to solve various problems that can [illegible] theory itself can be treated as an independent unit; however, some elementary concepts of set theory and algebraic structure are useful before going through Chapter 13.

1.2 INTRODUCTION TO PROPOSITIONAL LOGIC

We might have come across a situation wherein we ask a little child to do something, and in reply, the child demands something from us in lieu of performing that job. For example, when a child is asked to sing a song, he or she says, 'First give me a chocolate, then I will sing.' Unknowingly, he or she puts a conditional statement before us and uses a logical statement. This example shows the presence of logic in human mind.

Whenever we go to a shop, we ask the shopkeeper about the availability of the required commodity. For example, if we go to a bookshop and ask the shopkeeper about the availability of novels of Premchand, then he might answer that he has

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some books of our choice or he might say that there is no novel of Premchand. Generally, we never try to explore the reasoning that can be developed from such types of statements. Let us try to interpret these sentences in a different way. In the first case, the shopkeeper says that in his collection there exists at least one novel of Premchand, and in the second case, he says that all the novels available in his shop are authored by someone different from Premchand. This example shows that we can express a sentence in a different but logically equivalent ways.

Logic has close connections to computer science, and it works as the basis for automated reasoning. Computer programming, programming languages, and artificial intelligence are some of the examples of its practical applications. It is important to know what is a proposition or a statement and how new propositions can be constructed by combining different propositions. Propositional logic, also known as sentential logic or statement logic, is the branch of logic that studies ways of joining or modifying propositions or statements to form more complicated propositions or statements and also studies the logical relationships that are derived from these methods. In this chapter, we will study about propositions, connectives, arguments, predicates, quantifiers, and various related terms. Finally, we discuss different methods of proof.

1.3 PROPOSITION

Let us consider the following seven sentences:

1. The sum of five and three is eight.
2. The sum of two and four is seven.
3. The sum of x and three is five.
4. Delhi is the capital of India.
5. Cricket is the national game of India.
6. Go to the classroom.
7. The sun will rise tomorrow.

Is it possible for us to say whether each of these sentences is true or false? The first and fourth sentences are true, the second and fifth sentences are false, and the third and sixth sentences are neither true nor false. In the third sentence, x is not defined whereas the sixth sentence is an order. In both the cases, we cannot determine the truth value of the sentence. What can we say about the seventh sentence? Sun never rises or sets. Suppose we consider a person A who begins a train journey at 2 p.m. from Delhi. Say, after undertaking a journey of approximately 16 h, A reaches yet another city in India the next morning. Then, obviously, A will see the sunrise. However, it may be possible that A takes a flight from Delhi at 2 p.m., and after undertaking the same duration of journey, A reaches a different country where again A could witness bright sunshine or even a dark night. Therefore, there is no question of the sentence being right or wrong. Here, the question of the rising of the sun the next day becomes situational. This may be both true and false. Thus, it is not a proposition.

In this chapter, we shall consider only those sentences that are either true or false but not both. Remember

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A proposition is a sentence that is either true or false but not both.

The two values *true* and *false* are called the truth values of a sentence. Sometimes, we use 1 or T for the truth value *true* and 0 or F for the truth value *false*. Generally, we use the symbols $p, q, r, P, Q, R, \ldots$ to denote a proposition and these symbols are called propositional variables. For example,

P: Delhi is the capital of India
Q: The sum of five and three is eight

The area of logic that deals with propositions is called propositional logic or propositional calculus.

1.4 LOGICAL OPERATORS

Many times, we need to join two or more propositions to form a new proposition or we need to negate a proposition. Logical operators are used for such purposes. Here, we shall go through the different logical operators.

1.4.1 Negation (~)

The logical operator negation is used with only one statement. It is similar to *not*. The negation of the proposition P is denoted by $\sim P$ and is read as 'not P'. Look at the following propositions:

P: I am an Indian
$\sim P$: It is not the case that I am an Indian
or $\sim P$: I am not an Indian

The truth table for negation of a statement is shown in Table 1.1

Table 1.1 Truth Table of Negation

P	$\sim P$
T	F
F	T

1.4.2 Disjunction (OR/∨)

Let P and Q be two propositions. Then disjunction of P and Q is denoted by $P \vee Q$ and the proposition is 'P or Q'. The word *or* used here is inclusive; that is, $P \vee Q$ is true, which means either P is true or Q is true or both P and Q are true. However, sometimes we remove the phrase *or both*, but still it is interpreted as inclusive.

Let P: Five is greater than two
Q: The sum of five and three is eight

Then, $P \vee Q$: Five is greater than two or the sum of five and three is eight

If one of the two variables is true, then disjunction of the two variables is also true. Disjunction of two propositions is false if both the variables are false; it is true for every other case. The truth table for disjunction of two variables is given in Table 1.2.

Table 1.2 Truth Table of Disjunction

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE 1.1

Determine whether the following compound propositions are true or false:

(a) $2 + 3 = 5$ or $4 + 5 = 9$
(b) $2 + 5 = 8$ or $4 + 5 = 9$
(c) $3 + 4 = 8$ or $3 + 5 = 7$
(d) $5 + 6 = 11$ or $2 + 5 = 8$

Solution:

(a) True. As both the propositions are true, the disjunction is true.
(b) True. As one of the two propositions is true, the disjunction is true.
(c) False. As both the propositions are false, the disjunction is false.
(d) True. As one of the two propositions is true, the disjunction is true.

1.4.3 Exclusive OR

Let P and Q be two propositions. The exclusive OR of P and Q, denoted by $P \oplus Q$ is the proposition, which is true when exactly one of P and Q is true (the case when both the variables are true is excluded), and is false otherwise. In simple words, it means either P is true or Q is true but not both. The truth table for exclusive OR of two variables is shown in Table 1.3.

Table 1.3 Truth Table of Exclusive OR

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

1.4.4 Conjunction (and/∧)

Let P and Q be two propositions. Then conjunction of P and Q is denoted by $P \wedge Q$, which is the proposition 'P and Q'.

Let P: Five is greater than two.
Q: The sum of five and three is eight.
Then, $P \wedge Q$: Five is greater than two and the sum of five and three is eight.

The conjunction of two variables is true if and only if both the variables are true. If one of the variables is false, then the conjunction of the variables is also false. The truth table for conjunction of two variables is given in Table 1.4.

Table 1.4 Truth Table of Conjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

EXAMPLE 1.2

Determine whether the following compound propositions are true or false:

(a) $2 + 3 = 5$ and $4 + 5 = 9$
(b) $2 + 5 = 8$ and $4 + 5 = 9$
(c) $3 + 4 = 8$ and $3 + 5 = 7$
(d) $5 + 6 = 11$ and $2 + 5 = 8$

Solution:

(a) True. As both the propositions are true, the conjunction is true.
(b) False. As one of the two propositions is false, the conjunction is false.
(c) False. As both the propositions are false, the conjunction is false.
(d) False. As one of the two propositions is false, the conjunction is false.

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There are many ways to express a conjunction in English. Consider the following propositions:

P: Amit walks fast
Q: Mohan walks slowly

The following sentences represent the same compound proposition $P \wedge Q$.

Amit walks fast and Mohan walks slowly.
Amit walks fast but Mohan walks slowly.
Amit walks fast yet Mohan walks slowly.

In English, the meanings of *and*, *but*, and *yet* may be used in different ways, but here, we are concerned with the logical aspect of the propositions. Hence, we shall treat them as the same form of *and*. The same case is applicable to *neither–nor* kinds of statements. Its logical form is given in the following.

The Logical Form of Neither-nor

The English sentence 'Neither Amit walks fast nor Mohan walks fast' is equivalent to 'Amit does not walk fast and Mohan does not walk fast'. Thus, if we denote the propositions as P: Amit walks fast and Q: Mohan walks fast, then the logical form of this sentence is $\sim P \wedge \sim Q$.

1.4.5 Conditional (→)

Let P and Q be two propositions. Then, conditional of P and Q is denoted by $P \rightarrow Q$, which is the proposition 'if P, then Q'. Here, P is the hypothesis (or condition) and Q is the conclusion.

The conditional statement can be expressed in a number of ways. Following are the equivalent forms of the conditional $P \rightarrow Q$.

1. P implies Q
2. P is sufficient for Q
3. P only if Q
4. Q is necessary for P

Let P: I attend classes regularly
Q: I get first division
Then, $P \rightarrow Q$: if I attend classes regularly, then I get first division

The truth value of $P \rightarrow Q$ is false when P is true but Q is false; otherwise, its truth value is true.

Many times, it seems confusing to remember the truth values of the conditional. Instead of remembering the truth table directly, we shall try to construct the truth table by taking an example. This will help readers construct the truth table without remembering it. It is important to note that P is a sufficient condition and is not a necessary condition. Sufficient means more than necessary. For example, look at this statement: 'If I have ₹100, then I can buy a pen.' A simple pen costs less than or equal to ₹100. Here, the condition is sufficient as ₹100 is sufficient to buy a pen. I can still buy a pen without having ₹100. Thus, conclusion may be true without the sufficient condition being true.

Example showing the construction of truth table for conditional

Let P: I have 1l petrol in my bike
Q: I travel a distance of 35 km
$P \rightarrow Q$: If I have 1l petrol in my bike, then I travel a distance of 35 km

In this example, having 1l petrol is a sufficient condition (not necessary), which implies that there is a chance of traveling 35 km with less than 1l petrol. Now, let us consider the different cases.

1. If P is true and Q is also true, then the statement $P \rightarrow Q$ is also true.
2. P is true but Q is false. This shows that I have 1l petrol, but I do not travel 35 km. If the sufficient condition is fulfilled, then there is no reason for the conclusion to be false. Therefore, a true condition with false conclusion indicates that the statement $P \rightarrow Q$ is false.
3. P is false but Q is true. This shows that I do not have 1l petrol but I travel a distance of 35 km. The statement $P \rightarrow Q$ will be true because the conclusion may be true without the condition being true, as it is a sufficient condition.
4. P is false and Q is also false. This shows that neither do I have 1l petrol nor do I travel a distance of 35 km. The statement $P \rightarrow Q$ will be true because the false condition, and therefore, false conclusion do not indicate that the statement $P \rightarrow Q$ is false.

Table 1.5 Truth Table of Conditional Statement

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

We can summarize the results as shown in Table 1.5.

This example is given to construct the truth table of the conditional $P \rightarrow Q$ in propositional calculus. It should be noted that the conditional is a part of propositional calculus and English language is not exactly the same as propositional calculus. The *if–then* statement used in natural language has partial similarity to this one. In propositional calculus, English sentences have been used only to make the concepts easy to learn. In programming languages, the use of the *if–then* statement is different.

Example 1.3

Determine whether the following conditional statements are true or false.

(a) If $2 + 3 = 5$, then $4 + 5 = 9$
(b) If $2 + 5 = 8$, then $4 + 5 = 9$
(c) If $3 + 4 = 8$, then $3 + 5 = 7$
(d) If $5 + 6 = 11$, then $2 + 5 = 8$

Solution:

(a) True. As the condition is true as well as conclusion is true, the conditional statement is true.
(b) True. As the condition is false but conclusion is true, the statement is true.

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(c) True. As the condition is false and conclusion is also false, the conditional statement is true.
(d) False. As the condition is true but conclusion is false, the conditional statement is false.

1.4.6 Biconditional ($\leftrightarrow$)

Let P and Q be two propositions. Then biconditional of P and Q is denoted by $P \leftrightarrow Q$, which is the proposition 'P if and only if Q'.

There are some other ways to express the biconditional:

1. P is necessary and sufficient for Q
2. P iff Q

The word *iff* is the abbreviation for *if and only if*.

Let P: I participate in all the activities of the class

Q: I get grade A

Then $P \leftrightarrow Q$: I participate in all activities of the class if and only if I get grade A

Table 1.6 Truth Table of Biconditional Statement

P	Q	$P \to Q$	$Q \to P$	$P \leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The truth value of $P \leftrightarrow Q$ is true whenever both the variables have the same truth values; otherwise, it is false. The truth table of biconditional can also be constructed as the conjunction of $P \to Q$ and $Q \to P$. The truth table of $P \leftrightarrow Q$ is shown in Table 1.6.

Example 1.4

Determine whether the following biconditional statements are true or false.

(a) $2 + 3 = 5$ if and only if $4 + 5 = 9$.
(b) $2 + 5 = 8$ if and only if $4 + 5 = 9$.
(c) $3 + 5 = 9$ if and only if $4 + 3 = 8$.
(d) $5 + 6 = 11$ if and only if $2 + 5 = 8$.

Solution:

(a) True. As both the propositions are true, that is, both have the same truth values, the biconditional statement is true.
(b) False. As one of the two propositions is true and the other is false, that is, both have different truth values, the biconditional statement is false.
(c) True. As both the propositions are false, that is, both have the same truth values, the biconditional statement is true.
(d) False. As one of the two propositions is true and the other is false, that is, both have different truth values, the biconditional statement is false.

1.4.7 NAND ($\uparrow$)

Let P and Q be two propositions. Then NAND of P and Q is denoted by $P \uparrow Q$, which is equivalent to $\sim(P \wedge Q)$ that is, first we will calculate conjunction of P and Q and then its negation. The truth table of NAND is shown in Table 1.7.

Table 1.7 Truth Table of NAND

P	Q	$(P \wedge Q)$	$\sim(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

1.4.8 NOR ($\downarrow$)

Let P and Q be two propositions. Then NOR of P and Q is denoted by $P \downarrow Q$, which is equivalent to $\sim(P \vee Q)$; that is, first we will calculate disjunction of P and Q and then its negation. The truth table of NOR is shown in Table 1.8.

Table 1.8 Truth Table of NOR

P	Q	$(P \vee Q)$	$\sim(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

The following examples show the use of connectives in writing compound propositions. These examples will help readers convert English sentences to propositions and vice versa.

Examples showing writing compound proposition for English sentences

Example 1.5

Let P, Q, and R be the propositions.

P: You go to school
Q: You appear in the exam
R: You pass the exam

Write the propositions for the following sentences:

(a) You do not go to school and you do not appear in the exam.
(b) If you appear in the exam, then you pass the exam.
(c) You go to school, but you do not pass the exam.
(d) If you do not go to school and do not appear in the exam, then you do not pass the exam.
(e) Either you go to school or you pass the exam.
(f) You go to school and you appear in the exam, but you do not pass the exam.

Solution:

(a) $\sim P \wedge \sim Q$ (b) $Q \to R$ (c) $P \wedge \sim R$ (d) $(\sim P \wedge \sim Q) \to \sim R$
(e) $P \vee R$ (f) $P \wedge Q \wedge \sim R$

Example 1.6

Write propositions for the following sentences:

(a) If I go to Delhi, then I visit Red Fort.
(b) I go to Delhi but I do not visit Red Fort.
(c) I go to Delhi and I visit Rajghat but I do not visit Red Fort.
(d) If I go to Delhi and do not visit Rajghat, then I visit Red Fort.
(e) To visit Red Fort, it is necessary for me to go Delhi.
(f) I go to Delhi if and only if I visit Red Fort and Rajghat.
(g) To visit Red Fort and Rajghat, it is sufficient for me to go Delhi.

Solution: Let P: I go to Delhi.
Q: I visit Red Fort.
R: I visit Rajghat.

The propositions for the sentences are as follows:

(a) $P \to Q$ (b) $P \wedge \sim Q$ (c) $P \wedge R \wedge \sim Q$ (d) $(P \wedge \sim R) \to Q$
(e) $Q \to P$ (f) $P \leftrightarrow (Q \wedge R)$ (g) $P \to (Q \wedge R)$

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Example showing writing English sentences for compound propositions

EXAMPLE 1.7 ....

Let P, Q, and R be the propositions.

P: Hari is playing football
Q: Hari is reading his book
R: Hari is inside the room

Write the English sentences for the following propositions:

(a) $P \rightarrow {\sim}Q$ (b) ${\sim}P \wedge {\sim}Q$ (c) $P \vee Q$ (d) $R \rightarrow Q$
(e) $(R \wedge Q) \rightarrow {\sim}P$ (f) $P \rightarrow ({\sim}Q \wedge {\sim}R)$

Solution:

(a) If Hari is playing football, then he is not reading his book.
(b) Neither Hari is playing football nor is he reading his book.
(c) Either Hari is playing football or he is reading his book.
(d) If Hari is inside the room, then he is reading his book.
(e) If Hari is inside the room and he is reading his book, then he is not playing football.
(f) If Hari is playing football, then neither is he reading his book nor is he inside the room.

....

Examples showing constructing truth tables

Table 1.9 Truth Table of $P \rightarrow (P \wedge Q)$

P	Q	$P \wedge Q$	$P \rightarrow (P \wedge Q)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

EXAMPLE 1.8 ....

Construct the truth table for the proposition $P \rightarrow (P \wedge Q)$.

Solution: For the construction of the truth table of the proposition $P \rightarrow (P \wedge Q)$, firstly, we shall construct the truth table of $P \wedge Q$ and then the truth table of $P \rightarrow (P \wedge Q)$ as shown in Table 1.9.

EXAMPLE 1.9 ....

Construct the truth table for the proposition ${\sim}P \vee ({\sim}Q \wedge R)$.

Solution: The truth table of ${\sim}P \vee ({\sim}Q \wedge R)$ is given in Table 1.10.

Table 1.10 The Truth Table of ${\sim}P \vee ({\sim}Q \wedge R)$

P	Q	R	${\sim}P$	${\sim}Q$	$({\sim}Q \wedge R)$	${\sim}P \vee ({\sim}Q \wedge R)$
T	T	T	F	F	F	F
T	T	F	F	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

....

Now consider the following proposition: $P \vee {\sim}Q \rightarrow P \wedge R$

Trying to construct the truth table of this proposition is quite confusing. Which of the following should be assumed?

$$(P \vee {\sim}Q) \rightarrow (P \wedge R) \text{ or } P \vee ({\sim}Q \rightarrow P) \wedge R$$

Which part of the proposition is calculated first? If a proposition has parentheses to describe its different parts, then it is easy to construct the truth table and we can say that the proposition is well defined. However, if it is not the case, the proposition is not well defined; for such cases, we should have a particular order of precedence of these operators. In Sections 1.4.9 and 1.4.10, we define well-defined propositions and the rules of precedence of logical operators.

1.4.9 Well-formed Formula

A statement that cannot be broken down into smaller statements is called an atomic statement. For example, 'P: It is raining today' is an atomic statement and P is the variable of the statement. A statement formula is said to be a well-formed formula (wff) if it has following properties:

1. Every atomic statement is a wff.
2. If P is wff, then ${\sim}P$ is also wff.
3. If P and Q are wff, then $(P \wedge Q)$, $(P \vee Q)$, and $(P \rightarrow Q)$ are wff.
4. Nothing else is wff.

For example, $((P \wedge Q) \vee R)$ is a wff, whereas $P \vee Q \wedge R$ is not a wff.

1.4.10 Rules of Precedence

If a given formula is not a wff, then we can convert it into a wff by using the order of precedence of logical operators, which is as follows:

1. $\sim$
2. $\wedge$
3. $\vee$, $\oplus$
4. $\rightarrow$
5. $\leftrightarrow$

For example, the formula ${\sim}P \wedge Q \rightarrow R \vee Q$ can be converted to a wff using the rules of precedence as $((({\sim}P) \wedge Q) \rightarrow (R \vee Q))$.

Check Your Progress 1.1

State whether the following statements are true or false:

1. If P and Q both are false, then their conjunction is also false.
2. If P and Q both are false, then their disjunction is also false.
3. If conjunction of P and Q is true, then P may not be true.
4. If disjunction of P and Q is true, then Q may be false.
5. If negation of negation of P is false, then P is true.
6. If P is necessary for Q, then $Q \rightarrow P$.
7. If P is sufficient for Q, then $P \rightarrow Q$.

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8. If P is false and Q is true, then $P \leftrightarrow Q$ is true.
9. If P and Q both are false, then $P \leftrightarrow Q$ is true.
10. If $P \leftrightarrow Q$ is true, then both the variables must have different truth values.

Now, we shall discuss some special types of propositions. The truth values of some propositions remain always true or always false regardless of the truth values of the variables that form the proposition. These propositions are of special interest.

1.5 TAUTOLOGY

Table 1.11 Truth Table of $P \vee \sim P$

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

A statement is said to be a tautology if it is true for every possible combination of truth values of the variables included in it. A tautology is denoted by T.

For example, the compound proposition $P \vee \sim P$ is a tautology. Truth table of $P \vee \sim P$ is given in Table 1.11.

1.6 CONTRADICTION

Table 1.12 Truth Table of $P \wedge \sim P$

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

A statement is said to be a contradiction if it is false for every possible combination of truth values of the variables included in it. A contradiction is denoted by F.

For example, $P \wedge \sim P$ is a contradiction. The truth table of $P \wedge \sim P$ is given in Table 1.12.

1.7 LOGICAL EQUIVALENCE

Sometimes, two propositions might look different; however, if we find the truth values of both the propositions for every possible combination of the truth values of the variables that form the propositions, then the truth values may be identical. In this case, the form of the two propositions may be different but they are logically equivalent.

Two propositions are said to be logically equivalent or simply equivalent if both have the same truth values for every possible combination of truth values of the variables included in them. We shall use the notation ' ≡ ' to denote the equivalence of two propositions.

A simple way to check whether any two propositions are equivalent is to construct the truth table of the two propositions for all combinations of the truth values of the variables included in them. Compare the truth values of both the propositions. If the truth values are the same, then the two propositions are equivalent.

Example 1.10

Show that $P \rightarrow Q \equiv \sim P \vee Q$.

Solution: We shall construct the truth tables of $P \rightarrow Q$ and $\sim P \vee Q$.

From Table 1.13, it can be concluded that the truth values of $P \rightarrow Q$ and $\sim P \vee Q$ are the same. Hence $P \rightarrow Q \equiv \sim P \vee Q$.

Table 1.13 Truth Table $P \rightarrow Q$ and $\sim P \vee Q$

P	Q	$P \rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

It is also possible that two propositions have different variables but still the two propositions are equivalent. Let us consider the proposition $(P \vee \sim P) \wedge Q$. The truth value of the proposition is independent of the truth values of the variable P as $(P \vee \sim P)$ is a tautology. Thus, the proposition $(P \vee \sim P) \wedge Q$ is equivalent to the proposition Q.

The following are some logical equivalence expressions:

1. $P \vee P \equiv P$, $P \wedge P \equiv P$ (idempotent laws)
2. $P \wedge Q \equiv Q \wedge P$, $P \vee Q \equiv Q \vee P$ (commutative laws)
3. $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$, $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ (associative laws)
4. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$, $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (distributive laws)
5. $P \vee T \equiv T$, $P \wedge F \equiv F$ (domination laws)
6. $P \wedge T \equiv P$, $P \vee F \equiv P$ (identity laws)
7. $P \vee \sim P \equiv T$, $P \wedge \sim P \equiv F$ (negation laws)
8. $P \wedge (P \vee Q) \equiv P$, $P \vee (P \wedge Q) \equiv P$ (absorption laws)
9. $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$, $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ (De Morgan's laws)
10. $\sim(\sim P) \equiv P$ (double negation law)

Some other logical equivalence formulae involving conditionals and biconditionals are as follows:

1. $P \rightarrow Q \equiv \sim P \vee Q$
2. $P \rightarrow Q \equiv \sim Q \rightarrow \sim P$
3. $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$
4. $(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$
5. $(P \rightarrow Q) \vee (P \rightarrow R) \equiv P \rightarrow (Q \vee R)$
6. $(P \rightarrow R) \vee (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$
7. $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
8. $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$

All these logical equivalences can be proved using truth tables. For example, Table 1.14 proves De Morgan's law:

The truth values of $\sim(P \vee Q)$ and $\sim P \wedge \sim Q$ are the same; therefore, $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$.

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Table 1.14 Showing Logical Equivalence of $\sim(P \vee Q)$ and $\sim P \wedge \sim Q$

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$\sim P \wedge \sim Q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

All these logical equivalences can be used directly whenever necessary, and with the help of these equivalences, equivalence of two propositions can be proved without constructing the truth table.

Examples showing logical equivalence without constructing truth tables

Example 1.11

Without constructing the truth table, show that $\sim(P \to Q)$ and $P \wedge \sim Q$ are logically equivalent.

Solution: $\sim(P \to Q) \equiv \sim(\sim P \vee Q)$ (since $P \to Q \equiv \sim P \vee Q$)
$\equiv \sim(\sim P) \wedge \sim Q$ (using De Morgan's law)
$\equiv P \wedge \sim Q$ (using Double negation law)

Example 1.12

Without constructing the truth table, show that

$$P \to (Q \to R) \equiv (P \wedge Q) \to R.$$

Solution: $P \to (Q \to R) \equiv (P \to (\sim Q \vee R)$ (since $P \to Q \equiv \sim P \vee Q$)
$\equiv \sim P \vee (\sim Q \vee R)$ (since $P \to Q \equiv \sim P \vee Q$)
$\equiv (\sim P \vee \sim Q) \vee R$ (using associative law)
$\equiv \sim(P \wedge Q) \vee R$ (using De Morgan's law)
$\equiv (P \wedge Q) \to R$ (since $P \to Q \equiv \sim P \vee Q$)

Example 1.13

Without constructing the truth table, show that $(\sim P \wedge (P \vee Q)) \to Q$ is a tautology.

Solution: $(\sim P \wedge (P \vee Q)) \to Q \equiv \sim(\sim P \wedge (P \vee Q)) \vee Q$ (since $P \to Q \equiv \sim P \vee Q$)
$\equiv (P \vee \sim(P \vee Q)) \vee Q$ (using De Morgan's law)
$\equiv (P \vee (\sim P \wedge \sim Q)) \vee Q$ (using De Morgan's law)
$\equiv ((P \vee \sim P) \wedge (P \vee \sim Q)) \vee Q$ (using distributive law)
$\equiv (T \wedge (P \vee \sim Q) \vee Q$ (since $P \vee \sim P \equiv T$))
$\equiv (P \vee \sim Q) \vee Q$ (since $T \wedge P \equiv P$)
$\equiv P \vee (\sim Q \vee Q)$ (using associative law)
$\equiv P \vee T$ (since $Q) \vee \sim Q \equiv T$)
$\equiv T$ (since $P \vee T \equiv T$)

Hence $(\sim P \wedge (P \vee Q)) \to Q$ is a tautology.

In the truth table of $P \to Q$, we have observed that $P \to Q$ is false in only one case—where P is true but Q is false. If suppose for some propositions P and Q, $P \to Q$ is true. This indicates that if P is false, then there is no restriction for Q to be true or false, but whenever P is true Q is bound to be true. This provides another interesting logical form explained in Section 1.8.

1.8 TAUTOLOGICAL IMPLICATION

We say that a statement P tautologically implies a statement Q if and only if $P \to Q$ is a tautology. We shall denote it by $P \Rightarrow Q$ (read as P tautologically implies Q). In other words, $P \Rightarrow Q$ means Q will have truth value *true* whenever P is *true*. The following are some implications summarized:

1. $P \wedge Q \Rightarrow P$
2. $P \wedge Q \Rightarrow Q$
3. $P \Rightarrow P \vee Q$
4. $Q \Rightarrow P \vee Q$
5. $\sim P \Rightarrow P \to Q$
6. $Q \Rightarrow P \to Q$
7. $P \wedge (P \to Q) \Rightarrow Q$ (modus ponens)
8. $\sim Q \wedge (P \to Q) \Rightarrow \sim P$ (modus tollens)
9. $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$ (hypothetical syllogism)
10. $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$ (dilemma)

Example showing implication

Example 1.14

Explain the logical reasoning that shows the implications $P \wedge Q \Rightarrow P$ and $P \wedge Q \Rightarrow Q$.

Solution: Let us consider the proposition $P \wedge Q$. The conjunction of two variables will be true if and only if both the variables are true. Therefore, whenever $P \wedge Q$ is true, it means P and Q are also true. Therefore, it can be concluded that $P \wedge Q$ tautologically implies P as well as Q.

Examples showing implication using truth table

Example 1.15

Show that $(P \wedge Q) \Rightarrow (P \to Q)$.

Solution: We shall construct the truth table of $(P \wedge Q)$ and $(P \to Q)$ (Table 1.15).

Table 1.15 Truth Table of $(P \wedge Q)$ and $(P \to Q)$

P	Q	$P \wedge Q$	$P \to Q$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

From the table it can be observed that whenever $P \wedge Q$ is true, $P \to Q$ is also true.

Therefore, $(P \wedge Q) \Rightarrow (P \to Q)$.

Alternatively, $(P \wedge Q) \Rightarrow Q$ (using $P \wedge Q \Rightarrow P$)
$\Rightarrow \sim P \vee Q$ (using $Q \Rightarrow P \vee Q$)
$\Rightarrow P \to Q$ (using $P \to Q \equiv \sim P \vee Q$)

A logical implication can be proved without constructing the truth table. Remember that an equivalence is always an implication; thus, to prove an implication, we can use the equivalent

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form of a proposition. Furthermore, the summary of implications discussed earlier can also be used for such purpose.

Example showing implication without constructing truth table

EXAMPLE 1.16

Show the implication $P \to Q \Rightarrow P \to (P \wedge Q)$ without constructing the truth table.

Solution:
$$\begin{aligned} P \to Q &\Rightarrow \sim P \vee Q && (\text{using } P \to Q \equiv \sim P \vee Q)\\ &\Rightarrow T \wedge (\sim P \vee Q) && (\text{using } P \equiv P \wedge T)\\ &\Rightarrow (\sim P \vee P) \wedge (\sim P \vee Q) && (\text{using } T \equiv P \vee \sim P)\\ &\Rightarrow \sim P \vee (P \wedge Q) && (\text{using distributive law})\\ &\Rightarrow P \to (P \wedge Q) && (\text{using } P \to Q \equiv \sim P \vee Q) \end{aligned}$$

1.9 CONVERSE, INVERSE, AND CONTRAPOSITIVE

For any statement formula $P \to Q$, the following are the converse, inverse, and contrapositive:

Converse: $Q \to P$
Inverse: $\sim P \to \sim Q$
Contrapositive: $\sim Q \to \sim P$

EXAMPLE 1.17

Find the inverse, converse, and contrapositive of the following statement:
If I go to market, then I buy a pen.

Solution: Here P: I go to market and Q: I buy a pen.
Converse: If I buy a pen, then I go to market.
Inverse: If I do not go to market, then I do not buy a pen.
Contrapositive: If I do not buy a pen, then I do not go to market.

1.10 FUNCTIONALLY COMPLETE SET OF CONNECTIVES

So far, we studied the different types of connectives. All these connectives are not necessary to express a statement formula. For example, we can express the conditional in an equivalent formula consisting only negation and disjunction. Thus, we can find a proper subset of these connectives such that these connectives are sufficient to express any formula in an equivalent form.

A set of connectives is called functionally complete if every compound proposition can be expressed as a logically equivalent proposition involving only these connectives.

EXAMPLE 1.18

Show that the set $\{\sim, \wedge, \vee\}$ is a functionally complete set of connectives.

Solution: We know that
$$P \to Q \equiv \sim P \vee Q$$
and
$$\begin{aligned} P \leftrightarrow Q &\equiv (P \to Q) \wedge (Q \to P)\\ &\equiv (\sim P \vee Q) \wedge (\sim Q \vee P) \end{aligned}$$

Thus, for every formula that contains the connectives other than the connectives given in the set, we can find an equivalent formula involving the connectives of the set. Furthermore, we can also use the fact that every formula can be written in disjunction normal form that involves only these connectives. Thus, the set $\{\sim, \wedge, \vee\}$ is functionally complete.

EXAMPLE 1.19

Show that the sets $\{\sim, \wedge\}$ and $\{\sim, \vee\}$ are functionally complete.

Solution: In Example 1.18, we have shown that the set $\{\sim, \wedge, \vee\}$ is a functionally complete set. Now, using De Morgan's law, we have
$$P \vee Q \equiv \sim(\sim p \wedge \sim Q)$$
$$P \wedge Q \equiv \sim(\sim P \vee \sim Q)$$

This shows that any statement formula containing the connectives$\{\sim, \wedge, \vee\}$ can be expressed in an equivalent formula using only the connectives of either the set $\{\sim, \wedge\}$ or the set$\{\sim, \vee\}$. Thus, the sets $\{\sim, \wedge\}$ and $\{\sim, \vee\}$ are functionally complete.

Check your progress 1.2

State whether the following statements are true or false:

1. Conjunction of a tautology and a contradiction is always a tautology.
2. $(P \wedge Q) \wedge \sim P$ is a contradiction.
3. $P \to \sim P$ is a tautology.
4. Disjunction of tautology and contradiction is a tautology.
5. $P \to Q$ and $Q \to \sim P$ are logically equivalent.
6. $(\sim P \wedge P) \vee Q$ is equivalent to Q.
7. $(\sim P \vee P) \wedge Q$ is equivalent to Q.
8. $(P \to Q)$ tautologically implies P.
9. $\sim(P \to Q)$ tautologically implies $\sim Q$.
10. P tautologically implies $P \to Q$.

1.11 NORMAL FORMS

We use the truth table of a proposition to check the proposition for tautology or contradiction, but it is not always possible to construct the truth table for practical purposes, especially when the number of variables is large. We, therefore, consider other procedures known as normal forms. In our present discussion, we shall use the term *product* in place of conjunction and *sum* in place of disjunction.

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1.11.1 Elementary Product

A product of the variables and their negations in a formula is called an elementary product. Let P and Q be any two atomic variables. Then P, $\sim P$, $\sim P \wedge Q$, and $\sim P \wedge Q \wedge \sim Q$ are some examples of elementary product.

We know that for any variable P, $P \wedge \sim P$ is a contradiction. Hence, if $p \wedge \sim p$ appears in the elementary product, then the product is identically false. Thus, it is easy and straightforward to prove the statement 'a necessary and sufficient condition for an elementary product to be identically false is that it contains at least one pair of factors in which one is the negation of the other'.

1.11.2 Elementary Sum

A sum of the variables and their negations is called an elementary sum. Let P and Q be any two atomic variables. Then P, $\sim P$, $\sim P \vee Q$, and $\sim P \vee Q \vee \sim P$ are some examples of elementary sum.

We know that for any variable P, $P \vee \sim P$ is a tautology. Hence, if $P \vee \sim P$ appears in the elementary sum, then the sum is identically true. Thus, it is easy and straightforward to prove the statement 'a necessary and sufficient condition for an elementary sum to be identically true is that it contains at least one pair of factors in which one is the negation of the other'.

1.11.3 Disjunctive Normal Form

A formula that is equivalent to a given formula and consists of a sum of elementary products is called a disjunctive normal form (DNF) of the given formula. To reduce a formula into DNF, we replace condition and biconditional by $\wedge$, $\vee$, $\sim$, and apply distributive law and De Morgan's law as per requirement. The following are some example of DNF:

1. $P \vee Q$
2. $(P \wedge Q) \vee (\sim P \wedge Q)$
3. $(P \wedge Q \wedge R) \vee (P \wedge Q) \vee R$
4. $P \vee (P \wedge \sim Q)$
5. $(P \wedge \sim Q) \vee (\sim P \wedge Q) \vee \sim P$
6. $(P \wedge \sim P) \vee (P \wedge Q)$

The following examples show the reduction of a given statement formula into DNF with the help of logical equivalences:

Example 1.20

Obtain the DNF of $(P \wedge Q) \vee \sim(P \rightarrow Q)$.

Solution: $(P \wedge Q) \vee \sim(P \rightarrow Q) \equiv (P \wedge Q) \vee \sim(\sim P \vee Q)$ (since $P \rightarrow Q \equiv \sim P \vee Q$)
$\equiv (P \wedge Q) \vee (P \wedge \sim Q)$

Example 1.21

Obtain the DNF of $(\sim P \wedge Q) \wedge (P \rightarrow Q)$.

Solution: $(\sim P \wedge Q) \wedge (P \rightarrow Q) \equiv (\sim P \wedge Q) \wedge (\sim P \vee Q)$

$\equiv ((\sim P \wedge Q) \wedge \sim P) \vee ((\sim P \wedge Q) \wedge Q)$ (using distributive law)
$\equiv (\sim P \wedge \sim P \wedge Q) \vee (\sim P \wedge Q \wedge Q)$
$\equiv (\sim P \wedge Q) \vee (\sim P \wedge Q)$ (using idempotent law)

This is the required DNF.

For a given formula, more than one DNF is possible; thus, the DNF of a formula is not unique.

1.11.4 Conjunctive Normal Form

A formula that is equivalent to a given formula and consists of a product of elementary sums is called a conjunctive normal form (CNF) of the given formula. To reduce a formula into CNF, the same procedure can be applied as given in DNF. The following are some examples of DNF:

1. $P \wedge Q$
2. $(P \vee Q) \wedge (\sim P \vee Q)$
3. $P \wedge (P \vee \sim Q)$
4. $(P \vee \sim Q) \wedge (\sim P \vee Q) \wedge \sim Q$
5. $(P \vee Q \vee R) \wedge (P \vee \sim R)$
6. $(P \vee \sim P) \wedge (P \vee Q)$

Examples showing the reduction of a given statement formula into CNF with the help of logical equivalences

Example 1.22

Obtain the CNF of $P \rightarrow (P \wedge (Q \rightarrow P))$.

Solution: $P \rightarrow (P \wedge (Q \rightarrow P)) \equiv \sim P \vee (P \wedge (\sim Q \vee P))$ (since $P \rightarrow Q \equiv \sim P \vee Q$)
$\equiv (\sim P \vee P) \wedge (\sim P \vee (\sim Q \vee P))$ (using distributive law)
$\equiv (\sim P \vee P) \wedge (\sim P \vee \sim Q \vee P)$

Example 1.23

Obtain the CNF of $(P \wedge (P \rightarrow Q)) \rightarrow Q$.

Solution: $(P \wedge (P \rightarrow Q)) \rightarrow Q \equiv \sim(P \wedge (\sim P \vee Q)) \vee Q$ (since $P \rightarrow Q \equiv \sim P \vee Q$)
$\equiv \sim P \vee \sim(\sim P \vee Q) \vee Q$ (using De Morgan's law)
$\equiv \sim P \vee (P \wedge \sim Q) \vee Q$

This is the required CNF.

Again, for a given formula, more than one CNF is possible; thus, the CNF of a formula is not unique.

As already discussed, the two normal forms, DNF and CNF, of a given formula are not unique. Hence, we will introduce another two forms in order to get a unique normal form of a given formula.

1.11.5 Principal Disjunctive Normal Form

Let P and Q be two statement variables. If we construct all possible formulae that consist of conjunctions of P or $\sim P$ with Q or $\sim Q$ excluding the forms where

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a variable and its negation both appear and any form equivalent to previously obtained form, we get the following forms:

$$P \wedge Q, \sim P \wedge Q, P \wedge \sim Q, \text{ and } \sim P \wedge \sim Q$$

These forms are called *minterms* for the two variables P and Q. It can be observed that all minterms are different. If there are n variables in a statement formula, then there will be 2^n minterms. For three variables P, Q, and R, the minterms are $P \wedge Q \wedge \sim R$, $P \wedge Q \wedge R$, $P \wedge \sim Q \wedge R$, $\sim P \wedge Q \wedge \sim R$, $\sim P \wedge Q \wedge R$, $P \wedge \sim Q \wedge \sim R$, $\sim P \wedge \sim Q \wedge R$, and $\sim P \wedge \sim Q \wedge \sim R$.

For a given formula, an equivalent formula consisting of disjunctions of minterms alone is known as its principal disjunctive normal form (PDNF).

EXAMPLE 1.24

Write the PDNF of $P \vee (P \wedge Q)$.

Solution: $P \vee (P \wedge Q) \equiv (P \wedge T) \vee (P \wedge Q)$ (since $P \equiv P \wedge T$)
$\equiv (P \wedge (Q \vee \sim Q)) \vee (P \wedge Q)$ (since $P \vee \sim P \equiv T$)
$\equiv ((P \wedge Q) \vee (P \wedge \sim Q)) \vee (P \wedge Q)$ (using distributive law)
$\equiv (P \wedge Q) \vee (P \wedge \sim Q)$

Construction of Principal Disjunctive Normal Form Using Truth Table

We can also obtain the PDNF of a given formula using the truth table as follows. For every truth value T of the given formula in the truth table, write the minterm corresponding to the truth values of the variables included in it. Minterm consists of the variable itself if its truth value is *true* and negation of the variable if its truth value is *false*. The disjunction of these minterms is the PDNF of the given formula. The PDNF of $P \vee (P \wedge Q)$ can be obtained as shown in Table 1.16:

Table 1.16 PDNF of $P \vee (P \wedge Q)$

P	Q	$P \wedge Q$	$P \vee (P \wedge Q)$	Minterm
T	T	T	T	$P \wedge Q$
T	F	F	T	$P \wedge \sim Q$
F	T	F	F	
F	F	F	F	

From the truth table, it can be observed that only two truth values are true for the given formula. Hence, the PDNF is $(P \wedge Q) \vee (P \wedge \sim Q)$.

Similarly, PDNF of $P \to Q$ can be obtained as shown in Table 1.17:

Table 1.17 PDNF of $P \to Q$

P	Q	$P \to Q$	Minterm
T	T	T	$P \wedge Q$
T	F	F	
F	T	T	$\sim P \wedge Q$
F	F	T	$\sim P \wedge \sim Q$

The PDNF of $P \to Q$ is $(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$.

1.11.6 Principal Conjunctive Normal Form

Let P and Q be two statement variables. If we construct all possible formulae that consist of disjunctions of P or $\sim P$ with Q or $\sim Q$ excluding the forms where a variable and its negation both appear in any form equivalent to previously obtained form, we get the following forms:

$$P \vee Q, \sim P \vee Q, P \vee \sim Q, \text{ and } \sim P \vee \sim Q$$

These forms are called *maxterms*. For three variables P, Q, and R, the maxterms are $P \vee Q \vee \sim R$, $P \vee Q \vee R$, $P \vee \sim Q \vee R$, $\sim P \vee Q \vee \sim R$, $\sim P \vee Q \vee R$, $P \vee \sim Q \vee \sim R$, $\sim P \vee \sim Q \vee R$, and $\sim P \vee \sim Q \vee \sim R$.

For a given formula, an equivalent formula consisting of conjunction of maxterms alone is known as its principal conjunctive normal form (PCNF).

EXAMPLE 1.25

Write the PCNF of $P \wedge (P \vee Q)$.

Solution: $P \wedge (P \vee Q) \equiv (P \vee F) \wedge (P \vee Q)$ (since $P \vee F \equiv P$)
$\equiv (P \vee (Q \wedge \sim Q)) \wedge (P \vee Q)$ (since $Q \wedge \sim Q \equiv F$)
$\equiv ((P \vee Q) \wedge (P \vee \sim Q)) \wedge (P \vee Q)$ (using distributive law)
$\equiv ((P \vee Q) \wedge (P \vee Q)) \wedge (P \vee \sim Q)$ (using associative law)
$\equiv (P \vee Q) \wedge (P \vee \sim Q)$ (since $P \wedge P \equiv P$)

Construction of Principal Conjunctive Normal Form Using Truth Table

The PDNF of a given formula using the truth table can be obtained as follows. For every truth value F of the given formula in the truth table, write the maxterm corresponding to the truth values of the variables included in it. Maxterm consists of the variable itself if its truth value is *false* and negation of the variable if its truth value is *true*. The conjunction of these maxterms will be the PCNF of the given formula, an equivalent form of the given formula. The PCNF of $P \wedge (P \vee Q)$ can be obtained as given in Table 1.18.

Table 1.18 PCNF of $P \wedge (P \vee Q)$

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$	Maxterm
T	T	T	T	
T	F	T	T	
F	T	T	F	$P \vee \sim Q$
F	F	F	F	$P \vee Q$

Hence, the PCNF is $(P \vee Q) \wedge (P \vee \sim Q)$.

Let us consider the following statements:

I attend classes regularly.
I get a good score in my class.
I get a good job.

These three statements form a certain sequence. The first two statements form a ground for the last statement. This is an argument and useful to draw valid conclusions from a set of statements. In Section 1.12, we discuss an argument and its validity.

1.12 ARGUMENT

An argument is a sequence of statements called premises followed by a conclusion. Consider a set of premises $(H_1, H_2, \ldots, H_n)$ and another statement C, the conclusion. We say that the conclusion C follows logically from the

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set of premises $(H_1, H_2, \ldots, H_n)$ iff $H_1 \wedge H_2 \wedge \ldots \wedge H_n \Rightarrow C$ or $(H_1 \wedge H_2 \wedge \ldots \wedge H_n) \rightarrow C$ is a tautology; in other words, the argument is called a valid argument.

1.12.1 Checking the Validity of an Argument by Constructing Truth Table

For checking the validity of the argument, we can construct the truth table of $(H_1 \wedge H_2 \wedge \ldots \wedge H_n) \rightarrow C$ and verify whether it is a tautology or not. Another way is to make the truth table of all premises and the conclusion for all possible combinations of the truth values of the variables included in them. If for each row in which all premises have truth value T, the conclusion also has the truth value T, then the argument is a valid argument.

Examples showing checking validity of arguments

Example 1.26 ....

Determine whether the conclusion C follows logically from the premises H_1 and H_2.

$$H_1: P \rightarrow Q$$
$$H_2: P$$
$$\overline{}$$
$$C: Q$$

Solution: We shall construct the truth table of $(H_1 \wedge H_2) \rightarrow C$, that is, $((P \rightarrow Q) \wedge P) \rightarrow Q$ (Table 1.19).

Table 1.19 Truth Table of $((P \rightarrow Q) \wedge P) \rightarrow Q$

P	Q	$P \rightarrow Q$	$(P \rightarrow Q) \wedge P$	$((P \rightarrow Q) \wedge P) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

From the truth table, we observe that $((P \rightarrow Q) \wedge P) \rightarrow Q$ is a tautology. Hence, the argument is a valid argument.

Alternative way Another way to check the validity of the argument is to construct the truth table of all the premises and the conclusion. If all the premises and the conclusions are true, the argument is a valid argument. This is shown in Table 1.20.

From Table 1.20, it can be observed in the first row itself that both $H_1: P \rightarrow Q$ and $H_2: P$ are true and $C: Q$ is also true. Therefore, the argument is a valid argument.

Table 1.20 Truth Table of H_1, H_2, and C

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.27

Check the validity of the following argument:

If I take breakfast, then I go to school. I do not take breakfast. Therefore, I do not go to school.

Solution: Let P: I take breakfast.
Q: I go to school.

Then $H_1: P \rightarrow Q$
$$H_2: \sim P$$
$$\overline{}$$
$$C: \sim Q$$

We shall construct the truth table of $(H_1 \wedge H_2) \rightarrow C$, that is, $((P \rightarrow Q) \wedge \sim P) \rightarrow \sim Q$. From Table 1.21, it can be observed that $(H_1 \wedge H_2) \rightarrow C$ is not a tautology, and thus, the argument is not a valid argument.

Table 1.21 Truth Table of $(H_1 \wedge H_2) \rightarrow C$

P	Q	$\sim P$	$\sim Q$	$P \rightarrow Q$	$(P \rightarrow Q) \wedge \sim P$	$((P \rightarrow Q) \wedge \sim P) \rightarrow \sim Q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Alternative way We shall construct the truth table of premises and the conclusion (Table 1.22).

Table 1.22 Truth Table of H_1, H_2, and C

P	Q	$\sim P$	$\sim Q$	$P \rightarrow Q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

From Table 1.22, it can be observed that whenever $P \rightarrow Q$ and $\sim P$ are true, $\sim Q$ is not true. Thus, the argument is not a valid argument.

Example 1.28 ....

Check the validity of the following argument.

If I go to school, then I attend all classes. If I attend all classes, then I get A grade. I do not get grade A and I do not feel happy. Therefore, if I do not go to school then, I do not feel happy.

Solution: Let P: I go to school, Q: I attend all classes, R: I get grade A, and S: I feel happy. Thus, $H_1: P \rightarrow Q$, $H_2: Q \rightarrow R$, $H_3: \sim R \wedge \sim S$, and $C: \sim P \rightarrow \sim S$. The truth values of all premises and the conclusion are shown in Table 1.23.

Table 1.23 Truth table of H_1, H_2, H_3 and C

P	Q	R	S	$\sim P$	$\sim R$	$\sim S$	$P \rightarrow Q$	$Q \rightarrow R$	$\sim R \wedge \sim S$	$\sim P \rightarrow \sim S$
T	T	T	T	F	F	F	T	T	F	T
T	T	T	F	F	F	T	T	T	F	T
T	T	F	T	F	T	F	T	F	F	T
T	T	F	F	F	T	T	T	F	T	T
T	F	T	T	F	F	F	F	T	F	T
T	F	T	F	F	F	T	F	T	F	T
T	F	F	T	F	T	F	F	T	F	T
T	F	F	F	F	T	T	F	T	T	T

(Contd)

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Table 1.23 *(Contd)*

P	Q	R	S	~P	~R	~S	P→Q	Q→R	~R∧~S	~P→~S
F	T	T	T	T	F	F	T	T	F	F
F	T	T	F	T	F	T	T	T	F	T
F	T	F	T	T	T	F	T	F	F	F
F	T	F	F	T	T	T	T	F	T	T
F	F	T	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	T	T	F	T
F	F	F	T	T	T	F	T	T	F	F
F	F	F	F	T	T	T	T	T	T	T

We construct the truth table for the given premises and the conclusion. From Table 1.23, it can be observed that whenever the premises are true, conclusion is also true. Therefore, the argument is valid.

1.12.2 Checking the Validity of an Argument Without Constructing Truth Table

In the arguments where the number of variables is too large, we can also prove the validity of the argument by using rules of equivalences and implications and by analysing and combining the inference drawn from each premise.

EXAMPLE 1.29

Show that the following argument is a valid argument.

$H_1: P \to Q$

$H_2: P$

$C: Q$

Solution: $H_1: \sim P \vee Q$ (since $P \to Q \equiv \sim P \vee Q$) (1.1)

$H_1 \wedge H_2: P \wedge (\sim P \vee Q) \equiv (P \wedge \sim P) \vee (P \wedge Q)$ (using distributive law)

$\equiv F \vee (P \wedge Q)$

$\equiv P \wedge Q$ (since $F \vee P = P$) (1.2)

$H_1 \wedge H_2 \Rightarrow Q$ (since $P \wedge Q \Rightarrow Q$)

Thus, $H_1 \wedge H_2 \Rightarrow C$ and the argument is valid.

EXAMPLE 1.30

Show that the following argument is a valid argument.

$H_1: (P \wedge Q)$

$H_2: (P \wedge Q) \to (R \wedge S)$

$H_3: U \wedge V$

$C: R \wedge U$

Solution: $H_1 \wedge H_2: R \wedge S$ (since $P \wedge (P \to Q) \equiv Q$) (1.3)

$H_1 \wedge H_2 \Rightarrow R$ (since $R \wedge S \Rightarrow R$) (1.4)

$H_3 \Rightarrow U$ (since $U \wedge V \Rightarrow U$) (1.5)

Using Eqs (1.4) and (1.5), we get $H_1 \wedge H_2 \wedge H_3 \Rightarrow R \wedge U$.

Thus, $H_1 \wedge H_2 \wedge H_3 \Rightarrow C$ and the argument is valid.

1.13 PREDICATES

Before defining predicates, let us consider the following sentences:

1. Mohan is a student.
2. Shrikant is a student.
3. Shefali is a student.

If we write the propositions for these three sentences, we need three propositions. In the same way if we have a list of hundred students, then it is not appropriate to write hundred propositions because the part 'is a student' of the sentence is common in all these sentences. Hence, it is better to assign a variable (say x) in place of the name of the student and keep the remaining part same, and define a set X of students from where x can take its values.

The sentence can be written as 'x is a student' in which the part 'is a student' is called *predicate*, and the set X is called the *universe of discourse* for x. The complete sentence is called predicate on x. A predicate on x is denoted by the symbols P, Q, R, and so on, with x in braces, that is, $P(x)$, $Q(x)$, $R(x)$, and so on, respectively.

For example, $P(x)$: x is a student.

$Q(x)$: x is an animal.

If we assign a particular value to x, then the predicate is converted into a proposition. For example, consider the predicate

$P(x)$: x is less than five.

The universe of discourse for x is the set of real numbers. Thus, $P(2)$ is a proposition whose truth value is *true*.

A predicate can be defined without defining its universe of discourse. In this case, the variable can take any value from the universal set. A predicate can also be defined over more than one variable. For example, consider the predicate on two variables.

$P(x, y)$: x is greater than y.

If we replace x by 6 and y by 3, then it becomes a proposition '6 is greater than 3' whose truth value is *true*.

1.13.1 Quantifiers

Let us first consider the following sentence:

Rakesh is brilliant and Mohan is brilliant and Alka is brilliant.

If we form a set A of three students, then the sentence can be written as follows:

All the students of the set A are brilliant.

For writing a symbolic form of the sentence, we need a predicate on a variable x like $P(x)$: x is brilliant, and the domain of x (called universe of

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discourse) defined as the set A, and a symbol for the phrase 'for all'. The symbol is called quantifier. Thus, quantifier is a symbol that quantifies the variables. If we use a quantifier before a predicate, then the predicate becomes a proposition.

There are two types of quantifiers: universal quantifier and existential quantifier.

Universal Quantifier

The universal quantifier is used when a statement is true for all values given in the universe of discourse. It is denoted by the symbol $\forall$. The universal quantification of $P(x)$ is the statement

$P(x)$ for all values x in the universe of discourse and is denoted by $\forall x P(x)$. We read $\forall x P(x)$ as 'for all $xP(x)$' or 'for every $xP(x)$'.

Note that $\forall x P(x)$ is true when $P(x)$ is true for every x and is false when there is any x for which $P(x)$ is not true.

Example 1.31 ....

Let $P(x)$: x is even number and the universe of discourse for x is the set $\{1, 2, 3, 4\}$. Find the truth value of $\forall x P(x)$.

Solution: As every number in the set is not an even number, the statement $\forall x P(x)$ is false.

Example 1.32 ....

Let $P(x)$:$x \neq 5$ and the universe of discourse for x is the set $\{1, 2, 3, 4\}$. Find the truth value of $\forall x\, P(x)$.

Solution: As for every number x in the set $x \neq 5$, the statement $\forall x\, P(x)$ is true.

Existential Quantifier

The existential quantifier is used when a statement is true for some values given in the universe of discourse. It is denoted by the symbol $\exists$. The existential quantification of $P(x)$ is the statement

There exists some x in the universe of discourse such that $P(x)$ and it is denoted by the symbol $\exists x P(x)$.

Note that $\exists x P(x)$ is true when $P(x)$ is true for at least one value of x in the universe of discourse and is false when $P(x)$ is false for every x in the universe of discourse.

Example 1.33 ....

Let $P(x)$: x is even number and the universe of discourse for x is the set $\{1, 2, 3, 4\}$. Find the truth value of $\exists x P(x)$.

Solution: As some numbers in the set are even numbers, the statement $\exists x P(x)$ is true.

Example 1.34 ....

Let $P(x)$: $x > 5$ and the universe of discourse for x is the set $\{1, 2, 3, 4\}$. Find the truth value of $\exists x P(x)$.

Solution: As none of the number in the set is greater than 5, the statement $\exists x P(x)$ is false.

1.13.2 Free and Bound Variables

A variable in a predicate is said to be bound if a quantifier is used before it, or in other words, a variable is bounded if it is bounded by a quantifier. A variable is free if it is not bounded. The variable x is a bound variable in both $\forall x P(x, y)$ and $\exists x P(x, y)$ whereas y is a free variable. The scope of a quantifier is the formula immediately following the quantifier. $P(x, y)$ is the scope of the quantifier in both the cases.

Examples showing symbolic form of English sentences using predicates and quantifiers

Example 1.35 ....

Write the symbolic form of the following sentences.

(a) All students are clever. (b) Some students are clever.

Solution: Let $P(x)$: x is clever.

Let the universe of discourse for x is a set of students. The first sentence is true for all the students, which indicates the use of the symbol $\forall$ before the variable x and the second sentence is true for only some of the students; thus, the symbol $\exists$ shall be used before the variable x.

The symbolic forms of the first and second sentences are as follows:

(a) $\forall x P(x)$ (b) $\exists x P(x)$

We can also write the symbolic forms of the sentences without using the universe of discourse.

Let $P(x)$:x is a student.

$Q(x)$:x is clever.

The first sentence can be written as follows:

For every x, if x is a student, then x is clever.

Its symbolic form is as follows:

$\forall x(P(x) \to Q(x))$

The second sentence can be written as follows:

There exists x such that x is a student who is clever.

Its equivalent form is as follows:

There exists x such that x is a student and x is clever.

Its symbolic form is

$\exists x(P(x) \wedge Q(x))$

Note that here we cannot write $\exists x(P(x) \to Q(x))$ because if there is no such student, that is, the statement is false, the expression $\exists x(P(x) \to Q(x))$ is true, as for a particular a the proposition will be $P(a) \to Q(a)$. This is true in both the cases when $P(a)$ is false but $Q(a)$ is true and when $P(a)$ is false and $Q(a)$ is also false.

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Example 1.36 ....

Symbolize the sentence 'every integer is either positive or negative'.

Solution: Here we discuss the different ways of symbolizing the sentence.

(a) Let $x \in Z$ and $P(x)$:x be either positive or negative.
The symbolic form is $\forall x P(x)$.

(b) Let $x \in Z$,
$P(x)$: x is positive, and
$Q(x)$: x is negative
Then the symbolic form is $\forall x(P(x) \vee Q(x))$.

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(c) Let $P(x)$: x be an integer.
$Q(x)$: x is either positive or negative.
Then the symbolic form is $\forall x(P(x) \rightarrow Q(x))$.

(d) Let $R(x)$: x be an integer.
$P(x)$: x is positive
$Q(x)$: x is negative.
Then the symbolic form is $\forall x[R(x) \rightarrow (P(x) \vee Q(x))]$.

EXAMPLE 1.37 ....

Symbolize the following statements:
(a) Some real numbers are integers.
(b) All integers are real numbers.
(c) For every positive integer, there is a positive integer greater than it.
(d) Some tigers are white.

Solution:

(a) Let $P(x)$: x be a real number.
$Q(x)$: x is an integer.
The symbolic form of the statement is $\exists x(P(x) \wedge Q(x))$.

(b) Let $P(x)$: x be a real number.
$Q(x)$: x is an integer.
The symbolic form of the statement is $\forall x(Q(x) \rightarrow P(x))$.

(c) Let $P(x)$: x be a positive integer.
$Q(x, y)$: x is greater than y.
The statement can be written as 'for any x, if x is a positive integer, then there exists y such that y is a positive integer and y is greater than x'. Thus, the symbolic form of the statement is $\forall x[P(x) \rightarrow \exists y(P(y) \wedge Q(y, x))]$.
The given sentence can also be symbolized as follows:

Let the universe of discourse for x and y be the set of positive integers and $Q(x, y)$: x be greater than y.
Then the symbolic form is $\forall x(\exists y)\, Q(y, x)$.

A sentence can be symbolized in various ways. It depends on the way of defining predicates and universe of discourse. In statement (c), the first one is without using universe of discourse and the second one is using universe of discourse.

(d) Let $P(x)$: x be a tiger.
$Q(x)$: x is white.
The symbolic form of the statement is $\exists x(P(x) \wedge Q(x))$.

1.13.3 Negation of Quantifiers

We often use the negation of quantified expressions. For example, consider the following statement:

Every politician is clever.

To express this statement in symbolic form, we can write $P(x)$: x is clever and the universe of discourse is the set of politicians. Then the statement can be symbolized as $\forall xP(x)$.

The negation of this statement is as follows:
It is not the case that every politician is clever.

Alternatively, an equivalent form is as follows:
There is a politician who is not clever.
The symbolic form of this statement is $\exists x \sim P(x)$.
This example shows that the negation of $\forall xP(x)$ is $\exists x \sim P(x)$. Similarly, we can find the negation of $\exists xP(x)$. Thus, we have the following equivalences:

$$\sim\forall xP(x) \equiv \exists x \sim P(x)$$

$$\sim\exists xP(x) \equiv \forall x \sim P(x)$$

EXAMPLE 1.38 ....

Write the negation of the following statements:
(a) All states in India are highly populated.
(b) Some states in India are highly populated.

Solution:

(a) If we assume $P(x)$: x is highly populated and the universe of discourse for x is the set of the states of India, then the sentence can be symbolized as $\forall xP(x)$ and its negation is $\sim\forall xP(x) \equiv \exists x \sim P(x)$. Thus, the negation of the sentence is 'Some states in India are not highly populated'.
(b) The negation of the sentence is 'All states in India are not highly populated'.

1.13.4 Removing Quantifiers from Predicates

As already mentioned, the use of quantifiers converts a predicate into a proposition, and we can write an equivalent form of a formula by removing quantifiers from it. The quantifier $\forall$ signifies that the predicate is true for all values defined in the universe of discourse and the quantifier $\exists$ signifies that the predicate is true for either first or second or third or … or last value defined in the universe of discourse. Let $P(x)$ be a predicate on x and the universe of discourse of x is the set $\{x_1, x_2, x_3, \ldots, x_n\}$. Then,

$$\forall xP(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n) \tag{1.6}$$

$$\exists xP(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n) \tag{1.7}$$

With the help of Eqs (1.6) and (1.7) and De Morgan's law, we can derive following equivalences:

$$\sim\forall xP(x) \equiv \sim P(x_1) \vee \sim P(x_2) \vee \ldots \vee \sim P(x_n) \tag{1.8}$$

The right-hand side (RHS) of Eq. (1.8) is equivalent to $\exists x \sim P(x)$.
Similarly,

$$\sim\exists xP(x) \equiv \sim P(x_1) \wedge \sim P(x_2) \wedge \ldots \wedge \sim P(x_n) \tag{1.9}$$

The RHS of Eq. (1.9) is equivalent to $\forall x \sim P(x)$.

EXAMPLE 1.39 ....

Let us assume that $P(x)$ and $Q(x)$ are two predicates on x, where $x \in \{1, 2, 3\}$. Remove the quantifiers from the following statement formulae:
(a) $\exists xP(x)$
(b) $\forall xP(x)$
(c) $\exists x(P(x) \wedge Q(x))$
(d) $\exists xP(x) \wedge \forall xQ(x)$

Solution:

(a) $P(1) \vee P(2) \vee P(3)$

(b) $P(1) \wedge P(2) \wedge P(3)$

(c) $(P(1) \wedge Q(1)) \vee (P(2) \wedge Q(2)) \vee (P(3) \wedge Q(3))$

(d) $(P(1) \vee P(2) \vee P(3)) \wedge (Q(1) \wedge Q(2) \wedge Q(3))$

1.14 NESTED QUANTIFIERS

So far, we have studied the universal and existential quantifiers and their implementation in writing statements. We shall now study nested quantifiers. Two quantifiers can be nested if one is within the scope of the other. Let us consider the proposition $\forall x \exists y P(x, y)$; the proposition is the same as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$. To understand the use of nested quantifiers, let us go through some examples.

Example 1.40

Let the universe of discourse for the variables x and y be the set of positive integers and let $P(x, y)$: $x^2 = y$, then translate $\forall x \exists y P(x, y)$ into an English sentence.

Solution: The proposition $\forall x \exists y P(x, y)$ is translated as follows:

For every positive integer x, there exists a positive integer y such that $x^2 = y$.

The truth value of the proposition is true. We can also translate this proposition as 'the square of every positive integer is a positive integer'.

Example 1.41

Let the universe of discourse for the variables x and y is the set of positive integers and let $P(x)$: $x > 0$ and $Q(x, y)$: $xy > 0$. Translate the proposition $\forall x \forall y[P(x) \wedge P(y) \rightarrow Q(x, y)]$ into an English sentence.

Solution: The proposition $\forall x \forall y[(P(x) \wedge P(y)) \rightarrow Q(x, y)]$ is translated as 'For every integer x and for every integer y, if x is positive and y is positive, then the product $x\,y$ is also positive.' The proposition can also be written as 'the product of two positive integers is a positive integer'.

1.14.1 Effect of Order of Quantifiers

Let us now discuss the effect of order of quantifiers in the case of a predicate of more than one variable. Consider the following example.

Example 1.42

Let the universe of discourse for x is the set $A = \{1, 2, 3, 4\}$ and for y is the set $B = \{5, 6, 7, 8\}$ and the predicate $P(x, y)$ is defined as:

$P(x, y)$: x is less than y.

Find the truth values of the propositions $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$.

Solution: $\forall x \forall y P(x, y)$ denotes the following proposition:

For every element x of the set A, for every element y of the set B, x is less than y.

The truth value of the proposition is true.

$\forall y \forall x P(x, y)$ denotes the following proposition:

For every element y of the set B, for every element x of the set A, x is less than y.

The truth value of the proposition is true.

Example 1.43

Let the universe of discourse for x be the set $A = \{1, 2, 3, 4\}$ and for y be the set $B = \{3, 4, 5, 6\}$, and the predicate $P(x, y)$ is defined as:

$P(x, y)$: x is less than y.

Find the truth values of the propositions $\forall x \forall y\, P(x, y)$ and $\forall y \forall x(P(x, y)$.

Solution: By inspection, there are some pairs (x, y), where $x \in A$ & $y \in B$, for which $P(x, y)$ is false. Thus, it is clear that both the propositions $\forall x \forall y(P(x, y)$ and $\forall y \forall x P(x, y)$ have truth value false.

From Examples 1.42 and 1.43, it can be observed that in both the cases the two statements $\forall x \forall y\, P(x, y)$ and $\forall y \forall x P(x, y)$ have the same meaning and both have the same truth values. This illustrates the fact that the order of the nested universal quantifiers in a statement with no other quantifiers can be changed without changing the meaning of the quantified statement. Thus, we have the following equivalence:

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y) \tag{1.10}$$

Example 1.44

Let the universe of discourse for x is the set $A = \{1, 2, 3, 4\}$ and for y is the set $B = \{5, 6, 7, 8\}$ and the predicate $P(x, y)$ is defined as follows:

$P(x, y)$: x is less than y.

Find the truth values of the propositions $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$

Solution: $\exists x \forall y P(x, y)$ denotes the following proposition:

There is an element x of the set A such that for every element y of the set B, x is less than y.

The proposition $\exists x \forall y P(x, y)$ is true if for some $a \in A$, $P(a, y)$ is true for all $y \in B$. If no such $a \in A$ exists, then the statement is false.

Clearly, we can find an element x in A such that for every element y of the set B, $x < y$. For example $x = 1, 2, 3$, or 4. Thus, the truth value of the proposition is true.

$\forall y \exists x P(x, y)$ denotes the following proposition:

For every element y of the set B, there exists an element x in the set A such that x is less than y.

Clearly, for every element y of the set B we can find an element x in A such that $x < y$. Thus, the truth value of the proposition is true.

If we change the universe of discourse, then what will be the results? It is left as an exercise to readers. However, we give another example for checking other possibilities of truth values.

Example 1.45

Let the universe of discourse for x is the set $A = \{2, 3, 4\}$ and for y is the set $B = \{4, 9, 16\}$ and the predicate $P(x, y)$ is defined as follows:

$Q(x, y)$: $x^2 = y$ Find the truth values of the proposition $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$.

Solution: $\exists x \forall y P(x, y)$ denotes the proposition:

There exists an element x of the set A such that for every element y of the set B, $x^2 = y$.

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The truth value of this proposition is false, as there is no element a in the set A such that $a^2 = y$ for each value y in B.

$\forall y \exists x Q(x, y)$ denotes the following proposition:

For every element y of the set B, there exists an element x in the set A such that $x^2 = y$.

It is clear that for each $y \in B$, there exists $x \in A$, such that $x^2 = y$ The truth value of the proposition is true.

From Examples 1.44 and 1.45, it can be observed that the order of the two quantifiers has a significant role in determining the truth value of the proposition. Thus, the two statements, $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$, are not logically equivalent.

Let $\exists x \forall y P(x, y)$ be true, which means there exists $x = a$ such that $P(a, y)$ is true for all y.

This allows us to say that for all y, there exists some x such that $P(x, y)$ is true; that is, $\forall y \exists x P(x, y)$ is true. Thus, we have the following implication:

$$\exists x \forall y P(x, y) \Rightarrow \forall y \exists x P(x, y)$$

Let $\forall y \exists x P(x, y)$ be true. This shows that for each y, there is an element x such that $P(x, y)$. Let $P(x, y)$ be true for the pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. This does not imply that there exists x such that $P(x, y)$ is true for all y. Thus, $\exists x \forall y P(x, y)$ is not necessarily true whenever $\forall y \exists x P(x, y)$ is true.

Similarly, we can show the following implications.

$$\forall x \forall y P(x,y) \Rightarrow \exists y \forall x P(x,y) \tag{1.12}$$

$$\forall x \forall y P(x,y) \Rightarrow \forall x \exists y P(x,y) \tag{1.13}$$

$$\forall x \exists y P(x,y) \Rightarrow \exists y \exists x P(x,y) \tag{1.14}$$

$$\exists x \exists y P(x,y) \Rightarrow \exists y \exists x P(x,y) \tag{1.15}$$

$$\forall x \forall y P(x,y) \Rightarrow \exists x \exists y P(x,y) \tag{1.16}$$

1.15 INFERENCE THEORY OF PREDICATE CALCULUS

Inference theory of predicate calculus uses the rules of inferences for statement calculus. As quantifiers are used in predicate calculus, some additional rules are required that can deal with quantifiers. First, we eliminate the quantifiers from a predicate to get a statement and then derivation takes place as in case of statement calculus and thus, conclusion is reached. Second, if required, the conclusion is quantified to get the required inference in predicate calculus. Elimination of quantifiers from a predicate can be done using the rules of specifications defined in the following subsections.

1.15.1 Universal Specification

Let $P(x)$ be a predicate on x and the set A be the universe of discourse for x. If we assume $P(x)$ is true for all $x \in A$, then it can be concluded that the predicate P is also true for any arbitrary element $y \in A$, that is,

$$\forall x P(x) \Rightarrow P(y)$$

Here y is an arbitrary element; it can take any value of the set.

For example, let $A = \{3, 6, 9\}$ and $P(x)$: x be a multiple of 3.

Then $P(x)$ is true for all elements of A and therefore it can be concluded that

$$\forall x P(x) \Rightarrow P(y)$$

where y can take any value 3, 6 or 9.

1.15.2 Existential Specification

Let $P(x)$ be a predicate on x and the set A be the universe of discourse for x. If we assume $P(x)$ is true for some $x \in A$ and if $y \in A$ is the element for which the predicate is true, then we have $\exists x P(x) \Rightarrow P(y)$.

Here y is not an arbitrary element; it cannot take any value of the set. For example, let $A = \{5, 6, 7\}$ and $P(x)$: x is less than or equal to five.

The predicate is true only for element 5; thus

$$\exists x P(x) \Rightarrow P(y), \quad \text{where } y \text{ can take only one value, that is, 5.}$$

Therefore, whenever we use existential specification (ES) for two different predicates, each time a new variable should be chosen. For example, if $P(x)$ and $Q(x)$ are the two predicates defined on a set A, then

$$\exists x P(x) \wedge \exists x Q(x) \Rightarrow P(y) \wedge Q(z)$$

Quantification of a statement can be done using the rules of generalization defined in the following subsections.

1.15.3 Universal Generalization

Let $P(x)$ be a predicate on x and the set A be the universe of discourse for x. If $P(x)$ is true for any arbitrary element $y \in A$, then it can also be generalized for all the elements of the set A, that is,

$$P(y) \Rightarrow \forall x P(x)$$

For example, let $A = \{3, 4, 5\}$ and $P(x)$: x is greater than 2.

On using universal specification we get

$$\forall x P(x) \Rightarrow P(y)$$

Here, y can take any value of the set A; thus, y is an arbitrary element and the proposition $P(y)$ can be generalized for all values of A, that is,

$$P(y) \Rightarrow \forall x P(x)$$

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> **Note:** Universal generalization can be applied only if $y \in A$ is an arbitrary element.

1.15.4 Existential Generalization

Let $P(x)$ be a predicate on x and the set A be the universe of discourse for x. If $P(x)$ is true for an element $y \in A$, then it can be concluded that the predicate $P(x)$ is true for some $x \in A$, that is,

$$P(y) \Rightarrow \exists x P(x)$$

Here y is not an arbitrary element.

For example, let $A = \{3, 4, 5\}$ and P(x): x is divisible by 2.

Thus, $P(4)$ is true. If a proposition is true for at least one element of the universe of discourse, then it can be concluded that $P(x)$ is true for some $x \in A$, that is,

$$P(4) \Rightarrow \exists x P(x).$$

Example 1.46 ....

Show that $\forall x(P(x) \rightarrow Q(x)) \wedge \exists x P(x) \Rightarrow \exists x Q(x)$.

Solution: $\forall x(P(x) \rightarrow Q(x)) \wedge \exists x P(x) \Rightarrow (P(y) \rightarrow Q(y)) \wedge P(y)$ (using universal specification or US)

$\Rightarrow Q(y)$ (using modus ponens)

$\Rightarrow \exists x Q(x)$ (using existential generalization or EG)

Example 1.47 ....

Show that $\sim(\exists x P(x) \wedge Q(y)) \Rightarrow \exists x P(x) \rightarrow \sim Q(y)$

Solution: $\sim(\exists x(P(x) \wedge Q(y)) \Rightarrow \sim\exists x P(x) \vee \sim Q(y)$ (using De Morgan's law)

$\Rightarrow \forall x(\sim P(x)) \vee \sim Q(y)$

$\Rightarrow \sim P(z) \vee \sim Q(y)$ (using US)

$\Rightarrow P(z) \rightarrow \sim Q(y)$

$\Rightarrow \exists x P(x) \rightarrow \sim Q(y)$(using EG)

Example 1.48 ....

Show that $\forall x(P(x) \rightarrow Q(x)) \wedge \forall x(Q(x) \rightarrow R(x)) \Rightarrow \exists x(P(x) \rightarrow R(x))$

Solution: $\forall x(P(x) \rightarrow Q(x)) \wedge \forall x(Q(x) \rightarrow R(x)) \Rightarrow (P(y) \rightarrow Q(y)) \wedge (Q(y) \rightarrow R(y))$ (using US)

$\Rightarrow P(y) \rightarrow R(y)$ (hypothetical syllogism)

$\Rightarrow \exists x(P(x) \rightarrow R(x))$ (using EG)

Example 1.49 ....

Show that $\forall x(P(x) \vee Q(x)) \Rightarrow \forall x P(x) \vee \exists x Q(x)$.

Solution: To prove the implication, we shall use the indirect method of proof (see further points to understand the reason behind this). We know that $P \rightarrow Q \equiv \sim Q \rightarrow \sim P$; thus, we shall show $\sim(\forall x P(x) \vee \exists x Q(x)) \Rightarrow \sim\forall x(P(x) \vee Q(x))$ to prove the implication.

$\sim(\forall x P(x) \vee \exists x Q(x)) \Rightarrow \sim\forall x P(x) \wedge \sim\exists x Q(x)$ (De Morgan's law)

$\Rightarrow \exists x \sim P(x) \wedge \forall x \sim Q(x)$

$\Rightarrow \sim P(y) \wedge \sim Q(y)$(using ES)

$\Rightarrow \exists x(\sim P(x) \wedge \sim Q(x))$(using EG)

$\Rightarrow \sim\forall x(P(x) \vee Q(x))$

Therfore, $\forall x(P(x) \vee Q(x)) \Rightarrow \forall x P(x) \vee \exists x Q(x)$.

POINTS TO UNDERSTAND

In this example, we cannot write $\forall x(P(x) \vee Q(x)) \Rightarrow \forall x P(x) \vee \forall x Q(x)$ as it is not correct. For example, let the universe of discourse of x is the set $A = \{2, 3, 4, 5, 6, 7\}$ and

$P(x)$: x is less than five.

$Q(x)$: x is greater than four.

Then, $\forall x\,(P(x) \vee Q(x))$ will be interpreted as follows:

Every number of the set A is either less than five or greater than four.

On the other hand, $\forall x(P(x) \vee \forall x\, Q(x)$ has the following interpretation:

Every number of the set A is less than five or every number of the set A is greater than four.

The first one has truth value true whereas the second one has truth value false. Therefore,

$\forall x(P(x) \vee Q(x)) \Rightarrow \forall x P(x) \vee \forall x Q(x)$ is not valid. But it is interesting to note that the converse of the argument is valid, that is, $\forall x P(x) \vee \forall x Q(x) \Rightarrow \forall x(P(x) \vee Q(x))$.

1.15.5 Substitution

Let us consider the quantified predicate $\forall x P(x, y)$. Here, the variable x is a bound variable and y is a free variable. If we substitute z for y, then the predicate $\forall x P(x, z)$ will have same interpretation as $\forall x P(x, y)$ as z is free for substituting y. However, x is not free for substituting y; hence, we cannot write $\forall x P(x, x)$ because it will have different interpretation. A variable in a predicate formula can be substituted by another variable if the variable is free for substitution.

1.15.6 First-order and Second-order Logic

The predicate logic that we have discussed so far is also called *first-order logic* as quantification takes place over variables only; for example, we say $\forall x P(x)$ for any predicate $P(x)$ and its universe of discourse defined for x. In *second-order logic*, quantification takes place over predicates, for example, $\forall P \exists x P(x)$. In second-order predicate logic, a predicate is represented with predicate and functions as arguments.

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1.16 METHODS OF PROOF

In this section, we discuss the different methods of proof. Proving a theorem or a mathematical statement is basically proving the validity of an argument. So far, we have discussed equivalences and implications in propositional logic. We shall use some of these equivalences and implications to describe different methods of proof. Before defining the different methods of proof, we shall discuss some terminologies used to denote the statements.

A *theorem* is a statement, fact, or result that can be shown to be true. A *proposition* is considered as a less important theorem. Sometimes, to prove the theorem, we first prove some parts of the theorem separately, and then use those results to prove the theorem. A *lemma* is considered as a less important theorem that is used to prove other theorems. A *corollary* is a theorem that can be proved directly from a theorem that has been proved. In general, a theorem is a valid argument having some premises and a conclusion, or more specifically, it may be interpreted as the universal quantification of a conditional statement. In some cases, a theorem may be a logical statement as well.

Here, we discuss the different methods of proof to prove the statements like the conditional $P \to Q$ or any simple logical statement P.

1.16.1 Trivial Proof

Trivial proof is considered as one of the easiest way of proof. We know that the statement $P \to Q$ is true whenever the conclusion Q is true regardless of the truth values of P. Showing only Q is true to prove $P \to Q$ is known as trivial proof of the statement $P \to Q$.

EXAMPLE 1.50 ....

If a is an integer, then prove that $a^{2n} \geq 1$ for $n = 0$.

Solution: As $a^0 = 1$ (regardless the value of a), $a^{2n} \geq 1$ for $n = 0$.

This proves the statement.

1.16.2 Vacuous Proof

Vacuous proof is also considered as one of the easiest way of proof. We know that the statement $P \to Q$ is true whenever P is false. Thus, the statement $P \to Q$ can easily be proved by proving P is false and this method is known as vacuous proof.

EXAMPLE 1.51 ....

Let $P(n)$: If $n > 2$, then $n^2 \geq 2n$. Prove that $P(0)$ is true.

Solution: For $n = 0$, the condition $n > 2$ is false. Thus, the statement $P(n)$ is true for $n = 0$.

1.16.3 Direct Proof

In direct proof, we can rephrase the theorem or statement as a conditional statement $P \to Q$. We start with the assumption that P is true and then use the rules of inferences with given axioms and already proved theorems and definitions to show that Q is also true. (Note that in a conditional $P \to Q$ is always true whenever P is false, that is why start with assumption that P is true.)

EXAMPLE 1.52 ....

Show that the square of an even number is an even number.

Solution: First, we will restructure the sentence. We have to prove that 'if n is an even number, then n^2 is also even'.

Here P: n is an even number.

and Q: n^2 is an even number

Let us assume that n is an even number. Then we can write $n = 2k$, where $k \in Z$

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

This $\Rightarrow n^2$ is an even number.

EXAMPLE 1.53 ....

Show that the sum of two odd integers is an even number.

Solution: Let a and b be odd integers.

Here P: a is odd and b is odd.

Q: $a + b$ is even.

Let us assume P is true, that is, a and b are odd integers.

As a and b are odd integers, we can write $a = 2l + 1$ and $b = 2m + 1$ for some integers l and m.

Now
$$\begin{aligned} a + b &= 2l + 1 + 2m + 1 \\ &= 2(l + m) + 2 \\ &= 2(l + m + 1) \end{aligned}$$

This shows that $a + b$ is even number.

1.16.4 Proof by Contradiction

Proof by contradiction is based on the fact that a statement is either true or false but not both at the same time. We get at a contradiction when we arrive at a situation where we say that a statement is both true and false at the same time. This shows that our initial assumptions are inconsistent.

To prove that a statement P is true, we assume that $\sim P$ is true, and taking $\sim P$ as premise, we draw a contradiction F as the conclusion. $\sim P \Rightarrow F$ proves that $\sim P \to F$ is true; thus, $\sim P$ must be false, that is, P must be true. We can summarize the steps as follows:

1. Assume that P is false.
2. Using this assumption show a contradiction.

EXAMPLE 1.54 ....

Show that $\sqrt{2}$ is an irrational number.

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Solution: Here P: $\sqrt{2}$ is an irrational number.

We assume that $\sim P$ is true, that is, $\sqrt{2}$ is not an irrational number. This implies that $\sqrt{2}$ is a rational number. We know that every rational number can be expressed in the form of $\frac{p}{q} (q \neq 0)$, where p and q have no common factor (assuming these are the lowest terms).

Let $\sqrt{2} = \frac{p}{q}$ such that p and q have no common factor.

$\Rightarrow \sqrt{2}q = p$

$\Rightarrow 2q^2 = p^2$

$\Rightarrow p^2$ is an even number

$\Rightarrow p$ is an even number. (Since if p^2 is even, p must be even.)

$\Rightarrow p = 2k$ for some integer k.

$\Rightarrow p^2 = 4k^2$

$\Rightarrow q^2 = 2k^2$ (on substituting the value of p^2 in $2q^2 = p^2$)

$\Rightarrow q^2$ is an even number.

$\Rightarrow q$ is an even number.

$\Rightarrow 2$ is the common factor of a and b.

This is a contradiction that p and q have no common factor. Thus, the assumption '$\sim P$ is true', that is, '$\sqrt{2}$ is not an irrational number' is false. Hence, $\sqrt{2}$ is an irrational number.

EXAMPLE 1.55 ....

Prove that there is no largest integer that is a multiple of 5.

Solution: Let P: There is no largest integer that is a multiple of 5.

We assume that there is a largest integer that is a multiple of 5 and suppose that the integer is m. Thus, $m = 5k$ for some $k \in Z$.

Now consider the integer $m + 5$.

$$m + 5 = 5k + 5 = 5(k + 1)$$

This shows that $m + 5$ is also a multiple of 5 and also $m + 5$ is greater than m; thus, this is a contradiction that m is the largest integer that is a multiple of 5 and our assumption is not true. Thus, there is no largest integer that is a multiple of 5.

To prove the conditional statement $P \rightarrow Q$, we assume that both P and $\sim Q$ are true. Then considering $\sim Q$ as a premise, we draw the conclusion $\sim P$. Thus, we get the contradiction $P \wedge \sim P$. We say that our initial assumption is not true, that is, $\sim Q$ is false as P is assumed to be true. Finally, $\sim Q$ is false implies that Q is true and hence $P \rightarrow Q$. We summarize the steps as follows:

1. Assume both P and $\sim Q$ are true.
2. Use $\sim Q$ and show that P is false, which is a contradiction.

Alternatively, assuming both P and $\sim Q$ are true, we draw a contradiction F, that is, $P \wedge \sim Q \Rightarrow F$. This shows that $(P \wedge \sim Q) \rightarrow F$ is true and this is possible only if when $(P \wedge \sim Q)$ is false or $\sim(P \wedge \sim Q)$ is true. As $\sim(P \wedge \sim Q)$ is equivalent to $P \rightarrow Q$, $P \rightarrow Q$ is true.

EXAMPLE 1.56 ....

Prove the statement 'if $3n + 1$ is even, then n is odd' utilizing the method of proof by contradiction.

Solution: Here P: $3n + 1$ is even and Q: n is odd.

We shall assume that P is true and $\sim Q$ is true.

Let n is even and $3n + 1$ is even.

Let $n = 2k$ for some integer, then

$$3n + 1 = 3.2k + 1 = 6k + 1$$

since $6k = 2(3k)$

This implies that $6k$ is an even number.

$\Rightarrow 6k + 1$ is an odd number

$\Rightarrow 3n + 1$ is an odd number

This is a contradiction to the assumption that $3n + 1$ is even. Hence n is not even, that is, n is odd. This proves the statement 'if $3n + 1$ is even, then n is odd'.

EXAMPLE 1.57 ....

Prove that the sum of two consecutive integers is odd.

Solution: Let a and b be two consecutive integers.

Here P: a and b are two consecutive integers.

Q: $a + b$ is odd.

We shall assume that P is true and $\sim Q$ is true.

Thus, a and b are consecutive integers and the sum of a and b is even.

$a + b$ is an even number $\Rightarrow a$ and b both are even or a and b both are odd.

$\Rightarrow a$ and b are not consecutive integers.

This is a contradiction to the assumption that a and b are two consecutive integers.

Thus, $a + b$ is odd. This proves that the sum of two consecutive integers is odd.

Alternatively, we assume that the sum of two consecutive numbers is even; that is, a and b are two consecutive integers and $a + b$ is an even number. We can write $a = k$ and $b = k + 1$ for some integer k. This gives $a + b = 2k + 1$, which is an odd number—a contradiction. Thus, the sum of two consecutive numbers is not even, that is, odd.

EXAMPLE 1.58 ....

Prove that for all non-negative real numbers x, y and z if $x^2 + y^2 = z^2$, then $x + y \geq z$.

Solution: Here P: $x^2 + y^2 = z^2$ and Q: $x + y \geq z$.

We shall assume that P is true and $\sim Q$ is true.

Thus $x^2 + y^2 = z^2$ and $x + y < z$

$$x + y < z \Rightarrow (x + y)^2 < z^2 \quad \text{(since all are non-negative real numbers)}$$
$$\Rightarrow x^2 + y^2 + 2xy < z^2$$
$$\Rightarrow x^2 + y^2 < z^2 \quad \text{(since } 2xy \text{ is also a non-negative real number)}$$

This is a contradiction to the assumption $x^2 + y^2 = z^2$; thus, $x + y < z$ is not true, that is, $x + y \geq z$. This proves that for all non-negative real numbers x, y, and z, if $x^2 + y^2 = z^2$, then $x + y \geq z$.

1.16.5 Proof by Contraposition

In the method of contraposition, we use the fact that $P \rightarrow Q$ is equivalent to its contrapositive $\sim Q \rightarrow \sim P$. This shows that to prove the conditional $P \rightarrow Q$, we can

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also prove its contrapositive $\sim Q \to \sim P$. Thus, using the method of contraposition, to prove the statement $P \to Q$, we shall take $\sim Q$ as premise and using the rules of inferences together with definitions and already proven theorems, we will show that $\sim P$ is the conclusion.

Example 1.59 ....

Prove that if n^2 is odd, then n is odd.

Solution: If we consider this example, then using direct method of proof it is tedious to prove the statement. In this case, proof by contrapositive is quite easy.

Here P: n^2 is odd and Q : n is odd.

Thus, $\sim P$: n^2 is even and $\sim Q$: n is even.

To prove the statement $P \to Q$ using the method of contrapositive, we shall take $\sim Q$ as premise.

Let $\sim Q$ is true, that is, n is even.

n is even $\Rightarrow n = 2k$ for some integer k.

$\Rightarrow n^2 = 4k^2 = 2(2k^2)$

$\Rightarrow n^2$ is even.

This shows that $\sim Q \to \sim P$, hence the equivalent statement of this is $P \to Q$, that is, if n^2 is odd, then n is odd.

1.16.6 Proof by Cases

Sometimes we need to prove a conditional statement of the form

$$(P_1 \vee P_2 \vee \ldots \vee P_n) \to Q.$$

We can easily prove the equivalence

$$(P_1 \vee P_2 \vee \ldots \vee P_n) \to Q \equiv (P_1 \to Q) \wedge (P_2 \to Q) \wedge \ldots \wedge (P_n \to Q)$$

Sometimes it becomes very tough to prove such statements by a single argument that holds for all possible cases. Thus, using the equivalence, these types of statements can be proved by proving each of the conditional statements separately. This method of proof is known as proof by cases.

Example 1.60 ....

Prove that $|a+b| \le |a|+|b|$ for all real numbers a and b.

Solution: We shall consider different cases of a and b.

Case 1: When a and b are positive

Let $a = m$ and $b = n$, where m and n are positive integers.

Then $|a+b| = |m+n| = m+n = |a|+|b|$

Case 2: When a is positive and b is negative

Let $a = m$ and $b = -n$, where m and n are positive integers.

Then $|a+b| = |m-n| = \begin{cases} m-n \text{ if } m-n \ge 0 \\ n-m \text{ if } m-n \le 0 \end{cases}$

and $|a|+|b| = m+n$.

Thus, $|a+b| < |a|+|b|$.

Case 3: When a is negative and b is positive

Let $a = -m$ and $b = n$, where m and n are positive integers.

Then $|a+b| = |n-m| = \begin{cases} n-m \text{ if } n-m \ge 0 \\ m-n \text{ if } n-m \le 0 \end{cases}$

$|a|+|b| = m+n$.

Thus, $|a+b| < |a|+|b|$.

Case 4: When a and b are negative

Let $a = -m$ and $b = -n$, where m and n are positive integers.

Then $|a+b| = |-m-n| = m+n = |a|+|b|$

From all the case, we observe that $|a+b| \le |a|+|b|$.

1.16.7 Exhaustive Proof

Some of the theorems involve a limited number of examples; thus, to prove the theorem by exhausting all these possibilities is known as exhaustive proof.

Example 1.61 ....

Prove that $2^n < n^2 + 2$ for $n \le 4$.

Solution: Here, we have limited number of examples to prove.

For $n = 1$, $2^n = 2$ and $n^2 + 2 = 3$.

For $n = 2$, $2^n = 4$ and $n^2 + 2 = 6$.

For $n = 3$, $2^n = 8$ and $n^2 + 2 = 11$.

For $n = 4$, $2^n = 16$ and $n^2 + 2 = 18$.

Thus, $2^n < n^2 + 2$ for $n \le 4$.

1.16.8 Proof by Mathematical Induction

First, we shall discuss Peano's axioms, and then we will move to mathematical induction.

Peano's Axioms

Peano's axioms are a set of axioms for natural numbers given by the Italian mathematician Giuseppe Peano. He described the set of natural numbers as a non-empty set N with the following properties:

1. 1 is a natural number.
2. If $k \in N$, then there is an element $s(k) = k + 1 \in N$, called the successor of K.
3. No two elements of N have the same successor, that is, if two elements m, n have the same successor, then $m = n$.
4. No element has 1 as its successor.
5. If a subset A of N follows the following properties:
   (a) $1 \in A$
   (b) whenever $k \in A$, $s(k) \in A$.

Then, $A = N$.

The last property of Peano's axiom provides the basis for the principle of mathematical induction. Here, we study the two forms of mathematical induction and their utilization to prove the identities.

Principle of Mathematical Induction

Let $P(n)$ be a statement defined on positive integers $n \in N$ such that it has the following properties:

1. $P(1)$ is true.
2. $P(K+1)$ is true whenever $P(k)$ is true for some positive integer $k \geq 1$.

Then, $P(n)$ is true for every positive integer.

The step 1 is called the basis of induction ($n = 1$ is called the base value) and step 2 is called the induction step. Sometimes, we would like to prove that the statement is true for the set of integers $\{i, i+1, i+2, \ldots\}$, where i is an integer. In this case, 1 is replaced by i in either of the statements.

To understand how the principle of mathematical induction proves that a statement $P(n)$ is true for all positive integers, we need to understand the two steps. The first step proves that the statement is true for a base value; in most of the cases, the base value is 1. The second step proves that if $P(k)$ is true, then $P(k+1)$ is also true. Using the first step that the statement is true for the base value, for example, 1, by substituting $k = 1$ in the second step, we get that the statement is true for $k = 2$. Again substituting $k = 2$ in the second step, we get that the statement is true for $k = 3$, and so on. In this way, it can be shown that the statement $P(n)$ is true for all positive integers.

Examples showing proofs through mathematical induction

Example 1.62 ....

Using mathematical induction prove that for every natural number.

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

Solution: Let $P(n)$ be the statement that $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$

For $n = 1$

Left-hand side (LHS) = 1

$$\text{RHS}=\frac{1(1+1)}{2}=1$$

Hence LHS = RHS and $P(1)$ is true.

Let the statement $P(n)$ is true for $n = K$, then

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

Now for $n = k + 1$

$$\text{LHS}=1+2+3+\cdots+k+(k+1)$$
$$=\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}=\text{RHS}$$

$P(n)$ is true for $n = k + 1$.

Therefore, by principle of mathematical induction, $P(n)$ is true for all natural numbers.

Example 1.63 ....

Using mathematical induction prove that for every non-negative integer

$$1+2+2^2+\cdots+2^n=2^{n+1}-1$$

Solution: Let $P(n)$ be the statement that $1+2+2^2+\cdots+2^n=2^{n+1}-1$.

In this case, the base value is 0.

For $n = 0$

LHS = 1

RHS = 4! = 24

16 < 24, hence $P(4)$ is true.

Let the statement $P(n)$ is true for $n = k$, then

$$1+2+2^2+\cdots+2^k=2^{k+1}-1$$

Now for $n = k + 1$

$$\text{LHS}=1+2+2^2+\cdots+2^k+2^{k+1}$$
$$=2^{k+1}-1+2^{k+1}=2.2^{k+1}-1=2^{k+2}-1$$
$$=\text{RHS}$$

$P(n)$ is true for $n = k + 1$.

Therefore, by principle of mathematical induction, $P(n)$ is true for all non-negative integers.

Example 1.64 ....

Using mathematical induction, prove that for every positive integer $n \geq 4$, $2^n < n!$.

Solution: Let $P(n)$ be the statement that $2^n < n!$.

Here, the base value is 4.

For $n = 4$

$$\text{LHS} = 2^4 = 16$$
$$\text{RHS} = 4! = 24$$

16 < 24, hence $P(4)$ is true.

Let the statement $P(n)$ is true for $n = k$ ($k \geq 4$), then

$2^k < k!$

Now for $n = k + 1$

$$2^{k+1}=2.2^k<2.k!<(k+1)k!\,(\because k\geq 4)<(k+1)!$$

$P(n)$ is true for $n = k + 1$.

Therefore, by principle of mathematical induction, $P(n)$ is true for all positive integers $n \geq 4$.

Example 1.65 ....

Using mathematical induction, prove that $n^3 + 2n$ is divisible by 3 for $n \geq 1$.

Solution: Let $P(n)$ be the statement that $n^3 + 2n$ is divisible by 3.

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EXERCISES

Identifying a proposition

1.1. Which of these sentences are propositions?
   (a) Mumbai is the capital of India. P
   (b) Go to the class room. F
   (c) What a surprise! F
   (d) Do not take it. F

1.2 Which of the followings are propositions?
   (a) $2+6=8$ P
   (b) $3+7=9$ P
   (c) $x+4 \leq 5$ F
   (d) $x+y=10$ F

Writing simple compound propositions using negation, 'OR', and 'AND'

1.3 Write negation of the following statements:
   (a) I am playing chess.
   (b) $4>5$
   (c) $5 \leq 8$
   (d) $4 \neq 5$

1.4 Let P and Q be the proposition defined as follows:
   P: You play cricket
   Q: You miss the film
   Write the proposition for the following sentences:
   (a) You do not play cricket $\sim P$
   (b) You play cricket or you miss the film. $P \vee Q$
   (c) Either you play cricket or you miss the film. $P \vee Q$
   (d) Either you do not play cricket or you do not miss the film. $\sim P \vee \sim Q$

1.5 Let P and Q be the proposition defined as follows:
   P: I go to college
   Q: I attend all the lectures
   Write the proposition for the following sentences:
   (a) I go to college and I attend all lectures. $P \wedge Q$
   (b) I do not go to college and I do not attend all lectures. $\sim P \wedge \sim Q$
   (c) I go to college yet I do not attend all lectures. $P \wedge \sim Q$
   (d) Neither do I go to college nor do I attend all the lectures. $P \vee Q$

Conversion of English sentences into propositions

1.6 Let P and Q be the proposition defined as follows:
   P: You play cricket
   Q: You miss the film
   Write the proposition for the following sentences:
   (a) If you play cricket, then you miss the film.
   (b) If you do not play cricket, then you do not miss the film.
   (c) You either play cricket or miss the film, but you play cricket if you miss the film.

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(d) Playing cricket is necessary for you to miss the film.
(e) Playing cricket is sufficient for you to miss the film.

1.7 Let P, Q, and R be the proposition.
P: You have distinction in mathematics
Q: You get grade A in the final exam
R: You get excellent student award
Write the proposition for the following sentences:
(a) You get excellent student award whenever you have distinction in mathematics and grade A on the final exam.
(b) You have distinction in mathematics and you get grade A on final exam, but you do not get excellent student award.
(c) It is sufficient for you to have distinction in mathematics and grade A on final exam to get excellent student award.
(d) If you get excellent award, then either you have distinction in mathematics or you have grade A on the final exam.
(e) If you have distinction in mathematics, you get excellent award if and only if you get grade A on the final exam.

1.8 Write the proposition for the following sentences:
(a) You buy a car and you buy a bike.
(b) You buy a car and you go for a long drive.
(c) If you either buy a car or buy a bike, you go for a long drive.
(d) You go for a long drive if and only if you either buy a car or buy a bike.
(e) If neither you buy a car nor you buy a bike, you do not go for a long drive.

Conversion of propositions into English sentences

1.9 Let P and Q be the propositions defined as follows:
P: I am a computer science graduate
Q: I have a distinction in programming
Write English sentences for the following propositions:
(a) $P \vee Q$ (b) $P \vee \sim Q$ (c) $\sim P \vee Q$ (d) $\sim P \vee \sim Q$

1.10 Let P and Q be the propositions defined as follows:
P: You are a student of graduation course
Q: You have mathematics as a subject
Write English sentences for the following propositions:
(a) $P \wedge Q$ (b) $P \wedge \sim Q$ (c) $\sim P \wedge Q$ (d) $\sim P \wedge \sim Q$

1.11 Let P, Q, and R be the proposition.
P: Rakesh is working with TCS
Q: Rakesh is a computer programmer
R: Rakesh is M. Tech. in computer science
Write the sentences for the following propositions:
(a) $P \rightarrow R$ (b) $(P \wedge Q) \rightarrow R$ (c) $\sim P \rightarrow \sim Q$ (d) $(Q \vee R) \rightarrow P$
(e) $P \rightarrow (Q \vee R)$ (f) $\sim P \rightarrow \sim(Q \vee R)$ (g) $P \rightarrow (Q \rightarrow R)$
(h) $Q \rightarrow (P \leftrightarrow R)$

Determining the truth values of compound propositions

1.12 Determine whether the following compound propositions are true or false:
(a) $2 + 1 = 4$ or $4 + 3 = 7$
(b) $2 * 3 = 6$ or $7 - 5 = 3$
(c) $6 + 4 = 11$ or $3 + 5 = 7$
(d) $4 + 5 = 8$ or $2 + 5 = 7$

1.13 Determine whether the following compound propositions are true or false:
(a) $4 + 3 = 8$ and $4 + 6 = 10$
(b) $5 - 2 = 3$ and $4 * 5 = 20$
(c) $8 - 5 = 3$ and $3 + 5 = 9$
(d) $5 + 6 = 10$ and $2 + 3 = 6$

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1.14 Determine whether the following compound propositions are true or false:
(a) Delhi is the capital of India but Washington D.C. is not the capital of USA.
(b) Either $2 > 3$ or $5 > 7$.
(c) $4 > 2$ but $3 \neq 5$.
(d) Neither $5 < 4$ nor $7 > 5$.

1.15 Determine the truth values of the following propositions:
(a) If $1 + 1 = 2$, then $2 + 3 = 5$.
(b) If $2 > 3$, then $4 > 5$.
(c) If $3 + 4 = 7$, then lion can fly.
(d) If New Delhi is the capital of India, then Islamabad is the capital of Pakistan.
(e) $2 + 5 = 8$ if and only if 2 is a divisor of 8.
(f) Lion can fly if and only if cats can sing a song.

Constructing truth tables of compound propositions

1.16 Construct the truth table for the following:
(a) $(P \wedge Q) \vee R$
(b) $(P \vee Q) \vee \sim R$
(c) $(\sim P \wedge \sim Q) \vee R$
(d) $(\sim P \vee \sim Q) \wedge \sim R$

1.17 Construct the truth table for the following:
(a) $(P \wedge \sim Q) \rightarrow R$
(b) $(P \rightarrow Q) \rightarrow R$
(c) $(P \rightarrow Q) \wedge (P \rightarrow R)$
(d) $(P \vee Q) \wedge (P \rightarrow R)$

1.18 Construct the truth table for the following:
(a) $P \vee \sim Q \rightarrow R$
(b) $\sim P \wedge Q \vee \sim R$
(c) $Q \wedge P \rightarrow P \wedge R$
(d) $P \rightarrow Q \wedge \sim P$

Checking of propositions for being tautology or contradiction

1.19 Construct the truth table of each of the following propositions and check whether the given proposition is a tautology:
(a) $(P \vee Q) \rightarrow P$ (b) $(P \wedge Q) \rightarrow P$ (c) $P \rightarrow (P \vee Q)$ (d) $P \rightarrow (P \wedge Q)$

1.20 Show that each of the following statements is a tautology:
(a) $((P \vee Q) \vee \sim P) \rightarrow Q$
(b) $(\sim Q \wedge \sim(P \wedge \sim Q)) \rightarrow \sim P$
(c) $(P \leftrightarrow (Q \wedge R)) \rightarrow (R \vee \sim P)$
(d) $((P \rightarrow Q) \vee R) \leftrightarrow ((P \vee R) \rightarrow (Q \vee R))$

1.21 Show that each of the following statements is a contradiction:
(a) $(\sim P \wedge \sim Q) \wedge (P \vee Q)$
(b) $\sim((P \wedge Q) \rightarrow P)$
(c) $(\sim(P \rightarrow Q) \vee \sim(Q \rightarrow R)) \wedge \sim(P \rightarrow R)$
(d) $(P \rightarrow Q) \wedge (P \wedge \sim Q)$

Showing logical equivalence

1.22 Prove the following logical equivalences with or without constructing truth tables:
(a) $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$.
(b) $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$.
(c) $(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$.
(d) $\sim(P \vee Q) \vee (\sim P \wedge Q) \equiv \sim P$.
(e) $(P \wedge Q) \rightarrow R \equiv P \rightarrow (Q \rightarrow R)$.

Showing logical implication

1.23 Prove the following logical implications with or without constructing truth tables:
(a) $(\sim P \vee Q) \wedge (P \vee R) \wedge (\sim Q \vee R) \Rightarrow R$.
(b) $(P \wedge R) \wedge (R \rightarrow (P \rightarrow \sim Q)) \Rightarrow \sim Q$.

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(c) $P \wedge (P \rightarrow (Q \wedge R)) \Rightarrow (P \vee Q)$.
(d) $(\sim Q \vee \sim R) \wedge (\sim P \vee Q) \wedge (\sim P \vee R) \Rightarrow \sim P$.
(e) $(\sim P \vee \sim R) \wedge (P \vee \sim Q) \wedge (\sim Q \vee R) \Rightarrow \sim(P \wedge Q)$.

Writing converse, inverse, and contrapositive of statements

1.24 Write the converse, inverse and contrapositive of the following statements:
(a) If I go to Delhi, then I visit Rajghat.
(b) If I play, then I do not run.
(c) If I do not take breakfast, then I do not play.

1.25 Write the converse, inverse and contrapositive of the following statements:
(a) If I dance, then I feel happy and I sing.
(b) If I do not dance or I do not feel happy, then I sing.
(c) If I do not dance and I do not feel happy, then I do not sing.

Checking validity of the arguments

1.26 Check the validity of the following arguments:
(a) Either you study or you play. You do not study. Therefore, you play.
(b) If I do not play football, then I read. If I do not read, then I go to market. I play football. Therefore, I do not go to market.
(c) If I walk, then I reduce fat from my body. If I reduce fat from my body, then I am healthy. I walk but do not reduce fat from my body. Therefore, I am not healthy.
(d) If you get grade A in the exam, then you do not get gift from your father. If you get first place in the exam, then you get gift from your father. You get grade A or you get first place in the exam. Therefore, you get gift from your father.
(e) You go to school or you go for tuition. You do not go to school or you go to market. Therefore, either you go for tuition or you go to market.

Checking the validity of arguments without constructing truth table

1.27 Check the validity of the following arguments without constructing the truth table:

(a) $H_1: P \vee Q$
$H_2: Q \rightarrow R$
$C: P \vee R$

(b) $H_1: P \vee Q$
$H_2: P \rightarrow R$
$H_2: \sim Q \vee S$
$C: S \vee R$

(c) $H_1: (P \vee Q) \rightarrow R$
$H_2: R \rightarrow (U \wedge V)$
$H_3: \sim(U \wedge V)$
$C: \sim P$

Normal forms

1.28 Write the DNF of the following propositions:
(a) $P \rightarrow (P \vee Q)$ (b) $(P \wedge Q) \rightarrow P$

1.29 Write the CNF of the following propositions:
(a) $P \rightarrow \sim(P \rightarrow Q)$ (b) $P \rightarrow (P \wedge Q)$

1.30 Write the principal DNF and PCNF of the following propositions:
(a) $(P \wedge Q) \vee (P \vee \sim Q)$ (b) $(P \wedge Q) \vee (P \wedge R)$
(c) $(P \vee Q) \wedge (P \vee R)$

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1.31 Write the principal DNF and PCNF of the following propositions:

(a) $(P \to Q) \to R$

(b) $(P \to R) \wedge (Q \to R)$

(c) $P \leftrightarrow Q$

Writing predicates

1.32 If $P(x)$: x is a student

$Q(x)$: x is honest

Then write the predicates for the following sentences:

(a) All students are honest.

(b) Some students are not honest.

(c) All students are not honest.

(d) Some students are honest.

1.33 Let $B(x)$: x is black

$C(x)$: x is a cat

The universe of discourse for x is the set of animals. Then write sentences for the following predicates:

(a) $\forall x(C(x) \to \sim B(x))$

(b) $\exists x(C(x) \wedge B(x))$

(c) $\exists x C(x) \wedge \exists x B(x)$

(d) $\exists x(C(x) \to B(x))$

1.34 Let $x \in \{4, 5, 6, 7, 8,\}$ and $P(x)$: x is a multiple of 2. Then write the truth value of the following propositions:

(a) $\forall x P(x)$ (b) $\exists x P(x)$ (c) $P(6)$ (d) $P(4) \wedge P(5)$

(e) $P(6) \wedge P(8)$

1.35 Let $x \in \{1, 2, 3, 4\}$ and $P(x)$ be a predicate. Then remove the quantifiers from the predicates.

(a) $\forall x P(x)$ (b) $\exists x P(x)$ (c) $\exists x \sim P(x)$ (d) $\sim \exists x P(x)$

(e) $\sim \forall x P(x)$

1.36 Write the predicates for the following sentences:

(a) All advocates are clever.

(b) Some students are brilliant.

(c) Not every politician of the country is corrupt.

(d) No student is genius in the class.

(e) Every politician can cheat every person.

(f) Some students are rich and some students are poor.

(g) Some students are rich but not intelligent.

(h) Some students are poor but intelligent.

(i) For every real number, there is at least one integer greater than it.

(j) Sum of every two positive integers is a positive integer.

Finding the truth values of quantified predicates with nested quantifiers

1.37 Let the universe of discourse for x is the set $A = \{2, 3, 4, 5\}$ and for y is the set $B = \{3, 4, 5, 6\}$ and the predicate $P(x, y)$ is defined as:

$P(x, y)$: x is greater than y.

Find the truth values of the propositions $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$.

1.38 Let the universe of discourse for x and y is the set of real numbers and the predicate $P(x, y)$ is defined as:

$P(x, y)$: $x + y = 0$

Find the truth values of the propositions $\exists x \forall y P(x, y)$ and $\forall x \exists y P(x, y)$.

1.39 Let the universe of discourse for x and y is the set of integers and the predicate $P(x, y)$ is defined as:

$P(x, y)$: $x - y < 0$

Find the truth values of the propositions $\exists y \forall x P(x, y)$ and $\forall y \exists x P(x, y)$.

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1.40 Let the universe of discourse for x and y is the set of real numbers and the predic
$P(x, y)$ is defined as:
$P(x, y)$: $x = 2y$
Find the truth values of the propositions $\exists x \exists y P(x, y)$ and $\exists y \exists x P(x, y)$.

Drawing inference through predicates

1.41 Prove that
(a) $\forall x(P(x) \rightarrow Q(x)) \wedge \sim Q(y) \Rightarrow \sim \forall x P(x)$
(b) $\forall x(P(x) \vee Q(x)) \wedge \forall x \sim P(x) \Rightarrow \exists x Q(x)$
(c) $\forall x(P(x) \rightarrow Q(x)) \wedge \exists x(P(x) \wedge Q(x)) \Rightarrow \exists x P(x)$
(d) $\forall x (\sim P(x) \rightarrow Q(x)) \wedge \forall x \sim Q(x) \Rightarrow P(y)$

1.42 Convert the following arguments in the form of predicates and check whether th
conclusions are valid:
(a) All students are young.
Krishna is a student.
Therefore, Krishna is young.
(b) All students are intelligent.
All intelligent people are honest.
Therefore, all students are honest.
(c) Some persons are corrupt.
Some corrupts are politician.
Therefore, some persons are politician.
(d) All doctors are brilliant.
All brilliant are laborious.
Rajesh is not laborious.
Therefore, Rajesh is not a doctor.

Using different methods of proof

1.43 Using the direct method of proof, prove that the sum of two even integ
is even.
1.44 Using the method of contradiction, prove that if n^2 is an odd integer, then
is odd.
1.45 Using the method of contrapositive, prove that if the product of two numbers
even then the two numbers are also even.
1.46 Prove $|a-b| \leq |a|+|b|$ using the method proof by cases.
1.47 Let $X = \{a, b, c\}$ be a set and $R = \{(a, b), (a, c)\}$. Prove that R is transitive.
1.48 Prove that if n^2 is even, then n is even.

Using principle of mathematical induction

1.49 Using principle of mathematical induction, prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1.$$

1.50 Using principle of mathematical induction, prove that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n \cdot (n+1) \cdot (n+2) = \frac{1}{4} n(n+1)(n+2)(n+3), n \in N.$$

1.51 Using principle of mathematical induction, prove that

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n-1)}{3}, n \geq 1.$$

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1.61 Using principle of mathematical induction, prove that $n^2 + n$ is an even number for all $n \geq 1$.

1.62 Prove that for every natural number n, $n(n^2 + 5)$ is divisible by 6.

1.63 Using mathematical induction, show that if X is a finite set with n elements ($n \geq 0$), then X has 2^n subsets.

Using principle of strong mathematical induction

1.64 Let $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ with the initial conditions $a_1 = a_2 = 1$. Using principle of strong mathematical induction, prove that $2^{n-1} a_n \equiv n \pmod 5$ for all $n \geq 1$.

1.65 Let $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ with the initial conditions $a_1 = a_2 = 1$. Using principle of strong mathematical induction, prove that $a_n \leq \left(\dfrac{1+\sqrt{5}}{2}\right)^{n-1}$ for all $n \geq 1$.

MULTIPLE-CHOICE QUESTIONS

1.1 The statement 'Not all students play football' is equivalent to
(a) Some students play football.
(b) Some students do not play football.
(c) All students do not play football.
(d) All students play football.

1.2 The statement 'No place in the city is safe' is equivalent to
(a) All places in the city are safe.
(b) All places in the city are not safe.
(c) Some places in the city are safe.
(d) Some places in the city are not safe.

1.3 Negation of the statement 'Some integers are not even' is
(a) Not all integers are even.
(b) All integers are not even.
(c) All integers are even.
(d) Some integers are even.

1.4 Negation of the statement 'All integers are real numbers' is
(a) Some integers are not real numbers.
(b) Some integers are real numbers.
(c) All integers are not real numbers.
(d) No integer is real number.

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Use the following for questions 1.5–1.11

Let

$P(x)$: x is a graduate in computer science

$Q(x)$: x is a computer programmer

$R(x)$: x know 'C' language

Choose the correct interpretation of the following:

1.5 $\forall x(Q(x) \to \sim R(x))$
- (a) Every computer programmer does not know 'C' language.
- (b) Every computer programmer knows 'C' language.
- (c) Some computer programmers do not know 'C' language.
- (d) Some computer programmers know 'C' language.

1.6 $\forall x(P(x) \to (Q(x) \vee R(x)))$
- (a) Every graduate in computer science is a computer programmer and knows 'C' language.
- (b) Every graduate in computer science either is a computer programmer or knows 'C' language.
- (c) Every programmer is a graduate in computer science and knows 'C' language.
- (d) Some graduates in computer science either are computer programmers or know 'C' language.

1.7 $\exists x((P(x) \wedge Q(x)) \wedge \sim R(x))$
- (a) Some graduates in computer science do not know 'C' language and are computer programmers.
- (b) Some graduates in computer science who are computer programmers do not know 'C' language.
- (c) Some graduates in computer science and some programmers do not know 'C' language.
- (d) Some graduates in computer science are either computer programmers or do not know 'C' language.

1.8 $\sim\forall x(P(x) \to Q(x))$
- (a) Every graduate in computer science is not a computer programmer.
- (b) Some graduates in computer science are computer programmers.
- (c) Some graduates in computer science are not computer programmers.
- (d) Every non-graduate in computer science is not a computer programmer.

1.9 $\sim\exists x(P(x) \wedge Q(x))$
- (a) Every graduate in computer science is not a computer programmer.
- (b) Some graduates in computer science are not computer programmers.
- (c) Some non-graduates in computer science are not computer programmers.
- (d) Every non-graduate in computer science is not a computer programmer.

1.10 $\sim\forall x((Q(x) \wedge R(x)) \to \sim P(x))$
- (a) Every person who either knows 'C' or is a computer programmer is not a graduate in computer science.
- (b) Every person who either does not know 'C' or is not a computer programmer is a graduate in computer science.
- (c) Some computer programmers who know 'C' language are not graduates in computer science.
- (d) Some computer programmers who know 'C' language are graduates in computer science.

1.11 $\sim\forall x(P(x) \to (Q(x) \vee R(x)))$
- (a) Every graduate in computer science is not a computer programmer and does not know 'C' language.

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(b) Every graduate in computer science is either not a computer programmer or does not know 'C' language.
(c) Some graduates in computer science are either not computer programmers or do not know 'C' language.
(d) Some graduates in computer science are neither computer programmers nor know 'C' language.

Use the following for questions 1.12–1.17

Let
$I(x)$: x is an integer.
$R(x)$: x is a real number.
Choose the correct logical expression of the following sentences.

1.12 All real numbers are not integers.
(a) $\forall x(R(x) \rightarrow \sim I(x))$
(b) $\exists x(R(x) \rightarrow \sim I(x))$
(c) $\forall x(R(x) \wedge \sim I(x))$
(d) $\exists x(R(x) \wedge \sim I(x))$

1.13 All integers are real numbers.
(a) $\forall x(I(x) \wedge R(x))$
(b) $\forall x(R(x) \rightarrow I(x))$
(c) $\exists x(R(x) \wedge I(x))$
(d) $\forall x(I(x) \rightarrow \sim R(x))$

1.14 Not every real number is an integer.
(a) $\forall x(R(x) \rightarrow \sim I(x))$
(b) $\exists x(R(x) \rightarrow \sim I(x))$
(c) $\sim \forall x(R(x) \rightarrow \sim I(x))$
(d) $\exists x(R(x) \wedge I(x))$

1.15 Square of every negative integer is positive.
(a) $\forall x\lfloor (I(x) \wedge (x<0)) \rightarrow (x^2>0)\rfloor$
(b) $\forall x\lfloor (I(x) \wedge (x<0)) \wedge (x^2>0)\rfloor$
(c) $\forall x\lfloor (I(x) \wedge (x<0)) \rightarrow (x^2>0)\rfloor$
(d) $\exists x\lfloor (I(x) \wedge (x<0)) \wedge (x^2>0)\rfloor$

1.16 Square of every non-negative integer is non-negative.
(a) $\forall x\lfloor (I(x) \wedge (x>0)) \rightarrow (x^2>0)\rfloor$
(b) $\forall x\lfloor (I(x) \wedge (x\geq 0)) \rightarrow (x^2\geq 0)\rfloor$
(c) $\forall x\lfloor (I(x) \wedge (x<0)) \rightarrow (x^2>0)\rfloor$
(d) $\exists x\lfloor (I(x) \wedge (x\geq 0)) \wedge (x^2>0)\rfloor$

1.17 Which of the following is false?
(a) $\forall x\forall y P(x, y) \Leftrightarrow \forall y\forall x P(x, y)$
(b) $\exists x\forall y P(x, y) \Rightarrow \forall y\exists x P(x, y)$
(c) $\exists y\forall x P(x, y) \Rightarrow \exists x\forall y P(x, y)$
(d) $\exists x\exists y P(x, y) \Rightarrow \exists y\exists x P(x, y)$

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