ENPM-667 Control of Robotic Systems
Final Project Report

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Design of LQR and LQG controllers for Cart attached to Two Pendulums

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Part A: Dynamic model of the system

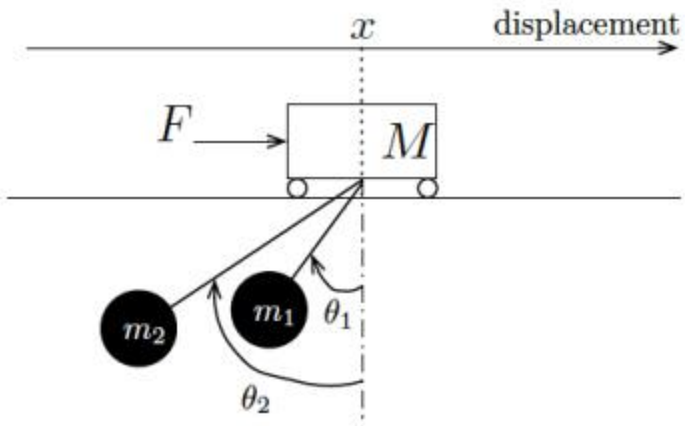


Figure 1:System Diagram

Symbols and Units used:

M_{cart}:	mass of cart	(kg)
m_1:	mass of ball 1	(kg)
m_2:	mass of ball 2	(kg)
l_1:	length or rod 1	(m)
l_2:	length of rod 2	(m)
θ_1:	angle made by rod of ball 1 with normal.	(rad)
θ_2:	angle made by rod of ball 2 with normal.	(rad)
$\dot{\theta}_1$:	angular velocity of rod 1.	(rad/s)
$\dot{\theta}_2$:	angular velocity of rod 2.	(rad/s)
$\ddot{\theta}_1$:	angular acceleration of rod 1.	(rad/s^2)
$\ddot{\theta}_2$:	angular acceleration of rod 2.	(rad/s^2)
$\dot{x}$:	velocity of cart.	(m/s)
$\ddot{x}$:	acceleration of cart.	(m/s^2)
g:	gravitational constant	(9.8 m/s^2)

The dynamic model of the system will be obtained using the Euler-Lagrange method. The coordinates we have here are (x, θ_1, θ_2).

The equations of motion are given by,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0$$

The Lagrangian (L) of the system describes the difference between the Kinetic Energy and the Potential Energy of the system. We will compute the Lagrangian in the next section.

1) Kinematics of the system

Let us start by defining the position of the objects in the system.

1. Position of the cart:

$$X_{cart} = x$$
$$Y_{cart} = o$$

2. Position of $Mass_1$:

$$x_1 = x - l_1 sin\theta_1$$
$$y_1 = -l_1 cos\theta_1$$

3. Position of $Mass_2$:

$$x_2 = x - l_2 sin\theta_2$$
$$y_2 = -l_2 cos\theta_2$$

4. Velocity of cart:

$$\dot{x}_{cart} = \dot{x}$$
$$\dot{y}_{cart} = 0$$

5. Velocity of $Mass_1$:

$$\dot{x}_1 = \dot{x} - l_1\dot{\theta}_1 cos\theta_1$$
$$\dot{y}_1 = l_1\dot{\theta}_1 sin\theta_1$$

6. Velocity of $Mass_2$:

$$\dot{x}_2 = \dot{x} - l_2\dot{\theta}_2 cos\theta_2$$
$$\dot{y}_2 = l_2\dot{\theta}_2 sin\theta_2$$

2) Calculation of the Lagrangian

The Lagrangian is given by the difference between Kinetic Energy and Potential Energy of the system. Kinetic Energy of the system is given by,

$$Kinetic\ Energy(KE) = \frac{1}{2}M_{cart}(\dot{x})^2 + \frac{1}{2}m_1\left(\dot{x}_1{}^2 + \dot{y}_1{}^2\right) + \frac{1}{2}m_2\left(\dot{x}_2{}^2 + \dot{y}_2{}^2\right)$$

$$KE = \frac{1}{2}M_{cart}(\dot{x})^2 + \frac{1}{2}m_1\left(\dot{x}^2 + l_1{}^2\dot{\theta}_1{}^2 - 2\dot{x}l_1\dot{\theta}_1 cos\theta_1\right) + \frac{1}{2}m_2\left(\dot{x}^2 + l_2{}^2\dot{\theta}_2{}^2 - 2\dot{x}l_2\dot{\theta}_2 cos\theta_2\right)$$

$$KE = \frac{1}{2}(M_{cart} + m_1 + m_2)\dot{x} - m_1 l_1 \dot{x}\dot{\theta}_1 cos\theta_1 + \frac{1}{2} m_1 {l_1}^2 {\dot{\theta}_1}^2 - m_2 l_2 \dot{x}\dot{\theta}_2 cos\theta_2 + \frac{1}{2} m_2 {l_2}^2 {\dot{\theta}_2}^2$$

The Potential Energy of the system is given by,

$$Potential\ Energy(PE) = -m_1 g l_1 cos\theta_1 - m_2 g l_2 cos\theta_2$$

The Lagrangian is given by,

$$L = Kinetic\ Energy - Potential\ Energy$$

$$L = \frac{1}{2}(M_{cart} + m_1 + m_2)\dot{x} - m_1 l_1 \dot{x}\dot{\theta}_1 cos\theta_1 + \frac{1}{2} m_1 {l_1}^2 {\dot{\theta}_1}^2 - m_2 l_2 \dot{x}\dot{\theta}_2 cos\theta_2 + \frac{1}{2} m_2 {l_2}^2 {\dot{\theta}_2}^2 - m_1 g l_1 cos\theta_1 + m_2 g l_2 cos\theta_2$$

3) Dynamic Equations of the system

Now we compute the equation of motion using this Lagrangian,

$$\frac{\partial L}{\partial \dot{x}} = M_{cart}\dot{x} - m_1 l_1 \dot{\theta}_1 cos\theta_1 - m_2 l_2 \dot{\theta}_2 cos\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M_{cart} + m_1 + m_2)\ddot{x} - m_1 l_1 \left(\ddot{\theta}_1 cos\theta_1 - {\dot{\theta}_1}^2 sin\theta_1\right) - m_2 l_2 \left(\ddot{\theta}_2 cos\theta_2 - {\dot{\theta}_2}^2 sin\theta_2\right)$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1\left(2{l_1}^2\dot{\theta}_1 - 2\dot{x} l_1 \dot{\theta}_1 cos\theta_1\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 {l_1}^2 \ddot{\theta}_1 + m_1 \dot{x}\dot{\theta}_1 l_1 sin\theta_1 - m_1 \ddot{x} l_1 cos\theta_1$$

$$\frac{\partial L}{\partial \theta_1} = m_1 \dot{x} l_1 \dot{\theta}_1 sin\theta_1 - m_1 g l_1 sin\theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2\left(2{l_2}^2\dot{\theta}_2 - 2\dot{x} l_2 \dot{\theta}_2 cos\theta_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 {l_2}^2 \ddot{\theta}_2 + m_2 \dot{x}\dot{\theta}_2 l_2 sin\theta_2 - m_2 \ddot{x} l_2 cos\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = m_2 \dot{x} l_2 \dot{\theta}_2 sin\theta_2 - m_2 g l_2 sin\theta_2$$

The first dynamic equation is given by,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

$$F = (M_{cart} + m_1 + m_2)\ddot{x} - m_1 l_1 \left(\ddot{\theta}_1 cos\theta_1 - \dot{\theta}_1{}^2 sin\theta_1\right) - m_2 l_2 \left(\ddot{\theta}_2 cos\theta_2 - \dot{\theta}_2{}^2 sin\theta_2\right)$$

Rearranging the equation,

$$\ddot{x} = \frac{m_1 l_1 \left(\ddot{\theta}_1 cos\theta_1 - \dot{\theta}_1{}^2 sin\theta_1\right) + m_2 l_2 \left(\ddot{\theta}_2 cos\theta_2 - \dot{\theta}_2{}^2 sin\theta_2\right) + F}{M_{cart} + m_1 + m_2}$$

The second dynamic equation is given by,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0$$

$$m_1 l_1{}^2 \ddot{\theta}_1 + m_1 \dot{x}\dot{\theta}_1 l_1 sin\theta_1 - m_1 \ddot{x} l_1 cos\theta_1 - \left(m_1 \dot{x} l_1 \dot{\theta}_1 sin\theta_1 - m_1 g l_1 sin\theta_1\right) = 0$$

$$l_1{}^2 \ddot{\theta}_1 - m_1 l_1 cos\theta_1 \ddot{x} + m_1 l_1 g.sin\theta_1 = 0$$

Rearranging the equation,

$$\ddot{\theta}_1 = \frac{m_1 l_1 cos\theta_1 \ddot{x} - m_1 l_1 g.sin\theta_1}{l_1{}^2}$$

The third dynamic equation is given by,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0$$

$$m_2 l_2{}^2 \ddot{\theta}_2 + m_2 \dot{x}\dot{\theta}_2 l_2 sin\theta_2 - m_2 \ddot{x} l_2 cos\theta_2 - \left(m_2 \dot{x} l_2 \dot{\theta}_2 sin\theta_2 - m_2 g l_2 sin\theta_2\right) = 0$$

$$l_2{}^2 \ddot{\theta}_2 - m_2 l_2 cos\theta_2 \ddot{x} + m_2 l_2 g.sin\theta_2 = 0$$

Rearranging the equation,

$$\ddot{\theta}_2 = \frac{m_2 l_2 cos\theta_2 \ddot{x} - m_2 l_2 g.sin\theta_2}{l_2{}^2}$$

The dynamic equations of the model are,

$$\ddot{x} = \frac{m_1 l_1 \left(\ddot{\theta}_1 cos\theta_1 - \dot{\theta}_1{}^2 sin\theta_1\right) + m_2 l_2 \left(\ddot{\theta}_2 cos\theta_2 - \dot{\theta}_2{}^2 sin\theta_2\right) + F}{M_{cart} + m_1 + m_2}$$

$$\ddot{\theta}_1 = \frac{m_1 l_1 cos\theta_1 \ddot{x} - m_1 l_1 g.sin\theta_1}{l_1{}^2}$$

$$\ddot{\theta}_2 = \frac{m_2 l_2 cos\theta_2 \ddot{x} - m_2 l_2 g.sin\theta_2}{l_2{}^2}$$

Simplifying these equations we get,

$$\ddot{x} = \frac{F - \left(\left(\frac{g}{2}\right)\left(m_1 sin(2\theta_1) + m_2 sin(2\theta_2)\right) - m_1 l_1 \dot{\theta}_1{}^2 sin\theta_1 - m_2 l_2 \dot{\theta}_2{}^2 sin\theta_2\right)}{(M_{cart} + m_1 sin^2\theta_1 + m_2 sin^2\theta_2)}$$

$$\ddot{\theta}_1 = \frac{1}{l_1}(\ddot{x}cos\theta_1 - gsin\theta_1)$$

$$\ddot{\theta}_2 = \frac{1}{l_2}(\ddot{x}cos\theta_2 - gsin\theta_2)$$

Part B: State-Space Representation of the system

We further calculate the state space using the Jacobian,

The state and state variables can be defined as,

$$\dot{x}(t) = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}, \quad x(t) = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

We substitute $\dot{x}(t) = 0$ in the Jacobian and try to linearize the system around $x = 0,\ \theta_1 = 0,\ \theta_2 = 0$

Hence the matrix A of the system is given by,

$$A = \nabla F_{x,\theta_1,\theta_2}\Big|_{(0,0,0)} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \theta_1} & \frac{\partial \dot{x}}{\partial \dot{\theta}_1} & \frac{\partial \dot{x}}{\partial \theta_2} & \frac{\partial \dot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial \theta_1} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_1} & \frac{\partial \ddot{x}}{\partial \theta_2} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{\theta}_1}{\partial x} & \frac{\partial \dot{\theta}_1}{\partial \dot{x}} & \frac{\partial \dot{\theta}_1}{\partial \theta_1} & \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \dot{\theta}_1}{\partial \theta_2} & \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{\theta}_1}{\partial x} & \frac{\partial \ddot{\theta}_1}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_1}{\partial \theta_1} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_1}{\partial \theta_2} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{\theta}_2}{\partial x} & \frac{\partial \dot{\theta}_2}{\partial \dot{x}} & \frac{\partial \dot{\theta}_2}{\partial \theta_1} & \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_1} & \frac{\partial \dot{\theta}_2}{\partial \theta_2} & \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{\theta}_2}{\partial x} & \frac{\partial \ddot{\theta}_2}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_2}{\partial \theta_1} & \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_2}{\partial \theta_2} & \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2} \end{bmatrix}_{(0,0,0)}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{-m_1 g}{M_{cart}} & 0 & \dfrac{-m_2 g}{M_{cart}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{(M_{cart} + m_1) g}{M_{cart} l_1} & 0 & \dfrac{-m_2 g}{M_{cart} l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-m_1 g}{M_{cart} l_2} 0 & 0 & \dfrac{(M_{cart} + m_2) g}{M_{cart} l_2} & 0 \end{bmatrix}$$

The matrix B can be obtained as follows,

$$B = \nabla F_u|_{(0,0,0)} = \begin{bmatrix} \dfrac{\partial \dot{x}}{\partial u} \\ \dfrac{\partial \ddot{x}}{\partial u} \\ \dfrac{\partial \dot{\theta}_1}{\partial u} \\ \dfrac{\partial \ddot{\theta}_1}{\partial u} \\ \dfrac{\partial \dot{\theta}_2}{\partial u} \\ \dfrac{\partial \ddot{\theta}_2}{\partial u} \end{bmatrix} \xrightarrow{yields} B = \begin{bmatrix} 0 \\ 1/M_{cart} \\ 0 \\ 1/M_{cart} l_1 \\ 0 \\ 1/M_{cart} l_2 \end{bmatrix}$$

The state-space representation of the linearized system is given by,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{-m_1 g}{M_{cart}} & 0 & \dfrac{-m_2 g}{M_{cart}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{(M_{cart} + m_1) g}{M_{cart} l_1} & 0 & \dfrac{-m_2 g}{M_{cart} l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-m_1 g}{M_{cart} l_2} 0 & 0 & \dfrac{(M_{cart} + m_2) g}{M_{cart} l_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/M_{cart} \\ 0 \\ 1/M_{cart} l_1 \\ 0 \\ 1/M_{cart} l_2 \end{bmatrix} F$$

Part C: Controllability Check of the Linearized System

First, we do a controllability check on the linearized system. We do this by computing the controllability matrix given by,

$$[B \quad AB \quad A^2B \quad \dots \quad \dots \quad A^{n-1}B]$$

Where $n = \ number\ of\ state\ variables$. If the rank of this matrix is equal to number of state variables, then the system is said to be controllable. For the current system, $n = 6$. The rank of this matrix is obtained using the *ctrb* function in MATLAB.

$$rank\,\{[B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B]\} = 6$$

Hence the system is controllable.

We observe that, the system is uncontrollable for the following conditions:

1. Very high values of M_{cart}:
   From the controllability matrix, we observe that as the value of M_{cart} increases, the value of determinant of the matrix approaches zero. Thus, the system becomes uncontrollable.

```
Controllability:

C =

[       0,      1/M,                                         0,            - (g*m1)/(M^2*l1) - (g*m2)/(M^2*l2),
[     1/M,        0,           - (g*m1)/(M^2*l1) - (g*m2)/(M^2*l2),                                         0,               ((g^2*m1*(M + m1))/(M^2*l1) + (g^2
[       0, 1/(M*l1),                                         0, - (g*(M + m1))/(M^2*l1^2) - (g*m2)/(M^2*l1*l2),
[1/(M*l1),        0, - (g*(M + m1))/(M^2*l1^2) - (g*m2)/(M^2*l1*l2),                                         0, ((g^2*m2*(M + m1))/(M^2*l1^2) + (g^2*m2*(M + m2))
[       0, 1/(M*l2),                                         0, - (g*(M + m2))/(M^2*l2^2) - (g*m1)/(M^2*l1*l2),
[1/(M*l2),        0, - (g*(M + m2))/(M^2*l2^2) - (g*m1)/(M^2*l1*l2),                                         0, ((g^2*m1*(M + m2))/(M^2*l2^2) + (g^2*m1*(M + m1))

Determinant:
-(g^6*(l1 - l2)^2)/(M^6*l1^6*l2^6)

Rank:

ans =

     6
```

2. $l_1 = l_2$:
   If this condition is substituted in the controllability matrix, the rank of the matrix reduces and thus fails the rank condition for controllability.

```
Conditional substitution: (l1=l2)

Ctl =

[       0,      1/M,                                         0,           - (g*m1)/(M^2*l2) - (g*m2)/(M^2*l2),
[     1/M,        0,           - (g*m1)/(M^2*l2) - (g*m2)/(M^2*l2),                                         0,              ((g^2*m1*(M + m1))/(M^2*l2) + (g^2*m1*
[       0, 1/(M*l2),                                         0, - (g*(M + m1))/(M^2*l2^2) - (g*m2)/(M^2*l2^2),
[1/(M*l2),        0, - (g*(M + m1))/(M^2*l2^2) - (g*m2)/(M^2*l2^2),                                         0, ((g^2*m2*(M + m1))/(M^2*l2^2) + (g^2*m2*(M + m2))/(
[       0, 1/(M*l2),                                         0, - (g*(M + m2))/(M^2*l2^2) - (g*m1)/(M^2*l2^2),
[1/(M*l2),        0, - (g*(M + m2))/(M^2*l2^2) - (g*m1)/(M^2*l2^2),                                         0, ((g^2*m1*(M + m1))/(M^2*l2^2) + (g^2*m1*(M + m2))/(

Rank :

ans =

     4

System is not controllable
>>
```

Part D: Design of LQR controller

We substitute the following values and design the controller for the nonlinear as well as the linearized system.

M_{cart}:	mass of cart	(1000kg)
m_1:	mass of ball 1	(100kg)
m_2:	mass of ball 2	(100kg)
l_1:	length or rod 1	(20m)
l_2:	length of rod 2	(10m)

The initial conditions for the system are set as:

$$x = 0, \dot{x} = 0, \theta_1 = \frac{\pi}{6}, \dot{\theta}_1 = 0, \theta_2 = \frac{\pi}{3}, \dot{\theta}_2 = 0$$

If the pair (A, B_K) is stabilizable, then we look for K that minimizes the following cost function,

$$J(K, \vec{x}(0)) = \int_0^{\infty} \vec{x}^T(t) Q \vec{x}(t) + \vec{U}_K{}^T R \vec{U}_k(t).dt$$

Where Q and R, are symmetric positive definite matrices. The Q matrix defines the weights for the state variables. The controller is guaranteed to stabilize the closed loop system and provides a way to trade off response with control effort. The optimal solution for the cost function is given by,

$$K = -R^{-1} B_K{}^T P$$

Here, the P is the symmetric positive solution to the Stationery Riccati equation,

$$A^T P + PA - PBR^{-}B^T P = -Q$$

1) Stability analysis using Lyapunov Indirect Method

Lyapunov's Indirect method offers a way to check the stability of the system. In this method take a non-linear system and linearize it about an equilibrium point (in this case $x = 0,\ \theta_1 = 0,\ \theta_2 = 0$). We check the Eigenvalues of system to determine the stability of the system.

The eigen values of matrix A are

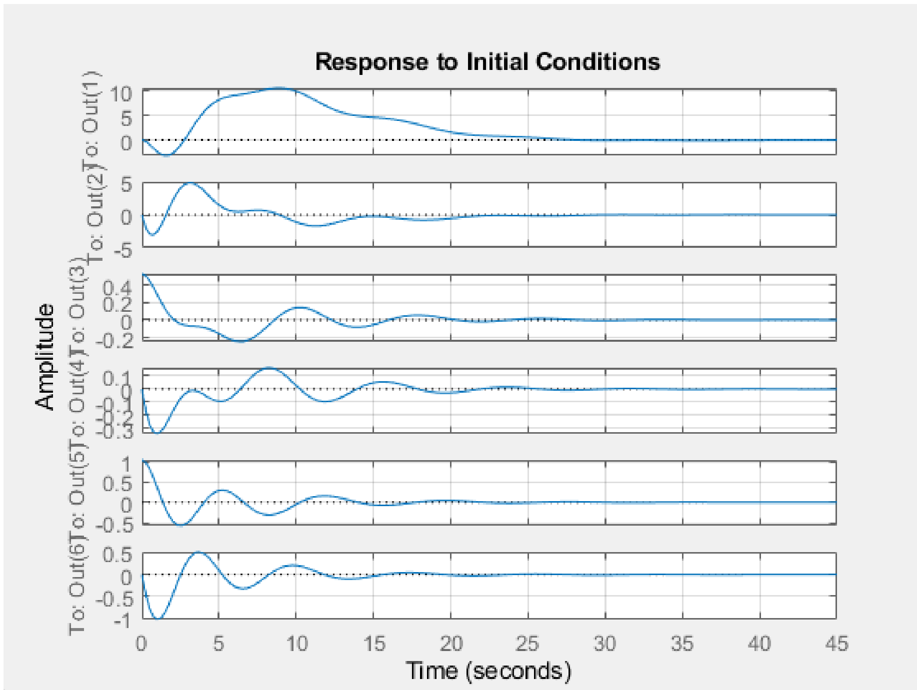
$$\lambda = 0, 0, -1.0344, -0.7407, 0.7407, 1.0344$$

Since two of the eigen values of the system are 0, the stability test of the system is **inconclusive**.

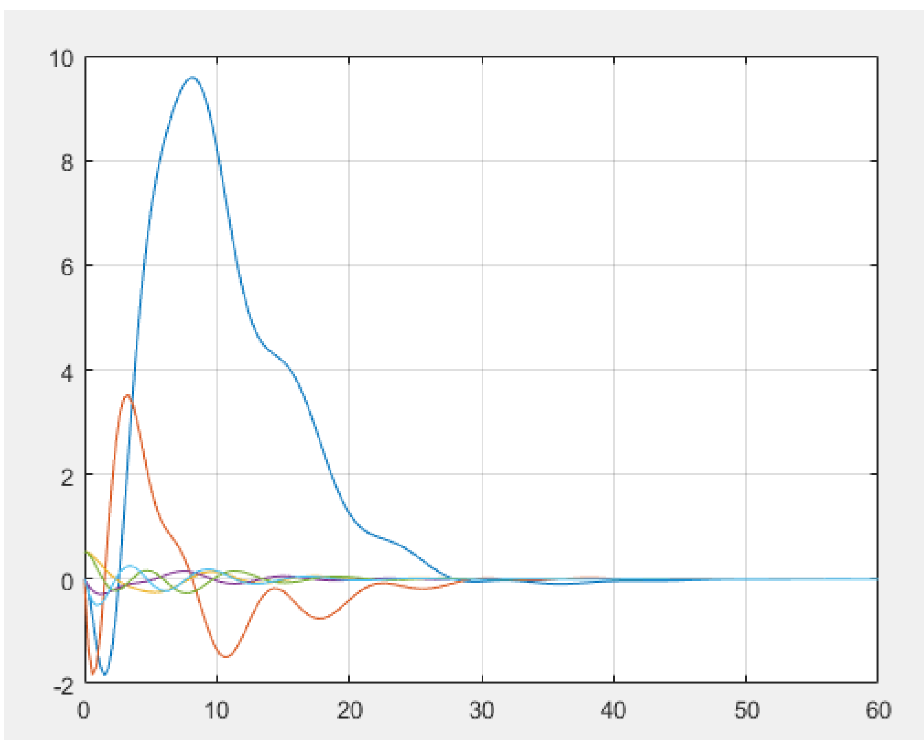
```
A = [0 1 0 0 0 0;
    0 0 -0.981 0 0.981 0;
    0 0 0 1 0 0;
    0 0 0.53955 0 0.04905 0;
    0 0 0 0 0 1;
    0 0 -0.0981 0 1.0791 0];
b=eig(A)
```

```
b = 6×1
         0
         0
   -1.0344
   -0.7407
    0.7407
    1.0344
```

2) LQR response for Linearized System



3) LQR response for the original nonlinear system



Part E: Observability of the System

We calculate the observability of the system for different output vectors. This determines the C matrix of the system. We calculate the Observability matrix and calculate the rank to determine whether the system is observable. The observability matrix is given by,

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

For an observable system, the rank of the observability matrix should be equal to n, where n is the number of state variables. In the current system $n = 6$.

<u>Case 1</u>: $\text{Output vector} = \mathrm{x}(t)$

For this case, C matrix is given by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The observability matrix can be computed in MATLAB using the <u>*obsv*</u> function. The rank of observability matrix in this case is,

$$rank\left\{\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix}\right\} = 6$$

The system is **observable.**

<u>Case 2</u>: $\text{Output vector} = \theta_1(t)\ and\ \theta_2(t)$

For this case, C matrix is given by

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability matrix can be computed in MATLAB using the <u>*obsv*</u> function. The rank of observability matrix in this case is,

$$rank\left\{\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix}\right\} = 4 < n$$

The system is **not observable.**

Case 3: $\text{Output vector} = \mathrm{x}(t)\ and\ \theta_2(t)$

For this case, C matrix is given by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability matrix can be computed in MATLAB using the *obsv* function. The rank of observability matrix in this case is,

$$rank\left\{\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix}\right\} = 6$$

The system is **observable.**

Case 4: $\text{Output vector} = \mathrm{x}(t), \theta_1(t)\ and\ \theta_2(t)$

For this case, C matrix is given by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability matrix can be computed in MATLAB using the *obsv* function. The rank of observability matrix in this case is,

$$rank\left\{\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix}\right\} = 6$$

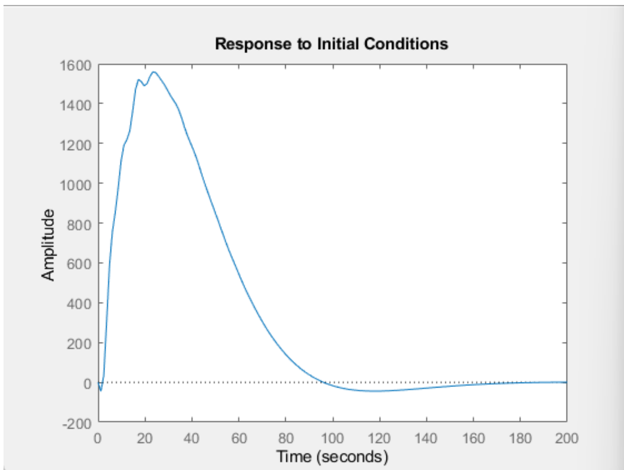
The system is **observable.**

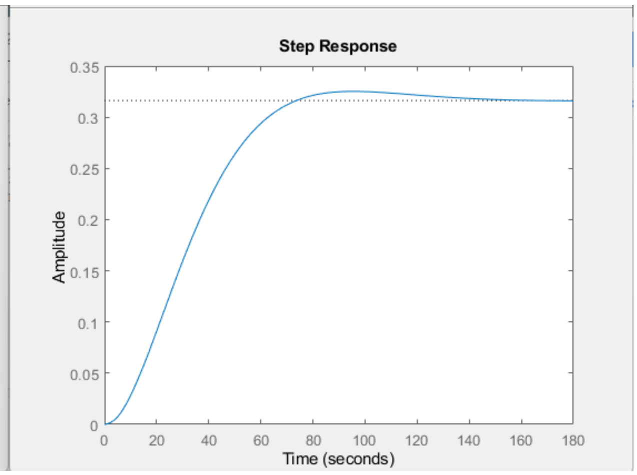
Part F: Design of Luenberger Observer for the system

In this section, we design a state observer for the original nonlinear system as well as the linearized system for all the observable output vectors from the previous section.

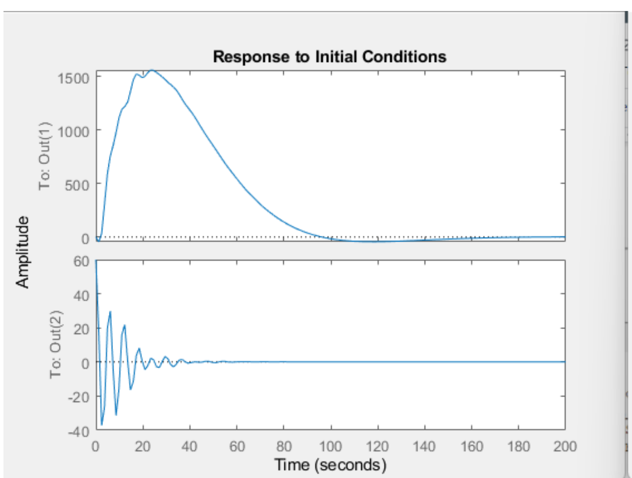
Luenberger Observer for the Linearized system

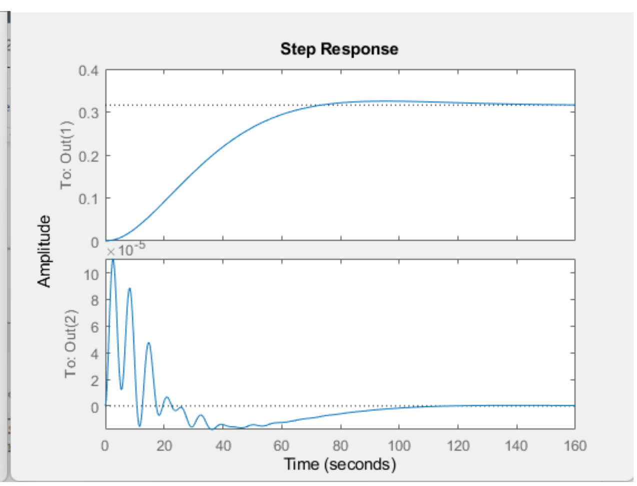
<u>Case 1</u>: Output vector $= x(t)$



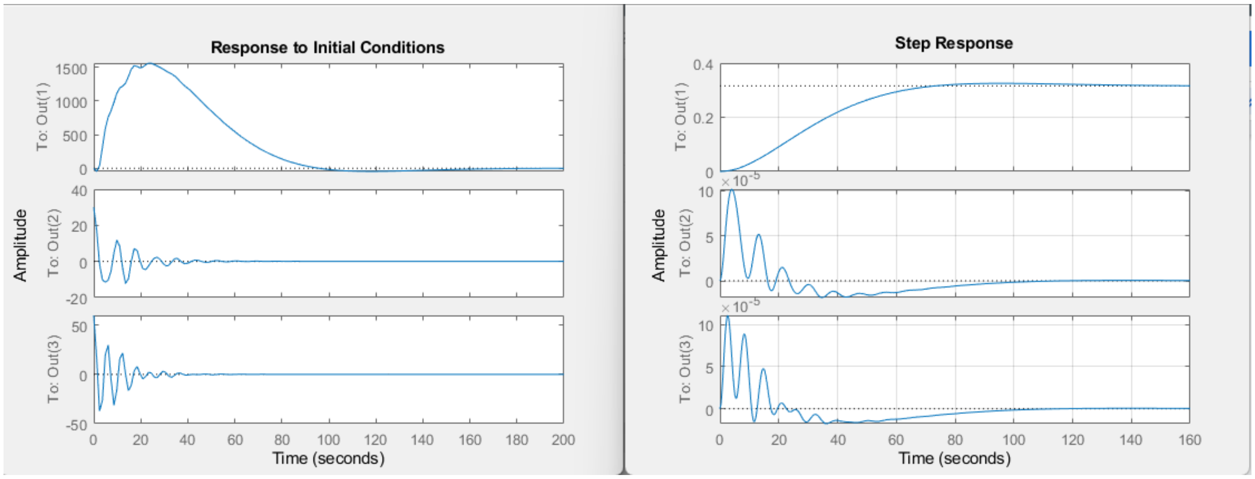


<u>Case 2</u>: Output vector $= x(t), \theta_2(t)$



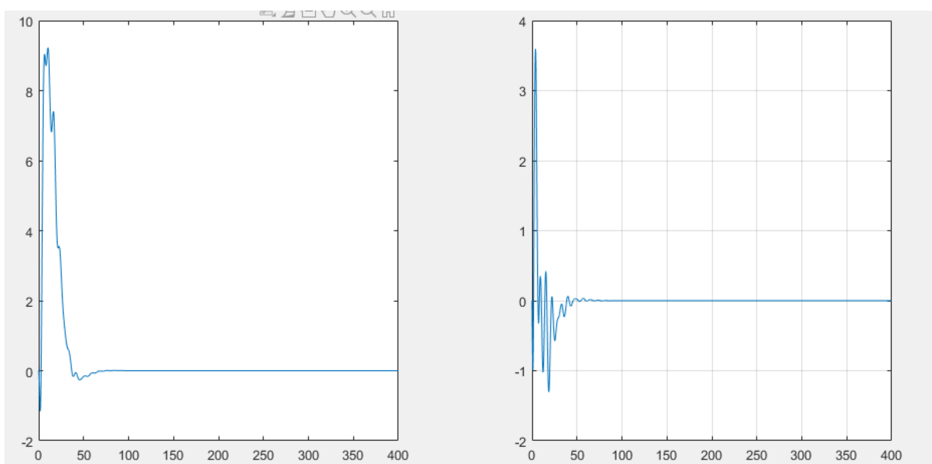


<u>Case 3</u>: Output vector $= x(t), \theta_1(t)\ and\ \theta_2(t)$

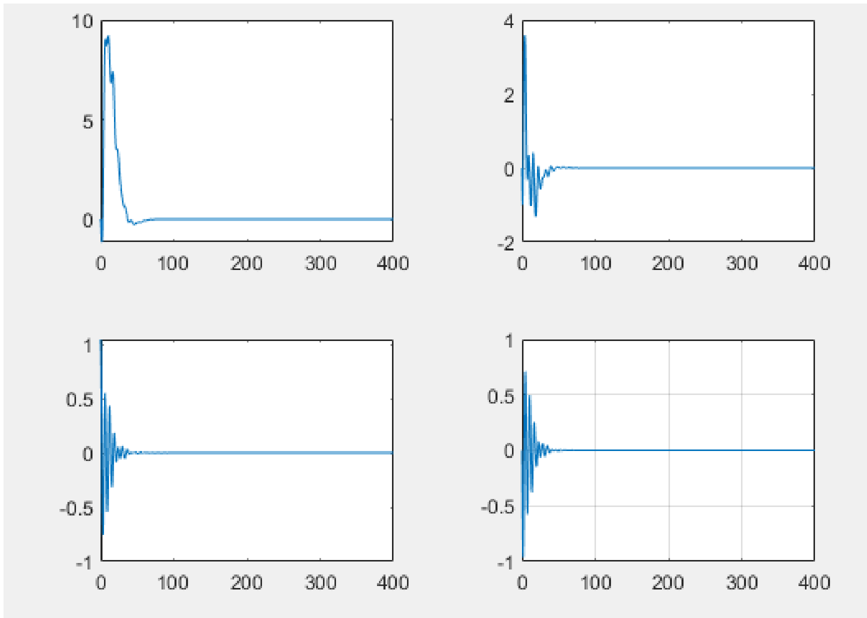


Luenberger Observer for Nonlinear system

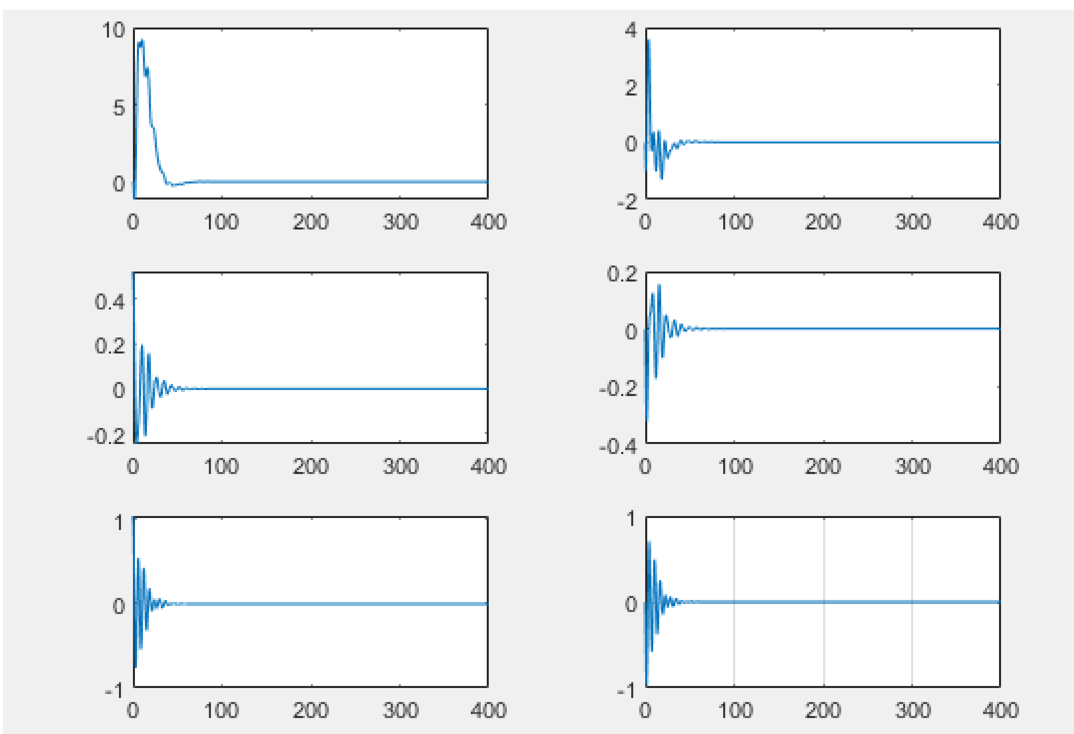
<u>Case 1</u>: Output vector $= x(t)$



<u>Case 2</u>: Output vector $= x(t), \theta_2(t)$

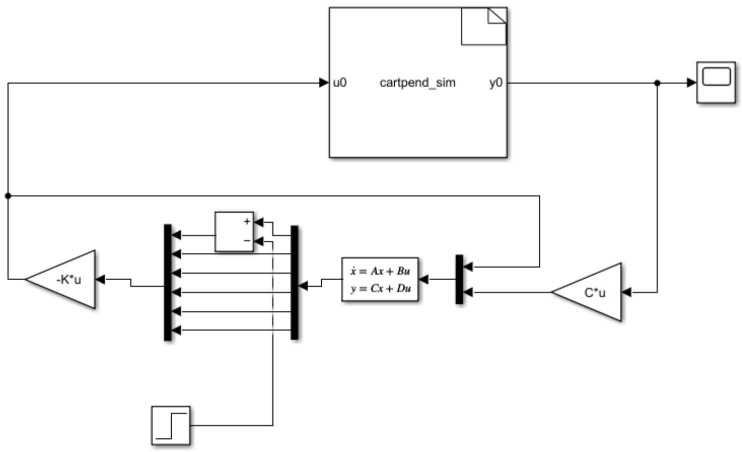


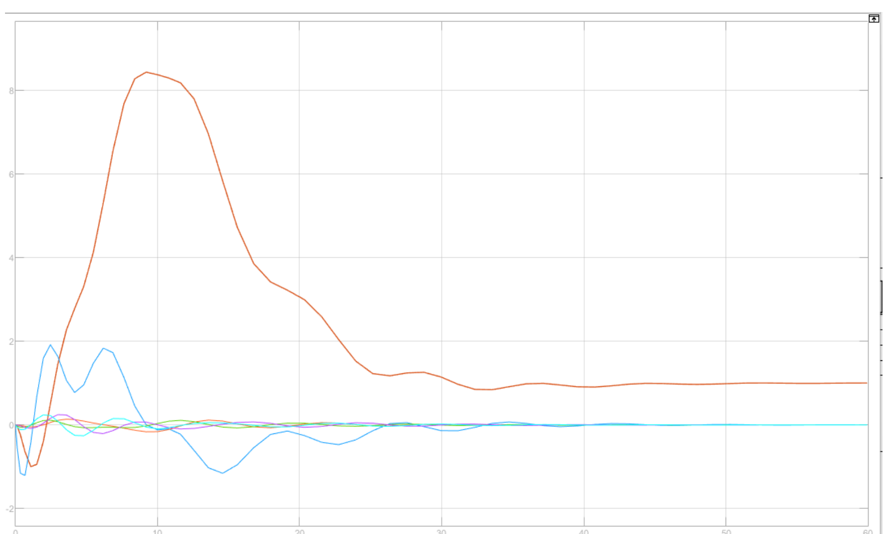
<u>Case 3</u>: Output vector $= x(t), \theta_1(t)\ and\ \theta_2(t)$



Part F: Design of LQG Controller

In this section we design a LQG controller for the nonlinear system. The out put vector chosen for this controller is $\text{Output vector} = x(t)$





Conclusions:

1. Tracking of constant reference on x can be tracked using closed loop observer embedded in the LQG controller.
2. The design does consider the external disturbances.

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