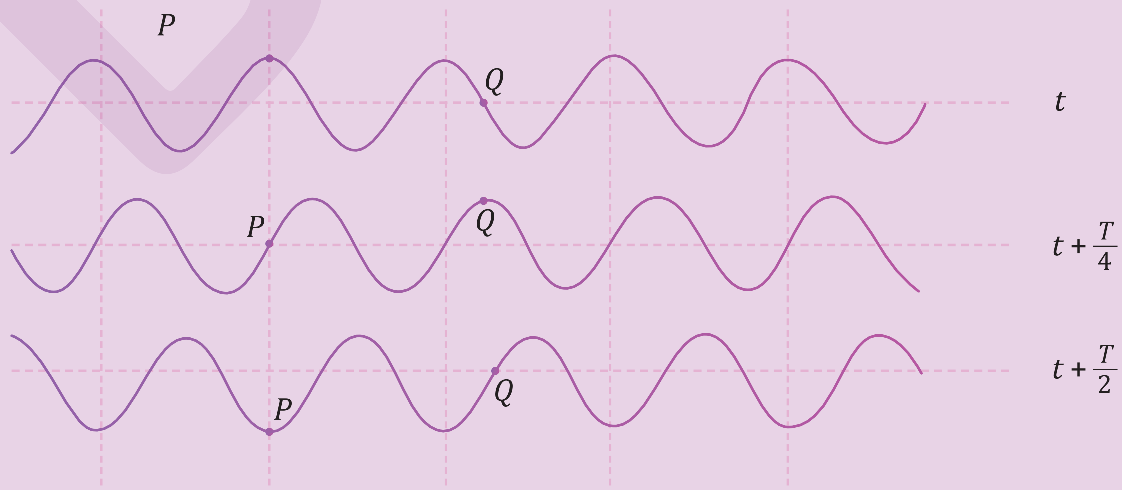
Consider two particles of a string P and Q at any time t on a wave travelling along the string which acquires the form of a sinusoidal wave due to the SHM of the particle on it at $x = 0$ (source executing SHM), as shown in the figure.



As we know that the smallest indivisible unit of time of SHM is $\frac{T}{4}$ (T is the time period of the source), at time $t + \frac{T}{4}$, the positions of particles P and Q will be as follows:



By this way, if we simultaneously increase the time by $\frac{T}{4}$, then the corresponding positions of both the particles will be as follows:



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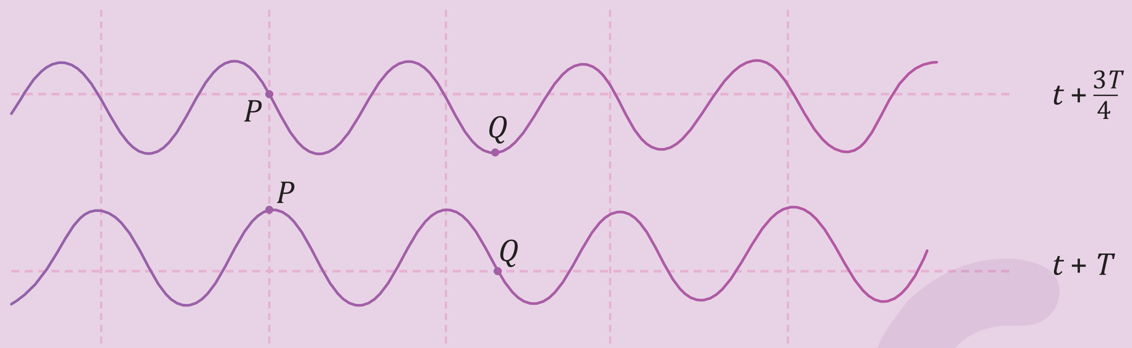


By carefully observing the picture, it can be said that particles P and Q reach their initial positions after one time period T. Hence, one can conclude that each particle on the string is also executing SHM with the same amplitude and time period as the source.

In short, it can be said every particle on the string wave has the same characteristics as that of the source. Hence, the wave equation is the equation of the source executing SHM.

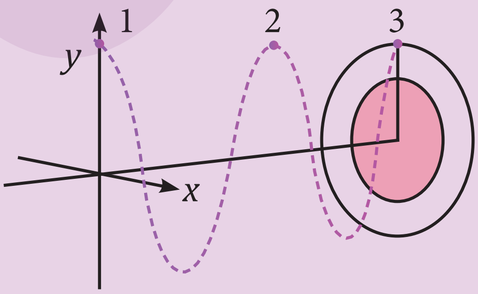
Phase

The general expression of the wave equation is, $y = A \sin(\omega t - kx + \phi)$

Mathematically, the argument of the sine function, i.e., $(\omega t - kx + \phi)$ is known as the total instantaneous phase of the wave, where ϕ is the epoch of the source. However, in general, the phase is a wave parameter and after every 2π value of the phase, the wave repeats itself.

A phase change of $2n\pi$ gives points oscillating in phase (i.e., behaviour of each particle will be the same), where n is an integer.

In the figure, points 1, 2, and 3 are the points of repetition of the wave.



Velocity and acceleration of a particle

Suppose that the general expression of the displacement of a particle on the string wave or sinusoidal wave for a particular x-coordinate is given by,

$$y_p = A \sin(\omega t - kx + \phi)$$

Then, the velocity of the particle is,

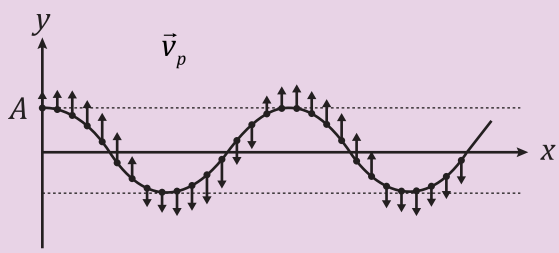
$$v_p = \left[\frac{dy}{dt}\right]_{x\,=\,Constant} = \frac{\partial y}{\partial t} = \omega A \cos(\omega t - kx + \phi)$$

And the acceleration of the particle is,

$$a_p = \left[\frac{d^2 y}{dt^2}\right]_{x\,=\,Constant} = \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow a_p = -\omega^2 A \sin(\omega t - kx + \phi)$$

$$\Rightarrow a_p = -\omega^2 (y_p)$$



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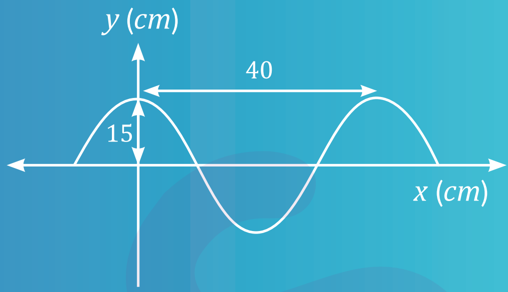
A sinusoidal wave travelling in the positive x-direction has an amplitude of 15 cm, a wavelength of 40 cm, and a frequency of 8 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15 cm.

(a) Find the angular wave number, period, angular frequency, and speed of the wave.

(b) Determine the phase constant (ϕ) and write a general expression for the wave function.



Solution

Given,

Amplitude of the wave, $A = 15\ cm$

Wavelength of the wave, $\lambda = 40\ cm$

Frequency of the wave, $f = 8\ Hz$

The vertical displacement of the medium at $t = 0$ and $x = 0$ is, $y(x = 0, t = 0) = 15\ cm$

Also, it is given that the wave is sinusoidal and is travelling in the positive x-direction.

(a) The angular wave number of the wave is, $k = \frac{2\pi}{\lambda} = \frac{2\pi}{40} = \frac{\pi}{20}\ cm^{-1}$

The time period of the wave is, $T = \frac{1}{f} = \frac{1}{8}\ s$

The angular frequency of the wave is, $\omega = 2\pi f = (2\pi) \times 8 = 16\pi\ s^{-1}$ (the unit of angular frequency is $rad\ s^{-1}$, but here, π is expressed in rad; so, the unit is written as s^{-1} only.)

The speed of the wave is,

$$v = \frac{\omega}{k} = \frac{16\pi}{\left(\frac{\pi}{20}\right)} = 320\, cm\ s^{-1}$$

(b) Since the vertical displacement of the medium at $t = 0$ and $x = 0$ is 15 cm and from the figure, it is seen that this is the positive extreme position. As the particle starts from the positive extreme position, the phase constant or the epoch is $\frac{\pi}{2}\ rad.$

Since the sinusoidal wave is travelling in the positive x-direction, the general expression of the wavefunction is, $y = A \sin(\omega t - kx + \phi)$

By substituting the values of the wave parameters, we get,

$$y = (15\ cm) \sin\left[(16\pi\ s^{-1})t - \left(\frac{\pi}{20}\ cm^{-1}\right)x + \frac{\pi}{2}\right]$$

$$\Rightarrow y = (15\ cm) \cos\left[(16\pi\ s^{-1})t - \left(\frac{\pi}{20}\ cm^{-1}\right)x\right]$$

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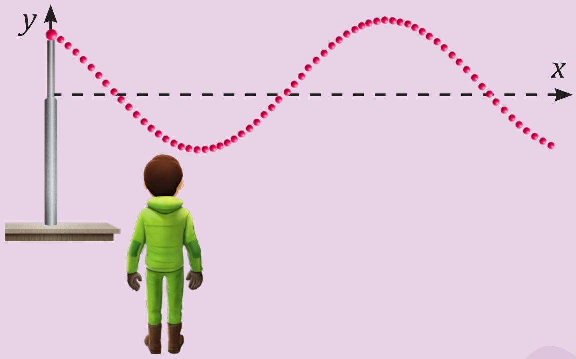
BOARDS MAIN

Velocity of a Wave on a String

Recall that in wave parameters, the velocity of a wave is the only parameter that depends on the properties (elasticity and inertia) of the medium.

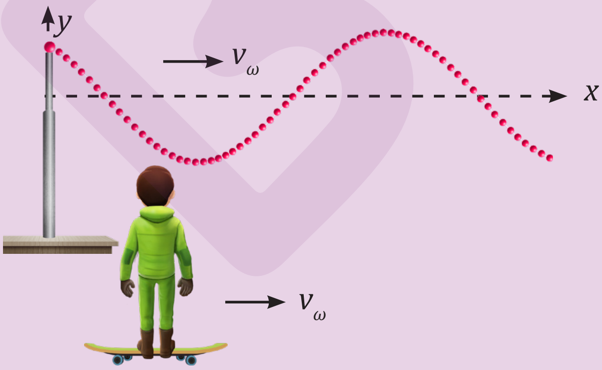
Consider a sinusoidal wave produced by a source executing SHM and an observer at rest with respect to the ground frame observing it, as shown in the figure.



Let us choose the x-axis and y-axis along the horizontal and vertical directions, respectively, and assume that the wave velocity is v_ω along the x-direction.

The observer will conclude that the velocity of the wave is $v_\omega \hat{i}$ and the velocity of the string is zero since the string itself does not go anywhere as a whole, although every particle of it executes SHM.

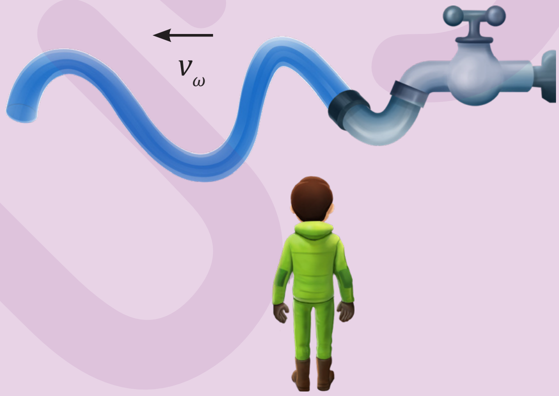
Now, suppose that the same observer is moving with the same velocity v_ω as that of the wave along the positive x-direction.



For this case, the observer will observe the following scenario:

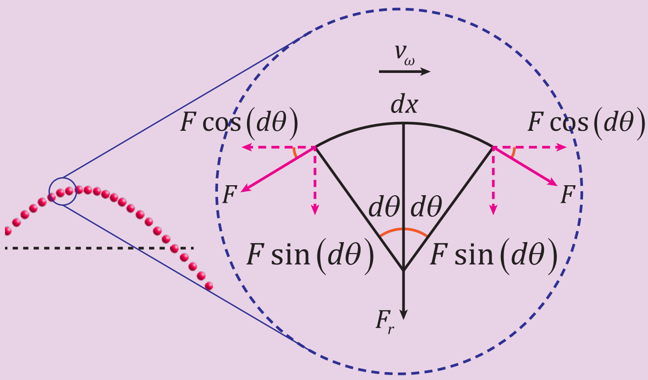
The velocity of the wave is, $v_\omega = 0$ (since the relative velocity between the wave and the observer is zero) and the velocity of the string is $-v_\omega \hat{i}$.

To clearly understand this, let us imagine a wave-shaped transparent glass tube connected with a tap and some coloured water flowing through the tube with velocity v_ω along the negative x-direction. The observer is at rest with respect to the tube. So, according to the observer, the velocity of the water is $-v_\omega \hat{i}$ and in analogy, it can be said that the tube here is nothing but the portion of the wave that the observer sees and the water flowing inside the tube is the string.



To find the velocity of a wave, let us focus on a small element dx of the wave (string) and we know that any infinitesimally small length dx can be considered as a vector dx or a part of a circle. Now, let us consider a very small element dx of the string which is a part of an imaginary circle of radius R, which is doing circular motion with tangential velocity v_ω and subtending an angle $2d\theta$ at the centre as shown in the figure.

Therefore, $dx = R(2d\theta)$ ....(i)



Also, consider that the tensile force, F, is acting on it. The horizontal component of the force gets cancelled and the resultant force on dx is the net vertical component of force $2F\sin d\theta$.

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If μ is the mass per unit length (linear mass density) of the string, then the mass of elemental length dx is,

$dm = \mu dx$ ....(ii)

The small element dx is a part of an imaginary circle of radius R and it is undergoing circular motion with tangential velocity v_ω. Therefore, the centripetal force acting on it is given by, $\frac{dm(v_\omega)^2}{R}$

Thus,

$$2F\sin(d\theta) = \frac{dm(v_\omega)^2}{R}$$

$$\Rightarrow 2F(d\theta) = \frac{(\mu dx)(v_\omega)^2}{R} \quad \text{[Since } d\theta \text{ is small, } \sin(d\theta) \approx d\theta \text{ and from equation } (ii), dm = \mu dx]$$

$$\Rightarrow 2F(d\theta) = \frac{[\mu R(2d\theta)](v_\omega)^2}{R} \quad \text{[From equation } (ii),\ dx = R(2d\theta)]$$

$$\Rightarrow v_\omega^2 = \frac{F}{\mu}$$

$$\boxed{\Rightarrow v_\omega = \sqrt{\frac{F}{\mu}}}$$

Let the volume density and area of cross section of the string be ρ and S, respectively.

The velocity of the wave will be, $v_\omega = \sqrt{\frac{F}{\rho S}}$

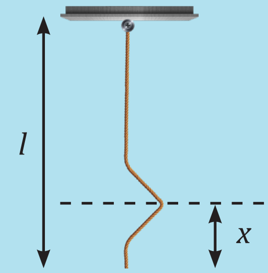
Hence, the velocity of the wave is,

$$v_w = \sqrt{\frac{F}{m}} = \sqrt{\frac{\text{Elastic property}}{\text{Inertial property}}} \text{ or } \sqrt{\frac{F}{rS}}$$

It is very important to notice that F is the tension in the string, which depends on the elastic property of it and μ is the linear mass density, which depends on the inertial property of the string. Therefore, the velocity of the wave is the property of the medium.

A very thin uniform string of length l hangs from a rigid support. Find the speed of the transverse wave on the string at a distance x from the lower end.

Solution



Wrong solution

Let the mass of the string be m.

Therefore, the tensile force on the string is, $F = mg$ and

the linear mass density of the string is, $\mu = \frac{m}{l}$

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Thus, the velocity of the wave that is produced in the string is,

$$v_{wave} = \sqrt{\frac{F}{\mu}} \Rightarrow v_{wave} = \sqrt{\frac{mg}{\left(\frac{m}{l}\right)}} \Rightarrow v_{wave} = \sqrt{gl}$$

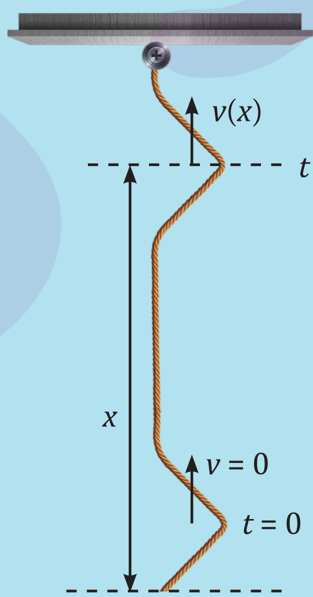
The time required for the wave to reach to the top is,

$$t = \frac{l}{v_{wave}} \Rightarrow t = \frac{l}{\sqrt{gl}} = \sqrt{\frac{l}{g}}$$

Since the string is the only body in consideration here, we cannot ignore the variation of the tension throughout the string. Thus, we cannot apply this method as we do not consider the variation of the tension. This type of method is only applicable when a mass is attached to the string and the mass of the string is negligible to it.

Correct solution

Suppose that the wave pulse is at the lower end of the string at time $t = 0$, and the tension in the lower end is zero. Therefore, the velocity of the wave pulse at time $t = 0$ is zero.



Now, consider that at any time t, the wave pulse is at a distance x from the lower end of the string. Therefore, the mass of the part of the string up to a distance x from the lower end is μx; hence, the force on the string at a distance x is, $F(x) = \mu xg$

Thus, the velocity of the wave pulse at this time is,

$$v_{wave} = \sqrt{\frac{F(x)}{\mu}} \Rightarrow v_{wave} = \sqrt{\frac{(\mu x)g}{\mu}} \Rightarrow v_{wave} = \sqrt{gx}$$

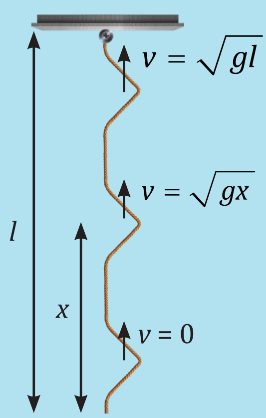
Hence, the speed of the transverse wave on the string at a distance x from the lower end is, $v = \sqrt{gx}$

A heavy, uniform rope of length l is suspended from a ceiling. If the rope is given a sudden sideways jerk at the bottom, how long will it take for the pulse to reach the ceiling?

Solution MAIN

Method 1

Suppose that at $t = t_0$, the wave pulse reaches the rigid support.



We know that the velocity of the wave pulse at any distance x from the lower end of the string is, $v = \sqrt{gx}$

$$v = \sqrt{gx}$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{gx} \Rightarrow \sqrt{g}\int_0^{t_0} dt = \int_0^{l} \frac{dx}{\sqrt{x}} \Rightarrow \sqrt{g}\,t_0 = \left[2\sqrt{x}\right]_0^l \Rightarrow t_0 = 2\sqrt{\frac{l}{g}}$$

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Method 2

We know that the velocity of the wave pulse at any distance x from the lower end of the string is, $v = \sqrt{gx}$

Therefore, the acceleration of the wave pulse is,

$$a = v\frac{dv}{dx} \Rightarrow a = \left(\sqrt{gx}\right)\left(\frac{\sqrt{g}}{2\sqrt{x}}\right) \Rightarrow a = \frac{g}{2}$$

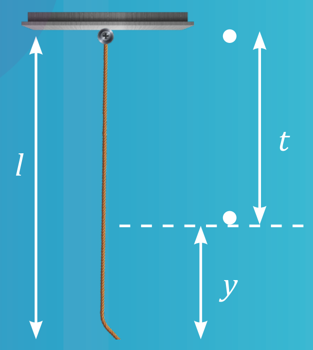
Therefore, the acceleration of the wave pulse is constant.
Now, we know that from the equation of motion in 1-D,

$$x = ut + \frac{1}{2}at^2$$

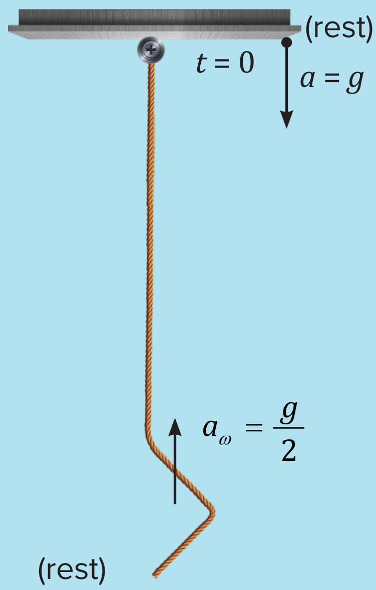
Here, initial velocity of the wave pulse is, $u = 0$ and the pulse covers distance l in time t_0.
Thus,

$$l = \frac{1}{2}\left(\frac{g}{2}\right)t_0^2 \Rightarrow t_0 = 2\sqrt{\frac{l}{g}}$$

A particle is dropped from the ceiling and at that instant, the bottom end is given a jerk. Where and when will the particle meet the pulse?



Solution ADVANCED



Suppose that at time $t = 0$, the particle is just dropped from the ceiling and the jerk is given at the bottom end of the string, as shown in the figure. Here, both have zero initial velocities.

Suppose that at time $t = 0$, the particle is just dropped from the ceiling and the jerk is given at the bottom end of the string, as shown in the figure. Here, both have zero initial velocities.

Therefore, the acceleration of the particle and the wave pulse will be g in the downward direction and $\frac{g}{2}$ in the upward direction, respectively. Therefore, the relative acceleration is $\frac{3g}{2}$.

Thus, by applying the equation of motion in one dimension in a relative frame, we get the following:

$$x_{rel} = u_{rel}t + \frac{1}{2}a_{rel}t^2$$

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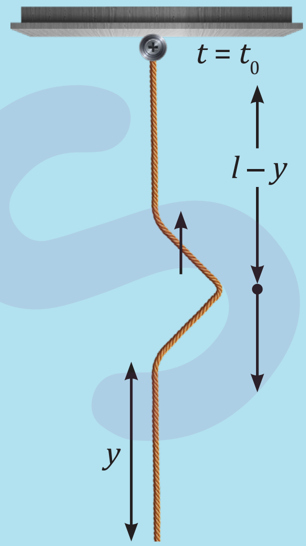
Here, relative initial velocity is, $u_{rel} = 0$ and the total distance covered by both wave pulse and particle is l in time t_0.

Thus,

$$l = \frac{1}{2}\left(\frac{3g}{2}\right)t_0^2$$

$$\Rightarrow t_0 = \sqrt{\frac{4l}{3g}}$$



Therefore, the particle will meet the pulse at, $t = \sqrt{\frac{4l}{3g}}$

Assume that at time $t = t_0$, the particle meets the wave pulse when the pulse is at a distance y from the bottom end.

Therefore, the distance from the bottom end where the pulse will meet the particle is,

$$y = \frac{1}{2}\left(\frac{g}{2}\right)t_0^2 \quad \left[\begin{array}{l}\text{Since the initial velocity of the pulse is zero} \\ \text{and the acceleration is } \frac{g}{2}\end{array}\right]$$

$$\Rightarrow y = \frac{1}{2}\left(\frac{g}{2}\right)\left(\sqrt{\frac{4l}{3g}}\right)^2$$

$$\Rightarrow y = \frac{l}{3}$$

Therefore, the particle will meet the pulse at a distance $\frac{2l}{3}$ from the ceiling.

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