Total Variational Image Denoising

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***Abstract*—This report presents an analysis of total variational image denoising, a convex optimization problem, by providing the mathematical formulation and exploring the existing solutions proposed in the literature. The significance of this problem in image processing applications is highlighted, and the mathematical intricacies of total variational image denoising are discussed in detail. The effectiveness of various techniques is validated by implementing them using basic Python libraries and presenting the results obtained. This analysis provides valuable insights into the strengths and limitations of each technique, shedding light on the performance of total variational image denoising.**

I. Introduction

Noise can be introduced in an image during various stages of image acquisition, compression, transmission, and storage due to several factors such as environmental conditions, acquisition devices, and transmission channels. The quality of an image is vital for several image processing tasks such as object detection, tracking, segmentation, medical image analysis, and video processing. Noisy images negatively impact the quality of these ancillary image and video processing activities. Hence, image denoising is a crucial pre-processing step for most image processing and analysis systems. The primary objective of image denoising is to remove noise while retaining the essential features and details of the image. A wide range of techniques for image denoising exists, from linear filtering methods, such as Gaussian smoothing, to more advanced techniques such as total variation denoising.

Total variation (TV) denoising is a popular technique due to its ability to remove noise effectively while preserving edges, making it superior to other denoising techniques that may blur edges or introduce artifacts. In this report, we explore the total variation denoising optimization problem and the algorithms used to solve the problem. The Algorithms were tested on the standard Lena Image and results have been discussed

The formulation of the problem as a convex optimization problem provides a solid theoretical foundation for total variation denoising. This motivates us to explore this topic despite the development of several image-denoising algorithms since Total Variation Denoising was introduced.

Total variation denoising was first introduced by Rudin, Osher, and Fatemi in their paper ”Nonlinear Total Variation Based Noise Removal Algorithms” in 1992 [1].

II. Formulating the Problem and Total Variation

A commonly used noise model for images is the additive white Gaussian noise (AWGN) model. Given a clean image $x : \Omega \to \mathbb{R}$, the noisy image $u_0 : \Omega \to \mathbb{R}$ is obtained by adding Gaussian noise η to x:

$$u_0(x, y) = u(x, y) + \eta(x, y) \tag{1}$$

where $\eta(x, y) \sim \mathcal{N}(0, \sigma^2)$ is a Gaussian random variable with mean 0 and variance σ^2.

Several approaches were taken to solve for the clean image. One such approach was the least-squares method which seeks to find the denoised image u by minimizing the sum of squared differences between the noisy image u_0 and the denoised image u.

$$\min_u \int_\Omega (u - u_0)^2 dx \tag{2}$$

However, this method tends to produce a denoised image that is overly smooth and loses important details, such as edges [1]. Another approach is to minimise the L1-Norm instead, i.e., minimizing the sum of the absolute differences between the noisy image u_0 and the denoised image u.

$$\min_u \int_\Omega |u - u_0| dx \tag{3}$$

While this method can preserve edges better than the least-squares method, it may still introduce artifacts in the denoised image.

The total variation (TV) of an image is a measure of its ”smoothness” and is defined as the integral of the magnitude of the gradient. Given an image $u : \Omega \to \mathbb{R}$, the total variation is given by:

$$TV(u) = \int_\Omega |\nabla u| dx \tag{4}$$

where ∇u denotes the gradient of u and $|\nabla u|$ is the magnitude of the gradient. The TV denoising problem as an energy minimization problem, subject to a constraint on the L2 norm difference between the denoised image u and the original noisy image u_0. The constraint was introduced to ensure the denoised image remains close to the original noisy image [2]. The problem can be formulated as:

$$\min_u \quad \text{TV}(u) \quad \text{s.t} \quad \frac{1}{2}||u - u_0||_2^2 \leq \delta \tag{5}$$

The problem is clearly a convex optimization problem, as both the objective and constraint are convex. To incorporate the constraint into the optimization problem, we introduce a Lagrange multiplier $\lambda > 0$. The Lagrangian for the constrained optimization problem is given by:

$$L(u, \lambda) = \int_\Omega |\nabla u| \, dx + \frac{\lambda}{2}(|u - u_0|^2 - \delta^2) \tag{6}$$

III. Iterative Algorithms

A. Gradient Descent-based Method

Gradient descent is a popular optimization algorithm that can be used to solve the Total Variation (TV) denoising problem. The method iteratively updates the image by moving in the direction of the negative gradient of the objective function. In this section, we present a detailed step-by-step overview of the gradient descent-based denoising algorithm and the required mathematical background.

Given a noisy image u_0, the TV denoising problem can be formulated as:

$$\min_u \text{TV}(u) + \frac{\lambda}{2}||u - u_0||_2^2, \tag{7}$$

where λ is a regularization parameter controlling the trade-off between denoising and fidelity to the original image. The Total Variation of an image is defined as:

The gradient descent-based denoising algorithm starts with an initial guess $u^{(0)} = u_0$ and iteratively updates u using the following update rule:

$$u^{(k+1)} = u^{(k)} - \alpha \nabla J(u^{(k)}), \tag{8}$$

where k is the iteration index, α is the step size, and $\nabla J(u)$ is the gradient of the objective function

$$J(u) = \text{TV}(u) + \frac{\lambda}{2}||u - u_0||_2^2 \tag{9}$$

The gradient of the objective function can be calculated as:

$$\nabla J(u) = \nabla \cdot \frac{\nabla u}{|\nabla u|} + \lambda(u - u_0), \tag{10}$$

where $\nabla \cdot$ denotes the divergence operator.

The gradient descent-based denoising algorithm converges to the optimal solution for the TV denoising problem under appropriate conditions on the step size α. If not chosen appropriately, convergence might be slower compared to other more advanced optimization algorithms like Chambolle's method [**?**]. However, gradient descent is easy to implement and can be a viable solution method for small to moderately-sized denoising problems. For large-scale problems or cases where faster convergence is required, more advanced optimization algorithms like Chambolle's method may be preferable.

B. Chambolle's Denoising Method

In Chambolle et al. [2], the author proposes a fast algorithm for solving the Total Variation (TV) denoising problem. The TV denoising problem is formulated as follows:

In the paper, the authors propose an algorithm for total variation minimization by solving the following problem:

$$\min_u \frac{1}{2\lambda}||u - g||_2^2 + J(u) \tag{11}$$

where u is the denoised image, g is the noisy image, and $J(u)$ is the total variation of the image u. The Euler equation for this problem can be rewritten as:

$$\frac{g}{\lambda} \in \frac{g-u}{\lambda} + \frac{1}{\lambda}\partial J^*(\frac{g-u}{\lambda}), \tag{12}$$

where ∂J^* is the subdifferential of the convex conjugate of J. (For a convex function $f : \mathbb{R}^n \to \mathbb{R}$, the subdifferential of f at a point $x \in \mathbb{R}^n$ is the set of all subgradients of f at x) The solution u can be obtained as $u = g - \pi_{\lambda K}(g)$, where $\pi_{\lambda K}(g)$ is the nonlinear projection of g onto the set λK.

The authors propose a semi-implicit gradient descent algorithm to compute the nonlinear projection $\pi_{\lambda K}(g)$. The algorithm involves choosing a step size $\tau > 0$, initializing $p^{(0)} = 0$, and updating $p^{(n+1)}$ using the following equation:

$$p^{(n+1)}i,j = \frac{p^{(n)}i,j + \tau(\nabla(\text{div}, p^{(n)} - \frac{g}{\lambda}))i,j}{1 + \tau|(\nabla(\text{div}, p^{(n)} - \frac{g}{\lambda}))i,j|}. \tag{13}$$

In this iterative process, the variable p represents the dual variable associated with the gradient of the denoised image. It is used to enforce the smoothness constraint while preserving important details like edges in the denoised image.

The authors prove that when $\tau \leq 1/8$, the sequence $\lambda\text{div}, p^{(n)}$ converges to $\pi_{\lambda K}(g)$ as $n \to \infty$ [2]. However, they note that in practice, the optimal constant for stability and convergence seems to be $1/4$, and they do not know the reason for this discrepancy.

IV. Implementation

In this section, we present our transition to the discrete domain and the subsequent implementation of three denoising methods. In the discrete domain, images are represented as a grid of pixels, and the definition of Total Variation (TV) is adapted accordingly.

The Total Variation in the digital domain is defined as the sum of the magnitudes of the gradient of the image, which can be mathematically represented as:

$$TV(u) = \sum_{i,j} \sqrt{(\nabla_x u_{i,j})^2 + (\nabla_y u_{i,j})^2} \tag{14}$$

where $u_{i,j}$ denotes the pixel intensity at position (i, j), and ∇_x and ∇_y represent the discrete gradient operators in the horizontal and vertical directions, respectively. On expanding,

$$TV(X) = \sum_{i,j} \sqrt{(u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2} \tag{15}$$

A. Denoising Method Implementations

All images were normalised to values between 0 to 1. Noise has been added to the normalised images.

We implemented three methods for TV denoising as follows:

- **CVX:** This method utilizes the CVxpy library for convex optimization. This library provides a powerful framework for solving various convex optimization problems. The cvxpy.tv() function assists in formulating the objective equation.

Fig. 1. Result of Image Denoising

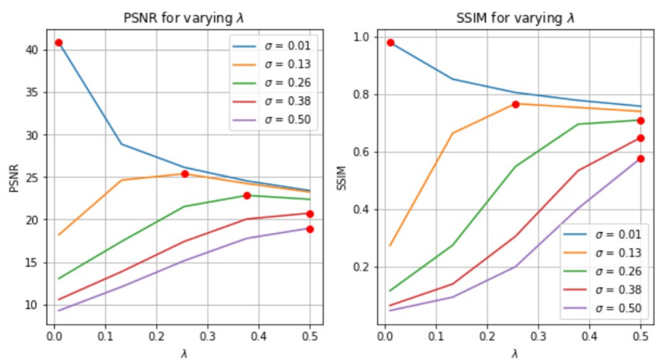

Fig. 2. Variation of image metrics for varying λ and σ

Noise std dev	PSNR (dB)	SSIM
0.01	39.77	0.97
0.06	29.05	0.84
0.12	26.20	0.77
0.17	24.54	0.73
0.23	23.37	0.70
0.28	22.14	0.68
0.34	21.28	0.68
0.39	20.42	0.67
0.45	19.64	0.66
0.50	18.65	0.65

TABLE I
SSIM AND PSNR FOR VARYING VALUES OF σ

- **Gradient Descent-Based:** This model was implemented using the gradient descent algorithm described in Section 3.1.
- **Chambolle's Denoising:** This method was implemented using the algorithm described in Section 3.2.

B. Image Quality Metrics: PSNR and SSIM

To quantitatively evaluate the performance of the denoising methods, two image quality metrics are commonly used, Peak Signal-to-Noise Ratio (PSNR) and Structural Similarity Index (SSIM).

PSNR measures the ratio of the maximum possible power of a signal to the power of corrupting noise that affects the fidelity of its representation. It is defined as:

$$PSNR = 10 \log_{10} \left(\frac{L^2}{MSE} \right) \tag{16}$$

where L is the maximum pixel intensity (e.g., for 8-bit representation, 255), and MSE is the mean squared error between the original and denoised images. In our implementation, the image has been normalized, so L =1.

SSIM measures the structural similarity between two images by comparing their luminance, contrast, and structure. It is defined as:

$$SSIM(x, y) = \frac{(2\mu_x \mu_y + c_1)(2\sigma_{xy} + c_2)}{(\mu_x^2 + \mu_y^2 + c_1)(\sigma_x^2 + \sigma_y^2 + c_2)} \tag{17}$$

where μ_x, μ_y, σ_x, σ_y, and σ_{xy} are the means, standard deviations, and cross-covariance of the luminance and contrast of the two images x and y, and c_1 and c_2 are constants to stabilize the division.

PSNR and SSIM values were computed for the denoised images using the original images as ground truth. The results are presented in Table 1.

C. Comparison of Methods

Out of all three implementations, the CVXpy-based implementation performs the best, as shown in Fig 1. Chambolle's Algorithm performs reasonably well. The performance of the Gradient-based method was considerably poorer. This could be attributed to an incorrect or suboptimal implementation of the gradient descent algorithm. It is also possible that the chosen step size or convergence criteria were not suitable for this particular problem, leading to subpar results.

D. Results and Observations

Figure 1 presents the results obtained from all three implementations for a noisy image with a PSNR of 20.04 dB. The CVXpy implementation demonstrates a significant improvement in the noisy image. The implementation of Chambolle's Algorithm produces satisfactory results, while the Gradient Descent-Based Implementation further degrades the image quality.

Figure 2 illustrates the variation of PSNR and SSIM for different values of λ (regularization parameter) and σ. The red dot's indicate the value of λ for which PSNR/SSIM is maximum. Empirically, it has been observed that setting $\lambda \approx 1.5\sigma$ yields the optimal PSNR/SSIM value. Table 1 displays the variation of PSNR and SSIM with varying values of σ. As anticipated, both the PSNR and SSIM decrease as the standard deviation of the noise increases.

V. Conclusion

In recent years, image denoising algorithms have advanced significantly, with modern methods such as BM3D (Block Match 3D Filtering), NLM (Non-Local Means), and DnCNN (feed-forward denoising convolutional neural network) outperforming Total Variation (TV) denoising by a considerable margin. Despite this, the formulation of TV denoising as a convex optimization problem continues to make it an intriguing subject for research and study.

Throughout this project, I have delved into the optimization problem and developed a comprehensive understanding of the iterative solutions discussed in Section 3. Additionally, I have implemented a CVXpy-based denoiser that demonstrates satisfactory performance. However, my implementation of both the Gradient Descent and Chambolle's algorithms requires further refinement.

As I move forward, my future work will focus on deepening my theoretical understanding of the algorithms and successfully implementing them to enhance the performance of the denoising process. This will allow me to further explore the potential of TV denoising in the context of contemporary image denoising techniques and assess its relevance in an ever-evolving field.

References

[1] Rudin, Leonid Osher, Stanley Fatemi, Emad. (1992). Nonlinear total variation-based noise removal algorithms. Physica D: Nonlinear Phenomena. 60. 259-268. 10.1016/0167-2789(92)90242-F.

[2] Chambolle, A. (2004). An algorithm for total variation minimization and applications. Journal of Mathematical Imaging and Vision, 20(1-2), 89-97.