BYJU'S Classes

PHYSICS

WAVES

INTRODUCTION TO WAVE MECHANICS

NOTES

What you already know

- Kinematics
- Newton's laws of motion
- Circular motion
- Rotation
- SHM and oscillations

What you will learn

- Introduction to waves
- Classification of waves
- String wave
- Travelling wave
- Wave parameters

BOARDS

Introduction to Waves

A wave is a disturbance that propagates in space and transfers energy and momentum without transferring the matter.

Classification of waves

Waves can be classified based on the requirement of the medium, the direction of motion of the particle, and the direction of propagation of the wave itself as follows:

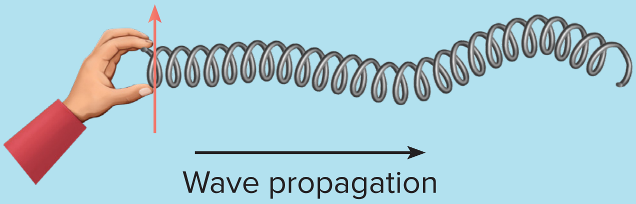
- **Based on the requirement of medium:** Depending on the requirement of the medium for the propagation of waves, the waves are classified into the following two categories:

    1. **Mechanical waves:** The waves that necessarily require a medium to propagate are known as mechanical waves. The medium needs to be elastic to return the particles to their mean positions and it needs to have inertia to store the energy and transfer it further.
    Example: Sound waves are mechanical waves as they require a medium to propagate. Hence, they cannot propagate through vacuum.

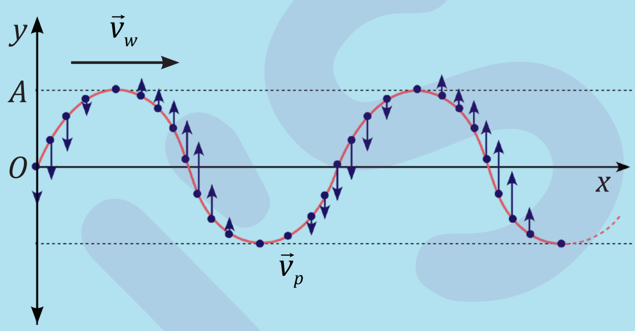
    2. **Non-mechanical waves:** The waves that do not necessarily require a medium to propagate are known as non-mechanical waves.
    Example: Light waves are non-mechanical waves as they can pass through vacuum.

- **Based on the direction of motion of particles:** The following are the most important categories of waves which are divided according to the direction of motion of the particles.

1. **Transverse waves:** If the particles of the medium vibrate in a direction perpendicular to the direction of wave propagation, i.e., if the displacement of the particles of the medium is perpendicular to the direction of propagation of waves, then the waves are known as transverse waves.



Examples: Electromagnetic waves, the Mexican wave which is usually formed by people in a crowd, the waves on a string, the ripples on the surface of water, etc.



Mathematically, if $\vec{v}_p$ is the velocity of the particle of the medium and $\vec{v}_w$ is the velocity of the wave, then for the case of transverse wave,

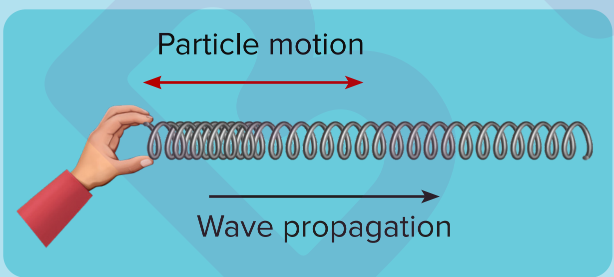
$$\vec{v}_p \perp \vec{v}_w \Rightarrow \vec{v}_p \cdot \vec{v}_w = 0$$

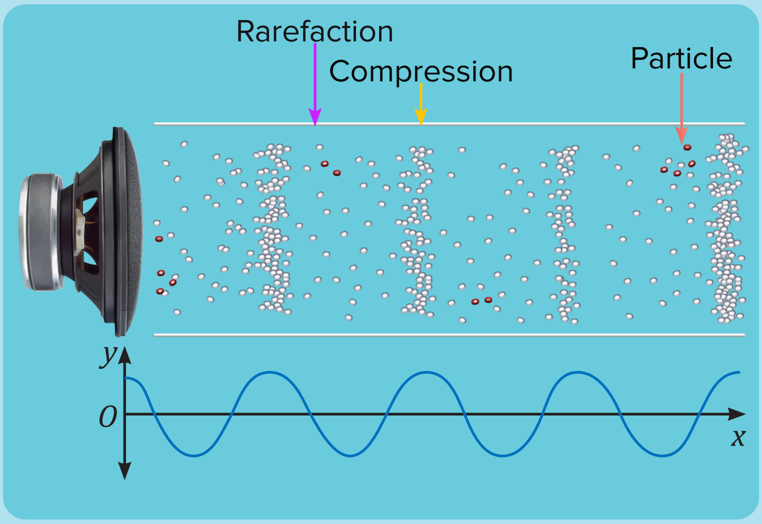
2. **Longitudinal waves:** If the particles of a medium vibrate in the direction of propagation of waves, i.e., if the displacement of the particles is parallel to the direction of the propagation of waves, then the waves are known as longitudinal waves. These waves propagate as compressions and rarefactions.

Examples: Sound wave in air, the vibration in a spring, etc.

Mathematically, if $\vec{v}_p$ is the velocity of the particle of the medium and $\vec{v}_w$ is the velocity of the wave, then for the case of longitudinal wave,

$$\vec{v}_p \| \vec{v}_w \Rightarrow \vec{v}_p \times \vec{v}_w = \vec{0}$$





- **Based on the direction of propagation of waves:** Depending on the direction of propagation of waves, the waves are divided into three categories.

1. **One-dimensional waves:** If the waves propagate in one direction, then they are known as linear waves or one-dimensional waves.

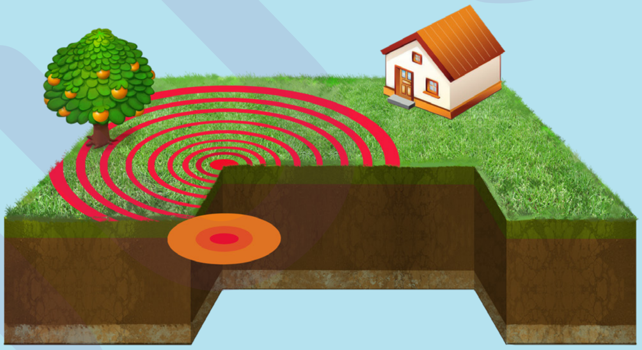
   Example: The waves on a string.

2. **Two-dimensional waves:** If the waves propagate in a two-dimensional space, then they are known as planar waves, surficial waves, or two-dimensional waves.

   Example: The ripples on the surface of water.

3. **Three-dimensional waves:** If the waves propagate through all over the space, then they are known as spatial waves or three-dimensional waves.

   Example: Seismic waves or earthquake waves.



Important Terminologies

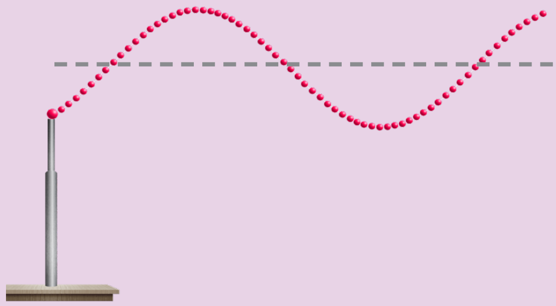
1. **Wave pulse:** A wave pulse is a single vibration or a non-periodic wave that has only one major crest.

   Example: Single disturbance in a string.

2. **Wave packet:** A continuous wave pulse produced by an external agency is known as the wave train or wave packet.

   Example: Continuous disturbance in a string.

   Practically, these wave packets are known as the so-called waves.



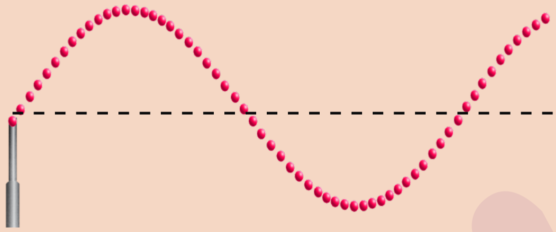
Our region of interests among all the categories of waves will be the following:

(i) String waves (One-dimensional, mechanical, and transverse)

(ii) Sound waves (Three-dimensional, mechanical, and longitudinal)

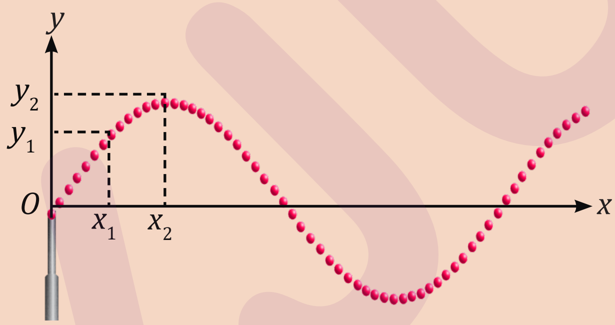
String Wave

Imagine that an external agent is continuously creating a wave pulse (up and down motion) in a string by holding the free end of the string while the other end is fixed. This will create a wave train and if we snapshot the scenario at an instant, it will look like the following figure:



A wave created in this way is known as a string wave which is one-dimensional, mechanical, and transverse in nature.

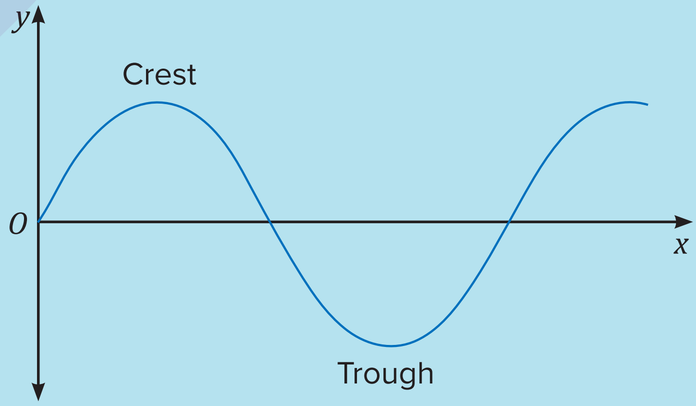
Now, if we set the x-axis along the horizontal direction (the direction of propagation of the string wave) and the y-axis along the vertical direction (the direction of vibration of the particles in the string), then the y-x graph at an instant gives the position of each of the particles at that instant.



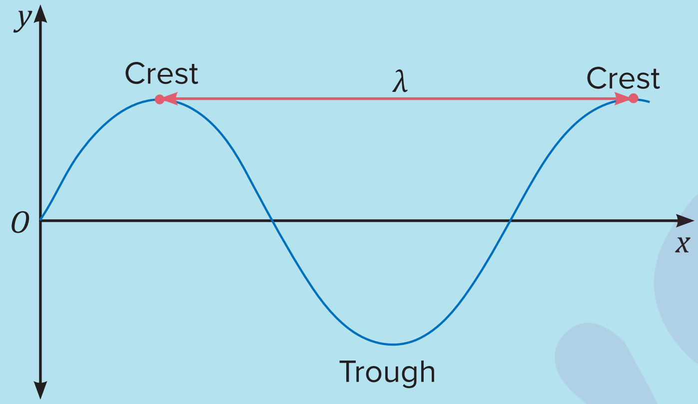
The given figure is the snapshot of the scenario at that instant where origin O is the end in which the vibration is created, and (x_1, y_1), (x_2, y_2), etc., are the coordinates of the position of the constituent particles of the string at that instant.

Wave Parameters

- **Crest and trough:** The topmost point of a wave is known as the crest, whereas the bottom-most point is known as the trough.

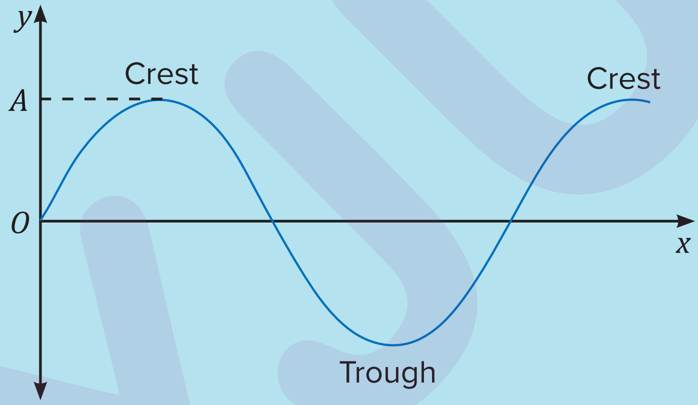


- **Wavelength:** The crest-to-crest or the trough-to-trough distance is known as the wavelength of the transverse wave. Sometimes, this distance is known as the peak-to-peak distance.



Symbolically, the wavelength is represented by λ.

- **Amplitude:** The magnitude of maximum displacement of each particle from its undisturbed position or the height of the crest from the horizontal axis passing through the origin is known as the amplitude of the transverse wave.



In the figure, OA is the amplitude of the wave. Every particle of the string will have an amplitude similar to that of the source.

- **Time period (T):** The time taken for one complete oscillation or the time interval between two successive wave pulses to come out from the source, or the time between two successive crests or troughs is known as the time period of the transverse wave.
- **Frequency:** The number of oscillations per unit time is the frequency (f) of the wave. It should be remembered that,

Frequency of the wave = Frequency of the source

Relationship between the wavelength, frequency, and velocity of a wave

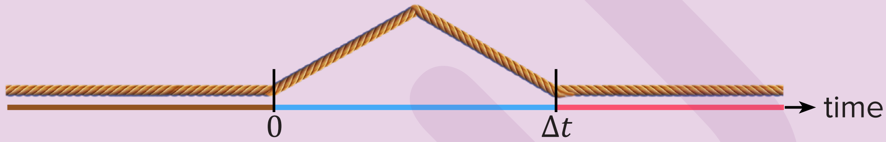
Let the wavelength, frequency, and velocity of a wave be λ, f, and v, respectively. Then, they are related as follows:

$$\text{Wavelength}(\lambda) = \frac{\text{Velocity}(v)}{\text{Frequency}(f)}$$

- A wave borrows every single property from the source except velocity because the wave propagates through a medium. Hence, the velocity of the wave depends on the property of the medium.
- Since the velocity (v) of a wave depends on the medium and the frequency (f) of the wave solely depends on the source, the wavelength (λ) depends on both the medium and the source.

Travelling Wave

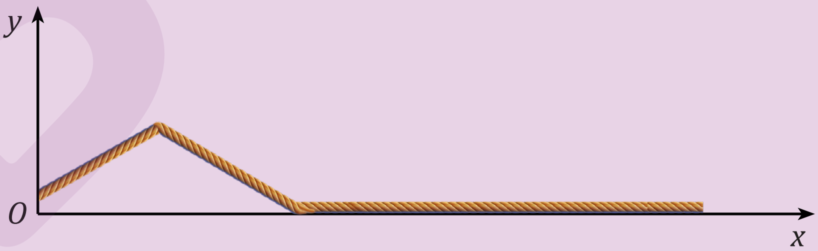
Consider that a wave pulse is created in a string by a source (external agent) which is at the origin during a time interval from 0 to Δt. The pulse is travelling along a horizontal line. A snapshot is also taken during this interval as shown in the figure. Assume that the source does not produce any disturbance before and after this time interval, which means that the source is doing nothing.



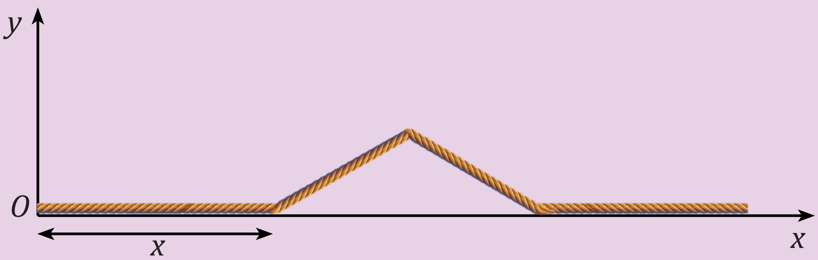
We can plot this scenario in the y-x graph for three time intervals as shown in the figures:

(i) **For any time, $t < 0$:** For this interval, the source does not give any disturbance. So, there is no disturbance along the y-axis and the y-coordinate of the source is zero (source at the origin).

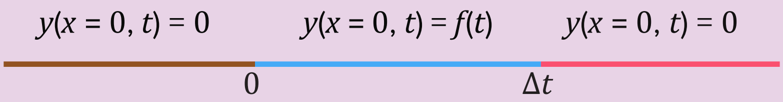
(ii) **For $0 < t < \Delta t$:** For this interval, the source produces a disturbance. Hence, there is a wave pulse at the origin and the y-coordinate of the source at the origin is non-zero. Mathematically, it is, $y\ (x = 0, t) = f(t)$. The pulse propagating along the x-axis is as shown in the figure:



(iii) **For any time, $t > \Delta t$:** For this interval as well, the source does not give any disturbance. Hence, there is no wave pulse at the origin and the y-coordinate of the source at the origin is zero.



If the disturbance along the y-axis, i.e., the disturbance produced by the source is denoted as $f(t)$ for the time interval $0 < t < \Delta t$, then the whole discussion can be depicted in a figure as shown:



Assume that the amplitude of the wave pulse is very small, and the string is uniform and homogeneous. The wave pulse can be justified to move forward with a constant velocity. Let that velocity be v.

Therefore, to cover a distance x, the wave pulse will take time equal to $\frac{x}{v}$. Thus, it can be said that,

(i) If at $t = 0$, the pulse is at $x = 0$, then at $t = \frac{x}{v}$, the pulse will be at $x = x$.

(ii) If at $t = t$, the pulse is at $x = 0$, then at $t = t + \frac{x}{v}$, the pulse will be at $x = x$.

(iii) If at $t = t$, the pulse is at $x = x$, then at $t = t - \frac{x}{v}$, the pulse will be at $x = 0$.

Hence, we can say that whatever happens at $x = x$ at time t has already happened at $x = 0$ at time $t - \frac{x}{v}$. Mathematically, this can be written as follows:

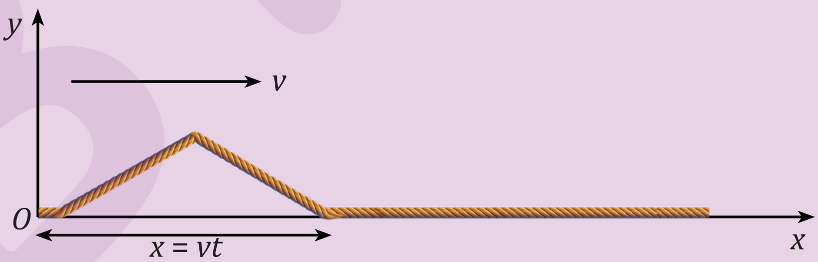
$$y(x, t) = y\left(x = 0, t - \frac{x}{v}\right)$$

Therefore, we can figure out the entire wave equation if the source equation is known.

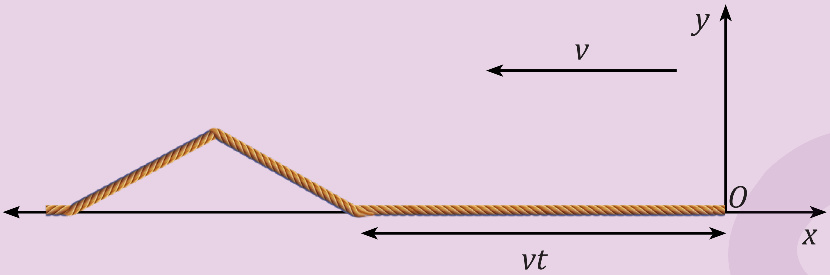
Since $y(x = 0, t) = f(t)$ is the equation of the source at any time t,

$$y(x, t) = y\left(x = 0, t - \frac{x}{v}\right) = f\left(t - \frac{x}{v}\right) = g(x - vt)$$

- If an equation is in the form of $f\left(t - \frac{x}{v}\right)$ or $g(x - vt)$, then the equation represents a travelling wave in the positive x-direction.



- If an equation is in the form of $f\left(t+\frac{x}{v}\right)$ or $g(x+vt)$, then the equation represents a travelling wave in the negative x-direction.



Definitions

1. **Travelling wave:** Any mathematical expression or formula that either is or can be manipulated as $g(x \pm vt)$ or $f\left(t \pm \frac{x}{v}\right)$ represents a travelling wave.
2. **Wavefunctions:** The functions that represent the waves are known as wavefunctions. The wavefunctions for the travelling waves moving with velocity v are $g(x \pm vt)$ and $f\left(t \pm \frac{x}{v}\right)$.
3. **Phase:** The quantity $(x \pm vt)$ is known as the phase of the wavefunction. For a wave pulse, the phase remains constant because the shape of the pulse does not change. In other words, a phase is nothing but the shape of the wave pulse.
4. **Phase velocity:** The phase velocity of a travelling wave is defined as, $v = \frac{dx}{dt}$. This is also known as wave velocity.

A wave is propagating on a long, stretched string along its length taken as the positive x-axis. The wave equation is given as, $y = y_0 e^{-\left(\frac{t}{T}-\frac{x}{\lambda}\right)^2}$, where, $y_0 = 4\ cm$, $T = 1\ s$, and $\lambda = 4\ cm$. Find the following:

(a) Velocity of the wave.

(b) Function $f(t)$ giving the displacement of the particle at $x = 0$.

(c) Function $g(x)$ giving the shape of the string at $t = 0$.

(d) Plot the shape $g(x)$ of the string at $t = 0$.

(e) Plot the shape $g(x)$ of the string at $t = 5\ s$.

Solution

Let us rewrite the given wave equation as follows:

$$y = y_0 e^{-\left(\frac{t}{T}-\frac{x}{\lambda}\right)^2}$$

$$\Rightarrow y = y_0 e^{-\frac{1}{T^2}\left(t-\frac{xT}{\lambda}\right)^2}$$

$$\Rightarrow y = y_0 e^{-\frac{1}{T^2}\left(t - \frac{x}{\left(\frac{\lambda}{T}\right)}\right)^2} \quad \text{........}(i)$$

(a) Equation (i) is in the form of $f\left(t - \frac{x}{v}\right)$. Hence, the given equation represents a travelling wave.

On comparing the forms of the phase parts, we get,

$$v = \frac{\lambda}{T}$$

$$\Rightarrow v = \frac{4\ cm}{1\ s}$$

$$\Rightarrow v = 4\ cm\ s^{-1}$$

Alternate

We have,

$$y = y_0\, e^{-\frac{1}{T^2}\left(t - \frac{x}{\left(\frac{\lambda}{T}\right)}\right)^2}$$

Therefore, the phase of the travelling wave is, $\left(t - \frac{x}{\left(\frac{\lambda}{T}\right)}\right)$, and we know that the phase of the travelling wave remains constant.

$$\frac{d}{dt}\left(t - \frac{x}{\left(\frac{\lambda}{T}\right)}\right) = 0$$

$$\Rightarrow 1 - \frac{T}{\lambda}\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{\lambda}{T}$$

$$\Rightarrow v = \frac{\lambda}{T}$$

$$\Rightarrow v = \frac{4\ cm}{1\ s}$$

$$\Rightarrow v = 4\ cm\ s^{-1}$$

(b) By putting $x = 0$ in equation (i), we get,

$$y = y_0 e^{-\frac{1}{T^2}(t-0)^2}$$

$$\Rightarrow y = y_0 e^{-\left(\frac{t}{T}\right)^2}$$

$$\Rightarrow f(t) = y_0 e^{-\left(\frac{t}{T}\right)^2}$$

(c) By putting $t = 0$ in equation (i), we get,

$$y = y_0 e^{-\left(0-\frac{x}{\lambda}\right)^2}$$

$$\Rightarrow y = y_0 e^{-\left(\frac{x}{\lambda}\right)^2}$$

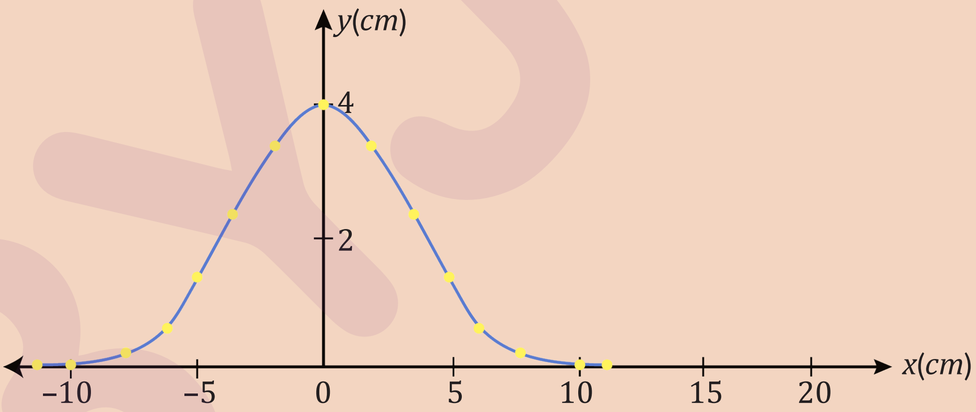
$$\Rightarrow g(x) = y_0 e^{-\left(\frac{x}{\lambda}\right)^2}$$

(d) We know that at $t = 0$, $g(x) = y_0 e^{-\left(\frac{x}{\lambda}\right)^2}$

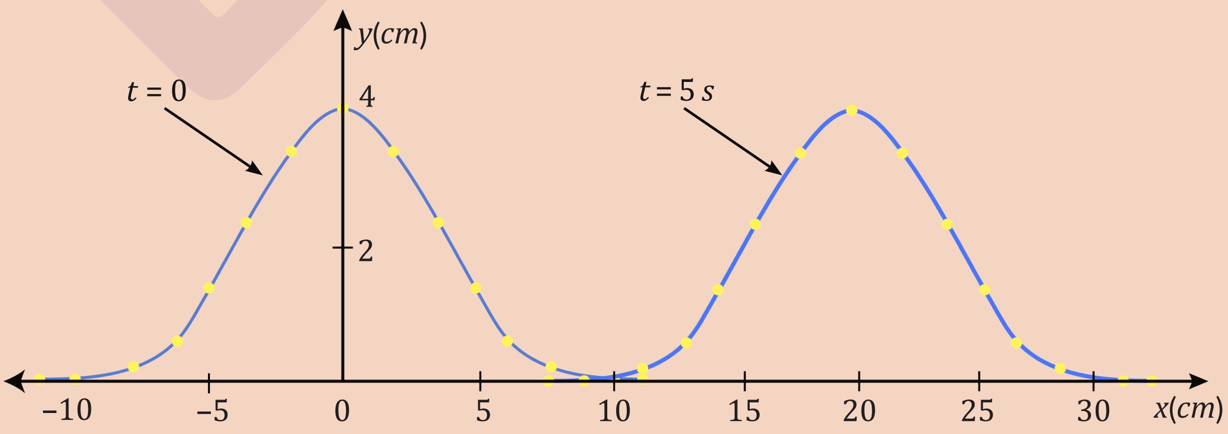
Now, by putting $x = 0$, we get,

$g(x = 0) = y_0 = 4\ cm$

Therefore, the peak of the graph will be at $x = 0$ with $g(x = 0) = 4\ cm$. The graph is as shown in the figure.



(e) At $t = 5\ s$, $g(x) = y_0 e^{-\left(5-\frac{x}{\lambda}\right)^2}$ and the graph is as following:



A sinusoidal wave is travelling along a rope. The oscillator that generates the wave completes 60 vibrations in 30 s. Also, a given pulse travels 425 cm along the rope in 10 s. What is the wavelength?

Solution

We know that frequency is defined as the number of oscillations per unit time. Therefore, the frequency is, $f = \frac{60}{30} = 2\,Hz$.

Given,

The wave pulse travels 425 cm along the rope in 10 s. Hence, the wave velocity is,

$$v = \frac{425}{10} = 42.5\ cm\,s^{-1}$$

Thus, the wavelength is, $\lambda = \frac{v}{f} = \frac{42.5}{2} = 21.25\ cm$.

A wave pulse is travelling on a string at 2 ms^{-1}. Displacement y of the particle at $x = 0$ at any time t is given by, $y = \frac{2}{t^2+1}$. Find the following:

(a) Expression of function $y = (x, t)$, i.e., the displacement of a particle at position x and time t.

(b) Shape of the pulse at $t = 0$ and $t = 1\ s$.

Solution

Given,

The displacement of the particle at $x = 0$ at any time t is, $y = \frac{2}{t^2+1}$.

The velocity of the wave pulse is, $v = 2\ ms^{-1}$.

(a) We know that if $y(x = 0, t) = f(t)$ is the equation of the source at any time t, then the displacement of the particle at position x and time t is as follows:

$$y(x,t) = y\left(x = 0,\ t - \frac{x}{v}\right)$$

$$\Rightarrow y(x,t) = \frac{2}{\left(t - \frac{x}{v}\right)^2 + 1}$$

$$\Rightarrow y(x,t) = \frac{2}{\left(t - \frac{x}{2}\right)^2 + 1}$$

(b) Now, we have,

$$y(x,t)=\frac{2}{\left(t-\frac{x}{2}\right)^2+1}$$

By putting $t=0$, we get,

$$y(x,t=0)=\frac{2}{\frac{x^2}{4}+1} \quad \text{..................}(i)$$

And by putting $t=1\ s$, we get,

$$y(x,t=1)=\frac{2}{\left(1-\frac{x}{2}\right)^2+1}$$

$$\Rightarrow y(x,t=1)=\frac{2}{\frac{(x-2)^2}{4}+1} \quad \text{..................}(ii)$$

From equations (i) and (ii), it is seen that there is a difference of 2 m between the peaks of the wave pulses along the x-direction.

Therefore, the shape of the wave pulse at t = 0 and t = 1 s is as follows: