SYLLABUS

Digital Communication - (304181)

Credit	Examination Scheme :
03	End Sem (Theory) : 70 Marks

Unit III Digital Modulation - II

Generation, Reception, Signal Space Representation and Probability of Error Calculation for Quadrature Amplitude Shift Keying (QASK), M-ary FSK (MFSK), Minimum Shift Keying (MSK), Pulse Shaping to reduce Interchannel and Intersymbol Interference, some Issues in transmission and reception, Orthogonal Frequency Division Multiplexing (OFDM), Comparison of digital modulation systems. **(Chapter - 3)**

Unit IV Spread Spectrum Modulation

Use of Spread Spectrum , Direct Sequence (DS) Spread Spectrum, Spread Spectrum and Code Division Multiple Access (CDMA), Ranging Using DS Spread Spectrum, Frequency Hopping (FH) Spread Spectrum,

Pseudorandom (PN) Sequences : Generation and Characteristics, Synchronization in Spread Spectrum Systems **(Chapter - 4)**

Unit V Information Theoretic Approach to Communication System

Introduction to information theory, Entropy and its properties, Source coding theorem, Huffman coding, Shannon-Fano coding, Discrete memory less channel, Mutual information, Channel capacity, Channel coding theorem, Differential entropy and mutual Information for continuous ensembles, Information Capacity theorem. **(Chapter - 5)**

Unit VI Error-Control Coding

Linear Block Codes : Coding, Syndrome and error detection, Error detection and correction capability, Standard array and syndrome decoding. Cyclic Codes : Coding & Decoding, Convolutional Codes: Coding & Decoding, Introduction to Turbo Codes & LDPC Codes. **(Chapter - 6)**

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IMPORTANT FORMULAE

Chapter - 3

i) *BPSK*, BW $= 2f_b$, $d = 2\sqrt{E_b}$

ii) *DPSK*, $BW = f_b$

iii) *QPSK*, $BW = f_b$, $d = 2\sqrt{E_b} = \sqrt{2E_s}$

iv) *M-ary PSK*, BW $= \dfrac{2f_b}{N}$, $d = 2\sqrt{E_s}\sin\dfrac{\pi}{M}$

v) *QASK* or *QAM* $BW = \dfrac{2f_b}{N}$, $d = 2\sqrt{E_s}\sin\dfrac{\pi}{M}$

vi) *BFSK*, BW $= 4f_b$, $d = 2\sqrt{E_b}$

vii) *M-ary FSK*, $BW = \dfrac{2^{N+1}f_b}{N}$, $d = \sqrt{2NE_b}$

viii) *MSK*, $f_H = (m+1)\dfrac{f_b}{4}$, and $f_L = (m-1)\dfrac{f_b}{4}$,

$f_H - f_L = \dfrac{f_b}{2}$ $BW = 1.5f_b$, $d = 2\sqrt{E_b}$

Error Probabilities

$P_e(ASK) = \dfrac{1}{2}erfc\sqrt{\dfrac{E}{4N_o}}$, $P_e(BPSK) = \dfrac{1}{2}erfc\sqrt{\dfrac{E}{N_o}}$

$P_e(BFSK) = \dfrac{1}{2}erfc\sqrt{\dfrac{0.6E}{N_o}}$, $P_e(DPSK) = \dfrac{1}{2}e^{-E_b/N_o}$

$P_e(QPSK) = erfc\sqrt{\dfrac{E}{2N_o}}$

$P_e \leq \dfrac{1}{2}erfc\sqrt{\dfrac{d_k^2}{4N_o}}$ using distance of signal points

$$P_e(M\text{-}ary\,PSK) = erfc\left(\frac{\sqrt{E_s}}{N_o}\sin\frac{\pi}{m}\right),$$

$$P_e(M\text{-}ary\,FSK) = \frac{M-1}{2}\,erfc\sqrt{\frac{E_s}{2N_o}}$$

$$P_e(M\text{-}ary\,FSK,\ non\ coherent) = \frac{M-1}{2}e^{-E_s/2N_o}$$

Chapter - 4

i) *Length of PN sequence,* $N = 2^m - 1,\qquad T_b = NT_c$

ii) *DS-BPSK SS :* $PG = \frac{T_b}{T_c},\qquad P_e = \frac{1}{2}erfc\sqrt{\frac{E_b}{JT_c}}$

Jamming margin, $\frac{J}{P_s} = (PG)_{dB} - \left(\frac{E_b}{N_o}\right)_{dB}$

iii) **FH-SS :** $PG = 2^t,\qquad R_c = R_s = \frac{R_b}{k}$ for slow hopping

$R_c = R_h$ for fast hopping, $\quad P_e = \frac{1}{2}e^{-\gamma_b/2},\ \gamma_b = \frac{E_b}{J_o}.$

Chapter - 5

Entropy, H = $\sum_{K=1}^{M} p_k \log_2 \frac{1}{p_k}$

Information rate, R = rH

Efficiency of source encoder, $\eta = \frac{H}{\overline{N}}$

$$H(X,Y) = H(X/Y)+H(Y) = H(Y/X)+H(X)$$

$$H(X/Y) = \sum_{i=1}^{M}\sum_{j=1}^{M} p(x_i,y_j)\log_2\frac{1}{p(x_i/y_j)}$$

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$$H(Y/X) = \sum_{i=1}^{M} \sum_{j=1}^{M} p(x_i, y_j) \log_2 \frac{1}{p(y_j/x_i)}$$

Average rate of information transmission,

$$D_t = [H(X) - H(X/Y)] \cdot r \text{ bits/sec}$$

'r' is symbols per second.

Capacity of binary symmetric channel,

$$C = 1 - h, \qquad \text{here } h = \sum_{j=1}^{2} p_j \log_2 \frac{1}{p_j}$$

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{M} p(x_i, y_j) \log_2 \frac{p(x_i/y_j)}{p(x_i)}$$

$$I(X;Y) = I(Y:X)$$

$$I(X:Y) = H(X) - H(X/Y) = H(Y) - H(Y/X)$$

$$I(X;Y) = H(X) + H(Y) - H(X, Y)$$

Capacity, $C = \max\limits_{p(x_i)} I(X;Y)$

Shannon Hartley law, $C = B \log_2 \left(1 + \frac{S}{N}\right)$ bits/sec

$$C_\infty = \lim_{B \to \infty} C = 1.44 \frac{S}{N_0}$$

Chapter - 6

Code efficiency, $\eta = \frac{k}{n}$ = Code rate

$$[X]_{1 \times n} = [M]_{1 \times k} [G]_{k \times n}$$

$$G = [I_k : p_{k \times q}]$$

$$H = [p^T : I_q]_{q \times n}$$

$$X = M : C$$

$$[S]_{1 \times q} = [Y]_{1 \times n} [H^T]_{n \times q}$$

$$X = Y + E$$

Non systematic form, $X(p) = M(p) \cdot G(p)$

Systematic form, $C(p) = rem\left[\dfrac{p^q M(p)}{G(p)}\right]$

t^{th} row of G, $p^{n-t} + R_t(p) = Q_t(p)G(p)$

Syndrome, $S(p) = rem\left[\dfrac{Y(p)}{G(p)}\right]$

Code rate $= \dfrac{k}{n}$ (Convolutional codes)

Code dimension $= (n, k)$

Output of convolutional code, $x^{(1)}(p) = g^{(1)}(p) \cdot m(p)$

$x^{(2)}(p) = g^{(2)}(p) \cdot m(p)$

Error Function and Q Function

Error Function Table

x	*erf (x)*	*x*	*erf (x)*
0.00	0.00000	1.10	0.88021
0.05	0.05637	1.20	0.91031
0.10	0.11246	1.25	0.92290
0.15	0.16800	1.30	0.93401
0.20	0.22270	1.35	0.94376
0.25	0.27633	1.40	0.95229

0.30	0.32863	1.45	0.95970
0.35	0.37938	1.50	0.96611
0.40	0.42839	1.55	0.97162
0.45	0.47548	1.60	0.97635
0.50	0.52050	1.65	0.98008
0.55	0.56332	1.70	0.98379
0.60	0.60386	1.75	0.98667
0.65	0.64203	1.80	0.98909
0.70	0.67780	1.85	0.99111
0.75	0.71116	1.90	0.99279
0.80	0.74210	1.95	0.99418
0.85	0.77067	2.00	0.99532
0.90	0.79691	2.50	0.99959
0.95	0.82089	3.00	0.99998
1.00	0.84270	1.15	0.89612
1.05	0.86244		

Q-Function Table

					$Q(x)$					
x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2168	0.2148
0.8	0.2169	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183

2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0094	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002

END...

Unit III

3 DIGITAL MODULATION - II

3.1 : QASK or QAM

Important Points to Remember

- System involving phase as well as amplitude shift keying is called QASK. (QAM).
- N-bits per symbol transmitted.
- Minimum BW is $\frac{2f_b}{N}$ and Euclidean distance is $\sqrt{0.4E_s}$ for M = 16.
- Noise immunity is increased. BW and PSD are similar to M-ary PSK for m = 4.

Q.1 What is QAM ? Draw and explain the transmitter and receiver of QAM. Draw its psd and give its BW requirement.

☞[SPPU : May-13, Marks 4, Dec.-11, Marks 10]

Ans. : QAM or QASK involves amplitude as well as phase shift keying. It is called quadrature amplitude and phase shift keying.

(i) QASK transmitter :

The equation for QASK signal is given by,

$$s(t) = k_1\sqrt{0.2P_s}\cos(2\pi f_0 t)+k_2\sqrt{0.2P_s}\sin(2\pi f_0 t) \quad \dots (Q.1.1)$$

Where k_1 and k_2 defined the amplitude of the modulated signal.

Fig. Q.1.1 shows the transmiter for 4 bit QASK.

Step 1 : Serial to Parallel Conversion :

The input bit stream is applied to a serial to parallel converter which gives the input stream in parallel at its output. Signal $b(t)$ is applied to input. The parallel bits b_k, b_{k+1}, b_{k+2}, b_{k+3}

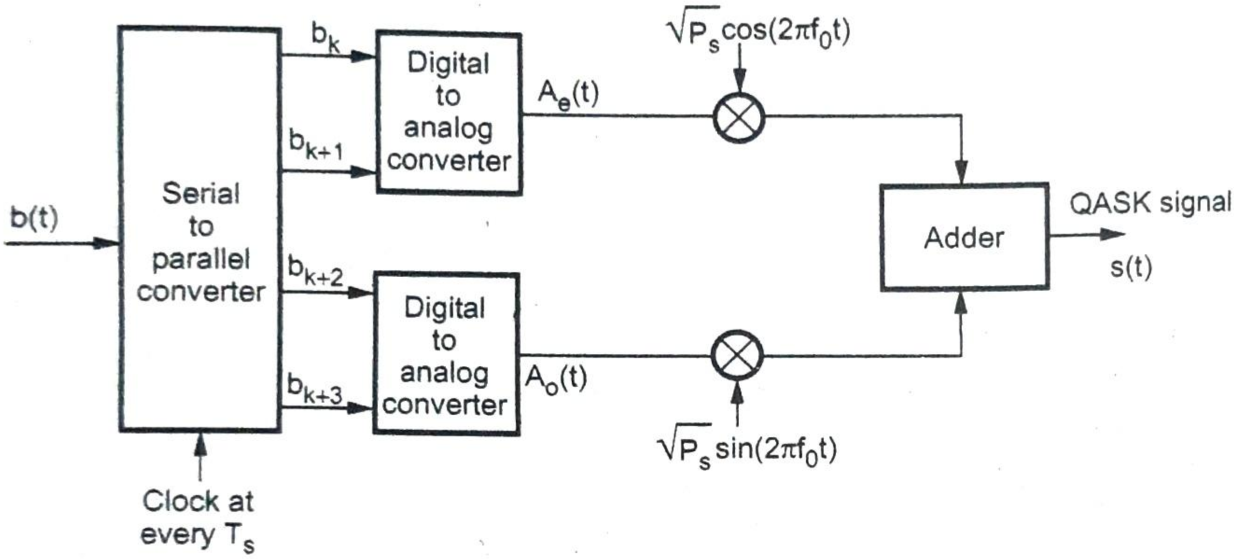


Fig. Q.1.1 Generation of QASK signal

Step 2 : Digital to Analog Conversion :

The four successive bits are applied to the digital to analog converters. These bits are applied after every T_s seconds. Here T_s is the symbol period and $T_s = 4T_b$.

Bits b_k and b_{k+1} are applied to upper DAC and b_{k+2}, b_{k+3} are applied to lower DAC. Depending on the input bits, the output of DAC takes four output levels.

Thus $A_e(t)$ and $A_o(t)$ takes 4 levels depending upon combination of two input bits.

Step 3 : Modulation of Carriers :

$A_e(t)$ modulates the carrier $\sqrt{P_s}\cos(2\pi f_0 t)$ and $A_o(t)$ modulates the carrier $\sqrt{P_s}\sin(2\pi f_0 t)$

Step 4 : Addition of Modulated Carries :

The adder combines two signals $A_e(t)$ and $A_o(t)$ to give QASK signal and it is given as.

$$s(t) = A_e(t)\sqrt{P_s}\cos(2\pi f_0 t) + A_o(t)\sqrt{P_s}\sin(2\pi f_0 t) \quad \ldots \text{(Q.1.2)}$$

ii) QASK Receiver :

Step 1 : Isolation of Carrier :

The input signal $s(t)$ is raised to 4^{th} power. It is then passed through a bandpass filler centered around the carrier frequency $4f_0$. The signal is then divided in frequency by four. It gives coherent carrier $\cos(2\pi f_0 t)$ and quadrature carrier $\sin(2\pi f_0 t)$

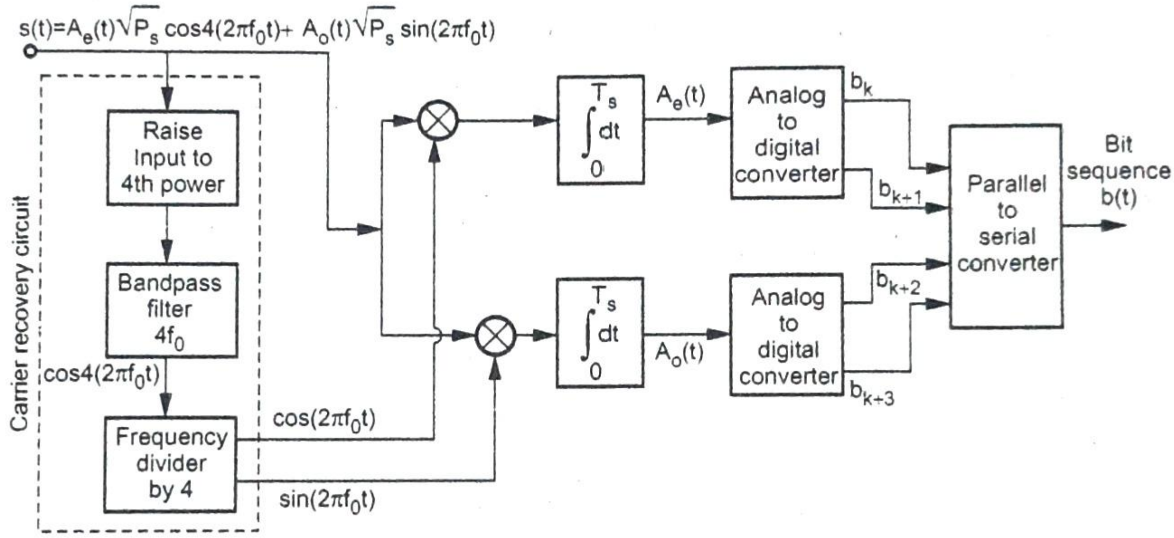


Fig. Q.1.2 4-bit QASK receiver block diagram

Step 2 : Synchronous Detection :

The in phase and quadrature coherent carriers are multiplied by QASK signal $s(t)$.

Now the amplitudes of $A_e(t)$ and $A_o(t)$ are bit constant and equal. The 4th power QASK signal is,

$$s^4(t) = P_s^2[A_e(t)\cos(2\pi f_0 t) + A_o(t)\sin(2\pi f_0 t)]^4$$

When this signal is passed through BPF of $4f_0$, then,

$$s^4(t) = \frac{P_s}{8}\left[A_e^4(t) + A_o^4(t) - 6A_e^2(t)A_o^2(t)\right]\cos 4(2\pi f_0 t)$$

$$+\frac{P_s}{2}\left[A_e(t)A_o(t)\{A_e^2(t) - A_o^2(t)\}\right]\sin 4(2\pi f_0 t)$$

The integrators integrate the multiplied signals over one system period.

The output of integrators at sampling period give $A_e(t)$ and $A_o(t)$. The analog to digital converters gives the four bits b_k, b_{k+1}, b_{k+2} and b_{k+3}.

The parallel to serial converter then generates the bit sequence $b(t)$.

iii) Bandwidth of QASK Signal :

The bandwidth can be given as,

$$BW = f_s - (-f_s) = 2f_s$$

$$= \frac{2}{T_s} \qquad \text{since } f_s = \frac{1}{T_s}$$

$$= \frac{2}{NT_b} \qquad (\text{since } T_s = NT_b) \qquad \text{... (Q.1.3)}$$

$$= \frac{2f_b}{N} \quad \left(\text{since } f_b = \frac{1}{T_b}\right) \quad \ldots \text{(Q.1.4)}$$

Thus the bandwidth and power spectral density of QASK is similar to that of M-ary PSK.

psd of QASK

The psd of QASK is given as,

$$S(f) = P_s T_s \left[\frac{\sin(\pi f T_s)}{\pi f T_s}\right]^2$$

The above equation gives power spectral density of $A_e(t)$ and $A_o(t)$. When they modulate the carrier, the main lobe given by above equation is shifted at carrier frequency f_0.

$$S(f) = \frac{P_s T_s}{2}\left[\frac{\sin \pi (f - f_0) T_s}{\pi (f - f_0) T_s}\right]^2 + \frac{P_s T_s}{2}\left[\frac{\sin \pi (f + f_0) T_s}{\pi (f + f_0) T_s}\right]^2 \quad \ldots \text{(Q.1.5)}$$

This equation gives power spectral density of QASK signal.

Q.2 Draw the constellation diagram of 16-ary PSK and 16 QAM. Compare them with respect to their Euclidean distance. What is the physical significance of Euclidean distance ?

☞[SPPU : Dec.-11, Marks 10]

Ans. : Fig. Q.2.1 shows the constellation diagram of 16-ary PSK.

Here Euclidean distance,

$$d = 2\sqrt{E_s} \sin \frac{\pi}{M}$$

$$= 2\sqrt{E_s} \sin \frac{\pi}{16}$$

$$= 0.39\sqrt{E_s}$$

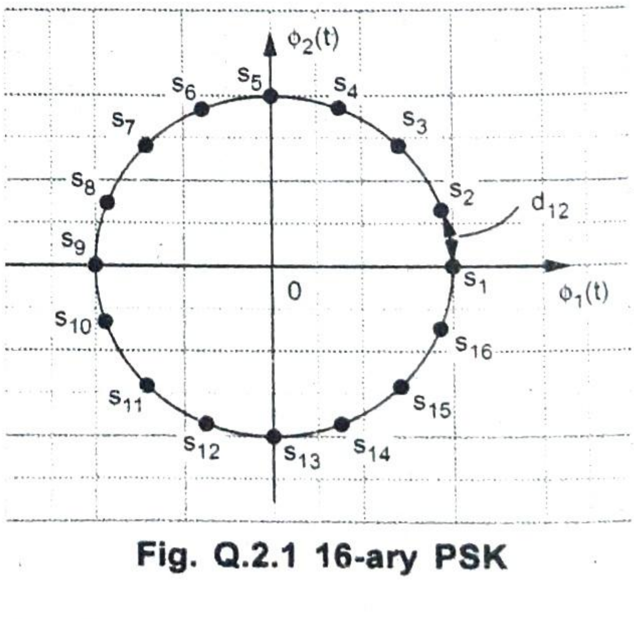


Fig. Q.2.1 16-ary PSK

Fig Q.2.2 shows the constellation diagram for 16 QAM. It's euclidean distance is given as,

$$d = \sqrt{0.4\, E_s}$$

Comment : The Euclidean distance is more in case of 16 QAM compared to 16-PSK.

Significance : The more Euclidean distance reduces error probability.

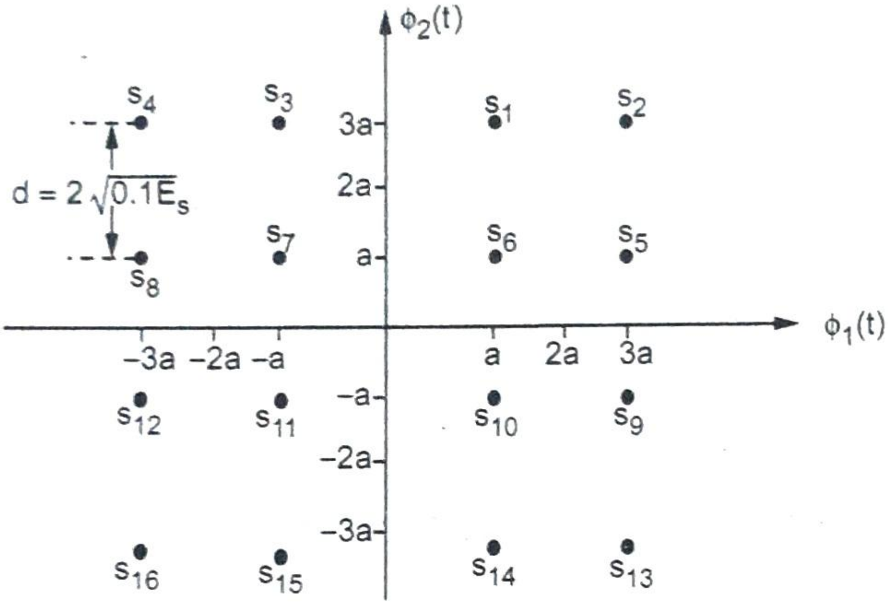


Fig. Q.2.2 Geometrical representation of 16 signals in QASK system

3.2 : M-ary FSK

Important Points to Remember

- There are $2^N = M$ different symbols. When every symbol uses separate frequency for transmission, it is M-ary FSK system.
- N bits per symbol transmitted.
- Minimum BW is $\frac{2^{N+1}}{N} f_b$ and Euclidean distance = $\sqrt{2NE_b}$

Q.3 What is M-ary FSK ? Explain the transmitter and receiver of M-ary FSK.

Ans. : M - symbols use seperate frequency for transmission. 'N' bits are used to form 2^N = M symbols. This is called M - ary FSK.

Transmitter and Receiver of M-ary FSK :

Transmitter : Fig. Q.3.1 shows the M-ary FSK transmitter. The 'N' successive bits are presented in parallel to digital to analog converter. These 'N' bits forms a symbol at the output of digital to analog converter.

There will be total $2^N = M$ possible symbols. The symbol is presented every $T_s = NT_b$ period. The output of digital to analog converter is given to a frequency modulator. Thus depending upon the value of symbol, the frequency modulator generates the output frequency. For every symbol, the frequency modulator produces different frequency output. This particular frequency signal remains at the output for one symbol duration. Thus for 'M' symbols, there are 'M' frequency signals at the output of modulator. Thus the transmitted frequencies are $f_0, f_1, f_2, \ldots f_{M-1}$ depending upon the input symbol to the modulator.

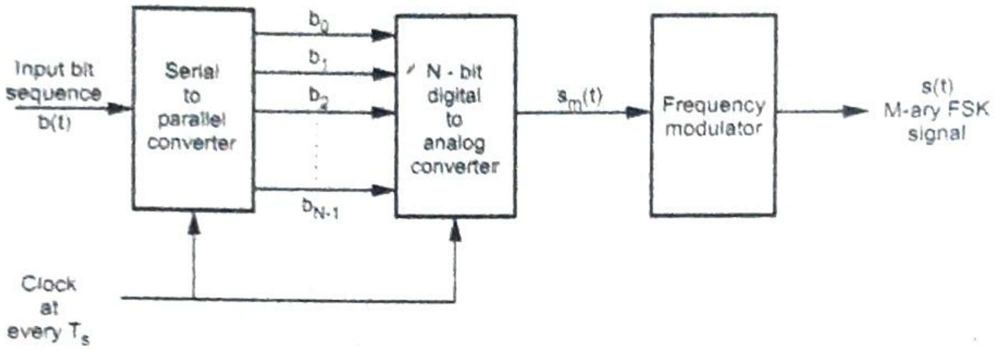


Fig. Q.3.1 M-ary FSK transmitter

Receiver : Fig. Q.3.2 shows block diagram of M-ary FSK receiver. It is the extension of BFSK receiver of Fig. Q.3.2. The M-ary FSK signal is given to the set of 'M' bandpass filters. The center frequencies of those filters are $f_0, f_1, f_2, \ldots., f_{M-1}$. These filters pass their particular frequency and attenuate others. The envelope detectors outputs are applied to a decision device. The decision device produces its output depending upon the highest input. Depending upon the particular symbol, only one envelope detector will have higher output. The outputs of other detectors will be very low. The output of the decision device is given to 'N' bit

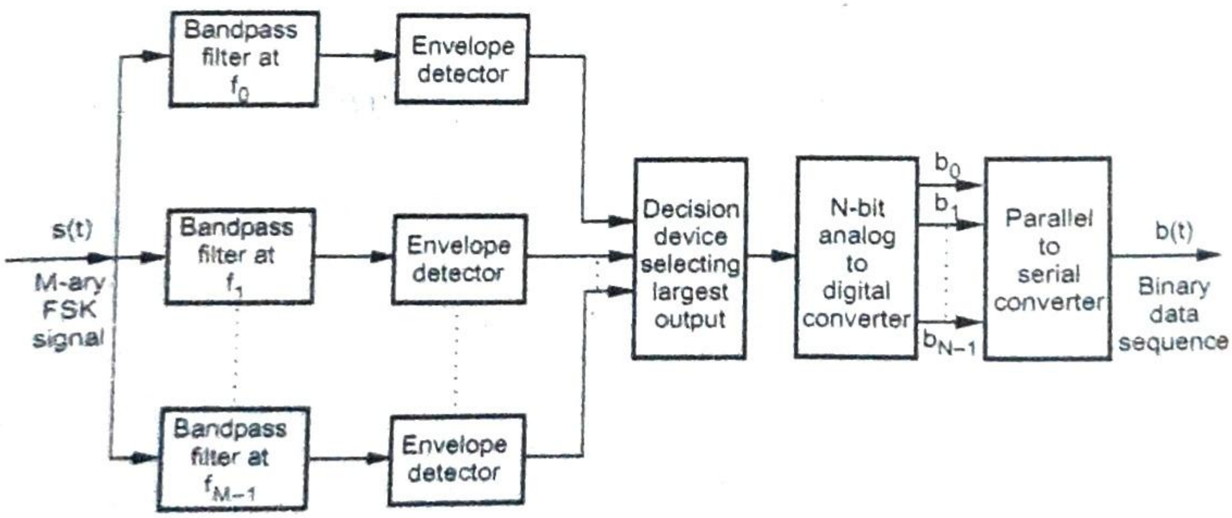


Fig. Q.3.2 Block diagram of M-ary FSK system

analog to digital converter. The analog to digital converter output is the 'N' bit symbol in parallel. These bits are then converted to serial bit stream by parallel to serial converter.

Q.4 Explain geometric representation, PSD and BW requirement of M-ary FSK. [SPPU : Dec.-11, Marks 2]

Ans. : i) Geometric representation : M - ary FSK uses manually orthogonal signals for transmission. $\phi_o(t)$, $\phi_1(t)$, $\phi_2(t)$, ... $\phi_{m-1}(t)$ are mutually orthogonal carries. And $S_0(t)$, $S_1(t)$, $S_2(t)$..... $S_{M-1}(t)$ are mutually orthogonal signals for M-symbols. Fig Q.4.1 shows signal space diagram for M = 3.

$$d = \sqrt{2NE_b}$$

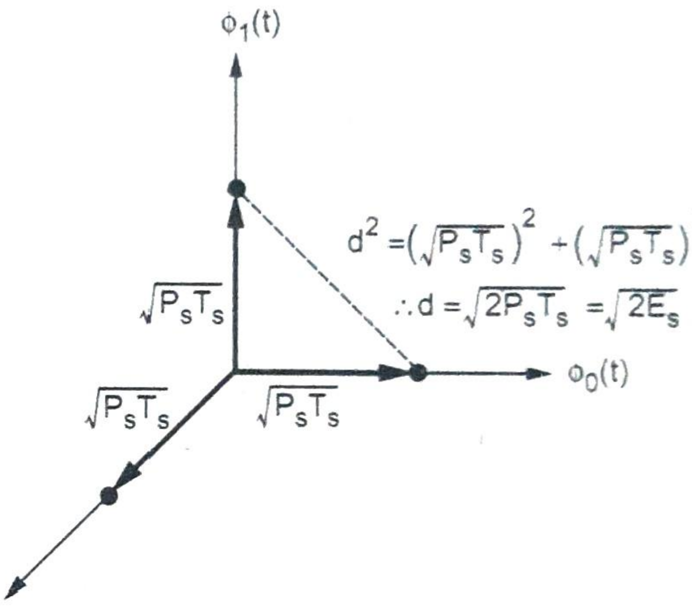


Fig. Q.4.1 Signal space (Geometrical) representation of M-ary FSK for M = 3

ii) Power spectral density :

For 'M' symbols, $f_o\ f_1\ f_2 \cdots f_{m-1}$ frequencies are used for transmission. These frequencies are selected as successive even harmonics of symbol frequency f_s. Then,

$$f_0 = k f_s$$

$$f_1 = (k+2)f_s \text{ and } f_2 = (k+4)f_s$$

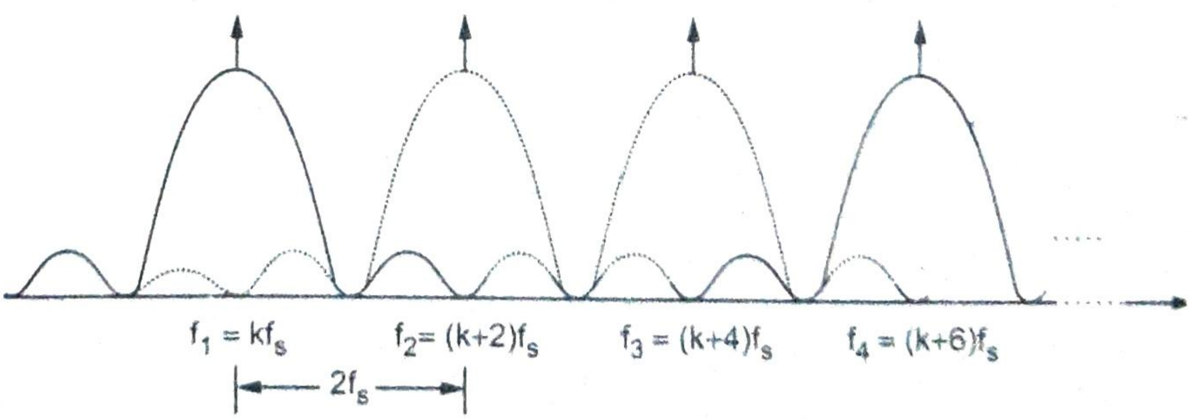


Fig. Q.4.2 Power spectral density M-ary FSK

Fig. Q.4.2 shows PSD of BFSK. It is simply extension of BFSK. The seperation between the two nearest main lobes is $2f_s$.

iii) Bandwidth of M-ary FSK :

From Fig. Q.4.2 it is clear that the width of one main lobe is $2f_s$. If there are M-symbols, then power spectral density spectrum will have M lobes. Therefore bandwidth of the system for M-symbols will be

$$BW = M \times (2f_s) = 2Mf_s \quad \ldots (Q.4.1)$$

We know that $2^N = M$ and $f_s = \frac{f_b}{N}$ we can write the above equations,

$$BW = 2 \cdot 2^N \cdot \frac{f_b}{N} = \frac{2^{N+1} f_b}{N} \quad \ldots (Q.4.2)$$

3.3 : Minimum Shift Keying (MSK)

Important Points to Remember

- In MSK, the output waveform is continous in phase and hence there are no abrupt changes in amplitude.
- 2 bit per symbol are transmitted.
- Minimum BW is $1.5 f_b$ and Euclidean distance $2\sqrt{E_b}$.
- MSK is also called continuous phase FSK or shaped QPSK.
- The difference between f_H and f_L is 'minimum' and at the same time they are orthogonal. Hence this technique is called MSK.

Q.5 Starting from signal expression of MSK, find suitable values of f_H and f_L. **[SPPU : Dec.-10, Marks 8]**

Ans. : The MSK signal is given as,

$$s(t) = \sqrt{2P_s}\,[b_e(t)\sin(2\pi t/4T_b)]\cos(2\pi f_0 t)$$
$$+ \sqrt{2P_s}\,[b_o(t)\cos(2\pi t/4T_b)]\sin(2\pi f_0 t) \quad \ldots (Q.5.1)$$

That is the product signal $b_e(t)\sin(2\pi t/4T_b)$ and $b_o(t)\cos(2\pi t/4T_b)$ modulate the quadrature carriers of frequency f_0.

$$s(t) = \sqrt{2P_s}\left[\frac{b_o(t)+b_e(t)}{2}\right]\sin 2\pi\left(f_0+\frac{1}{4T_b}\right)t$$

$$+\sqrt{2P_s}\left[\frac{b_o(t)-b_e(t)}{2}\right]\sin 2\pi\left(f_0-\frac{1}{4T_b}\right)t \quad \text{... (Q.5.2)}$$

$$= \sqrt{2P_s}\left[\frac{b_o(t)+b_e(t)}{2}\right]\sin 2\pi\left(f_0+\frac{f_b}{4}\right)t$$

$$+\sqrt{2P_s}\left[\frac{b_o(t)-b_e(t)}{2}\right]\sin 2\pi\left(f_0-\frac{f_b}{4}\right)t,\ f_b=\frac{1}{T_b} \quad \text{... (Q.5.3)}$$

Let $C_H(t) = \dfrac{b_o(t)+b_e(t)}{2}$ and $C_L(t)=\dfrac{b_o(t)-b_e(t)}{2}$

and Let $f_H = f_0+\dfrac{f_b}{4}$ and $f_L = f_0-\dfrac{f_b}{4}$ ... (Q.5.4)

With those substitutions equation (Q.5.3) becomes,

$$s(t)=\sqrt{2P_s}\,C_H(t)\sin(2\pi f_H t)+\sqrt{2P_s}\,C_L(t)\sin(2\pi f_L t) \quad \text{... (Q.5.5)}$$

If $b_o(t)=b_e(t)$ then $C_L(t)=0$ and $C_H(t)=\pm 1$ then above equation becomes,

$$s(t) = \sqrt{2P_s}\,C_H(t)\sin(2\pi f_H t) \quad \text{... (Q.5.6)}$$

Thus the transmitted frequency is f_H.

If $b_o(t)=-b_e(t)$ then $C_H(t)=0$ and $C_L(t)=\pm 1$. Then above equation becomes,

$$s(t) = \sqrt{2P_s}\,C_L(t)\sin(2\pi f_L t) \quad \text{... (Q.5.7)}$$

Thus the transmitted frequency is f_L.

The frequencies f_H and f_L are chosen such that $\cos(2\pi f_H t)$ and $\sin(2\pi f_L t)$ are orthogonal over the interval T_b. For orthogonality following relation should be satisfied i.e.,

$$\int_0^{T_b}\sin(2\pi f_H t)\sin(2\pi f_L t)\,dt = 0 \quad \text{... (Q.5.8)}$$

The above relation will be satisfied if we have integers '*m*' and '*n*' such that,

$$2\pi(f_H-f_L)T_b = n\pi \quad \text{... (Q.5.9)}$$

and $2\pi(f_H + f_L) T_b = m\pi$ ... (Q.5.10)

Let's put value of f_H and f_L from equation (Q.5.4) in above relations. From equation (Q.5.9) we get

$$2\pi\left(f_0 + \frac{f_b}{4} - f_0 + \frac{f_b}{4}\right)T_b = n\pi$$

$$\therefore \quad f_b T_b = n$$

$$\therefore \quad f_b \times \frac{1}{f_b} = n \quad \Rightarrow \quad n = 1 \qquad \text{... (Q.5.11)}$$

Similarly from equation (Q.5.10) we get,

$$2\pi\left(f_0 + \frac{f_b}{4} + f_0 - \frac{f_b}{4}\right)T_b = m\pi$$

$$\therefore \quad 4 f_0 T_b = m$$

$$\therefore \quad 4 f_0 \times \frac{1}{f_b} = m \quad \Rightarrow \quad f_0 = \frac{m}{4} f_b \qquad \text{... (Q.5.12)}$$

Putting this value of $f_0 = \frac{m}{4} f_b = m\frac{f_b}{4}$ in equation (Q.5.4),

$$f_H = f_0 + \frac{f_b}{4} = m\frac{f_b}{4} + \frac{f_b}{4} = (m+1)\frac{f_b}{4} \text{ and}$$

$$f_L = f_0 - \frac{f_b}{4} = m\frac{f_b}{4} - \frac{f_b}{4} = (m-1)\frac{f_b}{4}$$

Q.6 Explain the performance of MSK with suitable block schematic and explain how phase continuity is maintained ?

☞[SPPU : May-12, Dec.-11, Marks 8]

Ans. : MSK Transmitter and receiver :

MSK Transmitter : • Fig. Q.6.1 shows the block diagram of MSK Transmitter. The two sinusoidal signals $\sin(2\pi f_0 t)$ and $\cos(2\pi t / 4T_b)$ are mixed (multiplied). The bandpass filters then pass only sum and difference components of $f_0 + \frac{f_b}{4}$ and $f_0 - \frac{f_b}{4}$.

- The outputs of bandpass filters are then added and subtracted such that two signals $x(t)$ and $y(t)$ are generated. $x(t)$ is multiplied by $\sqrt{2P_s}\, b_o(t)$ and $y(t)$ is multiplied by $\sqrt{2P_s}\, b_e(t)$.

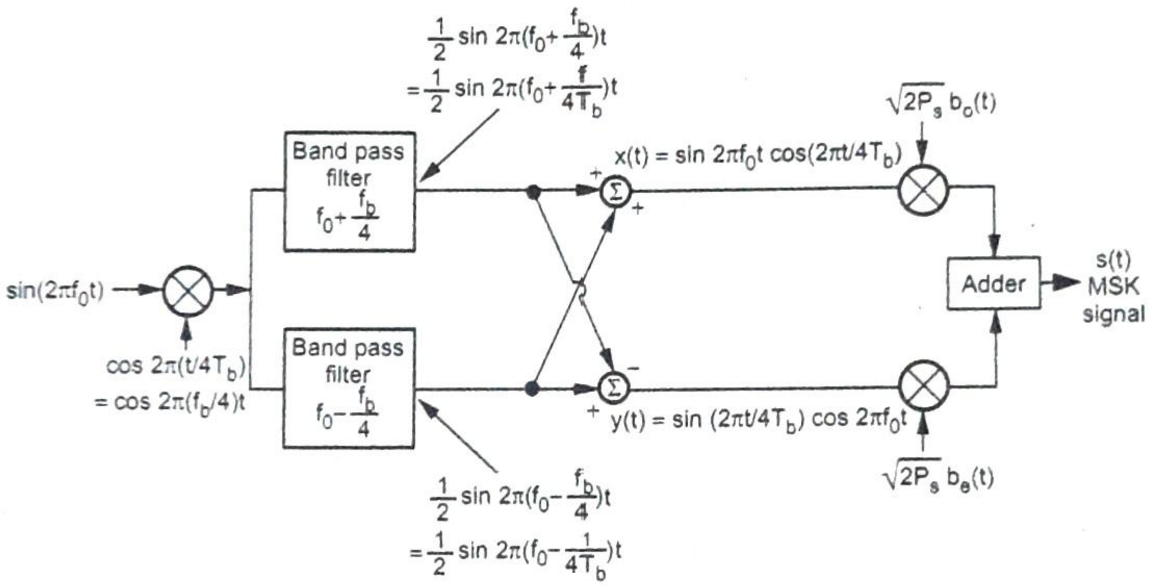


Fig. Q.6.1 MSK transmitter block diagram

- The outputs of the multipliers are then added to give final MSK signal.

MSK Receiver : • Fig. Q.6.2 shows the block diagram of MSK receiver. MSK uses synchronous detection. The signals $x(t)$ and $y(t)$ are multiplied with the received MSK signal.

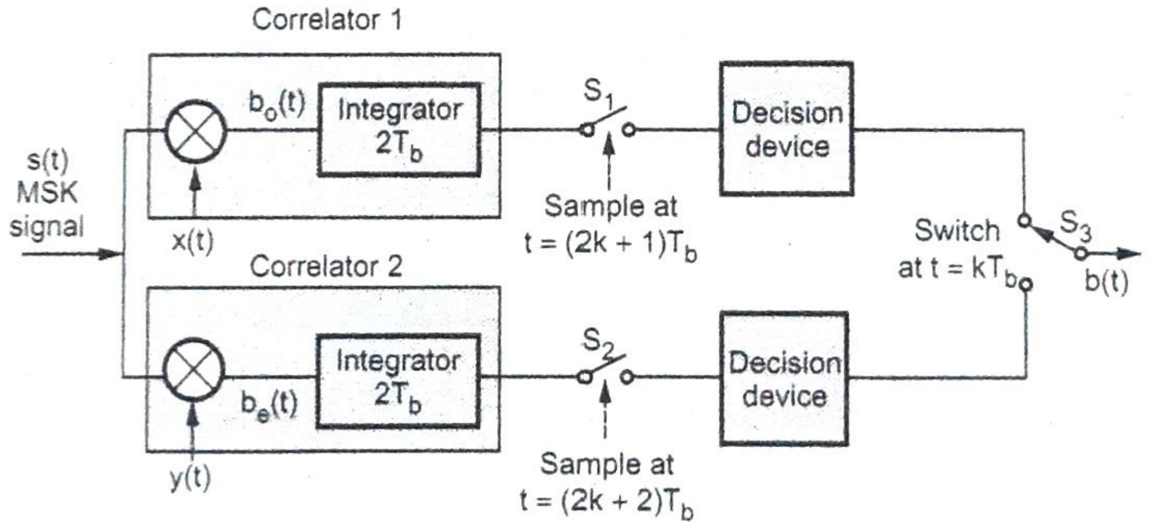


Fig. Q.6.2 MSK receiver block diagram

- $x(t)$ and $y(t)$ have same values as in the transmitter. The outputs of the multipliers are $b_e(t)$ and $b_o(t)$. The integrators integrate over the period of $2T_b$.
- For the upper correlator, the sampling switch samples output of integrator at $t=(2k+1)T_b$. Then the decision device decides whether $b_o(t)$ is + 1 or − 1. Similarly lower correlator output is $b_e(t)$.

- The outputs of two decision devices are staggered by T_b. The switch S_3 operates at $t = kT_b$ and simply multiplexes the two correlator outputs.

Phase Continuity in MSK : Consider the following MSK equation,

$$s(t) = \sqrt{2P_s}\left[\frac{b_o(t)+b_e(t)}{2}\right]\sin 2\pi\left(f_0+\frac{f_b}{4}\right)$$

$$t + \sqrt{2P_s}\left[\frac{b_o(t)-b_e(t)}{2}\right]\sin 2\pi\left(f_0-\frac{f_b}{4}\right)t \quad \ldots \text{(Q.6.1)}$$

Here, $b_e(t)$ and $b_o(t)$ does not change simultaneously. Let's find $\frac{s(t)}{\sqrt{2P_s}}$ for various combinations of $b_e(t)$ and $b_o(t)$. Table Q.6.1 shows these values.

$b_e(t)$	$b_o(t)$	$\frac{s(t)}{\sqrt{2P_s}}$
–1	–1	$-\sin 2\pi\left(f_0+\frac{f_b}{4}\right)t$
–1	1	$\sin 2\pi\left(f_0-\frac{f_b}{4}\right)t$
1	–1	$-\sin 2\pi\left(f_0-\frac{f_b}{4}\right)t$
1	1	$\sin 2\pi\left(f_0+\frac{f_b}{4}\right)t$

Table Q.6.1

Using Table Q.6.1 equation (Q.6.1) can be represented alternately as,

$$s(t) = b_o(t)\sqrt{2P_s}\sin 2\pi\left[f_0+b_e(t)\,b_o(t)\frac{f_b}{4}\right]t \quad \ldots \text{(Q.6.2)}$$

Here $b_o(t)$ decides the sign of MSK signal and product $b_e(t)\,b_o(t)$ decides frequency. We can write above equation as,

$$s(t) = b_o(t)\sqrt{2P_s}\sin\phi(t) \quad \ldots \text{(Q.6.3)}$$

This equation represents MSK signal with,

$$\phi(t) = 2\pi\left[f_0+b_e(t)\,b_o(t)\frac{f_b}{4}\right]t \quad \ldots \text{(Q.6.4)}$$

When $\quad b_e(t) \cdot b_o(t) = +1,$

$$\phi(t) = \phi_+(t) = 2\pi\left[f_0 + \frac{f_b}{4}\right]t \qquad \ldots \text{(Q.6.5)}$$

When $\quad b_e(t) \cdot b_o(t) = -1,$

$$\phi(t) = \phi_-(t) = 2\pi\left[f_0 - \frac{f_b}{4}\right]t \qquad \ldots \text{(Q.6.6)}$$

Since $b_e(t)$ and $b_o(t)$ are staggered by 'T_b' they can change at T_b. Therefore product $b_e(t) \cdot b_o(t)$ can change at integer multiples of 'T_b' i.e. kT_b. We can write equation (Q.6.5) and equation (Q.6.6) as,

$$\phi_+(t) = 2\pi f_0 t + \frac{\pi t}{2T_b} \text{ and } \phi_-(t) = 2\pi f_0 t - \frac{\pi t}{2T_b}$$

The phase change can be represented by difference of above two equations,

$$\phi_{diff}(t) = \phi_+(t) - \phi_-(t) = \frac{\pi t}{T_b} \qquad \ldots \text{(Q.6.7)}$$

$b_e(t)$ will change its sign at even bit times i.e. when $t = 2T_b, 4T_b, 6T_b$ etc. Therefore phase change at these times from above equation will be,

$$\phi_{diff}(t) = \frac{\pi t}{T_b} \text{ with } t = 2T_b, 4T_b, 6T_b, \ldots \text{ etc}$$

$$= 2\pi, 4\pi, 6\pi, \ldots \text{ etc.}$$

This shows that there is no phase shift in MSK signal.

$b_o(t)$ will change its sign only at odd bit times i.e. when $t = T_b, 3T_b, 5T_b, \ldots$ etc. Therefore phase changes at these bit times can be obtained from equation (Q.6.7) i.e.,

$$\phi_{diff}(t) = \frac{\pi t}{T_b} \text{ with } t = T_b, 3T_b, 5T_b, \ldots \text{ etc}$$

$$= \pi, 3\pi, 5\pi, 7\pi, \ldots \text{ etc.}$$

The above phase shift will change sign of sinusoidal term in equation (Q.6.2). But at the same time $b_o(t)$ also changes sign. Hence net sign of $s(t)$ remains unchanged. That is, there is no phase shift if $b_o(t)$ changes. *This means there is no abrupt phase change in MSK signal even if bit changes the sign. Thus phase is continuous in MSK.*

Q.7 Draw the signal space and spectral diagram for MSK. State the bandwidth requirement and mention advantages of MSK over QPSK.

[SPPU : May-13, Marks 6]

Ans. : **i) Signal space representation of MSK :**

- Fig. Q.7.1 shows the signal space representation of MSK. The carriers $\phi_H(t)$ and $\phi_L(t)$ are in quadrature. There are four signal points in $\phi_H - \phi_L$ plane.
- The Euclidean distance between nearest two points is,

$$d^2 = (\sqrt{P_s T_s})^2 + (\sqrt{P_s T_s})^2$$

$$\therefore\ d = \sqrt{2P_s T_s} = \sqrt{2E_s}$$

$$= \sqrt{4E_b} = 2\sqrt{E_b}$$

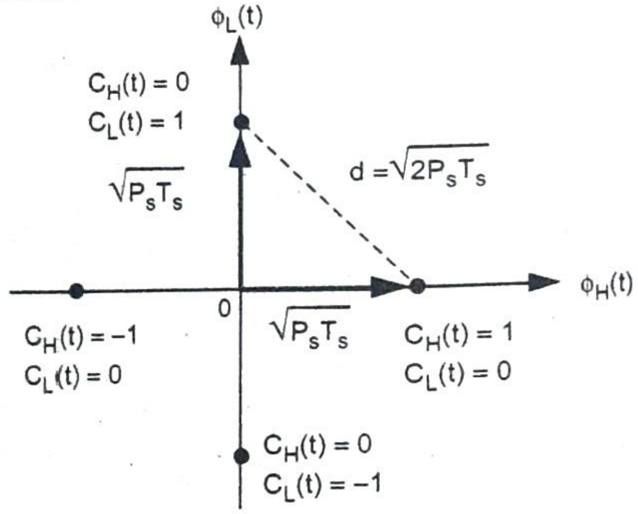


Fig. Q.7.1 Geometrical (Signal Space) representation of MSK signals

ii) Power spectral density of MSK :

Fig. Q.7.2 shows the PSD of MSK.

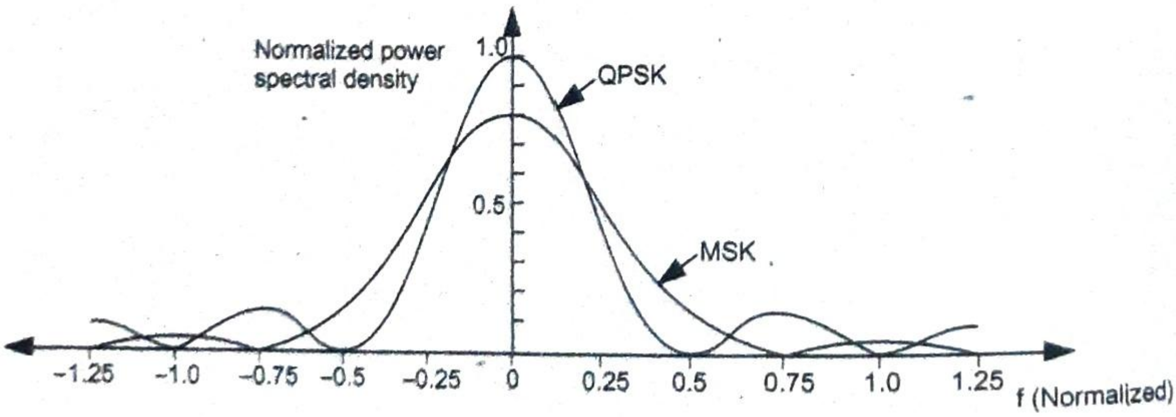


Fig. Q.7.2 Power spectral densities of MSK and QPSK

$$S(f) = \frac{8E_b}{\pi^2}\left\{\frac{\cos 2\pi(f - f_0)T_b}{1-[4(f - f_0)T_b]^2}\right\}^2 + \frac{8E_b}{\pi^2}\left\{\frac{\cos 2\pi(f + f_0)T_b}{1-[4(f + f_0)T_b]^2}\right\}^2 \quad \text{... (Q.7.1)}$$

iii) Bandwidth of MSK

$$\text{BW} = 0.75 f_b - (-0.75 f_b) = 1.5 f_b$$

iv) Advantages and disadvantages of MSK compared to QPSK ;

Advantages :

1. The MSK baseband waveforms are smoother compared to QPSK.
2. MSK signal have continuous phase in all the cases, whereas QPSK has abrupt phase shift of $\frac{\pi}{2}$ or π.
3. MSK waveform does not have amplitude variations, whereas QPSK signal have abrupt amplitude variations.
4. The main lobe of MSK is wider than that of QPSK. Main lobe of MSK contains around 99 % of signal energy whereas QPSK main lobe contains around 90 % signal energy.
5. Side lobes of MSK are smaller compared to that of QPSK. Hence interchannel interference is significantly large in QPSK.
6. To avoid interchannel interference due to sidelobes, QPSK needs bandpass filtering, where as it is not required in MSK.
7. Bandpass filtering changes the amplitude waveform of QPSK because of abrupt changes in phase. This problem does not exist in MSK.

The distance between signal points is same in QPSK as well as MSK. Hence the probability of error is also same.

Disadvantages :

1. The bandwidth requirement of MSK is 1.5 f_b, whereas it is f_b in QPSK.
2. The generation and detection of MSK is slightly complex. Because of incorrect synchronization, phase jitter can be present in MSK. This degrades the performance of MSK.

Q.8 Explain GMSK in detail. ☞ **[SPPU : May-15, Marks 6]**

Ans. : Fig. Q.8.1 shows the Gaussian MSK system.

- The inbut NRZ bit sequence is first passed through a lowpass filter. The spectrum of this lowpass filter has gaussian shape. It is given as,

$$H(f) = e^{\frac{-\log 2 f^2}{2W^2}}$$

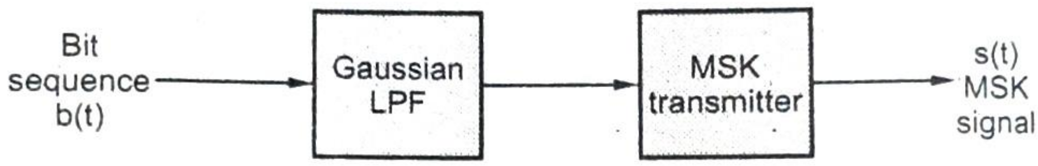


Fig. Q.8.1 Gaussian MSK system

- Here W is the 3 dB bandwidth of the baseband signal.
- When the input signal is passed through lowpass filter, its spectrum is made narrow and has sharp roll-off characteristics.
- When this signal is given to MSK, the MSK spectrum has sharp roll-off. It reduces adjacent channel interference.
- **Applications :** Gaussian MSK is used in multiuser communication systems.
- **Advantages :** Gaussian MSK has narrow power spectrum, sharp roll-off characteristics and reduced adjacent channel interference.
- **Disadvantages :** Lowpass filtering of NRZ signal is required. This increases complexity of MSK system.

Q.9 Sketch the waveform of the MSK signal for the sequence 101101. Assume that carrier frequency is 1.25 times the bit rate.

Ans. :

The bit rate is $f_b = \frac{1}{T_b}$. Then the carrier frequency is given as,

$$f_0 = 1.25 f_b$$

$$\therefore \quad f_H = f_0 + \frac{f_b}{4} = 1.25 f_b + \frac{f_b}{4} = 1.5 f_b$$

$$\therefore \quad f_L = f_0 - \frac{f_b}{4} = 1.25 f_b - \frac{f_b}{4} = 1.0 f_b$$

- Fig. Q.9.1 shows the waveforms of MSK.

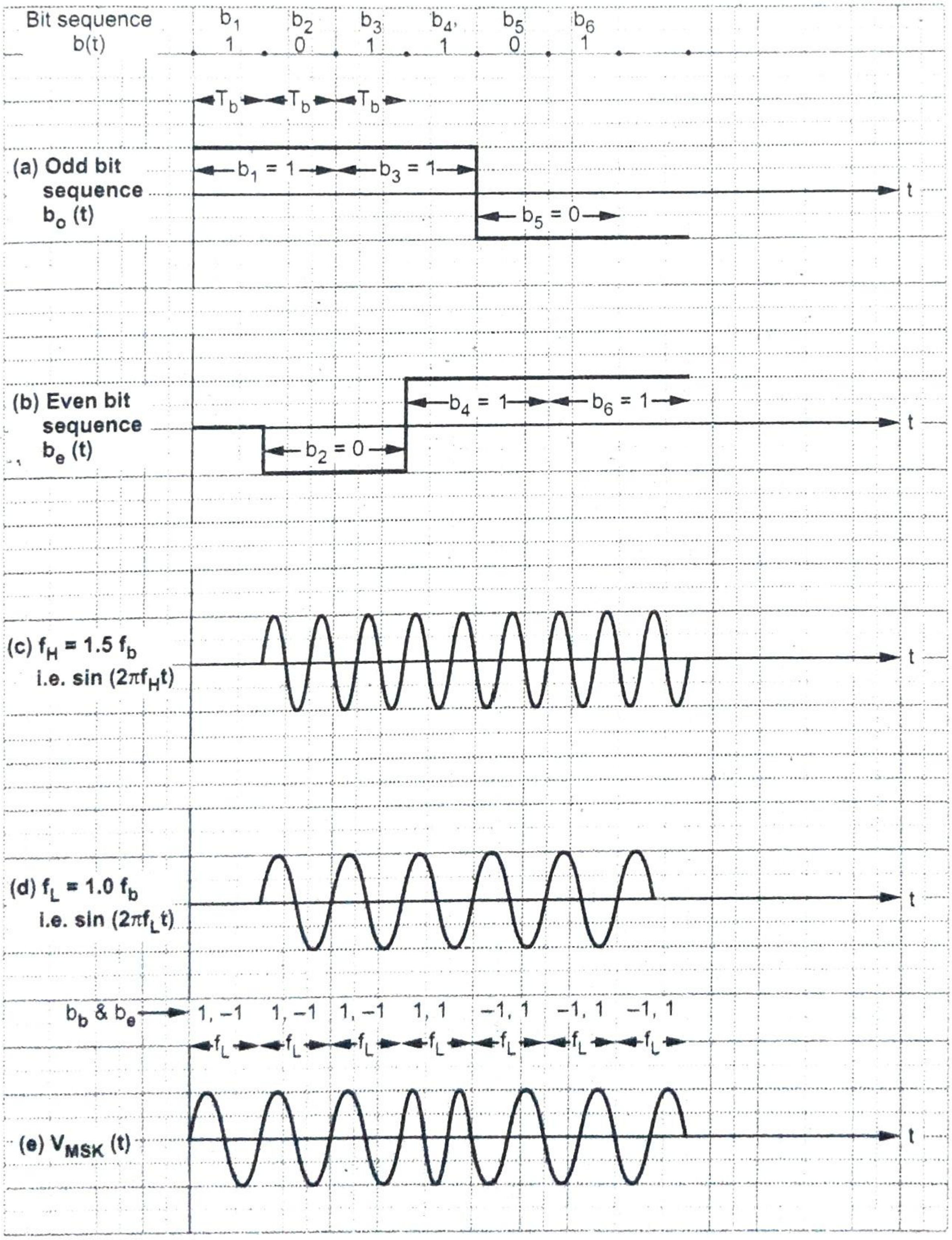


Fig. Q.9.1 Formation of MSK signal for the sequence 101101

Bit. No.	1	2	3	4	5	6	7	8	9	10
Bit sequence	1	0	1	1	0	1	0	1	0	1
Odd bit sequence $b_o(t)$	1	1	1	1	– 1	– 1	– 1	– 1	– 1	– 1
Even bit sequence $b_e(t)$		– 1	– 1	1	1	1	1	1	1	1
$C_H(t) = \frac{b_o(t)+b_e(t)}{2}$		0	0	1	0	0	0	0	0	0
$C_L(t) = \frac{b_o(t)-b_e(t)}{2}$		1	1	0	–1	–1	–1	–1	–1	–1
Transmitted MSK signal		$\sin(2\pi f_L t)$	$\sin(2\pi f_L t)$	$\sin(2\pi f_H t)$	$-\sin(2\pi f_L t)$	$-\sin(2\pi f_L t)$	$-\sin(2\pi f_L t)$	$-\sin(2\pi f_L t)$	$-\sin(2\pi f_L t)$	$-\sin(2\pi f_L t)$

Table Q.9.1 Synthesis of MSK waveform

n the above table observe that,

$$C_H(t) = \frac{b_o(t)+b_e(t)}{2} \quad \text{and} \quad C_L(t) = \frac{b_o(t)-b_e(t)}{2}$$

$b_o(t)$ and $b_e(t)$ are indicated in Table Q.9.1. In bit number 2, observe that $b_e(t)$ and $b_o(t)$ are of opposite sign. Hence $C_H(t)=0$ and $C_L(t)=1$. Hence transmitted signal is $s(t)=\sqrt{2P_s}\sin(2\pi f_L t)$. Fig. Q.9.2 shows the transmitted MSK waveform.

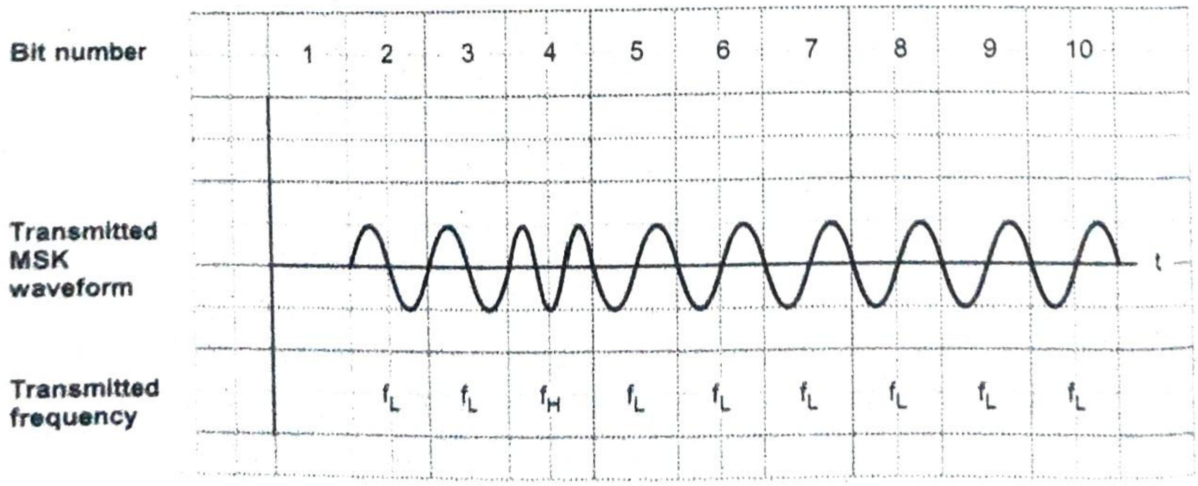


Fig. Q.9.2 MSK waveform

Q.10 If the digital message input data rate is 10 kbps and average energy per bit is 0.02 unit find bandwidth and euclidian distance for the following schemes. 1) BPSK 2) 16-MPSK 3) MSK 4) 16-QAM.

☞[SPPU : PU : May-11, Marks 6, May-16, End sem, Marks 8]

Ans. : $f_b = 10 \text{ kbps} = 10\times 10^3 \text{ bps} \quad E_b = 0.02$

1) BPSK $BW = 2f_b = 2\times 10\times 10^3 = 20 \text{ kHz.}$

$$d = 2\sqrt{E_b} = 2\sqrt{0.02} = 0.283$$

2) 16-ary PSK

$$M = 16,\ N = \log_2 M = \frac{\log_{10} M}{\log_{10} 2} = \frac{\log_{10}{}^{16}}{\log_{10} 2} = 4$$

$$BW = \frac{2f_b}{N} = \frac{2\times 10\times 10^3}{4} = 5 \text{ kHz}$$

$$d = 2\sqrt{E_s}\ \sin\frac{\pi}{M} = 2\sqrt{NE_b}\ \sin\frac{\pi}{M} \quad \text{Here } E_s = N\,E_b$$

$$= 2\sqrt{4\times 0.02}\ \sin\frac{\pi}{16} = 0.11$$

3) MSK $BW = 1.5f_b = 1.5\times 10\times 10^3 = 15 \text{ kHz}$

$$d = 2\sqrt{E_b} = 2\sqrt{0.02} = 0.283$$

4) 16-Q8AM

$$BW = \frac{2f_b}{N} = \frac{2\times 10\times 10^3}{4} = 5 \text{ kHz}$$

$$d = \sqrt{0.4\,E_s} = \sqrt{0.4\times 4\,E_b} = \sqrt{0.4\times 4\times 0.02} = 0.179$$

3.4 : Amplitude Shift Keying (OOK or ASK)

Important Points to Remember

- It is ON-OFF keying modulation technique. Carrier is transmitted only when binary '1' is transmitted.
- One bit per symbol transmitted.
- Minimum BW is $2f_b$ and Euclidean distance : $\sqrt{E_b}$

Q.11 What is ASK ? Explain it with the help of generator and detector.

Ans. : In ASK, carrier is transmitted for binary '1' and no carrier is transmitted for binary '0'.

Ans. : ASK Generator : Fig. Q.11.1 shows the ASK generator. The input binary sequence is applied to the product modulator. The product modulator amplitude modulates the sinusoidal carrier. It passes the carrier when input bit is '1'. It blocks the carrier (i.e. zero output) when intput bit is '0'.

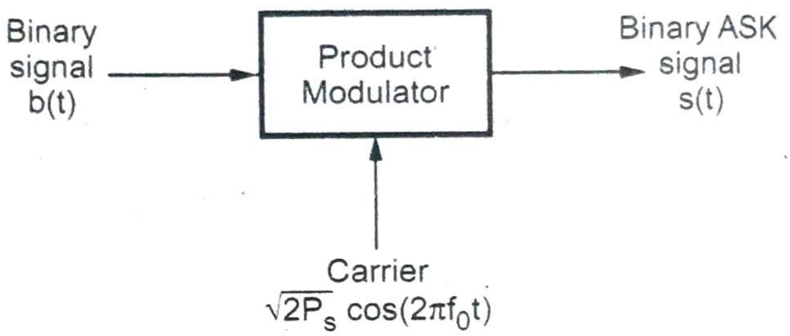


Fig. Q.11.1 Block diagram of ASK generator

ASK Detector : Fig. Q.11.2 shows the block diagram of coherent ASK detector. The ASK signal is applied to the correlator consisting of multiplier and integrator. The locally generated coherent carrier is applied to the multiplier. The output of multiplier is integrated over one bit period. The decision device takes the decision at the end of every bit period. It compares the output of integrator with the threshold. Decision is taken in favour of '1' when threshold is exceeded. Decision is taken as '0' if threshold is not exceeded.

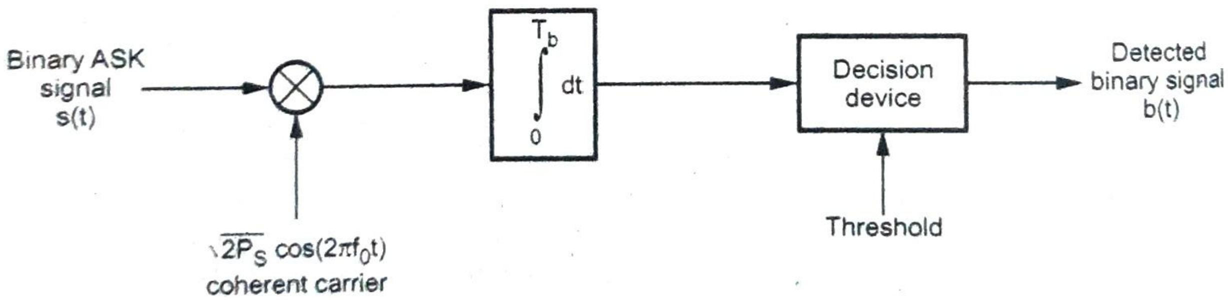


Fig. Q.11.2 Block diagram of coherent ASK detector

Q.12 Derive an equation of PSD for ON-OFF signalling.

[SPPU : May-11, Marks 6]

Ans. : The on-off signaling is shown in Fig. Q.12.1.

The NRZ pulse is used as a modulating signal in most of the digital modulation techniques. It's Fourier transform is given as,

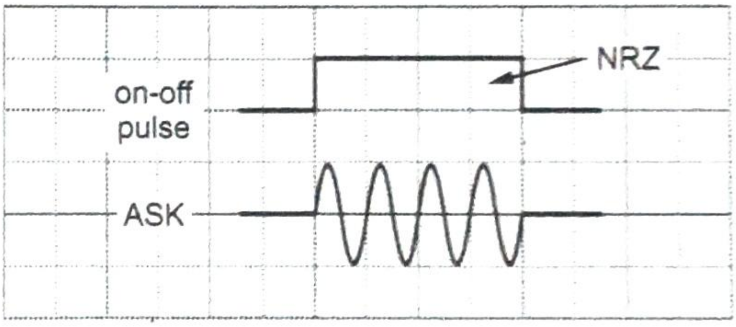


Fig. Q.12.1 On-off signaling pulse

$$X(f) = V_a T_b \frac{\sin(\pi f T_b)}{(\pi f T_b)} \qquad \text{... (Q.12.1)}$$

- When large number of such NRZ pulses are present, its PSD is given as,

$$S(f) = \frac{\overline{|X(f)|^2}}{T_s}$$

Here T_s is the symbol duration.

Putting from equation (Q.12.1) for $X(f)$,

$$S(f) = \frac{V_b{}^2 T_b{}^2}{T_s} \left[\frac{\sin(\pi f T_b)}{(\pi f T_b)}\right]^2$$

$$= V_b{}^2 T_b \left[\frac{\sin(\pi f T_b)}{(\pi f T_b)}\right]^2, \quad T_s = T_b$$

- When this signal modulates a sine wave in on-off signalling at frequency f_0, the spectral components are translated from f to $f_0 + f$ and $f_0 - f$. The magnitude of these components will be divided by two. Thus,

$$S_{ASK}(f) = \frac{V_b{}^2 T_b}{2} \left\{\left[\frac{\sin \pi(f_0 - f)T_b}{\pi(f_0 - f)T_b}\right]^2 + \left[\frac{\sin \pi(f_0 + f)T_b}{\pi(f_0 + f)T_b}\right]^2\right\}$$

This is the PSD of on-off signalling.

3.5 : Error Probabilities

Q.13 Derive an expression for error probability of ASK.

Ans. : In ASK, the transmitted signal for binary '1' is given as,

$$x_1(t) = \sqrt{2P_s}\cos(2\pi f_0 t)$$

and $x_2(t) = 0$ for binary '0'

Maximum signal to noise ratio is given as,

$$\rho_{max} = \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} x^2(t)\,dt$$

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty}[x_1(t)-x_2(t)]^2\,dt$$

$$= \frac{2}{N_0}\int_{-\infty}^{\infty} x_1^2(t)\,dt \quad since \quad x_2(t) = 0$$

$x_1(t) = \sqrt{2P_s}\cos(2\pi f_0 t)$ from 0 to 7. Then above equation will be,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T(\sqrt{2P_s}\cos(2\pi f_0 t))^2$$

$$= \frac{4P_s}{N_0}\int_0^T \cos^2(2\pi f_0 t)\,dt = \frac{4P_s}{N_0}\int_0^T \frac{1+\cos 4\pi f_0 t}{2}\,dt$$

$$= \frac{2P_sT}{N_0} \text{ By evaluating above integral} \qquad \dots (Q.13.1)$$

Error probability using matched filter detection is given as,

$$P_e = \frac{1}{2}\,erfc\left[\frac{1}{2\sqrt{2}}\cdot\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{max}$$

$$= \frac{1}{2}\,erfc\left[\frac{1}{2\sqrt{2}}\cdot\sqrt{\frac{2P_sT}{N_0}}\right] \qquad \text{from equation (Q.13.1)}$$

$$= \frac{1}{2}\,erfc\sqrt{\frac{P_sT}{4N_0}} = \frac{1}{2}\,efrc\sqrt{\frac{E}{4N_0}}, \qquad \text{since } E = P_sT \dots (Q.13.2)$$

Q.14 Derive the expression for error probability of BPSK system using matched filter. ☞[SPPU : May-15, 11, Dec.-11, 10, Makrs 8, May-16, End Sem Marks 8]

Ans. : In BPSK,

Binary '1' ⇒ $$x_1(t) = \sqrt{2P}\cos(2\pi f_0 t) \quad \text{... (Q.14.1)}$$

and Binary '0' ⇒ $$x_2(t) = -\sqrt{2P}\cos(2\pi f_0 t) \quad \text{... (Q.14.2)}$$

Thus, $$x_2(t) = -x_1(t)$$

$$P_e = \frac{1}{2}\, erfc\left\{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\,\sigma}\right\} \quad \text{... (Q.14.3)}$$

where,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)\,dt \quad \text{... (Q.14.4)}$$

Here $$x(t) = x_1(t)-x_2(t) = x_1(t)-[-x_1(t)] = 2x_1(t)$$

Hence equation (Q.14.4) becomes,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T 4x_1^2(t)\,dt = \frac{8}{N_0}\int_0^T x_1^2(t)\,dt \quad \text{...(Q.14.5)}$$

$$\int_0^T x_1^2(t)\,dt = \int_0^T 2P\cos^2(2\pi f_0 t)\,dt$$

$$= 2P\cdot\frac{1}{2}\int_0^T [1+\cos 4\pi f_0 t]\,dt = P\left[\int_0^T dt + \int_0^T \cos 4\pi f_0 t\,dt\right]$$

$$= P\cdot\int_0^T dt + 0 = PT = E \quad \text{...(Q.14.6)}$$

Putting the above result in equation (Q.14.5) we get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{8}{N_0}\cdot E$$

$$\therefore \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{max} = \sqrt{\frac{8E}{N_0}} \quad \text{... (Q.14.7)}$$

Putting this result in equation (Q.14.7) we get,

$$P_e = \frac{1}{2}\, erfc \left\{ \frac{1}{2\sqrt{2}} \sqrt{\frac{8E}{N_0}} \right\}$$

On simplification of above equation we get,

$$\therefore \qquad \boxed{\textbf{Error probability in PSK : } P_e = \frac{1}{2}\, erfc \sqrt{\frac{E}{N_0}}} \qquad \text{... (Q.14.8)}$$

Also, $\qquad P_e = Q\sqrt{\frac{2E}{N_0}}$, where, $erfc(u) = 2Q(\sqrt{2}\,u)$.

Q.15 Binary data has to be transmitted over a telephone link that has a usable bandwidth of 3000 Hz and a maximum achievable signal-to-noise power ratio of 6 dB at its output.

i) Determine the maximum signaling rate and probability of error of a coherent ASK scheme is used for transmitting binary data through this channel.

ii) If the data is maintained at 300 bits/sec calculate the error probability.

$$\mathbf{Q(3.4) = 0.0003,\ Q(6.4) = 10^{-10},\ Q(5.25) = 10^{-7}.}$$

Ans. : Here average signal power, $S = \frac{A^2}{4}$

The bandwidth of the signal is 3000 Hz. Hence average noise power, $N = 2 \times \frac{N_0}{2} \times 3000.$

Signal to noise ratio is given as 6 dB.

$$\therefore \qquad \frac{S}{N}\,\text{dB} = 10\log_{10}\frac{S}{N}$$

$$\therefore \qquad 6\text{ dB} = 10\log_{10}\frac{S}{N} \Rightarrow \frac{S}{N} \approx 4$$

And $\qquad \frac{S}{N} = \frac{\text{Average signal power}}{\text{Noise power}}$

$$\therefore \qquad 4 = \frac{A^2/4}{2\times\frac{N_0}{2}\times 3000}$$

$$\therefore \quad \frac{A^2}{N_0} = 48000$$

i) Signaling rate and probability of error

For ASK bandwidth and signaling rate,

$$BW = 3r_b$$

$$\therefore \quad r_b = \frac{BW}{3} = \frac{3000}{3} = 1000 \text{ bits/sec.}$$

For ASK, $$P_e = \frac{1}{2} erfc\sqrt{\frac{E}{4N_0}} = \frac{1}{2}\cdot 2\,Q\left(\sqrt{2}\cdot\sqrt{\frac{E}{4N_0}}\right) = Q\sqrt{\frac{E}{2N_0}}$$

$$= Q\sqrt{\frac{\frac{A^2}{2}T_b}{2N_0}}, \quad \text{since } E = \frac{A^2}{2}T_b$$

$$= Q\sqrt{\frac{A^2/N_0}{4r_b}}, \quad \text{since } T_b = \frac{1}{r_b} \qquad \text{... (Q.15.1)}$$

$$= Q\sqrt{\frac{48000}{4\times 1000}} \quad \text{putting values}$$

$$= Q(3.46) = 0.0003$$

ii) Error probability for r_b = 300 bits/sec

$$\therefore \quad P_e = Q\sqrt{\frac{A^2/N_0}{4r_b}} \qquad \text{By equation (Q.15.1)}$$

$$= Q\sqrt{\frac{48000}{4\times 300}} = Q(6.32) \approx Q(6.4) = 10^{-10}$$

3.6 : Pulse Shaping to Reduce Interchannel and Intersymbol Interference

Important Points to Remember

- Presence of outputs due to other symbols at the time of sampling the required symbol is called ISI.
- Raised cosine spectrum is used to reduce the effect of ISI.

Q.16 What is ISI ? Explain its causes and remedies to avoid it.

[SPPU : Dec.-15, In sem, Marks 8, Dec.-13, May-12, 14, Marks 10]

Ans. : ISI : The presence of outputs due to other bits (symbols) interfere with the output of required bit (symbol). This effect is called Intersymbol Interference (ISI)

The output of the receiver at t_i is given as,

$$y(t_i) = \underbrace{\mu A_i}_{\text{output due to } i^{th} \text{ bit}} + \underbrace{\mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} A_k \, p[(i-k)T_b]}_{ISI}, \quad i = 0, \pm 1, \pm 2, \ldots$$

Here μA_i is the contribution in output due to i^{th} transmitted bit.

- The second term in above equation represents residual effect of all other bits transmitted before and after the sampling instant 't_i'. It is ISI.
- **Causes of ISI**

i) Timing inaccuries : The ISI occurs if the rate of transmission of the transmitter is not same as the ringing and frequency of the channel.

ii) Insufficient BW : If the channel BW is reduced, then the timing error might increase in turn resulting ISI.

iii) Amplitude distortion : When the frequency characteristics of the communication channel differs from the expected one resulting pulse distortion leadings to ISI.

iv) Phase distortion : When various frequency components in the input undergo different amounts of time delay while travelling through the channel leads to phase distortion and becoming a cause for ISI

Remedies to reduce ISI

- Following are the conditions to be satisfied by the transmitted pulse to have zero ISI.

$$p[(i-k)T_b] = \begin{cases} 1 \text{ for } & i = k \\ 0 \text{ for } & i \neq k \end{cases}$$

or
$$\sum_{n=-\infty}^{\infty} p(f - n f_b) = T_b$$

- Above condition is satisfied by following pulse,

$$p(t) = \text{sinc}(2B_o t)$$

ut this pulse is physically unrealizable.

- Raised cosine spectrum gives rise to the time domain pulse that is physically realizable. i.e.,

$$\rho(t) = \text{sinc}\,(2B_0 t)\frac{\cos\,(2\pi\alpha B_0\, t)}{1-16\alpha^2 B_0{}^2 t^2}$$

This pulse becomes zero at the sampling instants of $\pm T_b$, $\pm 2T_b$ ... and so on. It eliminates ISI.

Q.17 State the Nyquist first criterion for zero ISI. Also explain raised cosine spectrum to reduce ISI.

☞[SPPU : Dec.-14, In sem, Marks 3]

Ans. : Nyquist gives criterion for zero ISI in time as well as frequency domains. It is given below :

$$\rho[(i-k)T_b] = \begin{cases} 1 \; for & i = k \\ 0 \; for & i \neq k \end{cases}$$

and $$\sum_{n=-\infty}^{\infty} P(f - nf_b) = T_b$$

Above time and frequency domain equations give Nyquist pulse shaping criterion for zero ISI.

- The pulse which satisfies above criteria is,

$$p(t) = \frac{\sin\,(2\pi B_0 t)}{2\pi B_0 t}$$

Here B_0 is the Nyquist bandwidth. And pulse $p(t)$ provides zero ISI. Note that this pulse is not physically realizable since it extends from $-\infty \leq t \leq \infty$.

- **Raised cosine spectrum :** In raised cosine spectrum, the frequency response P(f) decreases gradually towards zero. The corresponding time domain pulse is given as,

$$p(t) = \text{sinc}\,(2B_0 t)\frac{\cos\,(2\pi\alpha B_0 t)}{1-16\alpha^2 B_0{}^2 t^2}$$

Here roll off factor, $\alpha = 1 - \frac{f_1}{B_0}$ ... (Q.17.1)

The roll of takes place from f_1 to $2B_0 - f_1$

The bandwidth required for raised cosine spectrum is given as,

$$B = 2B_0 - f_1$$

From equation (Q.17.1) $f_1 = B_0 - B_0\alpha$. Putting this value in above equation,

$$B = 2B_0 - B_0 + B_0\alpha$$

$$\therefore \quad B = B_0(1+\alpha)$$

This is the bandwidth required by raised cosine spectrum.

Q.18 A computer gives a binary data at the rate of 56 kbps and its transmitted using baseband PAM system that is designed to have a raised cosine spectrum. Determine transmission BW required for roll off rates i) $\alpha = 0.25$ ii) $\alpha = 0.75$ **[SPPU : May-14, Marks 8]**

Ans. : Data rate is, $f_b = 56$ kbps

$$\therefore \quad BW,\ B_0 = \frac{f_b}{2} = 28 \text{ kbps}$$

Now BW required using raised cosine spectrum is given by,

$$B = B_0(1+\alpha)$$

i) for $\alpha = 0.25$,

$$B = 28\times 10^3(1+0.25) = 35 \text{ kHz}$$

ii) for $\alpha = 0.75$,

$$B = 28\times 10^3(1+0.75) = 49 \text{ kHz}$$

Thus as roll off factor increases, BW also increases

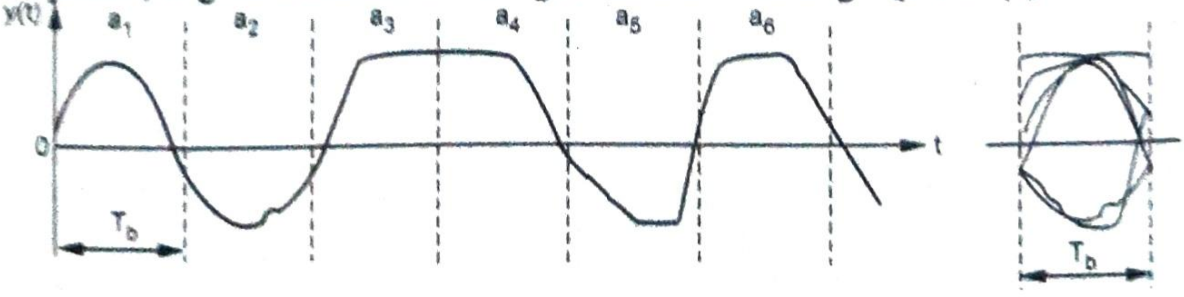
Q.19 Explain the use of eye diagram to measure ISI.

[SPPU : Dec.-13, Marks 4]

OR

Explain eye diagram. **[SPPU : May-11, Marks 4]**

Ans. : • Eye pattern is used to study the effect of ISI on baseband digital transmission. When the signal is transmitted over the channel in the form of pulses, it gets distorted. Such signal is shown in Fig. Q.19.1 (a)



(a) Signal waveform (b) Eye diagram

Fig. Q.19.1 : Eye diagram

- T_b is the interval of one bit. a_1, a_2, a_3, a_4, a_5 and a_6 are the transmitted bits. If we cut the waveforms of $a_1, a_2, ... a_6$ in their respective bit intervals and paste over one another, we set the diagram as shown in Fig. Q.19.1 (b). This diagram looks like 'eye', hence it is called eye diagram.
- This eye pattern indicates the effect of intersymbol interference.

Use of eye diagram

- Various important conclusions can be derived from eye pattern. The interval over which received waveform can be sampled without error can be obtained from eye pattern.
- Sensitivity of the system to timing error can be determined from eye diagram.
- The maximum clear eye opening indicates margin over noise.
- Fig. Q.19.2 indicates various parameters of eye diagram discussed above.

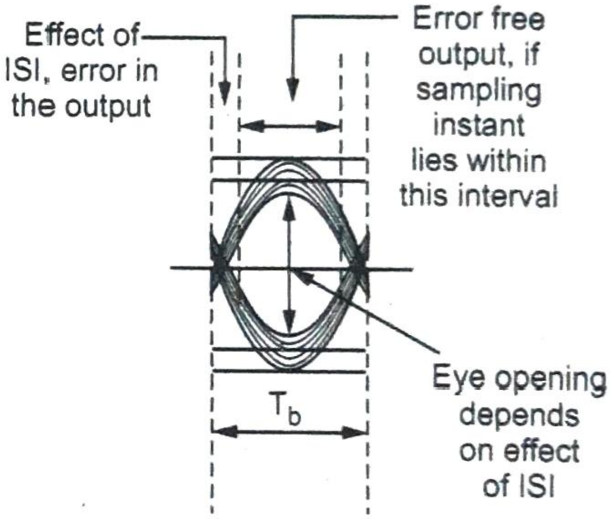


Fig. Q.19.2

3.7 : Some Issues in Transmission and Reception

Important Points to Remember

- There are some issues related to digital transmission and reception. These issues are :
  i) Line coding ii) Scrambling iii) Digital multiplexing.
- These issues address representation of binary data on tranmission line, radomization of binary data for efficient synchronization and multiplexing of digital data.

Q.20 What are the issues related to transmission and reception ?

Ans. : Line Coding

- The digital data consisting of two symbols '0' and '1' is represented by different waveforms using line coding.
- The factors like i) Power and bandwidth required for transmission, ii) Ability to extract timing information, iii) Presence of low frequency or DC component and iv) Error monitoring ability are addressed by variety of line codes.
- Unipolar NRZ, unipolar RZ, bipolar NRZ, bipolar RZ, AMI, HDB, split phase (Manchester), CMI are some of the types of line codes used.

Scrambling

- Synchronization issues are addressed by scrambling and unscrambling of data.
- The binary data is randomized using scrambling. Long strings of 1's and 0's are scrambled to avoid such long strings.
- The clock recovery and synchronization is improved. Scrambling is performed at the transmitter and unscrambling is performed at the receiver.

Digital Multiplexing

- The digital signals are multiplexed and transmitted over a telephone line.
- T_1 digital system multiplexes 24 analog signals sampled at 8 kHz in a single line of 1.544 Mbps.
- Multiple T_1 lines are further multiplexed into T_2, T_3, T_4 and so on. The bit rate of T_4 line is 274.176 Mbps.

3.8 : OFDM

Q.21 What is OFDM ? Explain its concept with relevent block diagram. **[SPPU : Dec.-14, End sem, Marks 3]**

Ans. : OFDM :

- The available information is split into N-parallel streams. These streams modulate N-distinct carriers. All these carriers are mutually orthogonal. They are also called subcarriers or tones.

- The duration of each stream on every carrier is increased. Since the carriers are orthogonal, the spacing between them can be as low as $\frac{W}{N}$. Here 'W' is the total available bandwidth and 'N' is the total number of carriers.
- Hence n^{th} frequency carrier in OFDM will have a frequency of $f_n = n \cdot \frac{W}{N}$. And the relationship between T_s and W will be $W = \frac{N}{T_s}$.

Block Diagram of OFDM

- Fig. Q.21.1 shows the block diagram of an analog OFDM transceiver. The original data stream is split into N parallel data streams by passing them through serial to parallel convertor.

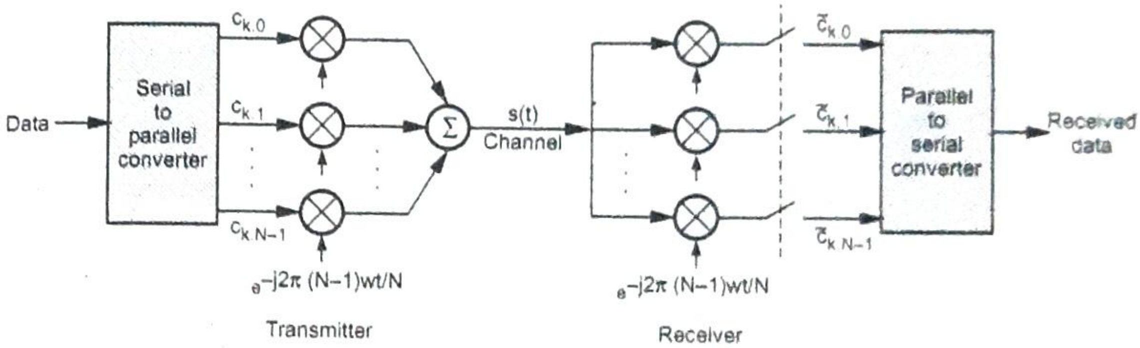


Fig. Q.21.1 Block diagram of analog transceiver of OFDM

- The parallel streams $c_{k,0}; c_{k,1}; \ldots; c_{k,N-1}$ have reduced data rate. These data streams modulate the 'N' orthogonal carriers. These carriers have the frequencies of $f_n = \frac{nW}{N}$, $n = 0, 1 \ldots, N-1$.
- The modulated carriers are then added to form the signal $s(t)$ and transmitted over the channel. The signal undergoes various distortions over the channel.
- At the receiver the signal is then given to various demodulates with same locally generated orthogonal carriers. The signal is sampled at every time interval and observed. The reconstructed signals are $\tilde{c}_{k,0}, \tilde{c}_{k,1} \ldots \tilde{c}_{k,N-1}$.
- The reconstructed signals are again given to parallel to serial convertor. The serial data stream is then the required received signal.

3.9 : Comparison of Digital Modulation Techniques

Q.22 Give the comparison of digital modulation techniques

☞[SPPU : Dec.-12, 14, May-15, Marks 6]

OR Compare the performance of BPSK, FSK, M - ary PSK, M - ary FSK with respect to following parameters.

☞[SPPU : May-16, End sem Marks 8]

Ans. :

Sr. No.	Parameter	BPSK	DPSK	QPSK	M-ary PSK
1	Modulation of	Phase	Phase	Phase	Phase
2	Equation of the transmitted signal $s(t)$	$s(t) = b(t)\sqrt{2P_s}$ $\cos(2\pi f_0 t)$	$s(t) = b(t)\sqrt{2P_S}$ $\cos(2\pi f_0 t)$ $b(t)$ differentially coded	$s(t) = \sqrt{2P_S}\cos$ $[2\pi f_0 t + (2m+1)\frac{\pi}{4}]$ $m = 0, 1, 2, 3$	$s(t) = \sqrt{2P_s}$ $\cos(2\pi f_0 t + \phi_m)$ $\phi_m = (2m+1)\frac{\pi}{M}$ $m = 0, 1, 2, \ldots, M-1$
3	Bits per symbol	One	One	Two	N
4	Number of possible symbols $M = 2^N$	Two	Two	Four	$M = 2^N$
5	Detection method	Coherent	Non-Coherent	Coherent	Coherent
6	Minimum Euclidean distance	$2\sqrt{E_b}$		$2\sqrt{E_b}$	$2\sqrt{E_s}\sin\frac{\pi}{M}$
7	Minimum bandwidth (BW)	$2f_b$	f_b	f_b	$\frac{2f_b}{N}$
8	Symbol duration (T_s)	T_b	$2T_b$	$2T_b$	NT_b
9	Error probability P_e	$\frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}}$	$\frac{1}{2}e^{-E/N_0}$	$\text{erfc}\sqrt{\frac{E}{2N_0}}$	$\text{erfc}\left(\sqrt{\frac{E_s}{N_0}}\sin\frac{\pi}{M}\right)$

Sr. No.	QASK	BFSK	M-ary FSK	MSK	ASK
1	Amplitude and phase	Frequency	Frequency	Frequency	Amplitude
2	$s(t) = k_1 \sqrt{0.2 P_s} \cos (2\pi f_0 t) + k_2 \sqrt{0.2 P_s} \sin (2\pi f_0 t)$ $k_1, k_2 = \pm 1$ or ± 3 for $M = 16$	$s(t) = \sqrt{2 P_s} \cos [2(\pi f_0 + d(t)\Omega) t]$ Ω is frequency shift.	$s(t) = \sqrt{2 P_s} \cos (2 \pi f_I t)$ $I = 1, 2, \ldots M$	$s(t) = b_o(t) \sqrt{2 P_s} \sin 2\pi \left[f_0 + b_e(t) b_o(t) \frac{f_b}{4} \right] t$ $b_e(t), b_o(t)$ = odd/even sequence	$s(t) = 2 \sqrt{2 P_s} \cos (2\pi f_0 t)$ for symbol '1' $= 0$ for symbol '0'
3	N	One	N	Two	One
4	$M = 2^N$	Two	$M = 2^N$	Four	Two
5	Coherent	Non-coherent	Non-coherent	Coherent	Coherent
6	$\sqrt{0.4 E_s}$ for $M = 16$	$\sqrt{2 E_b}$	$\sqrt{2 N E_b}$	$2 \sqrt{E_b}$	$\sqrt{E_b}$
7	$\frac{2 f_b}{N}$	$4 f_b$	$\frac{2^{N+1}}{N} f_b$	$1.5 f_b$	$2 f_b$
8	$N T_b$	T_b	$N T_b$	$2 T_b$	T_b
9.		$\frac{1}{2} \operatorname{erfc} \sqrt{\frac{0.6 E}{N_0}}$	$\left(\frac{M-1}{2} \right) e^{\frac{-E_s}{2 N_0}}$		$\frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{4 N_0}}$

END...

Unit IV

4 SPREAD SPECTRUM MODULATION

4.1 : Introduction

Important Points to Remember

- Spread spectrum modulation occupies much more bandwidth than required Bandwidth.
- Classification of spread spectrum systems.

  1) Averaging 2) Avoidance system

Q.1 Define spread spectrum modulation. Classify and explain in brief. Draw and explain the block diagram of spread spectrum digital communication system.

Ans. : Definition : 1. The transmitted data sequence occupies a much more bandwidth than the minimum required bandwidth and,

The Spectrum Spreading (i.e. increase of signal bandwidth) at the transmitter and despreading at the receiver is obtained by 'special code' which is independent of the data sequence (message signal).

Classification : 1. In *direct sequence modulation,* two stages of modulation are used. In the first stage, the incoming data sequence modulates the wideband code. This transforms the narrow band incoming data sequence into wide band signal. That is the spectrum of the signal is spreaded. This wideband signal then undergoes the second modulation using PSK.

2. In the *frequency hop spread spectrum* technique the spectrum of data modulated carrier is widened by changing the carrier frequency in a pseudo-random manner.

Model of Spread Spectrum Digital Communication

- Fig. Q.1.1 shows the model of spread spectrum digital communication system. The binary information sequence is input to the channel encoder on transmitter side. The channel encoder encodes this input sequence according to some error control coding technique.

- The coded sequence is then given to the modulator. The modulator gets pseudo-random or pseudo-noise (PN) sequence from the pseudo-random pattern generator. This pseudo-noise sequence spreads the signals randomly over the wide frequency band.
- The signal at the output of modulator is the spread spectrum modulated signal. This signal is then transmitted over some channel.
- At the receiver the demodulator gets coded signal back from the spread spectrum signal. For this purpose the demodulator requires the same pseudo-noise sequence which was used at the transmitting end.
- Hence the pseudo-random pattern generators at the transmitter and receiver side operate in synchronization with each other. The channel decoder at the receiver then gets the binary information sequence back.

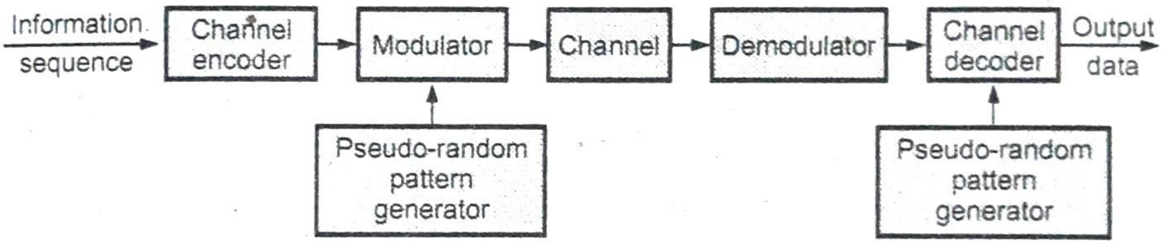


Fig. Q.1.1 Model of the spread spectrum digital communication system

- For any arbitrary receiver it is difficult to know the pseudo-noise sequence since it appears like noise.
- The pseudo-noise sequence at the modulator is used with the PSK modulation to shift the phase of the PSK signal pseudo-randomly. Such technique is called direct sequence (DS) spread spectrum modulation.
- When the pseudo-noise sequence in the modulator is used in conjunction with M-ary FSK to shift the frequency of FSK signal pseudo randomly, the technique is called frequency hopped (FH) spread spectrum method.

Q.2 What is the need for spread spectrum modulation technique ?

☞ [SPPU : May-16, Marks 9]

Ans. : • There are some other applications where it is necessary for the system to resist external interference and to make it difficult for unauthorized receivers to receive the message being transmitted. This type of communication is called secure communication such that noise interference and unwanted receivers should not detect the message.

- Such communication is very very important in military applications where techniques called *Spread Spectrum Modulation* is used.
- Even the spread spectrum modulation is used for non-military applications also. The interference in the transmission channel may be unintentional interference caused because the other user may be transmitting through that same channel.
- Sometimes the interference is created intentionally by a hostile transmitter to 'jam' the transmission.

4.2 : Pseudo-noise Sequences

Important Points to Remember

- The pseudo-random sequence is a high frequency noise binary signal which is generated by a feedback shift register and combinational logic.
- Length of maximum length PN sequence is $2^m - 1$.
- PN sequences here (i) Run (ii) Balance (iii) Correlation properties
- Shift register stages and taps for PN sequence generation are as follows :

  $m = 2 \Rightarrow (2, 1)$; $m = 3 \Rightarrow (3, 1)$; $m = 4 \Rightarrow (4, 1)$, $m = 5 \Rightarrow (5, 2), (5, 4, 3, 2), (5, 4, 2, 1)$

Q.3 What is PN sequence ? Explain the properties of maximum length sequences by giving the graphical representation of the autocorrelation function.

☞ [SPPU : May-15, Dec.-14, End Sem-10, 12, Marks 8]

Ans. : Pseudo Noise Sequence : • The pseudo-noise (or pseudo random) sequence is a noise like high frequency signal. This signal is binary in nature. Thus it looks like pulses. The sequence is not completely random but it is generated by a well defined logic. The same logic is used at transmitter and receiver. Since the sequence is generated by a well defined logic, it is rather 'pseudo' random. Hence it is called pseudo-random (or pseudo-noise) sequence.

Generation of PN Sequence : The generalized block diagram of this scheme is shown in Fig. Q.3.1. The shift register consists of '*m*' flip-flops. The pseudo-noise sequence is generated at the output of last flip-flop in the shift register. At each pulse of the clock, the state of the flip-flop is shifted to the next flip-flop and logic circuit output is shifted in the first flip-flop.

- The pseudo-noise sequence generated at the output of the flip-flop repeated after 2^m digits. 2^m is also called period of the output sequence.

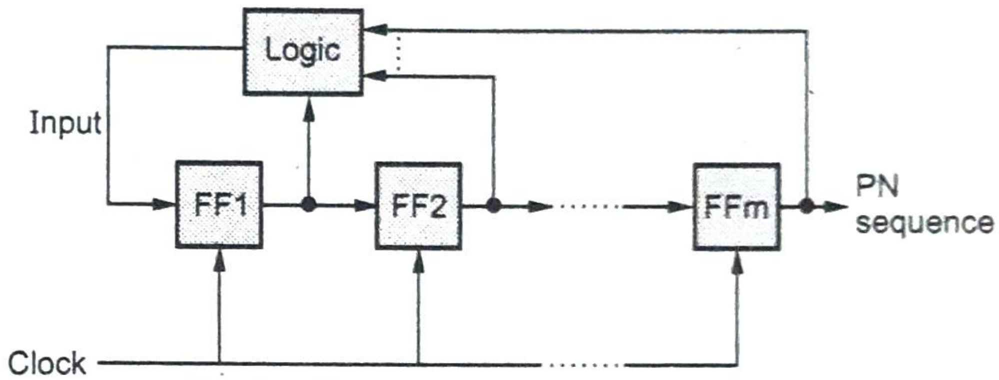


Fig. Q.3.1 Feed back shift register to generate the pseudo-noise sequence

Maximum Length Sequences : When the pseudo-noise sequence generated by linear feed back shift register has the length of $2^m - 1$, it is called maximum length sequence.

Properties of Maximum Length Sequences : 1. ***Balance property :*** The number of 1's is always one more than the number of zeros in each period of a maximum length sequence. Whereas in truly random binary sequence 1's and 0's are equally probable.

2. ***Run property :*** The run means subsequence of identical symbols i.e. 1's or 0's within one period of the sequence. The length of the run is equal to length of the subsequence. When the maximum length sequence is generated by feedback shift registor of length m, then the total number of runs is 2^{m-1}.

In each period of maximum length sequence, there are one half runs of 1's and 0's have length one. There can be one fourth runs of 1's and 0's of length two; or there can be one eighth runs of length three and so on.

3. ***Correlation property :*** The autocorrelation function of maximum length sequence is periodic and it is binary valued.

Autocorrelation function : Autocorrelation function of PN sequence is given as,

$$R_c(\tau) = \begin{cases} 1-\dfrac{N+1}{NT_c}|\tau| & \text{for}|\tau| < T_c \\ -\dfrac{1}{N} & \text{elsewhere} \end{cases}$$

The plot of autocorrelation function is given below :

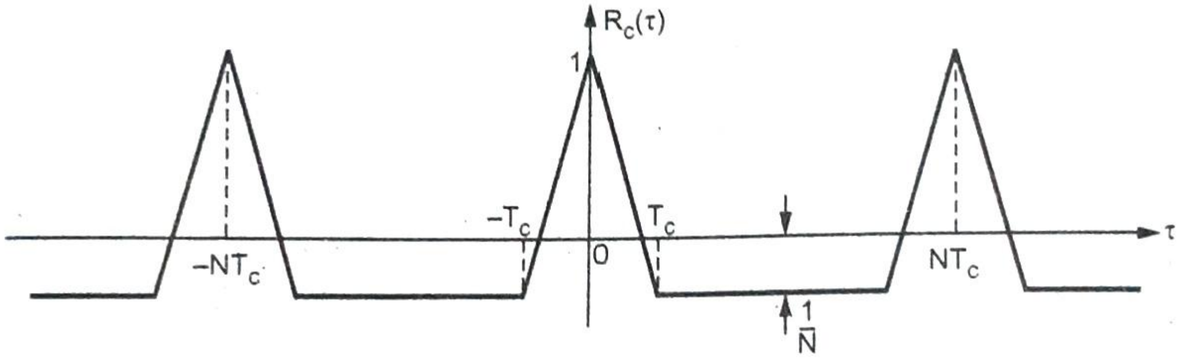


Fig. Q.3.2 Autocorrelation function of PN-sequence

Q.4 For a 4 stage shift register with feedback combination of (4,1), demonstrate the balance property and run property of PN sequence, also calculate and plot the autocorrelation of PN sequence produced by this shift register.

OR A PN sequence is generated using a feedback shift register of length 4 (i.e.4 stage). Find the generated output sequence if the initial contents of the shift register are 1000. If the chip rate is 10^7 chips/sec, calculate the chip and PN sequence duration, and period of output sequence. Draw its schematic arrangement.

☞ [SPPU : Dec.-07, 11, Marks 8; May-13, Marks 10, May-2000, 08, 10, 11, Marks 4; Dec.-06, Marks 8, May-16, End Sem, Marks 9]

Ans. : i) Schematic arrangement : Fig. Q.4.1 shows the 4-stage shift register. The feedback taps are taken out from 1st and 4th stage.

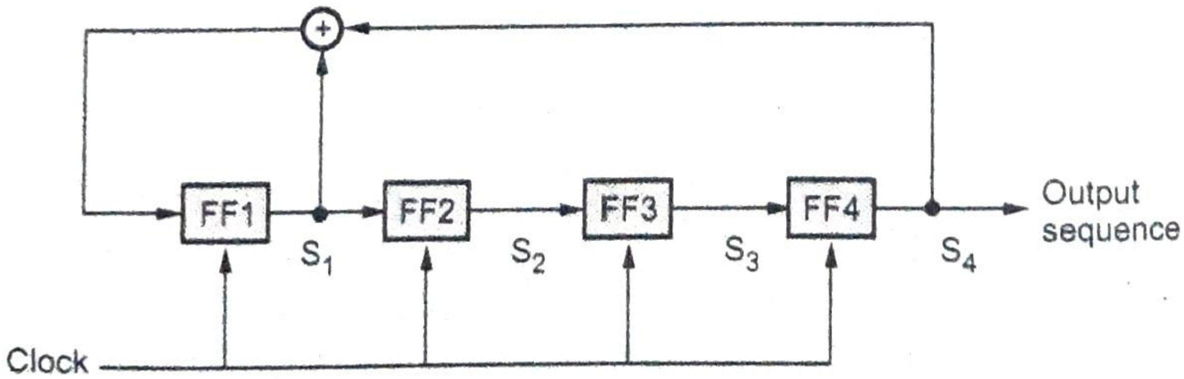


Fig. Q.4.1 A 4-stage shift register with (4,1) taps

ii) To obtain PN sequence : Assume that the initial contents of (s_1 s_2 s_3 s_4) = $\{1000\}$ following table shows the generated sequence.

Sr.No.	State of shift register s_1 s_2 s_3 s_4	MOD-2 adder output $s_1 \oplus s_4$	PN-sequence s_4	
1	1 0 0 0	$1 \oplus 0 = 1$	0	
2	1 1 0 0	$1 \oplus 0 = 1$	0	
3	1 1 1 0	$1 \oplus 0 = 1$	0	
4	1 1 1 1	$1 \oplus 1 = 0$	1	
5	0 1 1 1	$0 \oplus 1 = 1$	1	
6	1 0 1 1	$1 \oplus 1 = 0$	1	
7	0 1 0 1	$0 \oplus 1 = 1$	1	
8	1 0 1 0	$1 \oplus 0 = 1$	0	
9	1 1 0 1	$1 \oplus 1 = 0$	1	
10	0 1 1 0	$0 \oplus 0 = 0$	0	
11	0 0 1 1	$0 \oplus 1 = 1$	1	
12	1 0 0 1	$1 \oplus 1 = 0$	1	
13	0 1 0 0	$0 \oplus 0 = 0$	0	
14	0 0 1 0	$0 \oplus 0 = 0$	0	
15	0 0 0 1	$0 \oplus 1 = 1$	1	
16	1 0 0 0	$1 \oplus 0 = 1$	0	← sequence repeats here

Table Q.4.1 Generation of PN sequence of Fig. Q.4.1

Thus the produced PN sequence is ,

$$C_n = \{ 0\,0\,0 \;\; 1\,1\,1\,1 \;\; 0\,1\,0\,1\,1\,0\,0\,1 \}$$

iii) To verify properties of maximum length sequence

Balance property : In one period of the PN sequence there are 7 zeros and 8 ones. This satisfies balance property.

Run property : Here the number of runs will be

$$2^{m-1} = 2^{4-1} = 8 \text{ runs. These are as follows :}$$

$$C_n = \left\{ \underbrace{0\,0\,0}_{1} \;\; \underbrace{1111}_{2} \;\; \underbrace{0}_{3} \;\; \underbrace{1}_{4} \;\; \underbrace{0}_{5} \;\; \underbrace{11}_{6} \;\; \underbrace{00}_{7}\underbrace{1}_{8} \right\}$$

Thus there are total 8 runs.

1. There are 4 runs of length 1. These are run 3, run 4, run 5, and run 8.
2. There are 2 runs of length two. There are run 6 and run 7.
3. There is one run of length three. It is run 1.

Thus run property is satisfied.

iv) To obtain auto correlation function

Auto correlation function is given as,

$$R_c(\tau) = \begin{cases} 1 - \dfrac{N+1}{NT_c}|\tau| & \text{for} |\tau| < T_c \\ -\dfrac{1}{N} & \text{elsewhere} \end{cases}$$

Here $N = 2^m - 1 = 2^4 - 1 = 15$. Hence we have,

$$R_c(\tau) = \begin{cases} 1 - \dfrac{16|\tau|}{15T_c} & \text{for} |\tau| < T_c \\ -\dfrac{1}{15} & \text{elsewhere} \end{cases}$$

Fig. Q.4.2 shows the sketch of this auto correlation function.

Solution to the 'OR' example

i) To obtain chip duration :

The chip rate is $R_c = 10^7$ chips/sec. Hence the chip duration is

$$T_c = \frac{1}{R_c} = \frac{1}{10^7} \text{ sec}$$

$$= 0.1 \ \mu\text{sec.}$$

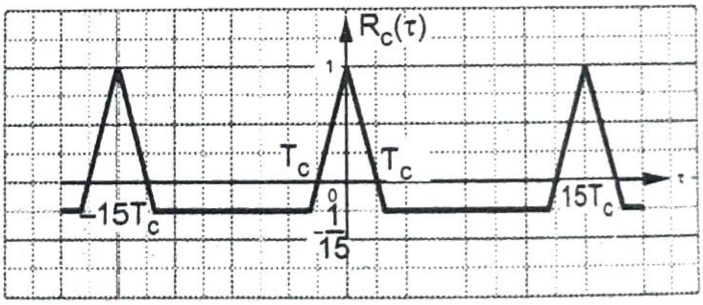


Fig. Q.4.2 Plot of auto correlation of PN sequence

ii) To obtain length of PN sequence : The length of the PN sequence is,

$$N = 2^m - 1 = 2^4 - 1 = 15 \text{ digits.}$$

iii) To obtain period of output sequence : Hence period of the output sequence is,

$$T_b = NT_c = 15 \times 0.1 \ \mu\text{sec} = 1.5 \ \mu\text{sec}$$

4.3 Direct Sequence Spread Spectrum

Important Points to Remember

- When the wide band signal (pseudo - noise signal) is multiplied with the narrow band data signal b(t), the modulated message signal has wide spectrum and hence it is called direct sequence spread spectrum.
- DS-SS is an essentially synchronous system. The receiver requires an exact knowledge of pseudo - noise sequence for proper detection of the data signal b(t).
- The performance of DS-SS can be evaluated on the basis of

  i) Processing gain ii) Probability of error iii) Jamming margin.

Q.5 Draw and explain baseband DS-SS transmitter and receiver.

[SPPU : May-11, Marks 6]

Ans. : DS-SS Transmitter (Encoder) : The data signal *b(t)* and the pseudo-noise signal *c(t)* are applied to the product modulator or modulator. The output of the modulator is the wide spectrum signal. The spectrum of this signal is quite high compared to that of narrowband data signal *b(t)* Fig. Q.5.1 shows this simplified direct sequence modulator.

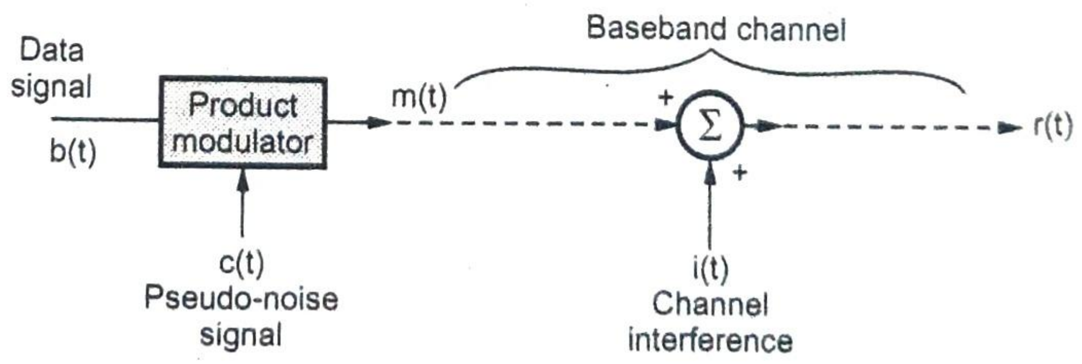


Fig. Q.5.1 Direct sequence spread spectrum system transmitter (base band transmission) or encoder

- The message signal is represented as,

$$m(t) = c(t) \cdot b(t) \quad \ldots (Q.5.1)$$

Here c(t) is a pseudo-noise signal and b(t) is data signal. The modulated message signal m(t) has wide spectrum as large as that of c(t).

- During transmission through channel, the noise i(t) interferes with the message signal. i.e.,

$$r(t) = m(t) + i(t) = c(t) \cdot b(t) + i(t) \qquad \text{from eq. (Q.5.1)}$$

DS-SS Receiver (Decoder) : Fig. Q.5.2 shows the block diagram of spread spectrum receiver of decoder for baseband transmission. The receiver consists of a multiplier and integrator. As shown in Fig. Q.5.2, a locally generated pseudo-noise signal is applied to the multiplier. This signal is an exact replica of that used in the transmitter. The output of the multiplier is equal to the received noisy signal *r(t)* and pseudo-noise signal *c(t)* i.e.

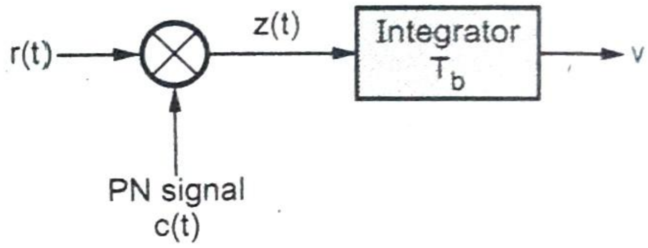


Fig. Q.5.2 Block diagram of spread spectrum receiver or decoder for baseband transmission

$$z(t) = c(t)\ r(t)$$

- Putting value of *r(t)* in the above equation we obtain,

$$z(t) = c^2(t)\ b(t) + c(t)\ i(t) \qquad \text{... (Q.5.2)}$$

The value of *c(t)* is either +1 or −1. Therefore

$$c^2(t) = +1 \qquad \text{... (Q.5.3)}$$

With this substitution equation (Q.5.2) becomes

$$z(t) = b(t) + c(t)\ i(t) \qquad \text{... (Q.5.4)}$$

The bandwidth of *c(t) i(t)* is large compared to that of *b(t)*. The signal *z(t)* is then passed through the integrator. This integrator acts as a low pass filter and removes the wide band noise *c(t) i(t)*.

Q.6 Explain DS-SS BPSK transmitter and receiver.

☞ [SPPU : May-11, 12, Dec.-10, 12,15, Marks 8]
[Dec.-14, (End Sem), Marks 2]

Ans. : DS-SS BPSK Transmitter : Fig. Q.6.1 shows the transmitter of Direct Sequence Spread Spectrum with BPSK.

The multiplier multiplies the two signals *b(t)* and *c(t)*. The output of multiplier is direct sequence spread signal *m(t)*. This signal is given as modulating signal to BPSK transmitter. The direct sequence BPSK (or DS/BPSK) signal is generated at the output (i.e. $x(t)$). he carrier is represented as,

$$\phi(t) = \sqrt{2P_s}\sin(2\pi f_c t) \quad \text{... (Q.6.1)}$$

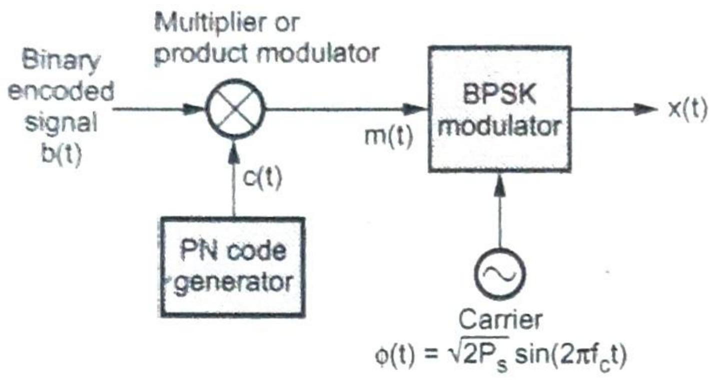


Fig. Q.6.1 Direct sequence spread spectrum BPSK transmitter or encoder

Then the transmitted signal is

$$x(t) = \sqrt{2P_s}\, m(t)\sin(2\pi f_c t) \quad \text{... (Q.6.2)}$$

Thus when $m(t)$ is positive, there is phase shift of '0' and if it is negative, there is phase shift of 180°.

DS-SS BPSK Receiver : • Fig. Q.6.2 shows the block diagram of DS/BPSK receiver. The received signal $y(t)$ is applied to the multiplier which is also supplied with locally generated coherent carrier. The output of the multiplier is then applied the low pass filter (integrator). The bandwidth of this low pass filter is equal to that of $m(t)$. This signal $\hat{m}(t)$ is applied to the second demodulator which despreads the signal. The local pseudo-noise signal is exact replica of that used in the transmitter.

- The integrator integrates the product of detected message signal and pseudo-noise signal over one bit period 'T_b'. The decision is then taken depending upon the polarity of output (v) of the integrator. Ouput is '1' if $v > 0$ and output is '0' if $v < 0$.

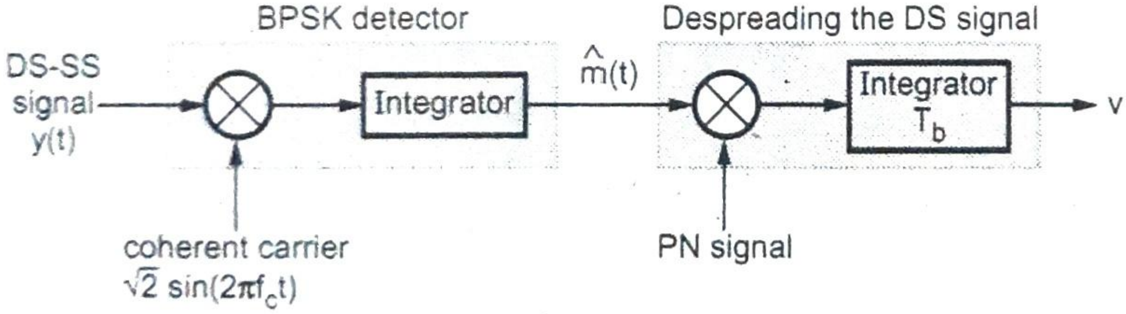


Fig. Q.6.2 Block diagram of DS/BPSK receiver or decoder

Q.7 Explain the i) Processing gain ii) Jamming margin.

[SPPU : May-15, Marks 4]

Ans. : i) Processing gain : It is defined as the ratio of the bandwidth of spread message signal to bandwidth of unspreaded data signal i.e.

$$\text{Processing gain} = \frac{\text{BW (spreaded signal)}}{\text{BW(unspreaded signal)}}$$

Now BW (spreaded signal) = $\frac{1}{T_c}$ and BW (data signal) = $\frac{1}{T_D}$

$$\therefore \quad \text{Processing gain} = \frac{1/T_c}{1/T_b} = \frac{T_b}{T_c}$$

$$\therefore \text{Processing gain,} \quad \boxed{PG = \frac{T_b}{T_c}}$$

ii) Jamming Margin : Now $E_b = P_s T_b$ where E_b is bit energy P_s is average signal power and T_b is one bit period of data signal.

Now consider the ratio of signal energy per bit to noise spectral density (E_b/N_0)

$$\text{i.e.} \quad \frac{E_b}{N_0} = \frac{P_s T_b}{N_0}$$

$$\text{put} \quad N_0 = JT_c$$

$$\therefore \quad \frac{E_b}{N_0} = \frac{P_s T_b}{JT_c} \text{ or } \frac{E_b}{N_0} = \left(\frac{P_s}{J}\right)\left(\frac{T_b}{T_c}\right)$$

$$\therefore \quad \frac{J}{P_s} = \frac{T_b}{T_c} \cdot \frac{1}{E_b/N_0}$$

$$\text{Now } \frac{T_b}{T_c} = \text{PG} \quad \therefore \frac{J}{P_S} = \frac{PG}{(E_b/N_0)}$$

Here ratio J/P_s is the **jamming margin** Thus, **jamming margin** is the ratio of average powers of interference (J) and data signal (P_s).

$$\left(J/P_s\right) = (PG)_{dB} - \left(E_b/N_0\right)_{dB}$$

Q.8 The direct sequence spread spectrum communication system has following parameters.

Data sequence bit duration, T_b = 4.095 ms

PN chip duration, T_c = 1 μs

$\frac{E_b}{N_0}$ = 10 for average probability of error less than 10^{-5}.

Calculate processing gain and jamming margin.

What is the number of shift registers required ?

[SPPU : Dec.-14, End Sem, Marks 6]

Ans. : Some times the one bit period of pseudo-noise sequence (i.e. PN sequence) is also calls as one '*Chip*'. Here one chip duration is T_c i.e. $T_c = 1\mu s$ and T_b = 4.095 ms. Processing gain is given as,

$$PG = \frac{T_b}{T_c} = \frac{4.095 \times 10^{-3}}{1 \times 10^{-6}} = 4095$$

Since PG = N, the length of the bit sequence is 4095.

The jamming margin is given as,

$$\text{Jamming Margin} = \frac{J}{P_s} = \frac{PG}{E_b/N_0} = \frac{4095}{10} = 409.5$$

This shows that information bits at the receiver output can be detected with the probability of error less than 10^{-5} even when noise interference is upto 409.5 times the received signal power. The jamming margin can also be calculated in dB i.e.,

$$(\text{Jamming Margin})_{\text{dB}} = (PG)_{\text{dB}} - 10\log_{10}\left(\frac{E_b}{N_0}\right)$$

$$= 10\log_{10}(4095) - 10\log_{10}(10) = 36.1 - 10 = 26.1 \text{ dB}$$

$$N = 2^m - 1$$

$$4095 = 2^m - 1 \Rightarrow 4096 = 2^m \Rightarrow 12.$$

Thus '12' shift registers are required.

Q.9 The information bit duration in DS-BPSK spread spectrum communication system is 4 ms while the chipping rate is 1 MHz. Assuming an average error probability of 10^{-5} for proper detection of message signal, calculate the jamming margin. Interpret your result.

Given Q(4.25) = 10^{-5}.

[SPPU : Dec.-10, Marks 6, May-16, End Sem, Marks 9]

Ans. : $T_b = 4\times10^{-3}$ sec, $T_c = \frac{1}{1\times10^6} = 1\times10^{-6}$

$$P_e = 10^{-5}$$

$$PG = \frac{T_b}{T_c} = \frac{4\times10^{-3}}{1\times10^{-6}} = 4000$$

$$P_e = \frac{1}{2}\, erfc\sqrt{\frac{E_b}{N_0}}$$

since $erfc\ (u) = 2Q\left(\sqrt{2}\,u\right)$, with $u = \sqrt{\frac{E_s}{N_0}}$,

$$P_e = \frac{1}{2}\times 2Q\left(\sqrt{2}\cdot\sqrt{\frac{E_b}{N_0}}\right)$$

$$\therefore \quad 10^{-5} = Q\sqrt{\frac{2E_b}{N_0}}$$

It is given that $Q(4.25) = 10^{-5}$. Hence,

$$\sqrt{\frac{2E_b}{N_0}} = 4.25$$

$$\therefore \quad \frac{2E_b}{N_0} = 4.25^2 \quad \text{or} \quad \frac{E_b}{N_0} = \frac{4.25^2}{2} = 9$$

Jamming margin, $\frac{J}{P_s} = \frac{PG}{(E_b/N_0)} = \frac{4000}{9} = 444.44$

$$= 10\log_{10} 444.44 = 26.48\ \text{dB}$$

Q.10 The information bit duration in DS-BPSK spread spectrum communication system is 10 msec while the chipping rate is 1 MHz. Assuming an average error probability is 10^{-6} for proper detection of message signal, calculate the Jamming margin.

[SPPU : May-11, Marks 6]

Ans. : Here $T_c = \frac{1}{1\times10^6} = 1\times10^{-6}$

$$T_b = \frac{1}{10\times10^3} = 1\times10^{-4} \quad \text{and} \quad P_e = 10^{-6}$$

For DS/BPSK, $P_e = \frac{1}{2}\ erfc\ \sqrt{\frac{E_b}{N_0}}$

$$\therefore \qquad 10^{-6} = \frac{1}{2}\ erfc\ \sqrt{\frac{E_b}{N_0}}$$

$$\therefore \qquad 2\times 10^{-6} = erfc\ \sqrt{\frac{E_b}{N_0}}$$

- From error function table on page (ix) observe that $erfc\ (3.3) \approx 2\times 10^{-6}$.

$$\therefore \qquad \sqrt{\frac{E_b}{N_0}} = 3.3$$

$$\therefore \qquad \frac{E_b}{N_0} = 3.3^2 = 10.89$$

Processing gain, $PG = \frac{T_b}{T_c} = \frac{1\times 10^{-4}}{1\times 10^{-6}} = 100$

Jamming margin, $\frac{J}{P_s} = \frac{PG}{E_b/N_0} = \frac{100}{10.89} = 9.18 = 10\log_{10} 9.18 = 9.63\,\text{dB}$

Q.11 In DSSS-BPSK system, the feedback shift register used to generate the PN sequence of length 15. The system is required to have an average probability of symbol error as 10^{-5}: Find (i) PG (ii) Antijam margin.

Given : erf (x) (3.01) = 0.00002074, erfc(3.02) = 0.00001947

erfc(3.03) = 0.00001827, erfc(3.04) = 0.00001714

☞ [SPPU : Dec.-13, Marks 8]

Ans. : Given : m = 15

$\therefore$Period of PN sequence is given by,

$$N = 2^m - 1 = 2^{15} - 1 = 32767$$

For DS-SS BPSK,

i) P.G. = N = 32767

ii) $(\text{Jamming margin})_{dB} = (PG)_{dB} - 10\log_{10}\left(E_b/N_0\right)$

$$\text{Now } P_e = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$$

$$10^{-5} = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$$

$$2\times 10^{-5} = erfc \sqrt{\frac{E_b}{N_0}}$$

$$\text{since erfc } (3.2) = 2\times 10^{-5}$$

$$3.2 = \sqrt{\frac{E_b}{N_0}} \Rightarrow \frac{E_b}{N_o} = 3.2^2 = 10.24$$

$$(\text{Jamming margin})_{dB} = \frac{PG}{E_b/N_0} = \frac{32767}{10.24} = 3200 = 35 \text{ dB.}$$

4.4 : Frequency Hop SS

Important Points to Remember

- Frequency hopping transmits the data bits in different frequency slots. The total BW of the output signal is equal to sum of all these frequency slots or hops.
- Two types of frequency hop spread spectrum :

  i) Slow frequency hopping ii) Fast frequency hopping
- When several symbols of data are transmitted in one frequency hop, then it is called slow-frequency hopping.
- When several frequency hops take place to transmit one symbol, then it is called fast hopping.

Q.12 Draw and explain FHSS spread spectrum with transmitter and receiver section. ☞ **[SPPU : May-11, 14, 15, Dec.-14, Marks 8]**

Ans. : Transmitter of FH / MFSK : Fig. Q.12.1 shows the block diagram of frequency hopping M-ary FSK transmitter.

The input binary data sequence is applied to the M-ary FSK modulator. This modulator output is the particular frequency (out of '*M*' frequencies) depending upon the input symbol. The output of FSK modulator is then applied to a mixer. The other input to the mixer is particular frequency from frequency synthesizer. The output of frequency synthesizer at

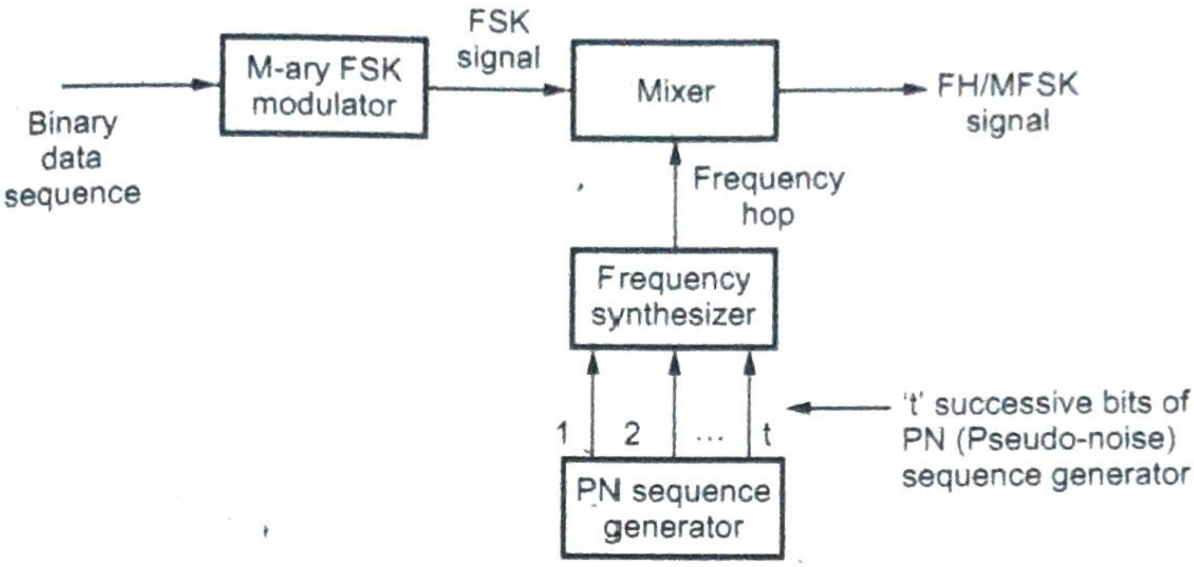


Fig. Q.12.1 Block diagram of transmitter of frequency hop spread spectrum system

particular instant is the frequency slot or 'hop'. This hop is mixed with the FSK signal. The output of the mixer is the 'sum' frequency component of FSK signal and frequency hop from synthesizer. This signal is FH/MFSK signal and is transmitted over the wideband channel.

The frequency hops (or slots) given to the mixer are generated by the frequency synthesizer. The inputs of the frequency synthesizer are controlled by pseudo-noise (PN) sequence generator. The 't' successive bits of PN sequence generator control the frequency hops generated by synthesizer.

Receiver of FH/MFSK : The received FH/MFSK signal is applied to the mixer. The output of frequency synthesizer is also given to the mixer. The sum and difference frequencies are generated by the mixer. Only difference frequencies are allowed to pass out of the mixer. Those difference frequencies are exactly the M-ary FSK signals. These signals are given to the non-coherent M-ary FSK detector. The detector detects particular symbol transmitted. (See Fig. Q.12.2 on next page).

Q.13 Explain the following terms : i) Slow frequency hopping ii) Fast frequency hopping. ☞ **[SPPU : May-15, End Sem, Marks 4]**

Ans. : i) Slow frequency hopping : When several symbols of data are transmitted in one frequency hop (slot), then it is called slow frequency hopping. This means symbol rate is higher than hop rate. Here hop rate is slower (than R_s), hence it is called slow frequency hopping.

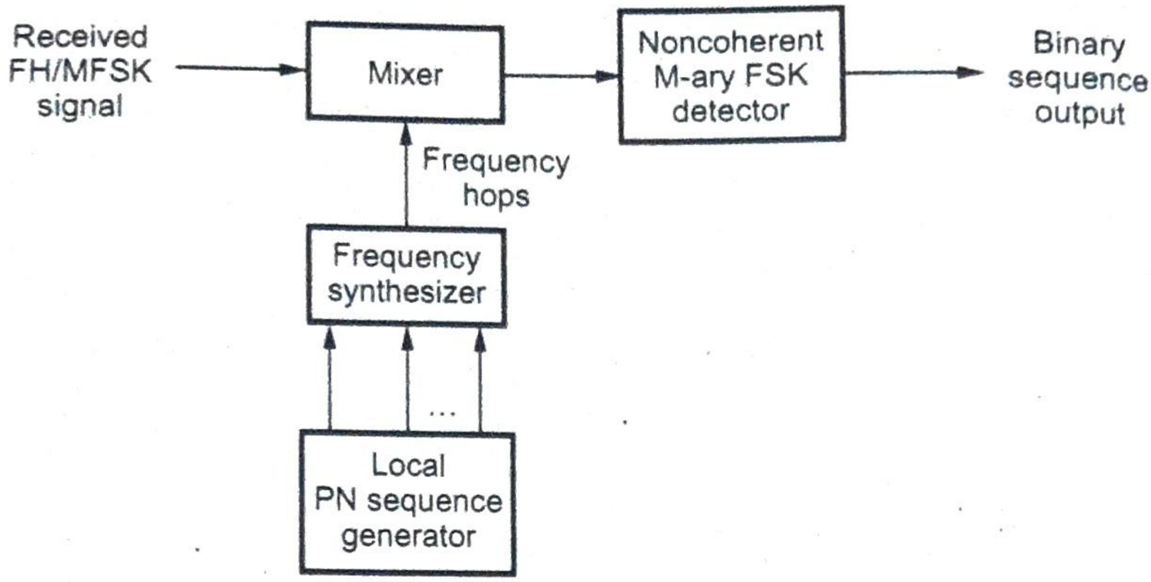


Fig. Q.12.2 Block diagram of receiver of frequency hop spread spectrum

$$\text{Chip rate } R_c = \frac{R_b}{k}$$

Here 'R_b' is input bit rate and 'k' is number of bits per symbol.

$$\text{Processing gain, PG} = 2^t$$

Here 't' are bits of PN sequence.

ii) Fast frequency hopping :

When several frequency hops take place to transmit one symbol, then it is called fast frequency hopping. Therefore the hop rate (R_h) is higher than symbol rate (R_s). The advantage of fast hopping is that, before the jammer tries to complete reception of one symbol, the carrier frequency is changed.

$$\text{Chip rate, } R_c = R_h$$

Here 'R_h' is hop rate

Q.14 Consider a slow hop SS system with binary FSK that transmits two symbols per frequency hop and has a PN generator with k = 3 outputs. For a binary message sequence [01 10 11 01 10 00]. Draw the spectral output (output frequency Vs data input). Determine the processing gain if $W_x = r_b = 3000$ and find the bit error probability in presence of white noise if $N_0 = 10^{-12}$W/Hz, $S_R = 5.4 \times 10^{-8}$ W.

☞ [SPPU : Dec.-05, Marks 10, May-13, Marks 8]

Ans. : To draw the spectral output.

Let the PN sequence be

001 111 011 001 110 101.

Fig. Q.14.1 shows the spectral output

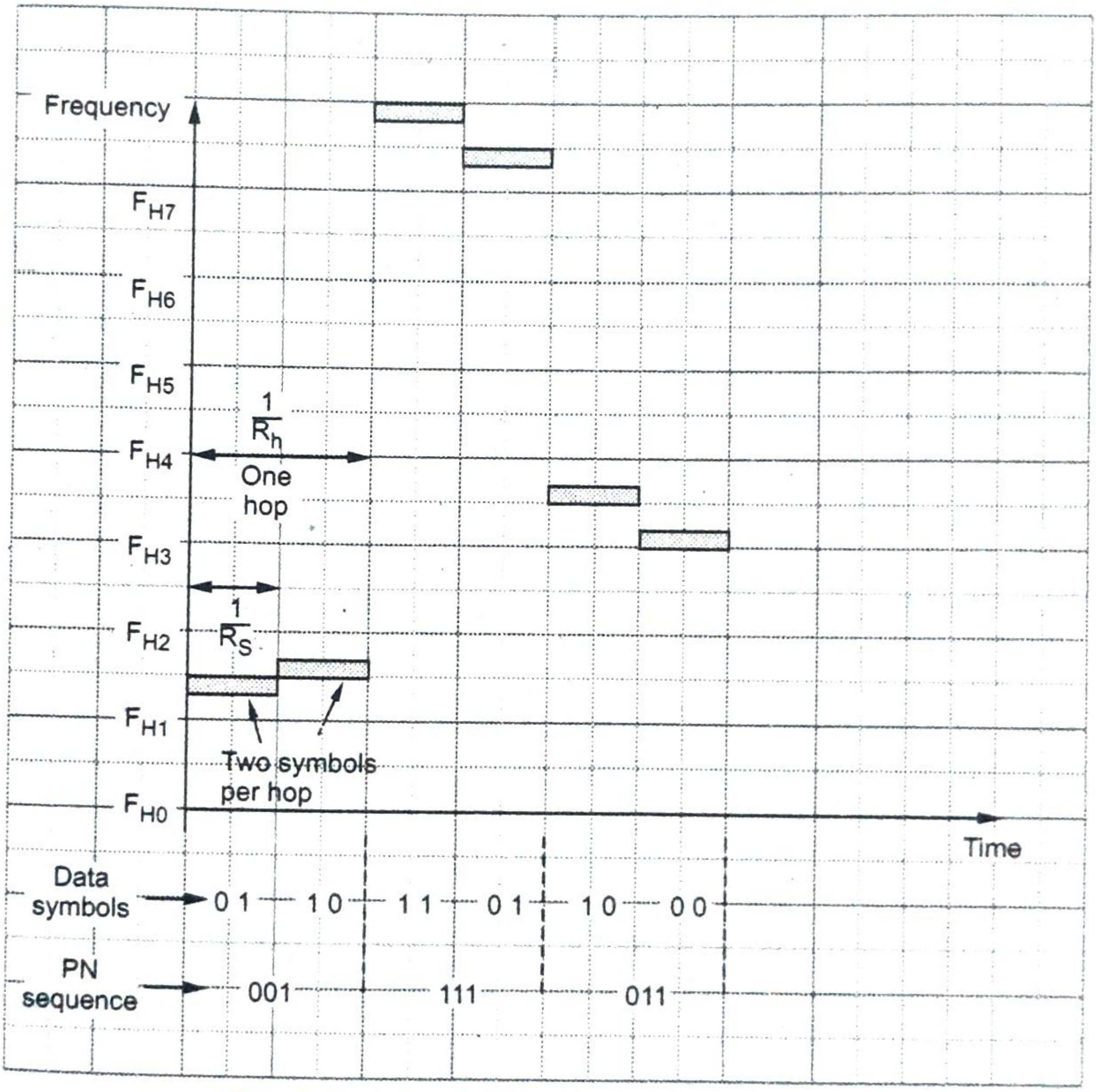


Fig. Q.14.1 Spectral output of slow FH-SS

To obtain PG

Processing gain is given as,

$$PG = 2^t$$

Here $t = k = 3$ is number outputs in PN sequence. Hence,

$$PG = 2^3 = 8.$$

To obtain error probability

Error probability for FH-SS orthogonal noncoherent FSK is given as,

$$P_e = \frac{1}{2} e^{-\gamma_b/2}$$

Here $\gamma_b = \frac{E_b}{N_0}$, hence $P_e = \frac{1}{2} e^{-\frac{E_b}{2N_0}}$

$$E_b = P_s T_b \text{, here } P_s = S_R = 5.4 \times 10^{-8} \text{ and } T_b = \frac{1}{r_b} = \frac{1}{3000}$$

$$\therefore \quad E_b = \frac{5.4 \times 10^{-8}}{3000} = 1.8 \times 10^{-11}$$

$$\therefore \quad P_e = \frac{1}{2} e^{-\frac{1.8 \times 10^{-11}}{2 \times 10^{-12}}} = 6.17 \times 10^{-5}$$

Q.15 Consider a fast hop spread spectrum system with binary FSK, 2 hops/symbol and a PN sequence generator with outputs with a binary message 010010010000. The message is transmitted using following PN sequence {010, 110, 101, 100, 000, 101, 011, 001, 001, 111, 011, 001, 110, 101, 101, 001, 110, 001, 011, 111, 100, 000, 110, 110}. Plot the output frequency for the input message.

☞[SPPU : Dec.-11, Marks 8]

Ans. : • Here number of bits per symbol are not mentioned. Hence 1 bit per symbol is transmitted. In other words, one bit is equal to one symbol.

• Two hops are used to transmit one bit (symbol). Fig. Q.15.1 shows the fast hopping output for this system. Note that first symbol '0' is transmitted in two hops (010 and 110). The corresponding transmitted frequencies are $(F_{H2} + F_1)$ and $(F_{H6} + F_1)$.

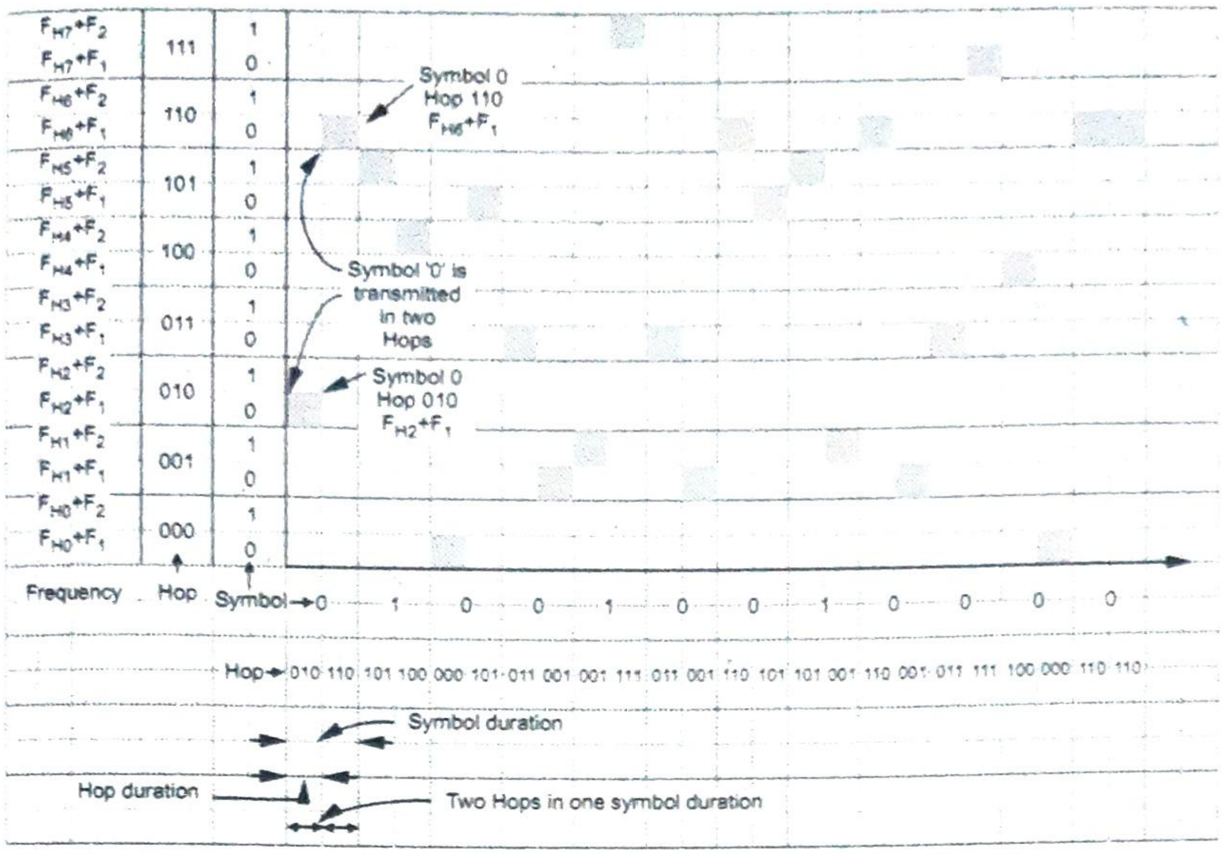


Fig. Q.15.1

Q.16 Represent variation of the frequency of an fast hop spread spectrum system with binary FSK, having following parameters. Number of bits per MFSK symbol K = 2, Number of MFSK tones M = 2^K = 4, length of PN segment per hop K = 3, total number of frequency hops 2^K = 8 for the binary message of 0111111000100111010.

☞[SPPU : May-15, End Sem, Marks 8, Dec.-15, End Sem, Marks 8, May-14, Dec.-12, Marks 10]

Ans. : Number of bits/symbol $K = 2,$

Number of MFSK tones, $M = 2^K = 4$

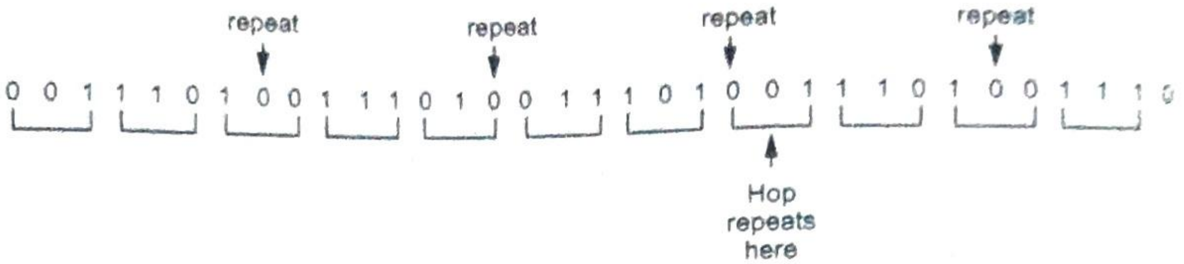
Length of PN segment per hop, $K = 3$

Number of frequency hop $2^K = 8$

Let the binary message be,

01 11 11 10 00 10 01 11 10 10

Let us consider the following three bit PN sequence, 0 0 1 1 1 0 1, ... and so on. This can be written repeated as follows :



Here observe that the PN sequence repeats after 7^{th} bit. Hops of 3-bits are formed from the repeated PN sequence. These hops are shown above. Observe that hops repeat after 7 hops. Fig. Q.16.1 shows the transmitted frequencies with shaded areas. (See Fig. Q.16.1 on next page)

4.5 : Comparison of SS

Q.17 Compare slow and fast FH-SS.

Ans. :

Sr. No.	Parameter	Slow frequency hopping	Fast frequency hopping
1.	Definition	Multiple symbols are transmitted in one frequency hop.	Multiple hops are taken to transmit one symbol.
2.	Chip rate	Symbol rate is equal to chip rate.	Hop rate is equal to chip rate.
3.	R_h and R_s	Hop rate is lower than symbol rate.	Hop rate is higher than symbol rate.
4.	Carrier frequencies	One or more symbols are transmitted over the same carrier frequency.	One symbol is transmitted over multiple carriers in different hops.
5.	Jammer interference	This signal can be detected by jammer if carrier frequency in one hop is known.	This signal is difficult to detect since one symbol is transmitted on multiple carrier frequencies.

Table Q.17.1 : Comparison of FH-SS methods

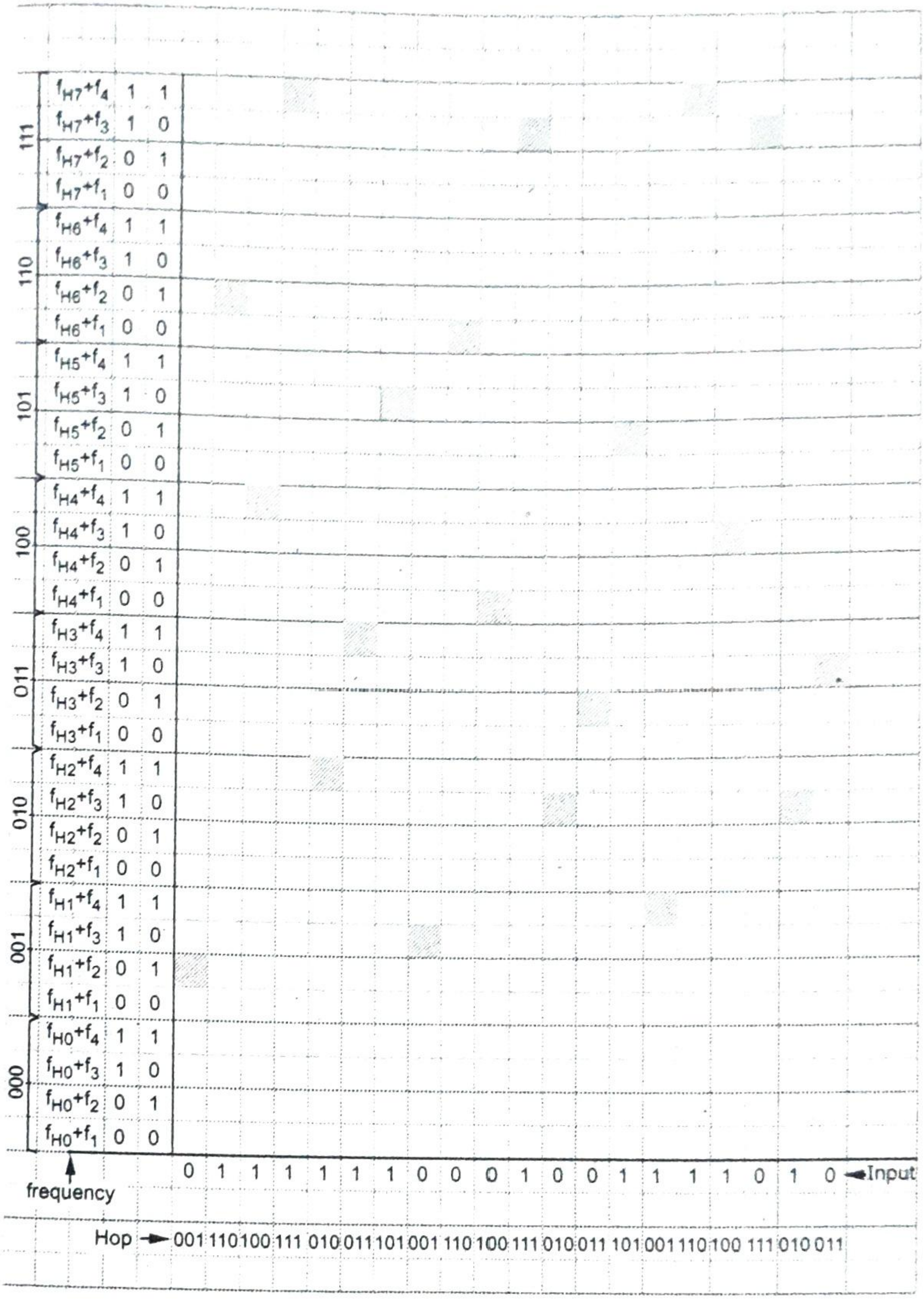


Fig. Q.16.1 Fast hoping

Q.18 Compare DS-SS and FH-SS.

[SPPU : Dec.-15, End Sem, Marks 8, May-16, End Sem, Marks 8]

Ans. :

Sr. No.	Parameter	Direct sequence spread spectrum	Frequency hop spread spectrum
1.	Definition	*PN* sequence of large bandwidth is multiplied with narrowband data signal.	Data bits are transmitted in different frequency slots which are changed by *PN* sequence.
2.	Spectrum of signal	Data sequence is spread over entire bandwidth of spread spectrum signal.	Data sequence is spread over small frequency slots of the spread spectrum signal.
3.	Chip rate R_c	Chip rate is fixed. It is the rate at which bits of *PN* sequence occur. $R_c = \frac{1}{T_c}$	Chip rate is maximum of hop rate or symbol rate. $R_c = \max(R_h, R_s)$
4.	Modulation technique	Normally uses BPSK modulation.	Normally uses M-ary FSK modulation.
5.	Processing gain	$PG = \frac{T_b}{T_c} = N$	$PG = 2^t$, here t is the bits in *PN* sequence.
6.	Probability of error	$P_e = \frac{1}{2} erfc \sqrt{\frac{E_b}{JT_c}}$	$P_e = \frac{1}{2} e^{-\gamma_b R_c / 2}$ Here $\gamma_b = \frac{E_b}{J_0}$
7.	Effect of distance	This system is distance relative.	Effect of distance is less in this system.
8.	Acquisition time	Acquisition time is long.	Acquisition time is short.

Table Q.18.1 : Comparison of direct sequence and frequency hopping spread spectrum

Q.19 State the advantages and disadvantages of DS-SS and HF-SS techniques. ☞[SPPU : Dec.-10, Marks 4]

Ans. : ***Advantages of direct sequence spread spectrum :***

1. This system has best noise and antijam performance.
2. Unrecognized receivers find it most difficult to detect direct sequence signals.
3. It has best discrimination against multipath signals.

Disadvantages of direct sequence systems :

1. It requires wideband channel with small phase distortion.
2. It has long acquisition time.
3. The pseudo-noise generator should generate sequence at high rates.
4. This system is distance relative.

Advantages of frequency hopping system :

1. These systems bandwidth (spreads) are very large.
2. They can be programmed to avoid some portions of the spectrum.
3. They have relatively short acquisition time.
4. The distance effect is less.

Disadvantages of frequency hopping systems :

1. Those systems need complex frequency synthesizers.
2. They are not useful for range and range-rate measurement.
3. They need error correction.

4.6 : Applications of Spread Spectrum System

Q.20 State the applications of spread spectrum techniques.

Ans. :

i) Military applications

ii) Low probability intercept

iii) Mobile communications

iv) Secure communications in military as well as commercial

v) Distance measurement.

vi) Selective calling.

vii)CDMA communication

Q.21 Explain Code Division Multiple Access (CDMA) with Direct Sequence SS (SSMA).

Ans. : In this application, many users transmit their signals on the same channel bandwidth. Each transmitter receiver pair has a distinct pseudo-noise (PN) sequence. Thus signals of a particular transmitter are received by its intended receiver only, even if many users are transmitting at the same time. This method is also called *spread spectrum multiple access (SSMA)*. The signals from other users appear as additive interference which are rejected by the spread spectrum decoder. The level of interference depends upon the number of users transmitting at any time. The main advantage of CDMA is that the number of users sharing the same channel can be increased or decreased very easily. Large number of users can transmit on the same channel if their messages are for short periods of time. For this method it is desirable that the PN sequences be mutually orthogonal.

Q.22 Explain the application ranging using Direct Sequence Spread Spectrum.

Ans. : In last subsection we discussed three basic types of applications where direct sequence spread spectrum can be used. Fig. Q.22.1 shows the block diagram of ranging using direct sequence spread spectrum. (See Fig. Q.22.1 on next page).

The pseudo-noise sequence generator generates the pseudo noise signal g(t). It modulates the carrier $\sqrt{2P_s}\cos\omega_o t$. Hence we get,

$$x(t) = \sqrt{2P_s}\,g(t)\cos\omega_0 t \qquad \text{... (Q.22.1)}$$

This signal is transmitted towards the spacecraft. The reflected signal from the spacecraft is,

$$y(t) = \alpha x(t - T_1 - T_2) \qquad \text{... (Q.22.2)}$$

Here α is the attenuation of the signal, T_1 is the transmit time and T_2 is the receive time.

The signal reaches to the receiver after time $T_1 + T_2$. Equation (Q.22.1) can also be written as,

$$y(t) = \alpha\sqrt{2P_s}\,g(t - T_1 - T_2)\cos(\omega_0 t + \theta) \qquad \text{... (Q.22.3)}$$

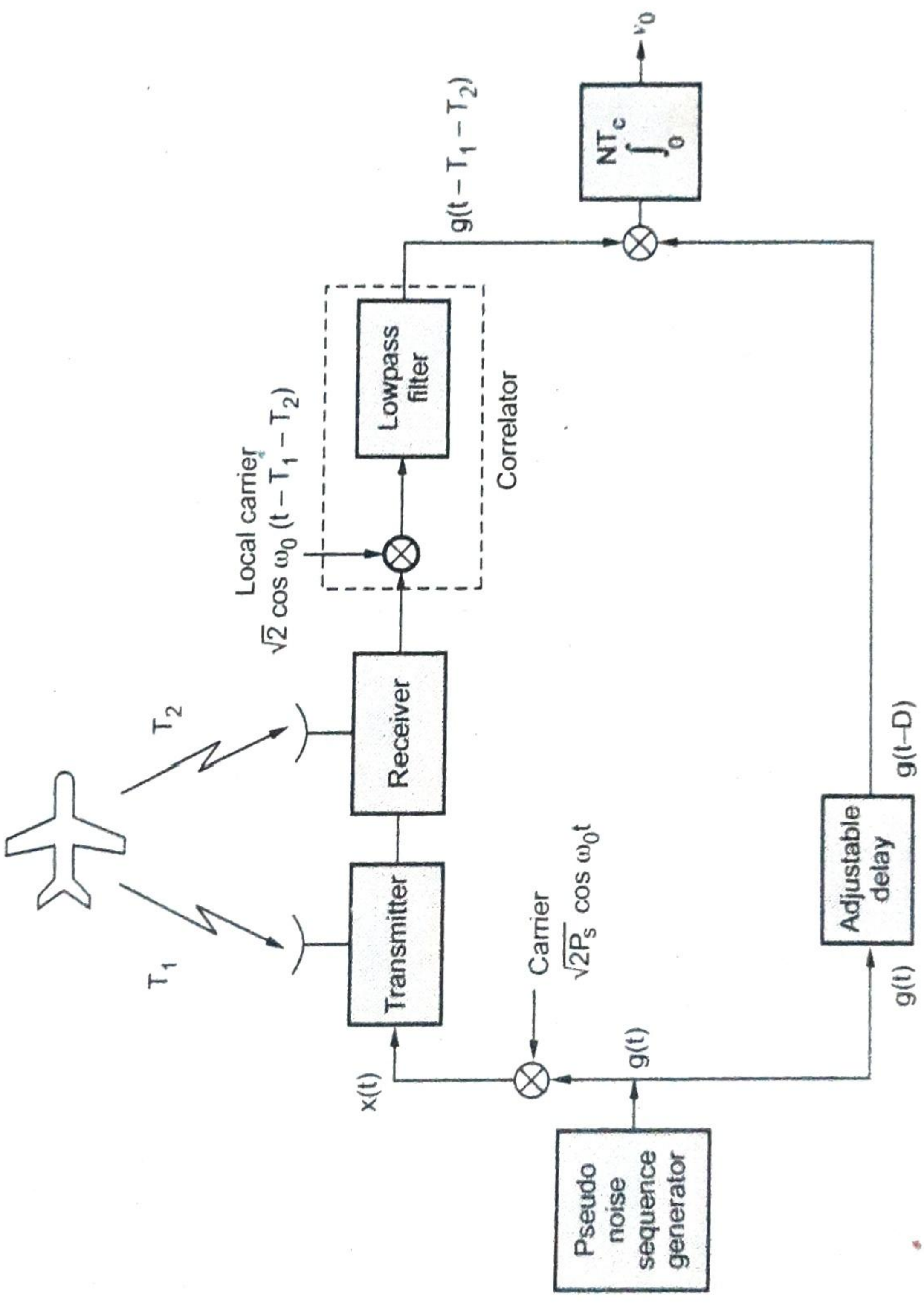


Fig. Q.22.1 Ranging using DS-SS

Note that delay in sinusoidal carrier does not carry any meaning, since it is periodic signal. The pseudo noise signal is delayed. 'θ' is the random phase shift caused by the delay. The correlator is supplied with locally generated carrier. The correlator extracts the pseudo-noise signal

$g(t-T_1-T_2)$. This received PN signal is correlated with transmitted PN signal, g(t). Hence PN signal is delayed by the delay 'D'. This delayed PN signal and received PN signal are multiplied and integrated. The multiplier and integrator at output performs correlation. The output of integrator is given as,

$$v_o = R(D) = \int_0^{NT_c} g(t-T_1-T_2)\, g(t-D)\, dt$$

Here $R(D)$ indicates correlation. N is the length of PN sequence.

The correlation coefficient $R(D)$ will be maximum when $D = T_1+T_2$. In other words, 'D' is varied to get maximum v_o. This value of 'D' is used to calculate range of the spacecraft. The signal travels at velocity of light, i.e. $c = 3\times10^8$ m/sec. Hence,

$$D \text{ in meters} = c \times D$$

And the range of the spacecraft is, $$d = \frac{c\times D}{2} = \frac{3\times10^8\times D}{2}$$

$$= 1.5D\times10^8 \text{ meters.}$$

$$\text{or} \quad d = 150D\times10^3 \text{ km}$$

The value of D can vary by one chip duration. Hence distance of the target will be,

$$d = 150\,(D\pm T_c)\times10^3 \text{ km}$$

Thus accuracy of the measurement depends upon chip duration T_c. Hence T_c should be as small as possible to get accurate distance. Small value of T_c means more length of the PN sequence.

END...

Unit V

5 INFORMATION THEORETIC APPROACH TO COMMUNICATION SYSTEM

5.1 : Introduction to Information Theory, Entropy and its Properties

Important Points to Remember

- Information conveyed by an event depends upon its probability of occurrence.
- Amount of information is, $I_k = log_2 \left(\frac{1}{p_k} \right)$
- Uncertain events carry more information.
- Entropy of the source, $H = \sum_{k=1}^{M} p_k \log_2 \left(\frac{1}{p_k} \right)$ bits/symbols
- If all the symbols are equally likely,

  $H = log_2 M$ Here 'M' are symbols
- Upper bound on entropy is given as $H_{max} = log_2 M$.
- Information rate, $R = rH$ where 'r' is the rate at which messages are generated.
- Entropy of n^{th} extension of discrete memoryless source is given as,

  $H(X^n) = nH(X)$

Q.1 Consider a DMS 'X' with two symbols x_1 and x_2 with probabilities $p(x_1) = 0.9$ and $p(x_2) = 0.1$. Find the efficiency and redundancy of this code and its second order extension.

[SPPU : Dec.-11, Marks 10]

Ans. :

i) To obtain entropy (H) : Entropy is given as,

$$H = \sum_{k=1}^{M} p_k \log_2 \frac{1}{p_k}$$

Here $p_1 = 0.9$ and $p_2 = 0.1$. Hence,

$$H = 0.9 \log_2 \left(\frac{1}{0.9}\right) + 0.1 \log_2 \frac{1}{0.1} = 0.1368 + 0.3322$$

$$= 0.469 \text{ bits/message.}$$

ii) To obtain second order extension : The source alphabet 'X' contains two symbols. Hence its second order extension will contain four symbols. These symbols, their probabilities and entropy calculations are given in Table Q.1.1.

i	Second order extension symbol σ_i	Probability of symbol σ_i $p(\sigma_i)$	$p(\sigma_i) \log_2 \frac{1}{p(\sigma_i)}$
1	$x_1 x_1$	$0.9 \times 0.9 = 0.81$	$0.81 \log_2 \frac{1}{0.81} = 0.2462$
2	$x_1 x_2$	$0.9 \times 0.1 = 0.09$	$0.09 \log_2 \frac{1}{0.09} = 0.3126$
3	$x_2 x_1$	$0.1 \times 0.9 = 0.09$	$0.09 \log_2 \frac{1}{0.09} = 0.3126$
4	$x_2 x_2$	$0.1 \times 0.1 = 0.01$	$0.01 \log_2 \frac{1}{0.01} = 0.0664$

Table Q.1.1

Entropy of the second order extension of the source can be obtained as,

$$H(X^2) = \sum_{i=1}^{4} p(\sigma_i) \log_2 \frac{1}{p(\sigma_i)}$$

Putting values from above table,

$$H(X^2) = 0.2462 + 0.3126 + 0.3126 + 0.0664$$

$$= 0.9378 \text{ bits/message}$$

Here note that $H(X^2) = 0.938 = 2H(X) = 2 \times 0.469$

iii) To obtain efficiency : Here are four symbols in extension of the source. Hence two bits will be required to code four symbols. Hence average number of bits per symbol will be two. i.e., $\overline{N} = 2$ bits/symbol.

$$\text{Code efficiency } \eta = \frac{H(X^2)}{\overline{N}} = \frac{0.938}{2} = 0.469$$

Q.2 What do you mean by measure of information ?

Ans. : Consider the communication system which transmits messages $m_1, m_2, m_3, \ldots,$ with probabilities of occurrence $p_1, p_2, p_3, \ldots$ The amount of information transmitted through the message m_k with probability p_k is given as,

$$\textbf{Amount of Information : } I_k = \log_2\left(\frac{1}{p_k}\right) \qquad \ldots \text{(Q.2.1)}$$

Unit of information :

In the above equation $\log_2 \frac{1}{p_k} = \frac{\log_{10}(1/p_k)}{\log_{10} 2}$. Normally the unit of information is *'bit'*.

Properties of Information

Following properties can be written for information.

i) If there is more uncertainty about the message, information carried is also more.

ii) If receiver knows the message being transmitted, the amount of information carried is zero.

iii) If I_1 is the information carried by message m_1 and I_2 is the information carried by m_2, then amount of information carried combinely due to m_1 and m_2 is $I_1 + I_2$.

iv) If there are $M = 2^N$ equally likely messages, then amount of information carried by each message will be N bits.

Q.3 Explain average information content of symbol in long independent sequence.

OR Explain entropy of the source.

Ans. : The average information per message is given as,

$$\text{Average information} = \frac{\text{Total information}}{\text{Number of messages}}$$

Average information is also called *Entropy*. It is represented by H. It is given as,

Entropy : $$H = \sum_{k=1}^{M} p_k \log_2 \left(\frac{1}{p_k}\right) \quad \text{... (Q.3.1)}$$

Properties of Entropy

1. Entropy is zero if the event is sure or it is impossible. i.e.,

$$H = 0 \quad \text{if} \quad p_k = 0 \text{ or } 1.$$

2. When $p_k = 1/M$ for all the 'M' symbols, then the symbols are equally likely. For such source entropy is given as $H = \log_2 M$.

3. Upper bound on entropy is given as,

$$H_{\max} = \log_2 M$$

Q.4 Show that if there are 'M' number of equally likely messages, then entropy of the source is $\log_2 M$.

Ans. : We know that for 'M' number of equally likely messages, probability is,

$$p = \frac{1}{M}$$

This probability is same for all 'M' messages. i.e.,

$$p_1 = p_2 = p_3 = p_4 = ... p_M = \frac{1}{M} \quad \text{... (Q.4.1)}$$

Entropy is given by equation (Q.4.1),

$$H = \sum_{k=1}^{M} p_k \log_2 \left(\frac{1}{p_k}\right)$$

$$= p_1 \log_2 \left(\frac{1}{p_1}\right) + p_2 \log_2 \left(\frac{1}{p_2}\right) + ... + p_M \log_2 \left(\frac{1}{p_M}\right)$$

Putting for probabilities from equation (Q.5.1) in above equation we get,

$$H = \frac{1}{M} \log_2 (M) + \frac{1}{M} \log_2 (M) + ... + \frac{1}{M} \log_2 (M)$$

(Add 'M' number of terms)

In the above equation there are 'M' number of terms in summation. Hence after adding these terms above equation becomes,

$$H = \log_2 (M) \quad \text{... (Q.4.2)}$$

Q.5 If 'X' is random variable assuming values $X_1, X_2, \dots X_k$; What should be probability density function of X to get maximum entropy $H(X)$. Determine the value of $H(X)$.

Ans. : The upper bound on entropy is given as,

$$H(X) \leq \log_2 M$$

Hence maximum value of entropy for 'K' messages will be,

$$H_{max}(X) = \log_2 K$$

For entropy to be maximum, all the symbols must be equally likely. Hence probability of each symbol will be,

$$P(X_1) = P(X_2) = \ldots\ldots P(X_k) = \frac{1}{K}$$

Above result shows that 'X' must have uniform probability density function. This is because all values of 'X' have same probability of occurrence.

$$\therefore \quad f_X(x) = \frac{1}{K} \quad \text{for} \quad x = X_1, X_2, \ldots\ldots X_k$$

Q.6 Consider a discrete memoryless source alphabet $A = \{S_0, S_1, S_2\}$ with respective probabilities. $P_0 = \frac{1}{4}$, $P_1 = \frac{1}{4}$ and $P_2 = \frac{1}{2}$. Find the entropy of the source.

Ans. :

$$H = \sum_{k=1}^{3} p_k \log_2 \frac{1}{p_k} = -\sum_{k=1}^{3} p_k \log_2 p_k,$$

$$\text{since } \log_2 \frac{1}{p_k} = -\log_2 p_k$$

$$= -\{p_1 \log_2 p_1 + p_2 \log_2 p_2 + p_3 \log_2 p_3\}$$

$$= -\left\{\frac{1}{4}\log_2 \frac{1}{4} + \frac{1}{4}\log_2 \frac{1}{4} + \frac{1}{2}\log_2 \frac{1}{2}\right\} = 1.5 \text{ bits/message}$$

Q.7 Consider a telegraph source having two symbols dot and dash. The dot duration is 0.2 sec ; and the dash duration is 3 times of the dot duration. The probability of the dots occurring is twice that of dash, and time between symbols is 0.2 seconds. Calculate information rate of the telegraph source.

Ans. : i) To calculate probabilities of dot and dash :

Let the probability of dash be 'p'. Then probability of dot will be '$2p$'. And,

$$p+2p = 1 \quad \therefore \quad p = \frac{1}{3}$$

Thus $p(dash) = \frac{1}{3}$ and $p(dot) = \frac{2}{3}$

ii) To calculate entropy of the source :

Entropy is given by equation (Q.4.1) as,

$$H = \sum_{k=1}^{M} p_k \log_2\left(\frac{1}{p_k}\right)$$

For dots and dash, entropy will be

$$H = \frac{1}{3}\log_2(3)+\frac{2}{3}\log_2\left(\frac{3}{2}\right)$$

$$= 0.9183 \text{ bits/symbol}$$

iii) To calculate average symbol rate :

It is given that,

$$\text{Dot duration} = T_{dot} = 0.2 \text{ sec.}$$

$$\text{Dash duration} = T_{dash} = 3 \times 0.2 = 0.6 \text{ sec.}$$

$$\text{Duration between symbols} = T_{symbols} = 0.2 \text{ sec.}$$

Now let us consider the string of 1200 symbols. On average the dots and dash will appear according to their probabilities in this string.

Hence,

$$\text{Number of dots} = 1200\times p(dots) = 1200\times\frac{2}{3} = 800$$

$$\text{Number of dash} = 1200\times p(dash) = 1200\times\frac{1}{3} = 400$$

Now let us calculate the total time for this string i.e.,

$$T = \text{Dots duration + Dash duration + (1200} \times \text{Time between symbols)}$$

$$= (800\times 0.2) + (400\times 0.6) + (1200\times 0.2)$$

$$= 640 \text{ sec.}$$

Hence average symbol rate will be,

$$r = \frac{1200}{T} = \frac{1200}{640} = 1.875 \text{ symbols/sec.}$$

iv) To calculate information rate :

Hence average information rate becomes

$$R = rH = 1.875 \times 0.9183$$
$$= 1.7218 \text{ bits/sec.}$$

Thus the average information rate of the telegraph source will be 1.7218 bits/sec.

Q.8 In conventional telegraphy we use dots and dashes to transmit messages. A dash is thrice as long as a dot and one third as probable as a dot. Find

i) Information in a dot and dash.

ii) Average information in dot-dash-code.

iii) If a dot takes 2 msec and same time is allowed between symbols, determine the information rate of this code.

Ans. : Given data :

Let 'p'_{dot} be the probability of dot. Then probability of dash 'p_{dash}' will be

$$p_{dash} = \frac{1}{3} p_{dot}$$

$$p_{dot} + p_{dash} = 1$$

$$p_{dot} + \frac{1}{3} p_{dot} = 1 \quad \Rightarrow \quad p_{dot} = \frac{3}{4}$$

And $$p_{dash} = \frac{1}{3} p_{dot} = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$$

i) To obtain information in dot and dash

$$I_{dot} = \log_2 \left(\frac{1}{p_{dot}} \right) = \log_2 \frac{1}{3/4} = 0.415 \text{ bits}$$

$$I_{dash} = \log_2 \left(\frac{1}{p_{dash}} \right) = \log_2 \frac{1}{1/4} = 2 \text{ bits}$$

ii) To obtain average information

$$H = p_{dot} \log_2 \frac{1}{p_{dot}} + p_{dash} \log_2 \frac{1}{p_{dash}}$$

$$= \frac{3}{4} \times 0.415 + \frac{1}{4} \times 2 = 0.81125 \text{ bits/message}$$

iii) To obtain information rate

Dot duration T_{dot} = 2 msec

$\therefore$ Dash duration, $T_{dash} = 3 \times T_{dot} = 3 \times 2$ msec = 6 msec

Duration between symbols, $T_{symbols}$ = 2 msec

Let us consider the string of 1000 symbols. On average the dots and dash will appear according to their probabilities in this string.

$$\text{Number of dots} = 1000 \times p_{dots} = 1000 \times \frac{3}{4} = 750$$

$$\text{Number of dash} = 1000 \times p_{dash} = 1000 \times \frac{1}{4} = 250$$

The time (T) for this string of 1000 symbols will be,

T = Dots duration + Dash duration + Duration between the symbols

$$= 750 \times 2 \times 10^{-3} + 250 \times 6 \times 10^{-3} + 1000 \times 2 \times 10^{-3} = 5 \text{ sec}$$

Hence average symbol rate will be,

$$r = \frac{1000}{T} = \frac{1000}{5} = 200 \text{ symbols/sec}$$

Information rate, $R = rH$

$$= 200 \times 0.81125 = 162.25 \text{ bits/sec}$$

5.2 : Source Coding Theorem

Important Points to Remember

- Efficiency of source encoder, $\eta = \frac{H}{\overline{N}}$, code redundancy $\gamma = 1 - \eta$
- Shannon fano coding and Huffman coding are variable length source coding methods.

Q.9 State Shannon's first theorem. [SPPU : May-13, Marks 2]

Ans. : *Given a discrete memoryless source of entropy H, the average codeword length $\overline{N}$ for any distortionless source encoding is bounded as,*

$$\overline{N} \geq H$$

Here the entropy H represents the fundamental limit on the average number of bits per symbol i.e. $\overline{N}$. This limit says that the average number of bits per symbol cannot be made smaller than entropy H.

Q.10 Why we need source coding techniques explain with example.

☞ [SPPU : April-17, Marks 4]

Ans. :

i) The encoding and bit allotment to the symbols must be optimum.

ii) Less frequently occurring words have more number of bits, whereas frequently occurring words have less number of bits.

iii) Entropy is the fundamental limit on the average codeword length.

iv) Code efficiency is increased with source coding.

v) It is possible to derive uniquely decodable codes using source coding.

Q.11 What is run length encoding ? Explain how it is used in bitmap file formats. **☞ [SPPU : May-15, Marks 4]**

Ans. : Runlength encoding is normally used for the data generated by scanning the documents, fax machine, typewriters etc. These information sources normally produce a data that contains long strings of 1's/0's and zeros. Those strings of 1s and 0's can be better compressed by Runlength encoding. Consider the following string generated due to document scanning :

1 1 1 1 1 1 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 ...

Then this string can be encoded using Runlength encoding as,

1, 6 ; 0, 5 ; 1, 4 ; 0, 6 ...

This means the string begins with six ones, then five zeros then four ones, then six zeros ... etc. Thus the binary symbol and its length is written in the compressed file. In above example the max number of zeros are 6 hence 3 bit binary code can be used. (1, 110) (0, 101) (1, 100) (0, 110) ... so on.

Q.12 Why Huffman encoding process is not unique ? Explain with suitable examples. **☞ [SPPU : Dec.-13, Marks 10]**

Ans. : A DMS have symbols $s_0, s_1, \ldots, s_4$ characterised by probability distribution as $0.4, 0.2, 0.1, 0.2$ respectively. Huffman coding can be implemented by placing the combined probability as high as possible or as low as possible. Above problem is solved with both combined probability in order to illustrate the non-uniqueness of Huffman encoding process.

I) Entropy of the source :

Entropy of the source is given as, $H = \sum_{k=0}^{4} p_k \log_2 \frac{1}{p_k}$

$$= 0.4 \log_2 \frac{1}{0.4} + 0.2 \log_2 \frac{1}{0.2} + 0.1 \log_2 \frac{1}{0.1} + 0.2 \log_2 \frac{1}{0.2} + 0.1 \log_2 \frac{1}{0.1}$$

$$= 2.122 \text{ bits/symbol}$$

II) Placing combined symbol as high as possible :

i) To obtain codeword :

Table Q.12.1 lists the coding. The combined symbol is placed as high as possible.

Symbol	Probability Stage - I	Stage - II	Stage - III	Stage - IV
s_0	0.4	0.4	0.4	0.6 0
s_1	0.2	0.2	0.4 0	0.4 1
s_3	0.2	0.2 0	0.2 1	
s_2	0.1 0	0.2 1		
s_4	0.1 1			

Table Q.12.1 : Huffman coding algorithm

As per the above table, the codes are listed below :

Symbol	Probability p_k	Digits obtained by tracing $b_2\ b_1\ b_0$	Codeword $b_0\ b_1\ b_2$	No. of bits per symbol n_k
s_0	0.4	0 0	0 0	2
s_1	0.2	0 1	1 0	2
s_2	0.1	0 1 0	0 1 0	3
s_3	0.2	1 1	1 1	2
s_4	0.1	1 1 0	0 1 1	3

Table Q.12.2

ii) To obtain average codeword length :

Average codeword length can be calculated as,

$$\bar{N} = \sum_{k=0}^{4} p_k\, n_k = 0.4\ (2) + 0.2\ (2) + 0.1\ (3) + 0.2\ (2) + 0.1\ (3) = 2.2$$

iii) To obtain code efficiency :

$$\eta = \frac{H}{N} = \frac{2.122}{2.2} = 0.9645$$

II) Placing combined symbol as low as possible :

i) To obtain codewords :

Table Q.12.3 shows the listing of Huffman coding algorithm. The combined symbol is placed as low as possible.

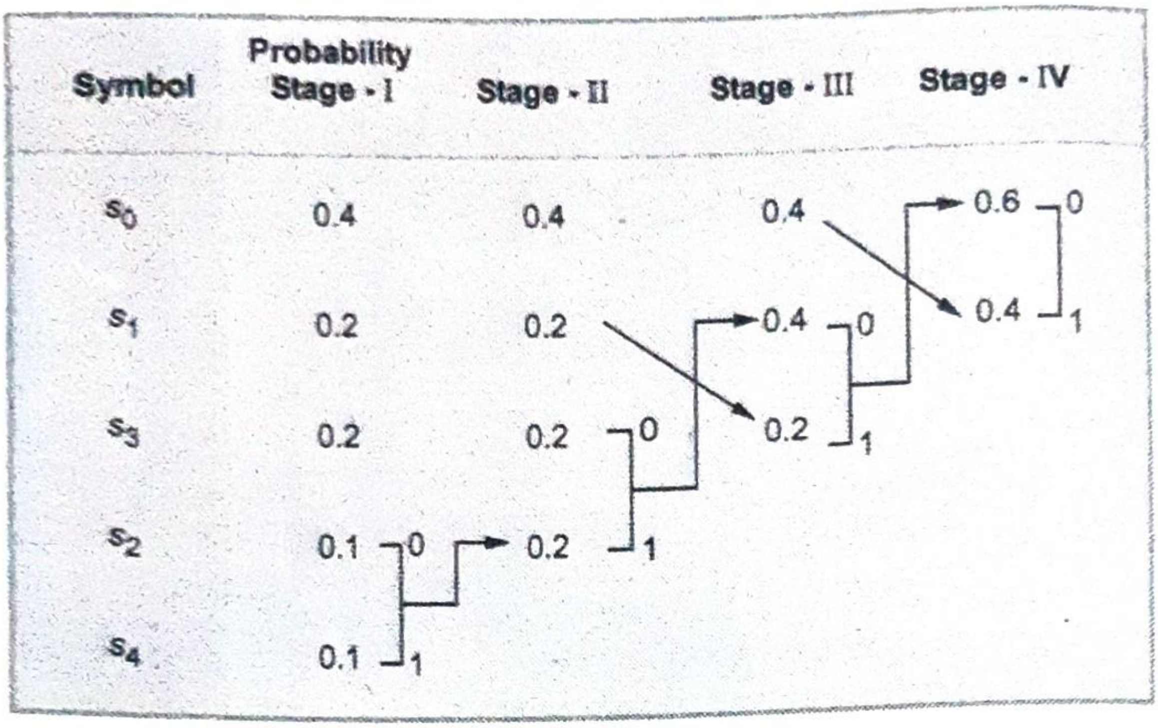


Table Q.12.3 : Huffman coding algorithm

As per the Table Q.12.3, the codes are listed below :

Symbol	Probability p_k	Digits obtained by tracing $b_2\ b_1\ b_0$	Codeword $b_0\ b_1\ b_2$	No. of bits per symbol n_k
s_0	0.4	1	1	1
s_1	0.2	1 0	0 1	2
s_2	0.1	0 1 0 0	0 0 1 0	4
s_3	0.2	0 0 0	0 0 0	3
s_4	0.1	1 1 0 0	0 0 1 1	4

Table Q.12,4

ii) To obtain average codeword length :

Average codeword length is given as,

$$\bar{N} = \sum_{k=0}^{4} p_k n_k = 0.4\ (1) + 0.2\ (2) + 0.1\ (4) + 0.2\ (3) + 0.1\ (4) = 2.2$$

iii) To obtain code efficiency :

$$\eta = \frac{H}{N} = \frac{2.122}{2.2} = 0.9645$$

Results :

Sr. No.	Method	Average length	Code efficiency
1	As high as possible.	2.2	96.45 %
2	As low as possible.	2.2	96.45 %

Above results show that average length of the codeword is same in both the methods though codewords are different.

Q.13 Design a Huffman code for a source generating 4 different types of messages with probabilities 0.3, 0.2, 0.4, 0.1. Find the coding efficiency. ☞ **[SPPU : Dec.-15, Marks 7]**

Ans. :

Step 1 : To obtain entropy of source :

$$H = \sum_{k=1}^{4} p_k \log_2 \frac{1}{p_k} = 0.3 \log_2\left(\frac{1}{0.3}\right) + 0.2 \log_2\left(\frac{1}{0.2}\right) + 0.4 \log_2\left(\frac{1}{0.4}\right)$$

$$+ 0.1 \log_2\left(\frac{1}{0.1}\right)$$

$$= 1.843 \text{ bits/symbol}$$

Step 2 : Huffman coding table :

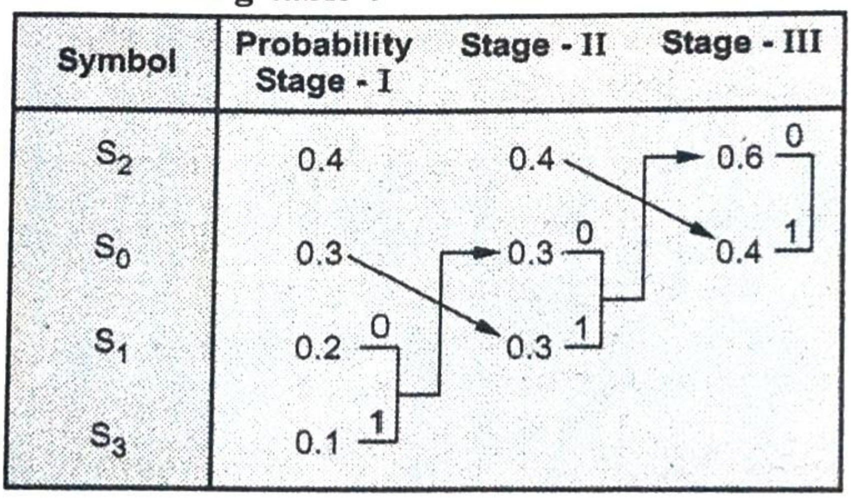


Table Q.13.1 : Huffman coding algorithm

As per above, the codes are listed below :

Symbols	Probability p_k	Digits obtained by tracing $b_2\ b_1\ b_0$	Codeword $b_0\ b_1\ b_2$	No of bits per symbol n_k
s_0	0.3	10	01	2
s_1	0.2	000	000	3
s_2	0.4	1	1	1
s_3	0.1	100	001	3

ii) To obtain average codeword length :

It is given by, $\overline{N} = \sum_{k=1}^{4} p_k n_k = (2\times0.3)+(3\times0.2)+(1\times0.4)+(3\times0.1) = 1.9$

iii) To obtain code efficiency :

$$\eta = \frac{H}{N} = \frac{1.843}{1.9} = 0.97$$

Q.14 A source generates four types of symbols A, B, C, D with probabilities $0.5, 0.25, 0.125$ and 0.125 respectively"
i) Using Huffman coding technique design source encoder
ii) Find coding efficiency
iii) If following message is generated by source, what will be output of the source encoder ABADCABC . ☞ **[SPPU : April.-15, Marks 6]**

Ans. : i) Designing source encoder using Huffman coding :

Huffman codes are obtained as shown in following table.

Symbol	Probability stage - I	Stage - II	Stage - III	Digits obtained by tracing $b_2\ b_1\ b_0$	Codeword $b_0\ b_1\ b_2$	No. of bits n_k
A	0.5	0.5	0.5 0	1	1	1
B	0.25	0.25 0	0.5 1	10	01	2
C	0.125 0	0.25 1		000	000	3
D	0.125 1			100	001	3

Thus the source encoder generates $A(1)$, $B(01)$, $C(000)$, $D(001)$.

ii) Coding efficiency :

Entropy of the source is given as,

$$H = \sum_{A,\,B,\ldots} p_k \log_2 \frac{1}{p_k} = -\sum_{A,\,B,\ldots} p_k \log_2 p_k$$

$$= -0.5 \log_2 0.5 - 0.25 \log_2 -0.25 - 0.125 \log_2 0.125 - 0.125 \log_2 0.125$$

$$= 1.75 \text{ bits/symbol.}$$

Average codeword length is given as,

$$\overline{N} = \sum_{A,\,B,} p_k n_k = 0.5\times 1 + 0.25\times 2 + 0.125\times 3 + 0.125\times 3$$

$$= 1.75 \text{ bits/symbol.}$$

$$\therefore \text{Coding efficiency, } \eta = \frac{H}{\overline{N}} = \frac{1.75}{1.75} = 1 \text{ or } 100\ \%$$

iii) Output for ABADCABA :

Input symbol :	A	B	A	D	C	A	B	A
Output sequence :	1	01	1	001	000	1	01	1

Q.15 Consider a DMS 'X' with two symbol x_1 and x_2 with probabilities $P(X_1) = 0.75$ and $P(X_2) = 0.25$. Find the efficiency and the redundancy of this code and its second order extension by using Huffman coding. **[SPPU : May-14, Marks 8]**

Ans. : i) To determine entropy of source

Entropy is given as,

$$H = \sum_{k=1}^{4} p_k \log_2\left(\frac{1}{p_k}\right) = 0.75 \log_2\left(\frac{1}{0.75}\right) + 0.25 \log_2\left(\frac{1}{0.25}\right)$$

$$= 1.811 \text{ bits/symbol}$$

ii) To prepare Huffman coding :

Symbol	Probability Stage - I
x_1	0.75 0
x_2	0.25 1

Table Q.15.1 : Huffman coding table

Symbols	Probability P_k	Digits obtained by tracing b_0 b_1	Huffman code b_1	No of bits per symbol n_k
x_1	0.75	0	0	1
x_2	0.25	1	1	1

Table Q.15.2

iii) To determine average codeward length ($\overline{N}$) :

It is given as,

$$\overline{N} = \sum_{k=1}^{2} p_k n_k = (0.75\times 1)+(0.25\times 1) = 1$$

iv) To determine code efficiency :

It is given as,

$$\eta = \frac{H}{\overline{N}} = \frac{0.811}{1} = 0.81$$

Second order extension of the source :

i) **To determine entropy of second order extension :** Here sequence of two symbols are considered. Since there are two symbols, second order extension will have 4 symbols. Following table lists the second order extension.

i	Second order extension symbol σ_i	Probability of symbol σ_i $p(\sigma_i)$	$p(\sigma_i)\, log_2 \left(\frac{1}{p(\sigma_i)}\right)$
1	x_1 x_1	$0.75\times 0.75 = 0.5625$	0.467
2	x_1 x_2	$0.75\times 0.25 = 0.1875$	0.453
3	x_2 x_2	$0.25\times 0.25 = 0.0625$	0.25
4	x_1 x_1	$0.25\times 0.75 = 0.1875$	0.453

Entropy of second order extension is given by,

$$H(X^2) = \sum_{i=1}^{4} p(\sigma i)\log\left(\frac{1}{p(\sigma i)}\right) = 1.623$$

ii) To determine Huffman code :

Symbol	Probability Stage I	Stage II	Stage III
1	0.5625	0.5625	0.5625 0
2	0.1875	0.25 0	0.4375 1
4	0.1875 0	0.1875 1	
3	0.0625 1		

Table Q.15.3

Symbol	Probability	Digits obtained by tracing	Huffman code	Bits/message n_k
1	0.5625	0	0	1
2	0.1875	10	01	2
3	0.625	1001	0101	3
4	0.1875	001	100	3

Table Q.15.4

iii) To determine average length :

$$\overline{N} = (1\times 0.5625)+(2\times 0.1875)+(3\times 0.0625)+(3\times 0.1875)$$

$$= 1.6875$$

iv) To determine efficiency :

$$\eta = \frac{H(x)^2}{N} = \frac{1.623}{1.6875} = 0.96$$

Q.16 Design a Shannon-Fano code for a source generating 5 different messages with probabilities 0.45, 0.3, 0.15, 0.05, 0.05. Find the coding efficiency. ☞ **[SPPU : May-15, Marks 7]**

Ans. : Following table illustrates Shannon-Fano coding.

Symbol	Probability p_k	Stage-I	Stage-II	Stage-III	Stage-IV	Codeword	No. of bits per symbol, n_k
s_1	0.45	0				0	1
s_2	0.3	1	0			10	2
s_3	0.15	1	1	0		110	3
s_4	0.05	1	1	1	0	1110	4
s_5	0.05	1	1	1	1	1111	4

i) Entropy of the source is given as,

$$H = -\sum_{k=1}^{5} p_k \log_2 p_k = -0.45 \log_2 0.45 - 0.3 \log_2 0.3$$

$$- 0.15 \log_2 0.15 - 0.05 \log_2 0.05 - 0.05 \log_2 0.05$$

$$= 1.882 \text{ bits/symbol.}$$

ii) Form above table **average number of bits per symbol** are given as,

$$\overline{N} = \sum_{k=1}^{5} p_k n_k = 0.45\times1+0.3\times2+0.15\times3+0.05\times4+0.05\times4$$

$$= 1.9 \text{ bits/symbol.}$$

iii) Coding efficiency, $\eta = \frac{H}{\overline{N}} = \frac{1.882}{1.9} = 0.99$ or 99 %

Q.17 A zero memory source emits six messages (N, I, R, K, A, T) with probabilities (0.30, 0.10, 0.02, 0.15, 0.40, 0.03) respectively. Given that 'A' is coded as '0' find :

i) Entroypy of source

ii) Determine Shannon Fan ocode and tabulate them

iii) What is the original symbol sequence of Shanon Fano signal (11001111011111110100)

☞ [SPPU : May-11, Marks 12, Dec.-17, Marks 8]

Ans. :

Step 1 : Entropy of source :

Entropy is given as,

$$H = \sum_{k=1}^{6} p_k \log\left(\frac{1}{p_k}\right)$$

$$= 0.3 \log_2\left(\frac{1}{0.3}\right) + 0.1\log_2\left(\frac{1}{0.1}\right) + 0.02 \log_2\left(\frac{1}{0.02}\right)$$

$$+ 0.15 \log_2\left(\frac{1}{0.15}\right) + 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.03 \log_2\left(\frac{1}{0.03}\right)$$

$$= 2.057 \text{ bits / symbol}$$

Step 2 : Shannon Fano coding :

Symbol Message	Probability	Stage I	Stage II	Stage III	Stage II	Stage IV	Code word
A	0.4	0					0
N	0.3	1	0				10
K	0.15	1	1	0			110
I	0.10	1	1	1	0		1110
T	0.03	1	1	1	1	0	11110
R	0.02	1	1	1	1	1	11111

Table Q.17.1 : Shannon Fano coding

iii) Original symbol sequence of Shannon Fano coded signal (110011110111111110100) :

Given :								
	Coded :	110 ↓	0 ↓	11110 ↓	11111 ↓	1110 ↓	10 ↓	0 ↓
	Decoded :	*K*	*A*	*T*	*R*	*I*	*N*	*A*

Q.18 Obtain efficiency of a Shannon Fano code for a zero memory source that emits six messages (A, E, H, N, G, S) with probabilities {0.19, 0.15, 0.02, 0.16, 0.4, 0.08} respectively. Given that A is coded as '0'.

☞ **[SPPU : May-13, Marks 9]**

Ans. : Step 1 : To obtain Shannon Fano code :

Symbol	Probability	Stage I	Stage II	Stage III	Stage IV	Codeword	No. of bits N_K Symbol
G	0.4	0				0	1
A	0.19	1	0	0		100	3
N	0.16	1	0	1		101	3
E	0.15	1	1	0		110	3
S	0.08	1	1	1	0	1110	4
H	0.02	1	1	1	1	1111	4

Step 2 : To calculate entropy :

It is given as,

$$H = \sum_{k=1}^{6} p_k \log_2\left(\frac{1}{p_k}\right) = 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.19 \log_2\left(\frac{1}{0.19}\right)$$

$$+ 0.16 \log_2\left(\frac{1}{0.16}\right)$$

$$+ 0.15 \log_2\left(\frac{1}{0.5}\right) + 0.08 \log_2\left(\frac{1}{0.08}\right)$$

$$+ 0.02 \log_2\left(\frac{1}{0.02}\right)$$

$$= 2.219 \text{ bits/symbols}$$

Step 3 : To find average codeword length :

It is given as,

$$\overline{N} = \sum_{k=1}^{6} P_k n_k = (1 \times 0.4) + 3 \times (0.19 + 0.16 + 0.15)$$

$$+ 4(0.08 + 0.02)$$

$$= 2.3$$

Step 4 : To calculate efficiency : It is given as,

$$\eta = \frac{H}{\overline{N}} = \frac{2.2219}{2.3} = 0.966$$

Q.19 A 3 bit PCM system generates 1000 samples per second. If the quantised samples produced by the system have probabilities $-\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16},\right\}$, find the rate of information if the samples are equiprobable. What will be rate of information ?

☞ **[SPPU : Dec.-15, Marks 7]**

Ans. : Given : Number of bits in PCM = 3 bits

Hence, quantised samples = 2^3 = 8 samples.

Case I :

Step 1 : To find entropy

It is given as,

$$H = \sum_{k=1}^{8} p_k \log\left(\frac{1}{p_k}\right) = \frac{1}{4}\log_2 4+\frac{1}{4}\log_2 4+\frac{1}{8}\log_2 8+\frac{1}{8}\log_2 8$$

$$+\frac{1}{16}\log_2 16+\frac{1}{16}\log_2 16+\frac{1}{16}\log_2 16+\frac{1}{16}\log_2 16$$

$$= 2.75 \text{ bits/symbol}$$

Step 2 : To find rate of information

It is given as, $R = rH$

where $\quad r = 1000 \text{ samples/second} \quad$ ... (given)

$\therefore \quad R = 2.75 \times 1000 = 2750 \text{ bits/sec}$

Case II : If samples are equiprobable,

$$H = \log_2 M = \log_2 8 = 3 \text{ bits/message}$$

Rate of information $R = rH = 1000 \times 3 = 3000$ bits/sec

Q.20 Obtain the coding efficiency of a Shannon Fano for a zero memory sources that emits eight messages with respective probabilities as given below. Use 3 letters for encoding such as −1, 0, 1.

P = [0.3 0.12 0.12 0.12 0.12 0.08 0.07 0.07]

X = [x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8]

☞ **[SPPU : May-17, (End Sem), Marks 6]**

Ans. : Obtain coding efficiency.

P = [0.3 0.12 0.12 0.12 0.12 0.08 0.07 0.07]

$X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8]$

Message	probability	step 1	step 2	step 3	No. of letters
x_1	0.3	–1			1
x_2	0.12	0	–1		2
x_3	0.12	0	0		2
x_4	0.12	0	1		2
x_5	0.12	1	–1		2
x_6	0.08	1	0		2
x_7	0.07	1	1	–1	3
x_8	0.07	1	1	0	3

$$\text{Coding efficiency} = \frac{H(X)}{\log M \times \bar{L}} \times 100\ \%$$

$$H(X) = \sum_{j=1}^{8} P(x_j) \cdot \log \frac{1}{P(x_j)}$$

$$= 0.03 \log \frac{1}{0.03} + 4 \times 0.12 \log \frac{1}{0.12} + 1 \times 0.08 \log \frac{1}{0.08} + 2 \times 0.07 \log \frac{1}{0.07}$$

$$= 2.841 \text{ bits/message}$$

$$\bar{L} = \sum_{j=1}^{8} n_j\, P(x_j) = 1 \times 0.3 + 4\ (2 \times 0.12) + 2 \times 0.08 + 3 \times 2 \times 0.07 = 1.84 \text{ bits/message}$$

$$M = 3$$

$$\eta = \frac{2.841}{1.84 \times \log 3} \times 100\ \% = 97.42\ \%$$

5.3 : Discrete Memoryless Channels and Mutual Information

Important Points to Remember

- Conditional entropy or equivocation is given as,

$$H(X/Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} p(x_i, y_j) \log_2 \frac{1}{p(x_i / y_j)}$$

and $$H(Y/X) = \sum_{i=1}^{M} \sum_{j=1}^{M} p(x_i, y_j) \log_2 \frac{1}{p(y_j / x_i)}$$

- Joint entropy. $H(X, Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)}$
- Rate of information transmission,
  $D_t = [H(X) - H(X/Y)] \cdot r$ bits/sec.
- Capacity of a discrete memoryless channel, $C = \max_{p(X)} \{Dt\}$
- Capacity by Muroga's method, $C = \log_2 \left\{ 2^{Q_1} + 2^{Q_2} \right\}$ bits/sec
- Mutual information,

$$I(X ; Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log_2 \frac{p(x_i / y_j)}{p(x_i)}$$

- Mutual information $I(X ; Y) = I(Y ; X)$ and $I(X ; Y) \geq 0$
- Channel capacity $C = \max_{p(x_i)} \{I(X ; Y)\}$

Q.21 Explain lossless, deterministic and binary symmetric channel with channel diagram and calculate capacity of each channel.

OR Derive the channal capacity of binary symmetric channel and find capacity of channel if $p(y_1/x_1) = 0.5$.

☞ [SPPU : April-17, Marks 5]

Ans. : i) Lossless channel : The channel shown in Fig. Q.21.1 is called as a lossless channel. The channel matrix of lossless channel consists only one non zero value in a column. There is zero information loss hence lossless channel.

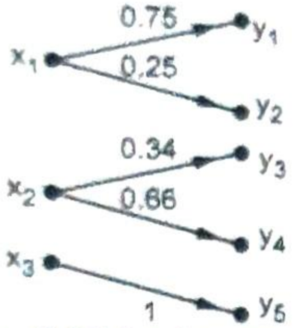


Fig. Q.21.1 : Lossless channel

$$[P(Y/X)] = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} 0.75 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0.34 & 0.66 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. Q.21.2 : Lossless channel matrix

Note that in each column of above matrix, there is only one non-zero element

In generalised form, it can be written as,

$$P(Y) = [p_1 \ p_2 \ p_3 \cdots p_n]$$

and $$P(X) = [p_1 + p_2 + p_3 + \ldots p_n]$$

$$\therefore \quad H(XY) = H(Y)$$

$$\therefore \quad H(X/Y) = H(XY) - H(Y) = 0$$

Also $$H(XY) \neq H(X)$$

$$\therefore \quad I(X;Y) = H(X)$$

and hence capacity $(C) = Max\, I[(X;Y)]$

$= Max[H(X)] = \log m$ bits/channel

ii) Deterministic channel : The deterministic channel is shown in Fig. Q.21.4 below. A deterministic channel has only one nonzero value in a row in channel matrix $(P(Y/X)]$. It is clear which output symbol will be received when a source symbol is transmitted on this channel.

$$[P(Y/X)] = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. Q.21.3 : Deterministic channel matrix

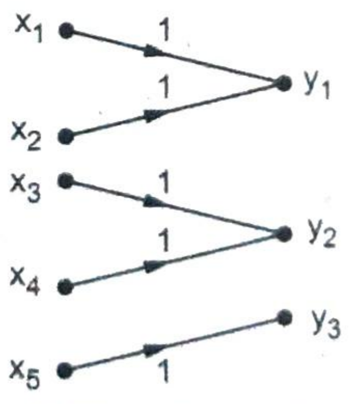


Fig. Q.21.4 : Deterministic channel

From above matrix and channel diagram in general, it can be written,

$$\therefore \quad P(X) = [p_1 \;\; p_4 \;\; p_3 \;\; p_2 \;\cdots\; p_m]$$

and $$P(Y) = [p_1 + p_4 + p_3 \quad p_2 + p_5 + p_6 \;\cdots\; p_{m-2} + p_{m-1} + p_m]$$

Thus, $$H(XY) = H(X)$$

$$\therefore \quad H(Y/X) = H(XY) - H(XY) = 0$$

$$\therefore \quad I(X;Y) = H(X) + H(Y) - H(Y) = H(Y)$$

Hence capacity $(C) = Max\, I[(X;Y)] = Max[H(Y)] = \log \text{n bits/channel}$ where, n is number receiver elements.

iii) Binary symmetric channel : The binary communication channel of Fig. Q.21.5 is said to be symmetric if $P(y_0/x_0) = P(y_1/x_1) = p$. Such channel is shown in Fig. Q.21.5.

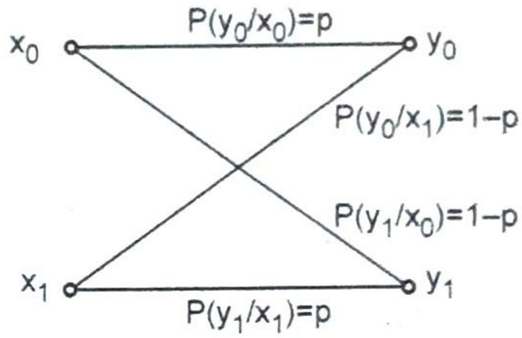


Fig. Q.21.5 Binary symmetric channel

For the above channel, we can write ,

$$\begin{bmatrix} P(y_0) \\ P(y_1) \end{bmatrix} = [P(x_0) \quad P(x_1)] \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \qquad \ldots (Q.21.1)$$

From Fig. Q.21.5 we can write the channel matrix as follows :

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Now calculate the quantity 'h' for any row of above matrix. i.e.,

$$h = \sum_{j=1}^{2} P(y_j/x_i) \log_2 \frac{1}{P(y_j/x_i)}$$

$$= P(y_1/x_1) \log_2 \frac{1}{P(y_1/x_1)} + P(y_2/x_1) \log_2 \frac{1}{P(y_2/x_1)}$$

$$= p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{(1-p)}$$

For this channel, there are two output symbols. Hence $M = 2$. Therefore from equation (Q.21.1),

$$C = [\log_2 M - h] \cdot r$$

For r = 1 message/sec and putting other values in above equation,

$$C = \log_2 2 - \left[p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{(1-p)} \right]$$

$$= 1 - \left[p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{(1-p)} \right]$$

$$= 1 - h, \quad \text{where } h = \sum_{j=1}^{2} P_j \log_2 \frac{1}{P_j} \qquad \text{with } P_1 = p \text{ and } P_2 = 1-p$$

This is the capacity of binary symmetric channel.

Q.22 Prove that the capacity of noisless channel is $log_2 M$ where M is number of symbols generated by the source.

[SPPU : April-15, Marks 6]

Ans. : For noiseless channel,

$$P(Y/X) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

Here, $P(y_j / x_i) = 1$ only when $i = j$

$$\therefore \quad P(x_i\, y_j) = P(y_j / x_i)\, P(x_i) = P(x_i) \text{ for i = j}$$

$$\therefore \quad H(X, Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} P(x_i\, y_j) \log_2 \frac{1}{P(x_i y_j)}$$

$$= \sum_{i=1}^{M} P(x_i) \log_2 \frac{1}{P(x_i)}, \quad \text{Since } P(x_i\, y_j) = P(x_i) \text{ for } i = j$$

$$= H(X) \qquad \text{Since } H(X) = \sum_{i=1}^{M} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$H(Y/X) = H(X, Y) - H(X) = 0 \qquad \text{Since } H(X, Y) = H(X)$$

Similarly $H(X/Y) = H(X, Y) - H(Y) = H(X, Y) - H(X)$,

Since $H(X) = H(Y)$ for noiseless channel

$= 0$ Since $H(X, Y) = H(X)$

$\therefore \quad I(X; Y) = H(X) - H(X/Y) \therefore I(X; Y) = H(X)$, Since $H(X/Y) = 0$

Capacity, $C = \underset{P(x_i)}{Max} \{I(X; Y)\} = \underset{P(x_i)}{Max} \{H(X)\}$

Note that the maximum value of source entropy for 'M' symbols is $\log_2 M$.

Hence, Capacity, $C = \log_2 M$ bits/symbol

Q.23 What is mutual information ? State and prove any one property of mutual information.

☞ [SPPU : April-15, May-11,13, Marks 4, May-17, Marks 3]

Ans. : i) Mutual Information : The mutual information is defined as the amount of information transferred when x_i is transmitted and y_j is received. It is represented by $I(x_i, y_j)$ and given as,

$$\boxed{I(x_i, y_j) = \log \frac{P(x_i / y_j)}{P(x_i)} \text{ bits}} \qquad \text{... (Q.23.1)}$$

Here $I(x_i, y_j)$ is the mutual information

$P(x_i / y_j)$ is the conditional probability that x_i was transmitted and y_j is received.

$P(x_i)$ is the probability of symbol x_i for transmission.

ii) To prove that the mutual information is always positive i.e. $I(X;Y) \geq 0$

Proof : Mutual information is given,

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i / y_j)}{P(x_i)} \qquad \text{... (Q.23.2)}$$

Also, $P(x_i / y_j) = \dfrac{P(x_i, y_j)}{P(y_j)}$

Putting above value of $P(x_i / y_j)$ in equation (Q.20.1),

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i, y_j)}{P(x_i)\, P(y_j)}$$

$\log_2 \frac{x}{y}$ can be written as, $-\log_2 \frac{y}{x}$. Hence above equation becomes,

$$I(X;Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i)\, P(y_j)}{P(x_i, y_j)}$$

This equation can be written as,

$$-I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i)\, P(y_j)}{P(x_i, y_j)} \quad \text{... (Q.23.3)}$$

Earlier we have derived one result given by equation (Q.23.3). It states that,

$$\sum_{k=1}^{m} p_k \log_2 \left(\frac{q_k}{p_k} \right) \leq 0$$

This result can be applied to equation (Q.23.3). We can consider p_k be $P(x_i \ y_i)$ and q_k be $P(x_i)\, P(y_j)$. Both p_k and q_k are two probability distributions on same alphabet. Then equation (Q.23.3) becomes,

$$-I(X;Y) \leq 0$$

i.e., $$I(X;Y) \geq 0 \quad \text{... (Q.23.4)}$$

The above equation is the required proof. It says that mutual information is always non negative.

Q.24 Find all entropies, mutual information of channel where channel matrix is given as

$$P[Y/X] = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} \text{ Take } p(x_i) = 0.6 \text{ and } p(x_2) = 0.4$$

[SPPU : April-15,17, Marks 6, Dec.-17, Marks 7]

Ans. : $P(Y/X) = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix}$

It can be compared with standard matrix as shown above.

and $P(x_1) = 0.6$ and $P(x_2) = 0.4$.

i.e. $P(x) = [0.6 \quad 0.4]$

Solution : i) To obtain input source entropy $H(X)$:

It is given as,

$$H(X) = P(x_1) \log_2 \frac{1}{P(x_1)} + P(x_2) \log \frac{1}{P(x_2)}$$

$$= 0.6 \log_2\left(\frac{1}{0.6}\right) + 0.4 \log_2\left(\frac{1}{0.4}\right)$$

$$H(X) = 0.9703 \text{ bits/symbol}$$

ii) To obtain output source entropy H(Y) :

$$\begin{bmatrix} P(y_1) \\ P(y_2) \end{bmatrix} = [P(x_1)\, P(x_2)] \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix}$$

$$= [0.6 \quad 0.4] \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} (0.6\times 0.7)+(0.4\times 0.3) \\ (0.6\times 0.3)+(0.4\times 0.7) \end{bmatrix}$$

$$= \begin{bmatrix} 0.54 \\ 0.46 \end{bmatrix}$$

$$\begin{bmatrix} P(y_1) \\ P(y_2) \end{bmatrix} = \begin{bmatrix} 0.54 \\ 0.46 \end{bmatrix} \quad \text{Thus } P(y_1) = 0.54 \text{ and } P(y_2) = 0.46$$

Entropy of output can be calculated as,

$$H(Y) = P(y_1)\log_2 \frac{1}{P(y_1)} + P(y_2)\log \frac{1}{P(y_2)}$$

$$= 0.54 \log_2\left(\frac{1}{0.54}\right) + 0.46 \log_2\left(\frac{1}{0.46}\right) = 0.9953 \text{ bits/symbol}$$

iii) To calculate joint entropy *H (X, Y)* :

For joint entropy we require joint probabilities $P(x_i, y_j)$

$$P(x_i, y_j) = P(y_j/x_i) \cdot P(x_i)$$

$$\therefore \quad P(x_1, y_1) = P(y_1/x_1) \cdot P(x_1) = 0.7 \times 0.6 = 0.42$$

$$P(x_1, y_2) = P(y_2/x_1) \cdot P(x_1) = 0.3 \times 0.6 = 0.18$$

$$P(x_2, y_1) = P(y_1/x_2) \cdot P(x_2) = 0.3 \times 0.4 = 0.12$$

$$P(x_2, y_2) = P(y_2/x_2) \cdot P(x_2) = 0.7 \times 0.4 = 0.28$$

$$\therefore \quad P(x, y) = \begin{bmatrix} 0.42 & 0.18 \\ 0.12 & 0.28 \end{bmatrix}$$

Now, Joint entropy is given by,

$$H(X, Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} P(x_i\, y_j) \log_2\left(\frac{1}{P(x_i\, y_j)}\right)$$

$$H(X, Y) = P(x_1, y_1)\log_2 \frac{1}{P(x_1, y_1)} + P(x_1, y_2)\log_2 \frac{1}{P(x_1, y_2)}$$

$$+ P(x_2, y_1)\log_2\left[\frac{1}{P(x_2, y_1)}\right] + P(x_2, y_2)\log_2\left[\frac{1}{P(x_1, y_2)}\right]$$

$$= 0.42 \log_2\left(\frac{1}{0.42}\right) + 0.18 \log_2\left(\frac{1}{0.18}\right)$$

$$+ 0.12 \log_2\left(\frac{1}{0.12}\right) + 0.28 \log_2\left(\frac{1}{0.28}\right) = 1.8521 \text{ bits/symbol}$$

iv) To obtain conditional entropies H (X / Y) and H (Y / X) :

$H(Y/X)$ is given by,

$$H(X/Y) = H(X, Y) - H(Y) = 1.8521 - 0.9953 = 0.8568 \text{ bits/symbol}$$

and $H(Y/X)$ is given by,

$$H(Y/X) = H(X, Y) - H(X) = 1.8521 - 0.9703 = 0.8818 \text{ bits/symbol}$$

v) To obtain mutual information I(X, Y) :

Mutual information is given as,

$$I(X, Y) = H(X) - H(X/Y) = 0.9703 - 0.8568 = 0.1135 \text{ bits/symbol.}$$

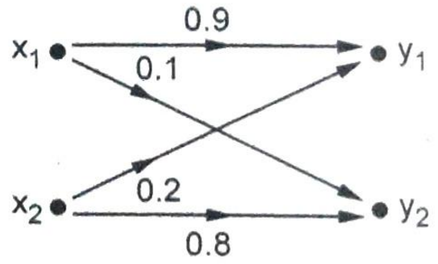
Also, $I(X, Y) = H(Y) - H(Y/X) = 0.9953 - 0.8818 = 0.1135$ bits/symbol.

Q.25 Consider the given binary channel

i) Construct the channel matrix

ii) Find out the value of $p(y)$; If the source is equiprobable.

iii) Calculate all entropies, mutual information and channel capacity.



[SPPU : Dec.-14, Marks 8]

Ans. : i) To construct channel matrix :

Channel diagram represents channel transition matrix, i.e.

$$p(y/x) = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

Given that inputs are equiprobable. Hence probabilities of two input symbols are,

$$p(x_1) = 0.5 \quad \text{and} \quad p(x_2) = 0.5$$

ii) To obtain values of $p(y)$:

Probabilities of output are given as,

$$\begin{bmatrix} p(y_1) \\ p(y_2) \end{bmatrix} = [p(x_1) \quad p(x_2)] \begin{bmatrix} p(y_1/x_1) & p(y_2/x_1) \\ p(y_1/x_2) & p(y_2/x_2) \end{bmatrix}$$

Putting values in above equation,

$$\begin{bmatrix} p(y_1) \\ p(y_2) \end{bmatrix} = [0.5 \quad 0.5] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$$

Thus $\quad p(y_1) = 0.55 \quad$ and $\quad p(y_2) = 0.45$

Now $\quad P[X, Y] = [P(X)][P(Y/X)] = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$

$$= \begin{bmatrix} 0.45 & 0.05 \\ 0.10 & 0.40 \end{bmatrix}$$

iii) To calculate all entropies :

a) $$H(X) = \sum_{k=1}^{M} p(x_k) \log_2 \frac{1}{p(x_k)}$$

$$= 0.5 \log_2 \left(\frac{1}{0.5} \right) + 0.5 \log_2 \left(\frac{1}{0.5} \right) = 1 \text{ bit/message}$$

b) $$H(Y) = \sum_{k=1}^{M} p(y_k) \log_2 \frac{1}{p(y_k)}$$

$$= 0.55 \log_2 \left(\frac{1}{0.55} \right) + 0.45 \log_2 \left(\frac{1}{0.45} \right)$$

$$= 0.9928 \text{ bits/message}$$

c) $$H(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log_2 \frac{1}{p(x_i / y_j)}$$

$$= 0.45 \log_2 \left(\frac{1}{0.45} \right) + 0.05 \log_2 \left(\frac{1}{0.05} \right) + 0.1 \log_2 \left(\frac{1}{0.1} \right)$$

$$= 0.4 \log_2 \left(\frac{1}{0.4} \right) = 1.5995 \text{ bits/message}$$

iv) To calculate mutual information :

It is given as,

$$I(X,Y) = H(X)+H(Y)-H(X,Y) = 1+0.9928-1.5955 = 0.3973$$

v) To calculate the capacity of BSC :

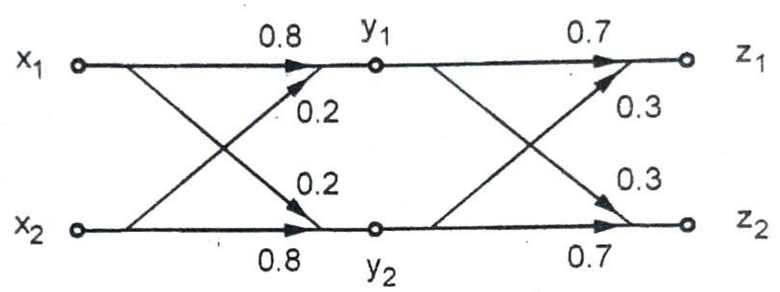
for $n = 2$. capacity is given as,

$$C = \log_2^2 + p \log p + (1-p) \log (1-p)$$

$$= \log_2^2 + 0.5 \log_2 (0.5) + 0.5 \log_2 0.5 = 1+0.5+0.5$$

$$= 2 \text{ bits/channel}$$

Q.26 Two BSC's are connected as shown in following figure.



i) Find channel matrix of resultant channel

ii) Find $p(z_1)$ and $p(z_2)$ if $p(x_1) = 0.6$, $p(x_2) = 0.4$

[SPPU : Dec.-12, Marks 8]

Ans. : i) To obtain channel matrix :

Here two matrices of the channel as follows :

$$P(Y/X) = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_2/x_2) & P(y_2/x_2) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

and $$P(Z/Y) = \begin{bmatrix} P(z_1/y_1) & P(z_2/y_1) \\ P(z_1/y_2) & P(z_2/y_2) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Hence resultant channel matrix is given as,

$$P(Z/X) = P(Y/X) \cdot P(Z/Y) = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

ii) To obtain $P(z_1)$ and $P(z_2)$:

The probabilities of Z_1 and Z_2 are given as,

$$P(Z) = P(X)\,P(Z/X) = [P(x_1)\ P(x_2)]\begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

$$= [0.6\ 0.4]\begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix} = [0.524\quad 0.476]$$

Thus $P(Z_1) = 0.524$ and $P(Z_2) = 0.476$

Q.27 A channel with three input x_1, x_2 and x_3 three output y_1, y_2, y_3 with noise matrix as given below :

$$p[Y/X] = \begin{bmatrix} \mathbf{0.6} & \mathbf{0.2} & \mathbf{0.2} \\ \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.2} \\ \mathbf{0.2} & \mathbf{0.2} & \mathbf{0.6} \end{bmatrix}$$

Calculate $H(X), H(Y), H(X, Y), H(X, Y), I(X, Y)$ where $P(x_1) = 1/8$, $P(x_2) = 1/8$ and $P(x_3) = 6/8$.

☞ [SPPU : Dec.-11, Marks 8]

Ans. : Given that $P(x_1) = 1/8$, $P(x_2) = 1/8$ and $P(x_3) = 6/8$

$$\therefore \quad P(X) = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{6}{8} \end{bmatrix}$$

and $$P(X,Y) = P(X) \cdot P(Y/X)$$

$$= \begin{bmatrix} 1/8 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & 0 & 6/8 \end{bmatrix}\begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.075 & 0.025 & 0.15 \\ 0.025 & 0.075 & 0.15 \\ 0.025 & 0.025 & 0.45 \end{bmatrix}$$

Taking the summation of column of matrix $P(X,Y)$, $P(Y)$ is obtained.

$$\therefore \quad P(Y) = [0.125\quad 0.125\quad 0.75]$$

Step 1 : To obtain $H(X), H(Y), H(X,Y)$ and $H(X/Y), H(Y/X)$:

a) $H(X)$ is given as,

$$H(X) = \sum_{k=1}^{m} p(x_k) \log_2 \frac{1}{p(x_k)}$$

$$= \frac{1}{8}\log_2(8) + \frac{1}{8}\log_2(8) + \frac{6}{8}\log_2\left(\frac{8}{6}\right)$$

$$= 1.0613 \text{ bits/message}$$

b) $H(Y)$ is given as,

$$H(Y) = \sum_{k=1}^{m} p(y_k)\log_2 \frac{1}{p(y_k)} = 0.125 \log_2\left(\frac{1}{0.125}\right)$$

$$+ 0.125 \log_2\left(\frac{1}{0.125}\right) + 0.75 \log_2\left(\frac{1}{0.75}\right) = 1.0613 \text{ bits/message}$$

c) $$H(X,Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i\ y_j)\log_2 \frac{1}{p(x_i, y_j)}$$

$$= 0.075 \log_2\left(\frac{1}{0.075}\right) + 0.025 \log_2\left(\frac{1}{0.025}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right)$$

$$+ 0.025 \log_2\left(\frac{1}{0.025}\right) + 0.075 \log_2\left(\frac{1}{0.075}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right)$$

$$+ 0.025 \log_2\left(\frac{1}{0.025}\right) + 0.025 \log_2\left(\frac{1}{0.025}\right) + 0.45 \log_2\left(\frac{1}{0.45}\right)$$

$$= 2.4322 \text{ bits/message}$$

d) $$H(X/Y) = H(XY) - H(X) = 2.4322 - 1.0613 = 1.3710$$

$$H(Y/X) = H(XY) - H(Y) = 2.4322 - 1.0613 = 1.3710$$

Step 2 :

To obtain $I(X,Y)$:

It is given as,

$$I(X,Y) = H(X) - H(X/Y) = 1.0613 - 1.3710 = -\ 0.3097$$

Q.28 The joint probability matrix representing transmitter and receiver is given below. Find all entropies and mutual information of the communication system.

$$\mathbf{P(X, Y)} = \begin{bmatrix} 0.3 & 0.05 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0.15 & 0.05 \\ 0 & 0.05 & 0.15 \end{bmatrix}$$

☞ **[SPPU : Dec.-13, May-17, (End Sem), Marks 6]**

Ans. : Joint probability matrix representing Tx and Rx is given.

The sum of rows will give $P(X)$ and sum of columns will give $P(Y)$

$$P(X) = [0.35 \quad 0.25 \quad 0.2 \quad 0.2]$$

$$P(Y) = [0.3 \quad 0.2 \quad 0.20]$$

- $$H(X) = \sum_{j=1}^{m} P(x_j) \cdot \log \frac{1}{P(x_j)}$$

$$= 0.35 \log \frac{1}{0.35} + 0.25 \log \frac{1}{0.25} + 0.2 \log \frac{1}{0.2} + 0.2 \log \frac{1}{0.2}$$

$$= 1.96 \text{ bits/message}$$

- $$H(Yy) = \sum_{k=1}^{n} P(y_k) \cdot \log \frac{1}{P(y_k)}$$

$$= 0.3 \log \frac{1}{0.3} + 0.5 \log \frac{1}{0.5} + 0.2 \log \frac{1}{0.2}$$

$$H(Y) = 1.49 \text{ bits/message}$$

$$H(X,Y) = \sum_{j=1}^{m} \sum_{k=1}^{n} p(x_i, y_k) \cdot \log \frac{1}{p(x_i, y_k)}$$

$$= 0.3 \log \frac{1}{0.3} + 0.05 \log \frac{1}{0.05} + 0.25 \log \frac{1}{0.25}$$

$$H(X,Y) = 2.49 \text{ bits/message}$$

$$H(X/Y) = H(XY) - H(Y) = 2.49 - 1.49$$

$$= 1 \text{ bit/message}$$

$$H(Y/X) = H(XY) - H(X) = 2.49 - 1.96$$

$$= 0.53 \text{ bits/message}$$

$$I(X;Y) = H(X) - H(X/Y) = 1.96 - 1$$

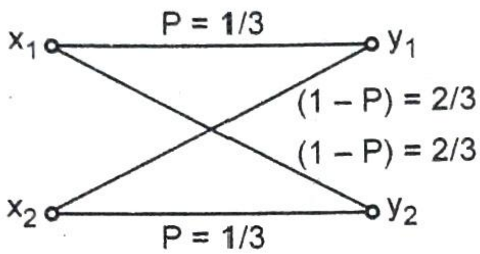
$$= 0.96 \text{ bits/message}$$

Q.29 A discrete source emits messages x_1 and x_2 with probabilities 3/4 and 1/4 with binary symmetric channels. Find $H(X), H(Y), H(X, Y]), H(X/Y), H(X), I(X:Y)$ if probabilities $p = 1/3$. Draw the channel diagram [SPPU : May-12 Marks 9]

Ans. :

Step 1 : To draw channel diagram :

The channel diagram for BSC is as shown below.



Here given that $P = \frac{1}{3} \therefore 1-P = 2/3$ and $p(x_1) = \frac{3}{4}, p(x_2) = \frac{1}{9}$

Now, $P[Y/X] = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$

$$P(X,Y) = P(X) \cdot P[Y/X] = \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$\therefore \quad P(X,Y) = \begin{bmatrix} 1/4 & 1/2 \\ 1/6 & 1/2 \end{bmatrix}$$

Now summation of column of $P(X,Y)$ give $P(Y)$

$$\therefore \quad P(Y) = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \end{bmatrix}$$

Step 2 : To find $H(X), H(Y), H(X/Y), H(Y/X)$ and $I(X/Y)$:

a) $$H(X) = \sum_{k=1}^{m} p(x_k) \log_2 \frac{1}{p(x_k)} = \frac{3}{4} \log_2 \left(\frac{3}{4}\right) + \frac{1}{4} \log_2 4$$

$$= 1.81 \text{ bits/message}$$

b) $$H(Y) = \sum_{k=1}^{m} p(y_k) \cdot \log_2 \left[\frac{1}{p(y_k)}\right]$$

$$= \frac{5}{12} \log_2 \frac{12}{5} + \frac{7}{12} \log_2 \frac{12}{7} = 0.98 \text{ bits/message}$$

c) $$H(X,Y) = \sum_{j=1}^{m} \sum_{i=1}^{n} p(x_i, y_j) \log_2 \frac{1}{p(x_j, y_i)}$$

$$= \frac{1}{4} \log_2 4 + \frac{1}{2} \log_2 2 + \frac{1}{6} \log_2 2 \frac{1}{12} \log_2 12$$

$$= 1.73 \text{ bits/message}$$

d) $$H(X/Y) = H(XY) - H(Y) = 1.73 - 0.98 = 0.75 \text{ bits/message}$$

e) $$H(Y/X) = H(XY) - H(Y) = 1.73 - 0.81 = 0.92 \text{ bits/message}$$

f) $$I(X,Y) = H(X) - H(X/Y) = 0.81 - 0.75 = 0.06 \text{ bits/message}$$

Q.30 Prove the following, mutual information
$I(X ; Y) = H(X) + H(Y) - H(X, Y)$

Ans. : Consider the standard relation,

$$H(X, Y) = H(X/Y) + H(Y)$$

$$\therefore \quad H(X/Y) = H(X, Y) - H(Y) \qquad \text{... (Q.30.1)}$$

Mutual information is given as,

$$I(X ; Y) = H(X) - H(X/Y)$$

Putting for $H(X/Y)$ in above equation from equation (Q.30.1),

$$I(X ; Y) = H(X) + H(Y) - H(X, Y) \qquad \text{... (Q.30.2)}$$

Thus the required relation is proved.

5.4 : Channel Capacity and Channel Coding Theorem

Important Points to Remember

- Maximum information rate is limited by channel capacity as per channel coding theorem.

Q.31 State Shannon second theorem. ☞[SPPU : April-15, Marks 2]

Ans. : Sharmon's second theorem : *Given a source of M equally likely messages, with $M >> 1$ which is generating information at a rate R. Given channel with channel capacity C. Then if,*

$$R \leq C,$$

there exists a coding technique such that the output of the source may be transmitted over the channel with a probability of error in the received message which may be made arbitrarily small.

5.5 : Differential Entropy and Mutual Information of Continuous Ensembles

Important Points to Remember

- Differential entropy $h(X) = \int_{-\infty}^{\infty} f_X(x) \log_2 \left[\frac{1}{f_X(x)}\right] d_x.$
- Mutual information of continuous ensemble,

$$I(X;Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) \log_2 \left[\frac{f_X(x/y)}{f_X(x)}\right] dx\, dy$$

Q.32 Explain differential entropy and mutual information for continuous ensembles. ☞[SPPU : Dec.-14, Marks 6]

Ans. : Differential Entropy : Let us consider the continuous random variable X having probability density function $f_X(x)$. Then,

$$h(X) = \int_{-\infty}^{\infty} f_X(x) \log_2 \left[\frac{1}{f_X(x)}\right] dx \qquad \dots (Q.32.1)$$

Here $h(X)$ is called the differential entropy of X. The entropy of the continuous random variable is infinitely large. In the interval $[-\infty, \infty]$, x takes infinite number of values. In other words, the uncertainty involved in the value of x is of the order of infinity. Hence differential entropy is defined for continuous variables.

Mutual Information

Mutual information for the continuous variables can be defined as

$$I(X;Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y) \log_2 \left[\frac{f_X(x/y)}{f_X(x)}\right] dx\, dy \qquad \dots (Q.32.2)$$

Here $f_{XY}(x, y)$ is the joint pdf of X and Y and $f_X(x/y)$ is conditional pdf of X given that $Y = z$. The mutual information has similar properties to those of continuous random variables.

i. $I(X;Y) = I(Y;X)$

ii. $I(X;Y) \geq 0$

iii. $I(X;Y) = h(X) - h(X/Y)$

iv. $I(X;Y) = h(Y) - h(Y/X)$

Here $h(X/Y)$ is the conditional differential entropy.

5.6 : Information Capacity Theorem

Important Points to Remember

- Information capacity theorem, C $= B\log_2\left(1+\frac{S}{N}\right)$ biks / sec.
- For infinite band width, $C_\infty = 1.44\frac{S}{N_0}$

Q.33 Explain Shannon third theorem (Information capacity theorem or Shannon Hartley theorem) and prove that when $B \to \infty$ then channel capacity $= \frac{S}{N_0} \log_2 e = 1.44 \frac{S}{N_0}$.

[SPPU : Dec.-11, 12, Marks 2, May-12, Marks 3, Dec.-14, Marks 6, April-17, Marks 4]

Ans. : Statement of theorem : *The channel capacity of a white bandlimited gaussian channel is,*

$$\boxed{C = B \log_2 \left(1+\frac{S}{N}\right) \text{ bits / sec.}} \quad \text{... (Q.33.1)}$$

Here B is the channel bandwidth,

$$S = \int_{-B}^{B} \text{Power spectral density}$$

Power spectral density of white noise is $\frac{N_0}{2}$. Hence noise power N becomes,

$$\text{Noise power } N = \int_{-B}^{B} \frac{N_0}{2} df, \quad \therefore N = N_0 B \quad \text{... (Q.33.2)}$$

Noise power is given as,

$$N = N_0 B \quad \text{... (Q.33.3)}$$

Putting this value in equation (Q.33.1) we get,

$$C = B \log_2 \left(1+\frac{S}{N_0 B}\right)$$

Rearrange the above equation as follows :

$$C = \frac{S}{N_0} \cdot \frac{N_0 B}{S} \log_2 \left(1+\frac{S}{N_0 B}\right) = \frac{S}{N_0} \log_2 \left(1+\frac{S}{N_0 B}\right)^{\frac{N_0 B}{S}}$$

$$= \frac{S}{N_0} \log_2 \left(1+\frac{S}{N_0 B}\right)^{1/\left(\frac{S}{N_0 B}\right)} \quad \text{... (Q.33.4)}$$

Apply the limits as $B \to \infty$,

$$C_\infty = \lim_{B \to \infty} C = \lim_{B \to \infty} \frac{S}{N_0} \log_2 \left(1 + \frac{S}{N_0 B}\right)^{1/\left(\frac{S}{N_0 B}\right)}$$

In the above equation put $x = \frac{S}{N_0 B}$. Then as $B \to \infty, x \to 0$ i.e.,

$$C_\infty = \frac{S}{N_0} \lim_{x=0} \log_2 (1+x)^{\frac{1}{x}}$$

Here use the standard relation, $\lim_{x=0} (1+x)^{\frac{1}{x}} = e$, then above equation becomes,

$$C_\infty = \frac{S}{N_0} \log_2 e = \frac{S}{N_0} \frac{\log_{10} e}{\log_{10} 2} = 1.44 \frac{S}{N_0} \qquad \dots (Q.33.5)$$

This is the required equation. It gives the upper limit on channel capacity as bandwidth B approaches infinity.

Q.34 State Shannon Hartley theorem with bandwidth efficiency diagram and show that Shannon limit is –1.6 dB.

☞[SPPU : May-14, Marks 8, May-13, 12, Dec.-11, Marks 6, May-17, Marks 3]

OR

For a Gaussian channel,

$C = B \log_2 \left(1 + (E_b / N_0)(C / B)\right)$

i) Find Shannon limit

ii) Draw the bandwidth efficiency diagram with (E_b / N_0) dB on horizontal axis and (R_b / B) on vertical axis. Mark different regions and Shannon limit on the graph.

☞[SPPU : May-11, Marks 10]

Ans. : i) Shannon Hartley theorem : Refer Q.33 for statement

ii) To prove that Shannon limit is – 1.6 dB :

For infinite bandwidth,

$$C_\infty = 1.44 \frac{S}{N_0}$$

Here $S = E_b C = E_b C_\infty$, then above equation can be further simplified as,

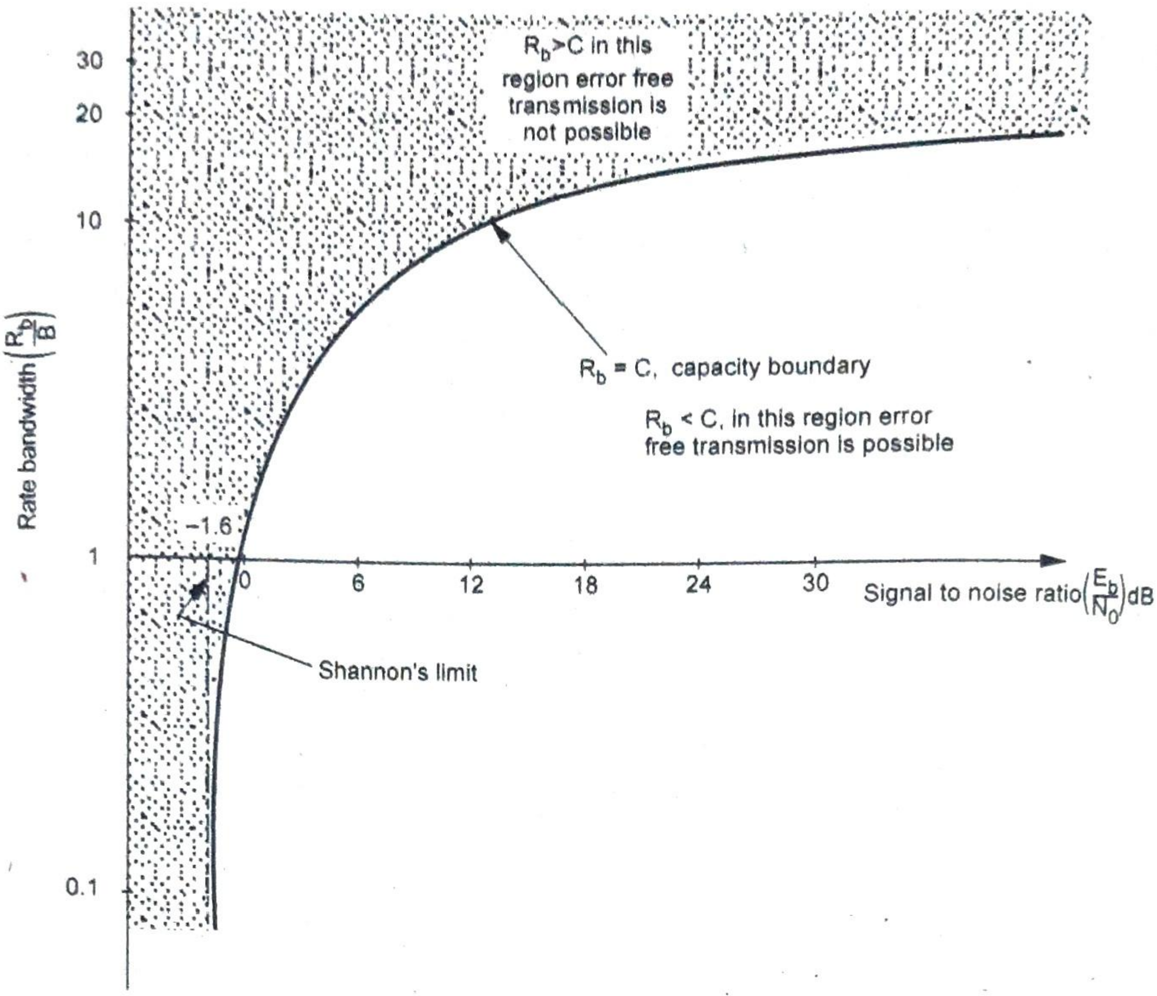


Fig. Q.34.1 Rate/bandwidth versus signal to noise ratio diagram

$$C_\infty = 1.44 \frac{E_b C_\infty}{N_0}$$

or
$$\frac{E_b}{N_0} = \frac{1}{1.44} = 0.693$$

$$\therefore \quad \left(\frac{E_b}{N_0}\right) dB = 10 \log \frac{E_b}{N_0} = 10 \log (0.693) = -1.6 \text{ dB}$$

Thus $\left(\frac{E_b}{N_0}\right) = -1.6$ dB for $B \to \infty$. This value of $\frac{E_b}{N_0}$ is called *Shannon's limit*. This limit is shown in Fig. Q.4.1. And the capacity at Shannon's limit is,

$$C_\infty = 1.44 \frac{S}{N_0}$$

Q.35 A Gaussian channel has 2 MHz bandwidth. Calculate the channel capacity if the signal power to noise spectral density ratio is 10^5 Hz. Also find the maximum information rate at which information can be transmitted. [SPPU : April-17, Marks 4]

Ans. : Here B $= 2\times 10^6$ Hz, $\frac{S}{N} = 10^5$

$$C = B\log_2\left(1+\frac{S}{N}\right) = 2\times 10^6 \log_2(1+10^5)$$

$$= 39.86 \text{ Mbps}$$

Q.36 A voice grade telephone channel has a bandwidth of 3400 Hz. If the signal-to-noise ratio is 30 dB. Determine the capacity of the channel. If above channel is used to transmit data at a rate 48 kbps. What is minimum required SNR ? [SPPU : May-15, Marks 4]

Ans. : i) Given data :

Channel BW, $B = 3400$ Hz, $\left(\frac{S}{N}\right)_{dB} = 30$ dB

$$\left(\frac{S}{N}\right)_{dB} = 10\log_{10}\left(\frac{S}{N}\right), \quad 30 = 10\log_{10}\left(\frac{S}{N}\right) \Rightarrow \frac{S}{N} = 1000$$

ii) To calculate capacity of the channel :

Capacity of channel is given as,

$$C = B\log_2\left(1+\frac{S}{N}\right) = 3400\log_2(1+1000) = 33.888 \text{ kbits/sec}$$

iii) To obtain minimum $\left(\frac{S}{N}\right)$ for 48 kbps data :

Here the data rate is 48 kbps,

From channel coding theorem,

$$R \leq C$$

Here $R = 48$ kbps and $C = B\log_2\left(1+\frac{S}{N}\right)$

and hence above equation becomes

$$48 \text{ kbps} \leq 3400\log_2\left(1+\frac{S}{N}\right), \quad 14.1176 \leq \log_2\left(1+\frac{S}{N}\right)$$

i.e. $\log_2\left(1+\frac{S}{N}\right) \geq 14.1176, \quad 1+\frac{S}{N} = 2^{14.1176} = 17775.5$

$$\therefore \quad \frac{S}{N} = 17774.5, \quad \frac{S}{N}\,dB = 10\log_{10} 17774.5 = 42.5 \text{ dB}$$

Q.37 In a fax transmission of picture, there are about 2.25×10^6. Picture elements per frame for good reproduction, 12 brightness level are necessary. Assuming all these levels in equiprobable.

i) Calculate the channel BW required to transmit 1 picture for every 3 minutes for SNR = 30 dB.

ii) If SNR requirement increases to 40 dB. Calculate BW.

iii) Explain the trade - off between BW and SNR.

☞[SPPU : Dec.-14, Marks 8]

Ans. : i) Given data

Number of picture elements/frame = 2.25×10^6

Source levels or brightness levels, $M = 12$

Picture transmission rate = 1/3 minutes

$\left(\frac{S}{N}\right) dB = 30 dB$ and 40 dB

Calculate 'B'.

ii) To calculate source entropy :

There are 12 distinct brightness levels. This means source has 12 different symbols of equal probability.

Hence $M = 12$ and $p_k = \frac{1}{M} = \frac{1}{12}$. For 'M' number of equally likely symbols, 'H' is given as,

$$H = \log_2 M$$
$$= \log_2 12 = 3.5849 \quad \text{bits/symbol (level)}$$

iii) To calculate message rate 'r' :

One picture consists of 2.25×10^6 picture elements. And such one picture is transmitted in every 3 minutes. Hence,

$$r = \frac{2.25\times 10^6}{3\ minutes} = \frac{2.25\times 10^6}{180} \text{ picture elements/sec}$$

$$= 12500 \text{ picture elements/sec}$$

Here each picture element is a symbol. Hence,

$$r = 12500 \text{ symbols/sec}$$

iv) To calculate information rate (R) :

Information rate is given as,

$R = rH$ = 12500 symbols/sec × 3.5849 bits/symbol = 44812 bits/sec

v) To calculate bandwidth (B) for $\frac{S}{N}$ = 30 dB :

$$\left(\frac{S}{N}\right)_{dB} = 10 \log_{10} \frac{S}{N}, \quad 30 \text{ dB} = 10 \log_{10} \frac{S}{N}$$

$$3 = \log_{10} \frac{S}{N}, \quad \frac{S}{N} = 10^3 = 1000$$

Channel capacity is given by,

$$C = B \log_2 \left(1+\frac{S}{N}\right)$$

From channel coding theorem,

$$R \leq C$$

$$\therefore \quad R \leq B \log_2 \left(1+\frac{S}{N}\right) \quad \text{... (Q.37.1)}$$

Putting values in above equation,

$$44812 \leq B \log_2 (1+1000), \quad B \geq \frac{44812}{\log_2 (1001)}$$

$$\therefore \quad B \geq 4495.93 \text{ Hz or } 4.5 \text{ kHz}$$

vi) To calculate bandwidth (B) for $\frac{S}{N}$ = 40 dB :

We know that,

$$\left(\frac{S}{N}\right)_{dB} = 10\log_{10} \frac{S}{N}, \quad 40 \text{ dB} = 10 \log_{10} \frac{S}{N}$$

$$\therefore \quad \frac{S}{N} = 10{,}000$$

Putting values in equation (Q.7.1),

$$44812 \leq B \log_2 (1+10{,}000), \quad B \geq 3372.4 \text{ Hz} \text{ or } 3.37 \text{ kHz.}$$

Tradeoff between bandwidth and SNR :

The results are shown below

$\frac{S}{N} = 30$ dB	$B = 4.5$ kHz
$\frac{S}{N} = 40$ dB	$B = 3.37$ kHz

From the above result it is clear that the increase in signal to noise ratio by 10 times reduces the bandwidth requirement by more than 1 kHz.

Q.38 A channel has Bandwidth of 5 kHz and a signal to noise power of 63. Determine the BW needed if the S/N power ratio is reduced to 31. What will be the signal power required if the channel BW is reduced to 3 kHz ? **[SPPU : Dec.-11, Marks 8]**

Ans. : Given : B = 5 kHz, SNR = 63

The capacity is given by,

$$C = B \log_2 \left(1+\frac{S}{N}\right) = 5 \log_2 (1+63) = 5 \log_2 64 = 30 \times 10^3 \text{ bit / sec.}$$

ii) Now, $\quad C = 30 \times 10^3, \frac{S}{N} = 31$

$$\therefore \quad B = \frac{C}{\log_2\left(1+\frac{S}{N}\right)} = \frac{30 \times 10^3}{\log_2 (1+31)} = \frac{30 \times 10^3}{\log_2 32} = 6 \text{ kHz.}$$

iii) Now, $\quad B = 3 \times 10^3$ bit / sec, $\quad C = 30 \times 10^3$ bits / sec

$$\therefore \quad C = B \log_2 \left(1+\frac{S}{N}\right)$$

$$30 \times 10^3 = 3 \times 10^3 \log_2 \left(1+\frac{S}{N}\right)$$

$$10 = \log_2\left(1+\frac{S}{N}\right) \Rightarrow 2^{10} = \left(1+\frac{S}{N}\right)$$

$$\therefore \quad \text{SNR} = 1024 - 1 = 1023 = \frac{S_2}{N_2}$$

and Let $\dfrac{S_1}{N_1} = 63$

Therefore signal power will

$$\frac{S_2 / N_2}{S_1 / N_1} = \frac{1023}{63} = 16,$$

Note that $N_1 = N_2$ since channel noise remains same.

$$\therefore \quad \frac{S_2}{S_1} = 16 \qquad \text{i.e. } S_2 = 16\, S_1$$

Hence, the signal power should be 16 times more than first case.

END...

Unit VI

6 ERROR CONTROL CODING

6.1 : Linear Block Codes and Hamming Codes

Important Points to Remember

- For linear block code C = MP, $G = [I_k | P_{k \times q}]$, Y = MG.
- Syndrome vector $S = RH^T$
- Hamming bound, $1 - r \geq \frac{1}{n} \log_2 \sum_{i=0}^{t} n_{C_i}$

Q.1 What are hamming codes ? Explain with suitable example.

[SPPU : Dec.-15, Marks 7, Dec.-12 Marks 2]

Ans. : Hamming codes are (n, k) linear block codes with following conditions :

i) Number of check bits $q \geq 3$

ii) Block length $n = 2^q - 1$

iii) Number of message bits $k = n - q$

iv) Minimum distance $d_{min} = 3$.

Consider (7,4) block code, where n = 7, q = 4, and $d_{min} = 3$

i) **Error correction capacity,** $t_c = \frac{d_{min} - 1}{2} = 1$

ii) **Block length** $n = 2^q - 1 = 7$

iii) **Number of message bits,** $k = n - q = 3$

iv) $d_{min} = 3$

Also, $\sum_{i=0}^{t} n_{C_i} = 7_{C_0} + 7_{C_1} = 1 + 7 = 8$

As $2^q = \sum_{i=0}^{t} {}^nC_i$

$\therefore$ It is a hamming code/perfect code.

Q.2 For a (6, 3) linear block code, following generator matrix is used.

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

i) **For error correction and detection capability of the code**

ii) **Is this perfect code ? Justify**

iii) **Is this linear code ? Justify** ☞[SPPU : April-15, 17, Marks 6]

Ans. : For this code $n = 6$ and $k = 3$, $\therefore \quad q = n - k = 6 - 3 = 3$

i) To obtain 'd_{min}' for this code :

To determine 'd_{min}' we have to find out all code words.

To obtain P submatrix :

Generrator matrix is given by,

$$G = [I_k : A_{k \times q}] = [I_{3 \times 3} : P_{3\times3}] = \begin{bmatrix} 1 & 0 & 0 & : & P_{11} & P_{12} & P_{13} \\ 0 & 1 & 0 & : & P_{21} & P_{22} & P_{23} \\ 0 & 0 & 1 & : & P_{31} & P_{32} & P_{33} \end{bmatrix}$$

Comparing above matrix with that given in the problem,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

To obtain equations for check bits :

We know that the check bits are given as,

$$C = MP$$

i.e. $$[C_1 C_2 C_3] = [m_1\, m_2\, m_3]_{1\times 3}[P]_{3\times 3} = [m_1\, m_2\, m_3]\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= [m_1 \oplus m_3 \quad m_1 \oplus m_2 \oplus m_3 \quad m_2 \oplus m_3]$$

Thus the equations for the check bits are,

$$C_1 = m_1 \oplus m_3, \; C_2 = m_1 \oplus m_2 \oplus m_3, \; C_3 = m_2 \oplus m_3$$

To determine the code vectors :

Since there are three message bits, there will be nine message vectors. Hence there will be nine code vectors table 1. lists the code vectors The check bits are calculated as per above equations.

As shown in Table Q.2.1, the message vector

of $m_1 m_2 m_3 = 001$, then check bits are calculated as,

$$C_1 = m_1 \oplus m_3 = 0 + 1 = 1$$

$$C_2 = m_1 \oplus m_2 \oplus m_3 = 0 + 0 + 1 = 1$$

$$C_3 = m_2 \oplus m_3 = 0 + 1 = 1$$

Similarly all other code words are found.

Sr. No.	Message vector M			Check bits C			Code vector X						Wt. of the code
	m_1	m_2	m_3	c_1	c_2	c_3	m_1	m_2	m_3	c_1	c_2	c_3	
1.	0	0	0	0	0	0	0	0	0	0	0	0	0
2.	0	0	1	1	1	1	0	0	1	1	1	1	4
3.	0	1	0	0	1	1	0	1	0	0	1	1	3
4.	0	1	1	1	0	0	0	1	1	1	0	0	3
5.	1	0	0	1	1	0	1	0	0	1	1	0	3
6.	1	0	1	0	0	1	1	0	1	0	0	1	3
7.	1	1	0	1	0	1	1	1	0	1	0	1	4
8.	1	1	1	0	1	0	1	1	1	0	1	0	4

Table Q.2.1 : Calculation of codewords

Weight of the code and d_{min} :

From Table Q.2.1, The minimum weight of the code is 3,

Hence, $d_{min} = [w(X)]_{min} = 3$

ii) Error correction and detection capabilities

$$d_{min} \geq S + 1, \;\; 3 \geq S + 1 \Rightarrow \mathbf{S \leq 2}$$

Thus, two errors will be detected.

and $d_{min} \geq 2t + 1, \; 3 \geq 2t + 1 \Rightarrow t \leq 1$

Thus, one error will be corrected.

This is a hamming code ($d_{min} = 3$) and it always detects double errors and corrects single errors.

ii) Whether perfect code or not ? :

This code can correct one error

$$\therefore \quad t = 1$$

$$\text{Now,} \quad 2^q = 2^3 = 8$$

$$\text{and} \sum_{i=0}^{t} {}^{n}C_i = \sum_{i=0}^{1} {}^{6}C_1 = {}^{6}C_0 + {}^{6}C_1 = \frac{6!}{(6-0)!\,0!} + \frac{6!}{(6-1)!\,1!} = 1 + 6 = 7$$

$$\text{Now,} \quad 2^q \neq \sum_{i=0}^{t} {}^{n}C_i$$

$\therefore$ (6, 3) is not perfect code as it does not satisfy Hamming bound condition.

iii) To verify whether it is linear or not :

If any two code vectors in Table Q.2.1 are added and resulting vector is code vector in Table Q.2.1 then it is a linear code.

$$\begin{array}{r} 0\ 0\ 1\ 1\ 1\ 1 \\ \oplus \quad 1\ 0\ 0\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 0\ 0\ 1 \end{array} \quad \text{is code vector in Table Q.2.1}$$

$\therefore$ It is linear code.

Q.3 For a systematic LBC, the three parity check digits C_4, C_5, C_6 are given by $C_4 = d_1 \oplus d_2 \oplus d_3$; $C_5 = d_1 \oplus d_2$; $C_6 = d_1 \oplus d_3$ Construct a matrix, find all possible code vectors. If received code vector is 1 0 1 1 0 0. Find out transmitted code vectors.

☞[SPPU : May-14, Marks 8, April-17, Marks 6]

Ans. : i) To obtain the generator matrix : The check bits, message bits and parity matrix are related as,

$$[C_4\ C_5\ C_6]_{1\times 3} = [d_1\ d_2\ d_3]_{1\times 3}\ [P]_{3\times 3} \qquad \text{... (Q.3.1)}$$

The above equation can be written as

$$[C_4\ C_5\ C_6] = [d_1\ d_2\ d_3] \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

Hence,

$$\left.\begin{aligned} C_4 &= d_1\ P_{11} \oplus d_2\ P_{21} \oplus d_3\ P_{31} \\ C_5 &= d_1\ P_{12} \oplus d_2\ P_{22} \oplus d_3\ P_{32} \\ C_6 &= d_1\ P_{13} \oplus d_2\ P_{23} \oplus d_3\ P_{33} \end{aligned}\right\} \quad \text{... (Q.3.2)}$$

Comparing the above equations with given check bit equations we get parity matrix as,

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{... (Q.3.3)}$$

The generator matrix is given as,

$$G = [I_k \ : \ P_{k \times q}] = [I_3 \ : \ P_{3 \times 3}]$$

$$= \begin{bmatrix} 1 & 0 & 0 & : & 1 & 1 & 1 \\ 0 & 1 & 0 & : & 1 & 1 & 0 \\ 0 & 0 & 1 & : & 1 & 0 & 1 \end{bmatrix} \quad \text{... (Q.3.4)}$$

ii) To obtain the code vectors :

In this code, there are 3 message bits and 3 check bits. Hence this is (6, 3) block code. Table Q.3.1 shows the message bits, check bits and code vectors for this code.

Sr. No.	**Message vector M**			**Check bits as per equation Q.11.2**			**Code vector or code word Y**						**Weight of code vector $w(Y)$**
	d_1	d_2	d_3	C_4	C_5	C_6	d_1	d_2	d_3	C_4	C_5	C_6	
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	0	1	0	0	1	1	0	1	3
3	0	1	0	1	1	0	0	1	0	1	1	0	3
4	0	1	1	0	1	1	0	1	1	0	1	1	4
5	1	0	0	1	1	1	1	0	0	1	1	1	4
6	1	0	1	0	1	0	1	0	1	0	1	0	3
7	1	1	0	0	0	1	1	1	0	0	0	1	3
8	1	1	1	1	0	0	1	1	1	1	0	0	4

Table Q.3.1 : Code vectors

iv) To prepare the decoding table :

The parity check matrix (H) is given as,

$$H = [P^T \; : \; I_q]_{q \times n}$$

Hence transpose of above matrix becomes,

$$H^T = \begin{bmatrix} P \\ \cdots \\ I_q \end{bmatrix}_{n \times q}$$

From equation (Q.3.3), above matrix will be,

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{... (Q.3.5)}$$

The syndrome vector (S) can be calculated from error vector (E) and H^T.

$$S = EH^T$$

Here E is the 1×6 size error vector. Let us calculate syndrome for 2^{nd} bit in error. The E will be,

$$E = [0\ 1\ 0\ 0\ 0\ 0]$$

Hence syndrome will be (from $S = EH^T$),

$$S = [0\ 1\ 0\ 0\ 0\ 0] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1\ 1\ 0]$$

Thus the above syndrome vector corresponds to 2^{nd} row of H^T. Similarly other syndromes can be obtained directly from rows of H^T. Table Q.3.2 shows the error patterns and corresponding syndrome vectors.

Sr. No.	Error vector 'E' showing single bit error patterns						Syndrome vector 'S'			Comments
1	0	0	0	0	0	0	0	0	0	
2	1	0	0	0	0	0	1	1	1	← First row of H^T
3	0	1	0	0	0	0	1	1	0	← Second row of H^T
4	0	0	1	0	0	0	1	0	1	← Third row of H^T
5	0	0	0	1	0	0	1	0	0	← Fourth row of H^T
6	0	0	0	0	1	0	0	1	0	← Fifth row of H^T
7	0	0	0	0	0	1	0	0	1	← Sixth row of H^T

Table Q.3.2 Decoding table

To decode 101100 :

Here observe that the received word 101100 is not standard code vector from Table Q.3.1. Hence there is an error in received word. Let,

$$R = [1\ 0\ 1\ 1\ 0\ 0]$$

The syndrome can be calculated for this word as,

$$S = RH^T$$

Putting the values,

$$S = [1\ 0\ 1\ 1\ 0\ 0]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [1\oplus 0\oplus 1\oplus 1\oplus 0\oplus 0 \quad 1\oplus 0\oplus 0\oplus 0\oplus 0\oplus 0 \quad 1\oplus 0\oplus 1\oplus 0\oplus 0\oplus 0]$$

$$= [1\ 1\ 0]$$

Note that [1 1 0] is second syndrome in Table Q.3.2, and the corresponding error pattern is,

$$E = [0\ 1\ 0\ 0\ 0\ 0]$$

The correct word is obtained as,

$$Y = R \oplus E = (1\ 0\ 1\ 1\ 0\ 0) \oplus (0\ 1\ 0\ 0\ 0\ 0) = 1\ 1\ 1\ 1\ 0\ 0$$

This is the correct word.

Q.4 The Parity check matrix of a (7, 4) hamming code is given as follows :

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

i) **Find generator matrix**
ii) **Find out all possible code words**
iii) **Determine error correcting capability**
iv) **Prepare error decoding table.**

☞[SPPU : Dec.-13, Marks 8, May-17, Marks 7]

Ans. : Step 1 : To obtain generator matrix :

Here n = 7, k = 4, and q = n – k = 3.

The parity check matrix is, $H = [p^T : I_q]_{q \times n}$

Above given H can be written as

$$H = [\underbrace{\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{matrix}}_{P^T\ 4\times 3} : \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_{3\times 3}}] \quad \therefore P_T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Hence matrix P will be, $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Now generator matrix is given as, $q = [I_k ; P_{k \times q}]_{k \times n}$

$$\therefore \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Step 2 : To find all code words :

The check bits can be obtained as, C = MP

$$\therefore \quad [C_1\, C_2\, C_3] = [m_1\, m_2\, m_3\, m_4] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

upon solving matrix, we get

$\therefore C_1 = m_1 \oplus m_2 \oplus m_3, \quad C_2 = m_2 \oplus m_3 \oplus m_4, \quad C_3 = m_1 \oplus m_2 \oplus m_4$

Now, message vector $m_1\, m_2\, m_3\, m_4 = 0\,0\,0\,1$

$\therefore \quad C_1 = 0 \oplus 0 \oplus 0 = 0, C_2 = 0 \oplus 0 \oplus 1 = 1, C_3 = 0 \oplus 0 \oplus 1 = 1$

code word : $(m_1\, m_2\, m_3\, m_4 : C_1\, C_2\, C_3) = (000\,1 : 01\,1)$

Similar all other codewords are obtained given below in table.

Sr.No	Message Vector M				Check bit C			Code vector X							wt of code
	m_1	m_2	m_3	m_4	C_1	C_2	C_3	m_1	m_2	m_3	m_4	C_1	C_2	C_3	
1.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.	0	0	0	1	0	1	1	0	0	0	1	0	1	1	3
3.	0	0	1	0	1	1	0	0	0	1	0	1	1	0	3
4.	0	0	1	1	1	0	1	0	0	1	1	1	0	1	4
5.	0	1	0	0	1	1	1	0	1	0	0	1	1	1	4
6.	0	1	0	1	1	0	0	0	1	0	1	1	0	0	3
7.	0	1	1	0	0	1	0	0	1	1	0	0	1	0	3
8.	0	1	1	1	0	1	0	0	1	1	1	0	1	0	4
9.	1	0	0	0	1	0	1	1	0	0	0	0	0	1	3
10.	1	0	0	1	1	1	0	1	0	0	1	0	1	0	4
11.	1	0	1	0	0	1	1	1	0	1	0	1	1	1	4
12.	1	0	1	1	0	0	0	1	0	1	1	1	0	0	3
13.	1	1	0	0	0	1	0	1	1	0	0	1	1	0	3
14.	1	1	0	1	0	0	1	1	1	0	1	1	0	1	4
15.	1	1	1	0	1	1	0	1	1	1	0	0	1	0	4
16.	1	1	1	1	1	1	1	1	1	1	1	0	1	1	7

Table Q.4.1 : Code vectors

iii) From the minimum Hamming distance :

$$d_{min} = 3$$

Error correcting capability is,

$$d_{min} \geq 2t+1 \Rightarrow 3 \geq 2t+1$$

$$\therefore \quad t \leq 1$$

Thus one error can be corrected

iv) Error decoding table :

Here Table Q.4.1 shows 7 bit error vector table, The syndrome vector is same as corresponding rows of H^T matrix.

Following table shows syndromes for (7, 4). Hamming code of single bit error.

Sr. No.	Error Vector 'E' showing single bit error pattern							Syndrome Vector 'S			
1.	0	0	0	0	0	0	0	0	0	0	$\leftarrow 1^{ST}$ row of H^T
2.	1	0	0	0	0	0	0	1	0	1	$\leftarrow 2^{nd}$ row of H^T
3.	0	1	0	0	0	0	0	1	1	1	$\leftarrow 3^{rd}$ row of H^T
4.	0	0	1	0	0	0	0	1	1	0	$\leftarrow 4^{th}$ row of H^T
5.	0	0	0	1	0	0	0	0	1	1	$\leftarrow 5^{th}$ row of H^T
6.	0	0	0	0	1	0	0	1	0	0	$\leftarrow 6^{th}$ row of H^T
7.	0	0	0	0	0	1	0	0	1	0	$\leftarrow 7^{th}$ row of H^T
8.	0	0	0	0	0	0	1	0	0	0	$\leftarrow 8^{th}$ row of H^T

Table Q.4.2 : Syndromes (7, 4) Hamming code

$$\text{Since } H^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.2 : Cyclic Code

Important Points to Remember

- Every cyclic shift of a code vector produces another valid code vector.
- $Y(x) = M(x) \cdot G(x)$
- $C(x) = \text{rem}\left[\frac{x^q M(x)}{G(x)}\right]$

Q.5 Write the procedure for coding of cyclic code.

[SPPU : Dec.-15, Marks 6]

Ans. : The code words can be represented by a polynomial.

Generation of code vectors in nonsystematic form

Let $M = \{m_{k-1}, m_{k-2}, \ldots m_1, m_0\}$ be 'k' bits of message vector. Then it can be represented by the polynomial as,

$$M(x) = m_{k-1} x^{k-1} + m_{k-2} x^{k-2} + \ldots + m_1 x + m_0 \quad \text{... (Q.5.1)}$$

Let $Y(x)$ represent the codeword polynomial. It is given as,

$$\boxed{Y(x) = M(x)\, G(x)} \quad \text{... (Q.5.2)}$$

Here $G(x)$ is the *generating polynomial* of degree 'q'. For an (n, k) cyclic code, $q = n - k$ represent the number of parity bits. The generating polynomial is given as,

$$G(x) = x^q + g_{q-1} x^{q-1} + \ldots. + g_1 x + 1 \quad \text{... (Q.5.3)}$$

Here $g_{q-1}, g_{q-2}, \ldots. g_1$ are the parity bits.

If M_1, M_2, M_3 .... etc are the other message vectors, then the corresponding code vectors can be calculated as,

$$Y_1(x) = M_1(x)\, G(x)$$
$$Y_2(x) = M_2(x)\, G(x)$$

$$Y_3(x) = M_3(x)G(x) \text{ and so on} \quad \dots (Q.5.4)$$

All the above codevectors Y_1, Y_2, Y_3 .... are in nonsystematic form and they satisfy cyclic property. Note the generator polynomial $G(x)$ remains the same for all codevectors.

Generation of codevectors in systematic form

Now systematic cyclic codes. The systematic form of the block code is,

$$Y = (k \text{ message bits} : (n-k) \text{ check bits}) \quad \dots (Q.5.5)$$

$$= \left(m_{k-1}\, m_{k-2} \dots m_1\, m_0 : c_{q-1}\, c_{q-2} \dots c_1\, c_0\right) \quad \dots (Q.5.6)$$

Here the check bits form a polynomial as,

$$C(x) = c_{q-1}\, x^{q-1} + c_{q-2}\, x^{q-2} + \dots c_1\, x + c_0 \quad \dots (Q.5.7)$$

The check bit polynomial is obtained by,

$$\boxed{C(x) = rem\left[\frac{x^q\, M(x)}{G(x)}\right]} \quad \dots (Q.5.8)$$

Above equation means -

i) Multiply message polynomial by x^q.

ii) Divide $x^q\, M(x)$ by generator polynomial.

iii) Remainder of the division is $C(x)$.

Q.6 What are cyclic hamming codes ? Give one example of cyclic hamming code. ☞ **[SPPU : May-15, Marks 4]**

Ans. : Hamming code is defined by following

Codeword length, n = $2^q - 1$

Message length, k = $2^q - 1 - q$

Check bits length, q = $n - k$

Minimum distance, $d_{\min} = 3$

Error correcting capability = 1

Error defecting capability = 2

The cyclic hamming codes are generated using the generator polynomial which is constructed from following primitive polynomials of degree 'q'.

The primitive polynomials are listed below :

q	Primitive polynomial	q	Primitive polynomial
3	x^3+x+1	8	$x^8+x^4+x^3+x^2+1$
4	x^4+x+1	9	x^9+x^4+1
5	x^5+x^2+1	10	$x^{10}+x^3+1$
6	x^6+x+1	11	$x^{11}+x^2+1$
7	x^7+x^3+1	12	$x^{12}+x^6+x^4+x+1$

Example of cyclic hamming code : The (7, 4) cyclic code with primitive polynomial $p(x) = x^3+x+1$ (since $q = 7 - 4 = 3$) is cyclic hamming code. It has $d_{min} = 3$. It detects double errors and corrects single errors.

Q.7 Using generator polynomial $g(x) = x^3+x+1$ generate systematic cycle code for following message :
i) 1011 ii) 1010 iii) 1110 ☞ **[SPPU : April-15, Dec.-15, Marks 6]**

Ans. : Here $M_1 = \{1\ 0\ 1\ 1\}$ and $M_2 = \{1\ 0\ 1\ 0\}$

Hence number of message bits, k = 4

The generator polynomial has degree '3'. Hence q = 3.

Therefore n = k + q = 4 + 3 = 7.

(i) Code for $M_1 = \{1\ 0\ 1\ 1\}$

$$M_1(x) = 1\times x^3+0\times x^2+1\times x+1 = x^3+x+1$$

$$\therefore \quad x^q M_1(x) = x^3 M_1(x) \quad \text{since } q = 3$$

$$= x^3(x^3+x+1) = x^6+x^4+x^3$$

Now perform division $\dfrac{x^2 M_1(x)}{G(x)}$

$$\begin{array}{r|l} & \quad\quad x^3 \\ x^3+x+1 & x^6+x^4+x^3 \\ & \oplus x^6+x^4+x^3 \\ \hline & 0 \quad 0 \quad 0 \leftarrow \text{remainder} \end{array}$$

Check bit polynomial, $C(x) = rem\left[\dfrac{x^3 M_1(x)}{G(x)}\right] = 0x^2+0x+0$

or $C = \{c_2\ c_1\ c_0\} = \{0\ 0\ 0\}$

Hence cyclic code in systematic form can be written as

$$Y_1 = \{M_1 : C\} = \{1\ 0\ 1\ 1 : 0\ 0\ 0\}$$

(ii) Code for $M_2 = \{1\ 0\ 1\ 0\}$

$$M_2(x) = x^3+x$$

$$\therefore \quad x^q M_2(x) = x^3 M_2(x) = x^3(x^3+x) = x^6+x^4$$

Now perform division $\dfrac{x^q M_2(x)}{G(x)}$

$$\begin{array}{r|l} & \quad\quad\quad x^3+1 \\ x^3+x+1 & x^6+x^4 \\ & \oplus x^6+x^4+x^3 \\ \hline & \quad\quad\quad x^3 \\ & \quad\quad\quad \oplus x^3+x+1 \\ \hline & \quad\quad\quad\quad x+1 \leftarrow \text{remainder} \end{array}$$

$\therefore$ Check bit polynomial, $C(x) = rem\left[\dfrac{x^3 M_2(x)}{G(x)}\right] = x+1$

or $C = \{c_2\ c_1\ c_0\} = \{0\ 1\ 1\}$

Hence cyclic code in sysematic form can be written as,

$$Y_2 = \{M_2 : C\} = \{1\ 0\ 1\ 0 : 0\ 1\ 1\}$$

(iii) Code for 1110 :

$$M_3(x) = x^3+x^2+x$$

$$\therefore \quad x^q M_3(x) = x^3(x^3+x^2+x) = x^6+x^5+x^4$$

Now perform division $\frac{x^q M_2(x)}{G(x)}$

$$
\begin{array}{r l}
 & \quad x^3 + x^2 \\
x^3 + x + 1 \;\big) & x^6 + x^5 + x^4 \\
\oplus & x^6 \quad\;\; + x^4 + x^3 \\
\hline
 & \qquad x^5 + x^3 \\
\oplus & \qquad x^5 + x^3 + x^2 \\
\hline
 & \qquad\qquad\quad x^2 \leftarrow \text{remainder}
\end{array}
$$

Check bit polynomial, $C(x) = \text{rem}\left[\frac{x^3 M_2(x)}{G(x)}\right] = x^2$

or $C = \{c_2\ c_1\ c_0\} = [1\ 0\ 0\ ]$

Hence cyclic code in systematic form can be written as,

$$Y_2 = [M_2 : C] = [1110 : 100].$$

Q.8 Find a generator polynomial g(x) for a systematic (7, 4) Cyclic code and find the code vectors for the following data vectors : 1010, 1111, 0001 and 1000.

Given that $x^7 + 1 = (x+1)(x^3+x+1)\ (x^3+x^2+1)$

[SPPU : May-11, Marks 10]

Ans. : Step 1 : To obtain the generator polynomial :

The generator polynomial will be the factor of $(x^n + 1)$ i.e.

$$(x^7+1) = (x+1)(x^3+x+1)(x^3+x^2+1).$$

The generator polynomial must be of the degree 'q'.

Hence two generator polynomials are possible : x^3+x+1 and x^3+x^2+1

$$\therefore \qquad G(x) = x^3 + x + 1$$

Step 2 : To determine the code vectors :

i) Code vector for M = 1010

Refer Q.7 (ii)

ii) Code vector for M = 1111

$$\therefore \qquad M_2(x) = x^3 + x^2 + x + 1$$

$$\therefore \quad x^q M_2(x) = x^3 (x^3 + x^2 + x + 1) = x^6 + x^5 + x^4 + x^3$$

Perform division $\dfrac{x^q M_2(x)}{G(x)}$

$$\begin{array}{r}
x^3 + x^2 + 1 \\
x^3 + x + 1 \overline{) \; x^6 + x^5 + x^4 + x^3} \\
\oplus \; x^6 \qquad + x^4 + x^3 \\
\hline
x^5 \qquad\qquad \\
\oplus \; x^5 + x^3 + x^2 \\
\hline
x^3 + x^2 \\
\oplus \; x^3 + x + 1 \\
\hline
x^2 + x + 1 \leftarrow \text{remainder}
\end{array}$$

$$\therefore \quad C = \{c_2 \, c_1 \, c_0\} = [111]$$

$$\therefore \quad Y_2 = [M_2 : C] = [1111 : 111]$$

iii) Code vector for M = 0001

$$\therefore \quad M_3(x) = 1$$

$$\therefore \quad x^q M_2(x) = x^3 (1) = x^3, \text{ Perform division } \frac{x^q M_2(x)}{G(x)}$$

$$\begin{array}{r}
1 \qquad\quad \\
x^3 + x + 1 \overline{) \; x^3 \qquad\quad} \\
\oplus \; x^3 + x + 1 \\
\hline
x + 1 \leftarrow \text{remainder}
\end{array}$$

$$\therefore \quad C_3 = \{c_2 \, c_1 \, c_0\} = \{0 \; 1 \; 1\}$$

$$\therefore \quad Y_2 = [M_3 : C] = [0001 : 011]$$

iv) Code vector for M = 1000

$$\therefore \quad M_1(x) = x^3$$

$$\therefore \quad x^q M_2(x) = x^3 (x^3) = x^6, \text{ Perform division } \frac{x^q M_2(x)}{G(x)}$$

$$
\begin{array}{r}
x^3 + x + 1 \\
x^3 + x + 1 \,\overline{\big)\, x^6 \qquad\qquad\quad} \\
\oplus \quad x^6 + x^4 + x^3 \\
\hline
x^4 + x^3 \\
\oplus \quad x^4 + x^2 + x \\
\hline
x^3 + x^2 + x \\
x^3 + x + 1 \\
\hline
x^2 + 1 \leftarrow \text{remainder}
\end{array}
$$

$$\therefore \quad C_4 = \{1\ 0\ 1\}$$

$$\therefore \quad Y_3 = [M_4 : C_4] = [1000 : 101]$$

6.3 : Encoding, Decoding and Circuit Implementation of Cyclic Code

Important Point to Remember

- Syndrome in cyclic code, $S(x) = \text{rem}\left[\dfrac{r(x)}{G(x)}\right]$

Q.9 Write the procedure for decoding a cyclic code.

[SPPU : May-15, Marks 6]

Ans. : Procedure for decoding a cyclic code :

i) The received vector $R(x)$ is given.

ii) Syndrome can be calculated by equation,

$$S(x) = \text{rem}\left[\frac{R(x)}{G(x)}\right]$$

If the syndrome is non-zero, then there is an error in received vector.

iii) Obtain the decoding table. The decoding table can be easily prepared from H^T. For the block code, each row of H^T represents a syndrome and unique error pattern.

Here, for a above obtained syndrome, error pattern vector E will be obtained from decoding table.

iv) The correct code is given as,

$$Y = R \oplus E$$

Q.10 Construct a systematic (7, 4) cyclic code using this generator polynomial g(x) = $x^3 + x + 1$. What are error correcting capabilities of this code ? Construct the decoding table and for received code word 1101100, determine the transmitted data word.

[SPPU : Dec.-13, Marks 10]

Ans.: For this code $n = 7$, $k = 4$ and $q = 3$.

i) To determine generator matrix (G)

The t^{th} row of the generator matrix is given as,

$$x^{n-t} + r_t(x) = q_t G(x) \quad \text{and} \quad t = 1, 2, \ldots\ldots k$$

For the generator polynomial g(x) = $x^3 + x + 1$, the generator matrix is given as,

$$\therefore \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & : & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & : & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & : & 0 & 1 & 1 \end{bmatrix} \quad \ldots \text{(Q.10.1)}$$

Hence P submatrix can be obtained from generator matrix,

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \ldots\text{(Q.10.2)}$$

Hence the check bits can be obtained by

$$C = MP$$

$$\therefore \quad [C_1\, C_2\, C_3] = [m_0 \;\; m_1 \;\; m_2 \;\; m_3] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad C_1 = m_0 \oplus m_1 \oplus m_2$$

$$C_2 = m_1 \oplus m_2 \oplus m_3$$

$$C_3 = m_0 \oplus m_1 \oplus m_3$$

The check bits can be obtained for all the codevectors with the help of above equations. Table Q.10.1 lists all the systematic code vectors.

Sr. No.	Message bits $M = m_3\ m_2\ m_1\ m_0$	Systematic code vectors $Y = m_3\ m_2\ m_1\ m_0\ c_2\ c_1\ c_0$
1	0 0 0 0	0 0 0 0 0 0 0
2	0 0 0 1	0 0 0 1 0 1 1
3	0 0 1 0	0 0 1 0 1 1 0
4	0 0 1 1	0 0 1 1 1 0 1
5	0 1 0 0	0 1 0 0 1 1 1
6	0 1 0 1	0 1 0 1 1 0 0
7	0 1 1 0	0 1 1 0 0 0 1
8	0 1 1 1	0 1 1 1 0 1 0
9	1 0 0 0	1 0 0 0 1 0 1
10	1 0 0 1	1 0 0 1 1 1 0
11	1 0 1 0	1 0 1 0 0 1 1
12	1 0 1 1	1 0 1 1 0 0 0
13	1 1 0 0	1 1 0 0 0 1 0
14	1 1 0 1	1 1 0 1 0 0 1
15	1 1 1 0	1 1 1 0 1 0 0
16	1 1 1 1	1 1 1 1 1 1 1

Table Q.10.1 Code vectors of a (7, 4) cyclic code for $G(x) = x^3 + x^2 + 1$

ii) Error correcting capability

It is clear from Table Q.10.1 that,

$$d_{min} = [w(X)]_{min} = 3$$

Hence this code can detect upto two errors and correct one error.

iii) To obtain parity check matrix

The parity check matrix is given as,

$$H = [P^T : I_q]$$

$$\therefore \quad H^T = \begin{bmatrix} P \\ \cdots \\ I_q \end{bmatrix} \qquad \text{... (Q.10.3)}$$

Hence from equation (Q.10.3) we can write above matrix as follows :

$$H^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{... (Q.10.4)}$$

iv) To obtain decoding table

The decoding table can be easily prepared from H^T. For the block code, each row of H^T represents a syndrome and unique error pattern. This we have discussed earlier in linear block codes. Table Q.10.2 shows the error patterns and the syndrome vectors.

Sr. No.	Error vector '*E*' showing single bit error patterns							Syndrome vector			Comments
1	0	0	0	0	0	0	0	0	0	0	
2	1	0	0	0	0	0	0	1	0	1	$\leftarrow 1^{st}$ row of H^T
3	0	1	0	0	0	0	0	1	1	1	$\leftarrow 2^{nd}$ row of H^T
4	0	0	1	0	0	0	0	1	1	0	$\leftarrow 3^{rd}$ row of H^T
5	0	0	0	1	0	0	0	0	1	1	$\leftarrow 4^{th}$ row of H^T
6	0	0	0	0	1	0	0	1	0	0	$\leftarrow 5^{th}$ row of H^T
7	0	0	0	0	0	1	0	0	1	0	$\leftarrow 6^{th}$ row of H^T
8	0	0	0	0	0	0	1	0	0	1	$\leftarrow 7^{th}$ row of H^T

Table Q.10.2 : Decoding table

v) To decode 1 1 0 1 1 0 0

Determine syndrome

Let the received codeword be,

$$R = [1\ 1\ 0\ 1\ 1\ 0\ 0]$$

$$\therefore \qquad R(x) = x^6 + x^5 + x^3 + x^2$$

The syndrome vector is given,

i.e.,

$$S(x) = rem\left[\frac{R(x)}{G(x)}\right]$$

$G(x) = x^3 + x + 1$. Hence perform the division of above equation. $R(x)$ can be written as,

$$R(x) = x^6 + x^5 + 0x^4 + x^3 + x^2 + 0x + 0$$

And $G(x)$ can be written as,

$$G(x) = x^3 + 0x^2 + x + 1$$

The division is as shown below :

$$\begin{array}{r|l}
 & x^3 + x^2 + x + 1 \\
\hline
x^3 + 0x^2 + x + 1 & x^6 + x^5 + 0x^4 + x^3 + x^2 + 0x + 0 \\
 & x^6 + 0x^5 + x^4 + x^3 \\
 & \oplus \quad \oplus \quad \oplus \quad \oplus \\
\hline
 & x^5 + x^4 + 0x^3 + x^2 \\
 & x^5 + 0x^4 + x^3 + x^2 \\
 & \oplus \quad \oplus \quad \oplus \quad \oplus \\
\hline
 & x^4 + x^3 + 0x^2 + 0x \\
 & x^4 + 0x^3 + x^2 + x \\
 & \oplus \quad \oplus \quad \oplus \quad \oplus \\
\hline
 & x^3 + x^2 + x + 0 \\
 & x^3 + 0x^2 + x + 1 \\
 & \oplus \quad \oplus \quad \oplus \quad \oplus \\
\hline
\end{array}$$

Remainder $\rightarrow \quad x^2 + 0x + 1$

Thus the remainder is,

$$S(x) = x^2 + 0x + 1$$

i.e. $$S = [1\ 0\ 1]$$

The syndrome is non zero. Hence there is an error in the received codeword.

Determine error pattern for $S = 101$ and correct the codeword

Table Q.10.2 indicates that there is error in the first bit, i.e.,

$$E = [1\ 0\ 0\ 0\ 0\ 0\ 0]$$

Hence correct codevector is,

$$Y = R \oplus E = (1\ 1\ 0\ 1\ 1\ 0\ 0) \oplus (1\ 0\ 0\ 0\ 0\ 0\ 0) = [0\ 1\ 0\ 1\ 1\ 0\ 0]$$

Thus the transmitted codeword is $Y = 0\ 1\ 0\ 1\ 1\ 0\ 0$. It is also one of the codevector in Table Q.10.1.

Q.11 Find out the generator matrix for a systematic (7, 4) cyclic code if $G(x) = x^3 + x + 1$. Also find out the parity check matrix.

Ans. : I) To obtain generator polynomial

The t^{th} row of generator matrix is given as,

$$x^{n-t} + r_t(x) = q_t(x) G(x) \quad \text{and} \quad t = 1, 2, \ldots . k$$

We are given that $n = 7, k = 4$ and $q = n - k = 3$.

The above equation will be,

$$x^{7-t} + r_t(x) = q_t(x)(x^3 + x + 1) \quad \text{and} \quad t = 1, 2, 3, 4 \quad \ldots \text{(Q.11.1)}$$

With $t = 1$, the above equation becomes,

$$x^6 + r_t(x) = q_t(x)(x^3 + x + 1) \quad \ldots \text{(Q.11.2)}$$

i) To obtain $r_t(x)$ and $q_t(x)$ for 1^{st} row

The RHS or LHS of this equation represents 1^{st} row of systematic generator matrix. We have to find $q_t(x)$. We know that $q_t(x)$ is obtained by dividing x^{n-t} by $G(x)$. Here to obtain $q_t(x)$ we have to divide x^6 by $G(x) = x^3 + x + 1$

$$
\begin{array}{rl}
 & x^3 + x + 1 \quad \leftarrow \text{Quotient} \\
x^3 + x + 1 \,\big) & x^6 + 0 + 0 \\
 & x^6 + x^4 + x^3 \\
 & \oplus \;\; \oplus \;\;\; \oplus \\
\hline
 & 0 + x^4 + x^3 + 0 + 0 \\
 & \quad x^4 + 0 + x^2 + x \\
 & \quad \oplus \;\; \oplus \;\; \oplus \;\; \oplus \\
\hline
 & \qquad x^3 + x^2 + x + 0 \\
 & \qquad x^3 + 0 + x + 1 \\
 & \qquad \oplus \;\; \oplus \;\; \oplus \;\; \oplus \\
\hline
 & \qquad\quad x^2 + 1 \quad \leftarrow \text{Remainder}
\end{array}
$$

Denotes mod-2 addition

Here $\quad q_t(x) = x^3 + x + 1$

and $\quad r_t(x) = x^2 + 1$

Putting those values in equation (Q.11.2) we get,

$$x^6 + x^2 + 1 = (x^3 + x + 1)(x^3 + x + 1)$$

The RHS or LHS (actually both are same) of the above equation represents 1^{st} row of generator matrix i.e.

1^{st} row polynomial = $x^6 + x^2 + 1$ ... (Q.11.3)

ii) Other row polynomials

Using the same procedure as discussed above, other row polynomials are obtained and they are given below

$$
\left.
\begin{array}{l}
t = 2 \Rightarrow 2^{nd} \text{ row polynomial} = x^5 + x^2 + x + 1 \\
t = 3 \Rightarrow 3^{rd} \text{ row polynomial} = x^4 + x^2 + x \\
t = 4 \Rightarrow 4^{th} \text{ row polynomial} = x^3 + x + 1
\end{array}
\right\} \qquad \text{... (Q.11.4)}
$$

iii) Conversion of row polynomials into matrix

The above equation can be transformed into generator matrix as shown below

$$G = \begin{array}{c} \\ Row\,1 \\ Row\,2 \\ Row\,3 \\ Row\,4 \end{array} \begin{array}{c} \begin{array}{ccccccc} x^6 & x^5 & x^4 & x^3 & & x^2 & x^1 & x^0 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & : & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & : & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & : & 0 & 1 & 1 \end{bmatrix}_{4 \times 7} \end{array} \quad \ldots \text{(Q.11.5)}$$

This is the required generator matrix in systematic form. The code vector can be obtained as

$$Y = MG$$

Let's take any 4 bit message vector and find corresponding code vector. Let's take

$$M = (m_3 \; m_2 \; m_1 \; m_0) = (1 \; 1 \; 0 \; 0)$$

$$Y = MG = [1 \; 1 \; 0 \; 0] \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$= (1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0)$$

This code vector is obtained by performing matrix multiplication and mod-2 additions. Observe that the same systematic code vector is listed in Table Q.11.1.

Using the same procedure other code vectors can be obtained.

II) To obtain parity check matrix (H)

We know that

$$G = \left[I_k \; : \; P_{k \times q} \right]_{k \times n}$$

The X submatrix can be obtained from equation (Q.11.5) as

$$x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{4 \times 3} \qquad \ldots \text{(Q.11.6)}$$

The parity check matrix is given by as,

$$H = \left[P^T \; : \; I_q\right]_{q \times n}$$

Here P^T is the transpose of P submatrix and

I_q is the $q \times q$ identity matrix.

By taking transpose of P submatrix of equation (Q.11.6), the parity check matrix will be,

$$H = \left[\begin{array}{cccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ \underbrace{1 \;\; 1 \;\; 0 \;\; 1}_{P^T} & & & & \underbrace{0 \;\; 0 \;\; 1}_{I_3} & & \end{array}\right]_{3 \times 7}$$

This is the required parity check matrix for (7, 4) cyclic code in systematic form.

Q.12 For a (7, 4) cyclic code, generator polynomial g(x) = $x^3 + x + 1$ is used. Draw the circuit for generating syndrome. Find syndrome for received code word 0011000

☞ [SPPU : April-15, Marks 6; Dec.-17, Marks 8]

Ans. : Syndrome calculator : Here $n = 7$ and $k = 4$.

Hence $q = n - k = 7 - 4 = 3$

The given generator polynomial is,

$$G(x) = x^3 + 0x^2 + x + 1$$

and $$G(x) = x^3 + g_2 x^2 + g_1 x + 1 \text{ generalised equation}$$

On comparison of above two equations we obtain.

$$g_1 = 1 \quad \text{and} \quad g_2 = 0$$

With these values the block diagram of syndrome calculator (7, 4) cyclic code is as shown below in Fig. Q.12.1.

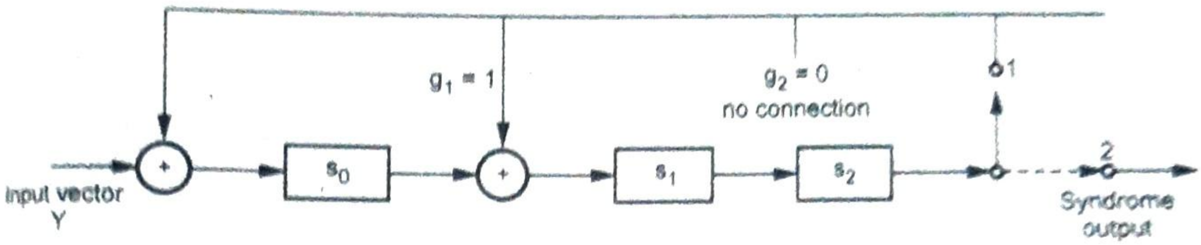


Fig. Q.12.1 Block diagram of syndrome calculator for (7, 4) cyclic code with $g(x) = x^3 + x + 1$

Syndrome for $Y = \{0\ 0\ 1\ 1\ 0\ 0\ 0\}$

The following table illustrates the operation of syndrome calculator or received code vector

$R = (0\ 0\ 1\ 1\ 0\ 0\ 0)$.

The table shows the contents of flip flops with every shift.

At the end of last shift, the register contents are $(s_2\ s_1\ s_0) = (1\ 0\ 1)$.

Shift	Received vector i.e. bits of Y	Contents of FF in shift register		
		$s_0 = y \oplus s_2$	$s_1 = s_0 \oplus s_2$	$s_2 = s_1$
-	-	0	0	0
1	0	$0 \oplus 0 = 0$	$0 \oplus 0 = 0$	0
2	0	0	0	0
3	1	1 + 0 = 1	0 + 0 = 0	0
4	1	1 + 0 = 1	1 + 0 = 1	0
5	0	0 + 0 = 0	1 + 0 = 1	1
6	0	0 + 1 = 1	0 + 1 = 1	1
7	0	0 + 1 = 1	1 + 1 = 0	1

Table Q.12.1 : Calculation of syndrome for R = (0 0 1 1 0 0 0)

Q.13 Sketch the encoder and syndrome calculator for the generator polynomial g(x) = $x^3 + x^2 + 1$, (7, 4) cyclic code and obtain the syndrome for the received code word 1001011.

[SPPU : Dec.-12, Marks 8]

Ans. : The generator polynomial is,

$$G(x) = x^3 + 0x^2 + 0x + 1$$

$$\text{and } G(x) = x^3 + g_2 x^2 + g_1 x + 1$$

On comparison of two equation,

$$g_1 = 0 \text{ and } g_2 = 1$$

$$\text{and} \quad n - k = 3 \therefore (n, k) \equiv (7, 4)$$

$\therefore$ The encoder is as shown below :

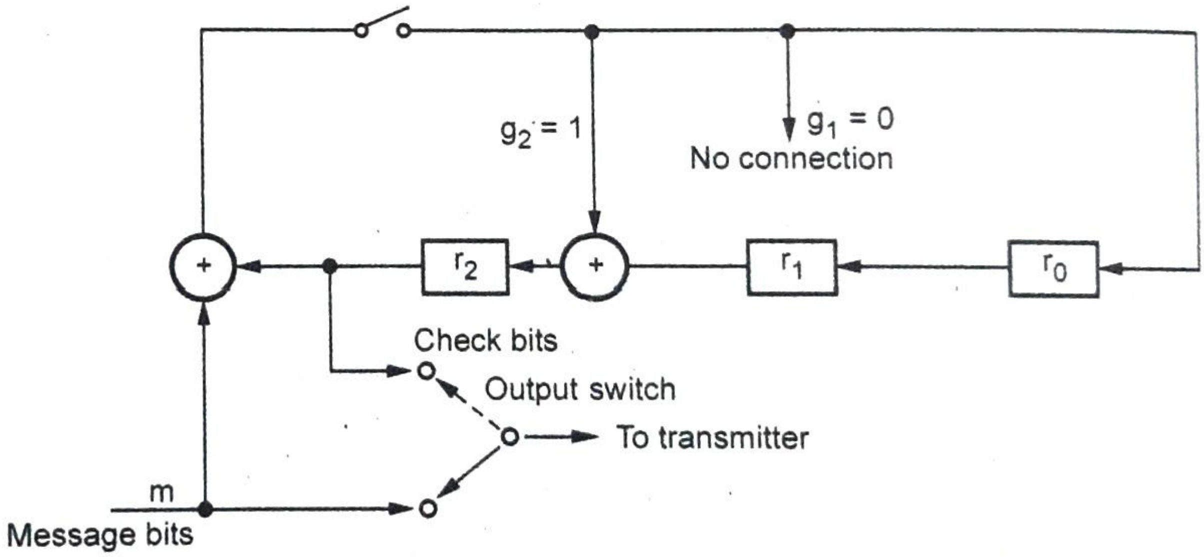


Fig. Q.13.1 Encoder for (7, 4) cyclic code for g(x) = $x^3 + x^2 + 1$

Syndrome calculator : With above values of syndrome calculator for cyclic code *(7, 4)* is as below :

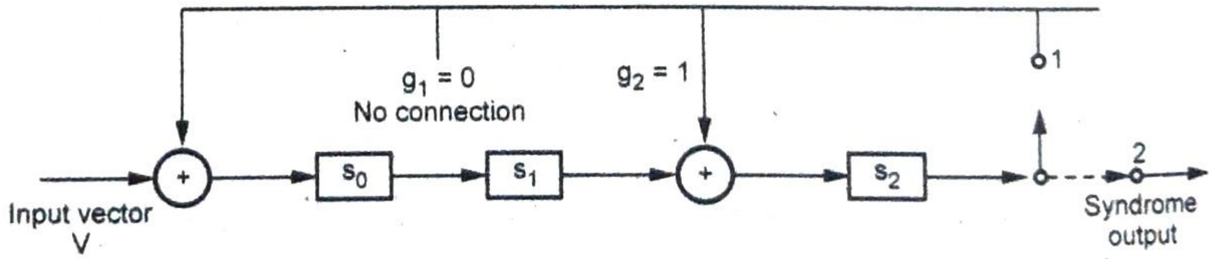


Fig. Q.13.2 Block diagram of syndrome calculator (7, 4)

Syndrome for Y = {1 0 0 1 0 1 1}

The following table illustrates operation of syndrome calculator

R = (1001011)

The table shows the contents of flipflops with every shift.

At the end of last shift, the register content $(s_2\ s_1\ s_0) = (000)$

$\therefore$ There is no error in received code word.

Shift	Received vector i.e bits of Y	Contents of FF in shift register		
		$S_0 = Y+S_2$	$S_1 = S_0$	$S_2 = S_1 \oplus S_2$
-	-	0	0	0
1	1	$1+0=1$	0	$0 \oplus 0 = 0$
2	0	$0+0=0$	1	$0 \oplus 0 = 0$
3	0	$0+0=0$	0	$1 \oplus 0 = 1$
4	1	$1 \oplus 1 = 0$	0	$0 \oplus 1 = 1$
5	0	$0 \oplus 1 = 1$	0	$0 \oplus 1 = 1$
6	1	$1 \oplus 1 = 0$	1	$0 \oplus 1 = 1$
7	1	$1 \oplus 1 = 0$	0	$1 \oplus 1 = 0$

Q.14 Consider (7, 4) cyclic code with g(x) = $x^3 + x + 1$

i) Draw the hardware arrangement of cyclic encoder.

ii) Draw the hardware arrangement of syndrome calculator, calculate syndrome for R = 1001101.

☞ [SPPU : May-14, Marks 8; April-17, Marks 4; May-17, Marks 7]

Ans. : i) Hardware arrangement of cyclic encoder :

The generator polynomial is,

$$G(x) = x^3 + 0x^2 + x + 1$$

and $$G(x) = x^3 + g_2 x^2 + g_1 x + 1$$

On comparison of the two equation we obtain,

$$g_1 = 1 \quad \text{and} \quad g_2 = 0$$

and $$q = n - k = 7 - 4 = 3$$

With these values the block diagram is shown in Fig. Q.14.1.

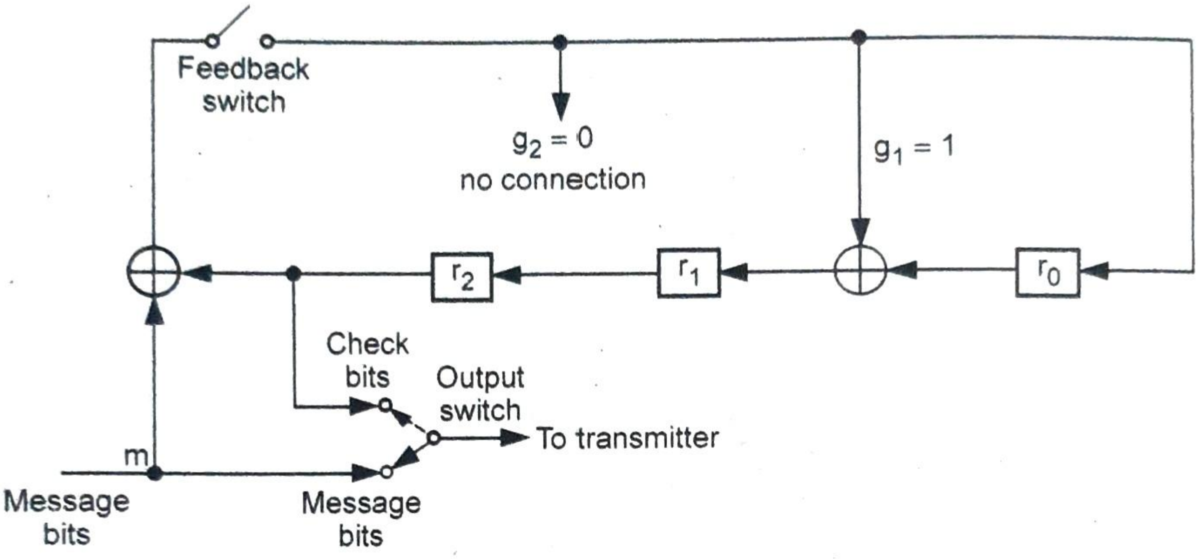


Fig. Q.14.1 Encoder for (7, 4) cyclic code for $G(x) = x^3 + x + 1$

ii) Syndrome calculator :

With these values the block diagram of a syndrome calculator for (7, 4) cyclic code will be as shown in Fig. Q.14.2.

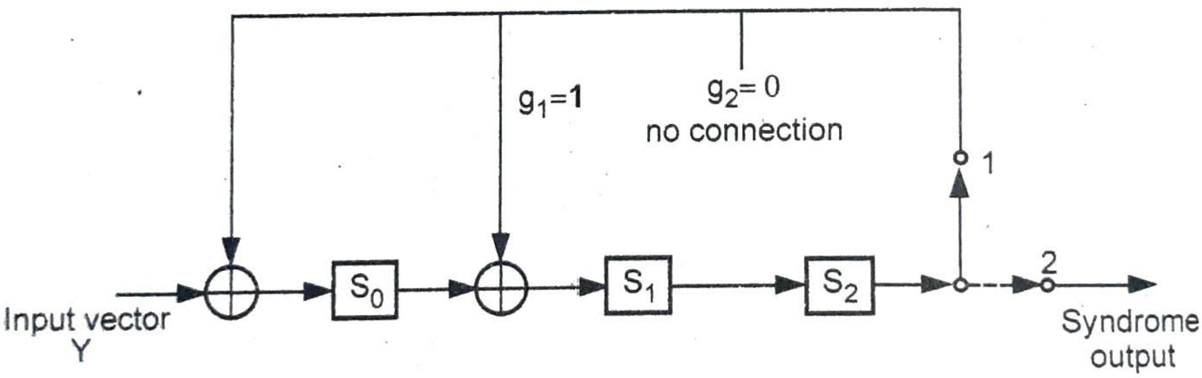


Fig. Q.14.2 Block diagram of a syndrome calculator for (7, 4) cyclic code with $G(x) = x^3 + x + 1$

Operation

The following table illustrates the operation of this syndrome calculator for received vector $R = (1\ 0\ 0\ 1\ 1\ 0\ 1)$. The table shows the contents of flip-flops with every shift.

The table shows that at the end of last shift the register contents are $(s_0\ s_1\ s_2) = (110)$.

Hence the calculated syndrome is,

$$S = (s_2\ s_1\ s_0) = (011)$$

Shift	Received vector i.e. bits of Y	Contents of flip flops in shift register $s_0 = y \oplus s_2$	$s_1 = s_0 \oplus s_2$	$s_2 = s_1$
–	–	0	0	0
1	1	$1 \oplus 0 = 1$	0	0
2	0	0	$1 \oplus 0 = 1$	0
3	0	0	0	1
4	1	0	1	0
5	1	1	0	1
6	0	1	0	0
7	1 Syndrome	1	1	0

Table Q.14.1 Calculation of syndrome for R = (1001101)

6.4 : Introduction to Convolutional Codes

Important Points to Remember

- Code rate is the ratio of a number of message inputs to number of output bits transmitted for each message bit.
- Constraint length is the number of shifts over which a single message bit influences encoder output.
- Code tree is the node diagram representation of convolutional encoder.
- Code trellis and state diagram are compact representation of code tree.

Q.15 Explain with suitable example generator polynomial description of convolutional codes. **[SPPU : May-15, Dec.-15, Marks 8]**

Ans. : Polynomial description of convolutional codes :

Let the impulse responses be represented by polynomials. i.e.,

$$g^{(1)}(x) = g_0^{(1)} + g_1^{(1)}x + g_2^{(1)}x^2 + \ldots\ldots + g_M^{(1)}x^M \quad \ldots \text{(Q.15.1)}$$

Similarly,

$$g^{(2)}(x) = g_0^{(2)} + g_1^{(2)}x + g_2^{(2)}x^2 + \ldots\ldots + g_M^{(2)}x^M \quad \ldots \text{(Q.15.2)}$$

Thus the polynomials can be written for other generating sequences. The variable 'x' is unit delay operator in above equations. It represents the time delay of the bits in impulse response.

Similarly we can write the polynomial for message polynomial i.e.,

$$m(x) = m_0 + m_1 x + m_2 x^2 + \ldots\ldots + m_{L-1} x^{L-1} \quad \ldots (Q.15.3)$$

Here L is the length of the message sequence. The convolution sums are converted to polynomial multiplications in the transform domain. i.e.,

$$\boxed{\left.\begin{aligned} v^{(1)}(x) &= g^{(1)}(x) \cdot m(x) \\ v^{(2)}(x) &= g^{(2)}(x) \cdot m(x) \end{aligned}\right\}} \quad \ldots (Q.15.4)$$

The above equations are the output polynomials of sequences $v_i^{(1)}$ and $v_i^{(2)}$.

All additions in above equations are as per mod-2 addition rules.

Consider a convolution encoder as shown as below in Fig. (Q.15.1).

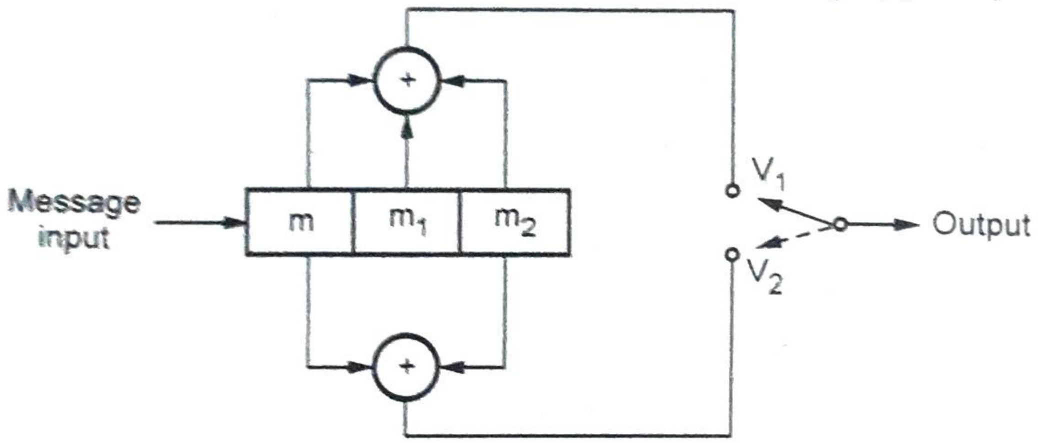


Fig. Q.15.1 : Convolutional encoder

Let us find the output sequence for message sequence of $m = \{1011\}$

a) To obtain generating polynomial for adder-1 :

The first generating sequence is given by

$$g_i^{(1)} = \{1\ 1\ 1\}$$

Hence its polynomial can be obtained (equation Q.15.1) as follows :

$$g^{(1)}(x) = 1 + 1 \times x + 1 \times x^2 = 1 + x + x^2 \quad \ldots (Q.15.5)$$

b) To obtain generating polynomial for adder-2

The second generating sequence is given by

$$g_i^{(2)} = \{1\ 0\ 1\}$$

Hence its polynomial can be obtained as follows :

$$g^{(2)}(x) = 1 + 0 \times x + 1 \times x^2 = 1 + x^2 \quad \ldots (Q.15.6)$$

c) To obtain message polynomial

The message sequence is,

$$m = (1\ 0\ 0\ 1\ 1)$$

Hence its polynomial can be obtained as,

$$m(x) = 1+0\times x+0\times x^2+1\times x^3+1\times x^4 = 1+x^3+x^4 \quad \dots (Q.15.7)$$

d) To determine the output due to adder-1

Now $v^{(1)}(x)$ can be obtained as

$$v^{(1)}(x) = g^{(1)}(x)\cdot m(x) = (1+x+x^2)(1+x^3+x^4)$$

$$= 1+x+x^2+x^3+x^6$$

The above polynomial can also be written as,

$$v^{(1)}(x) = 1+(1\times x)+(1\times x^2)+(1\times x^3)+(0\times x^4)+(0\times x^5)+(1\times x^6)$$

Thus the output sequence $v_i^{(1)}$ is, $v_i^{(1)} = \{1\ 1\ 1\ 1\ 0\ 0\ 1\}$

e) To determine the output due to adder-2

Similarly polynomial $v^{(2)}(x)$ can be obtained as,

$$v^{(2)}(x) = g^{(2)}(x)\cdot m(x) = (1+x^2)(1+x^3+x^4) = 1+x^2+x^3+x^4+x^5+x^6$$

Thus the output sequence $v_i^{(2)}$ is,

$$v_i^{(2)} = \{1\ 0\ 1\ 1\ 1\ 1\ 1\}$$

f) To determine the multiplexed output sequence

The multiplexed output sequence will be as follows :

$$\{v_i\} = \{1\ 1,\ 1\ 0,\ 1\ 1,\ 1\ 1,\ 0\ 1,\ 0\ 1,\ 1\ 1\}$$

Here note that very few calculations are involved in transform domain.

Q.16 Draw state diagram for the following convolutional encoder.

[SPPU : May-15, Marks 6]

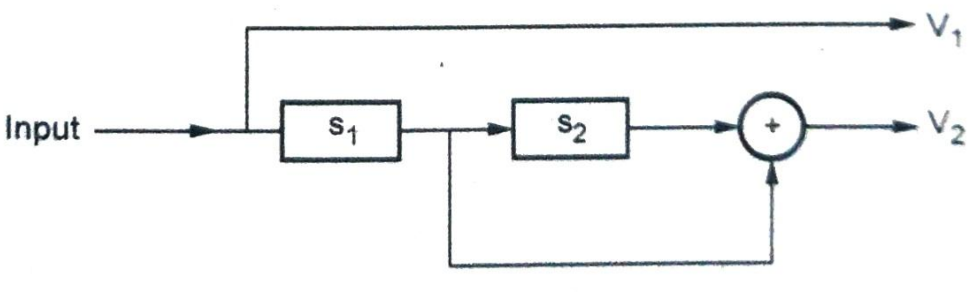


Fig. Q.16.1

Ans. : No. of input message bits = 1, $\therefore k = 1$

For one input bit, we get two output bits, $\therefore n = 2$

$\therefore$ Code rate, $\therefore R = \frac{k}{n} = \frac{1}{2}$

Now, in Fig. Q.16.1 encoder diagram, here, a single message bit influences ouput for three successive shifts.

At the fourth shift, the message bit is lost and it has no effect on the output., $\therefore K = 3$

The encoder can be redrawn as shown in Fig. Q.16.2 (a).

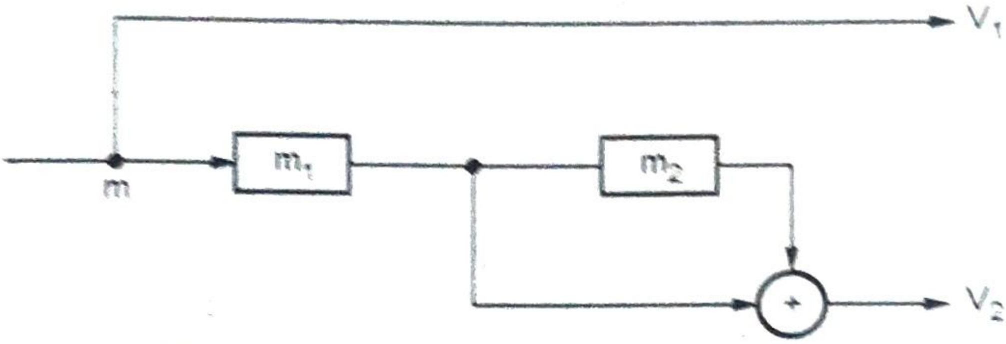


Fig. Q.16.2 (a) : Convolutional encoder

Logic table :

Sr. No.	Current state		input	Outputs		Next state	
	m_2	m_1	m	$v_1 = m$	$v_2 = m_1 \oplus m_2$	m_1	m
1.	a = 0	0	0	0	0	0	0 i.e. a
			1	1	0	0	1 i.e. b
2.	b = 0	1	0	0	1	1	0 i.e. c
			1	1	1	1	1 i.e. d
3.	c = 1	0	0	0	1	0	0 i.e. a
			1	1	1	0	1 i.e. b
4.	d = 1	1	0	0	0	1	0 i.e. c
			1	1	0	1	1 i.e. d

Table Q.16.1 : Logic table of encoder

State diagram :

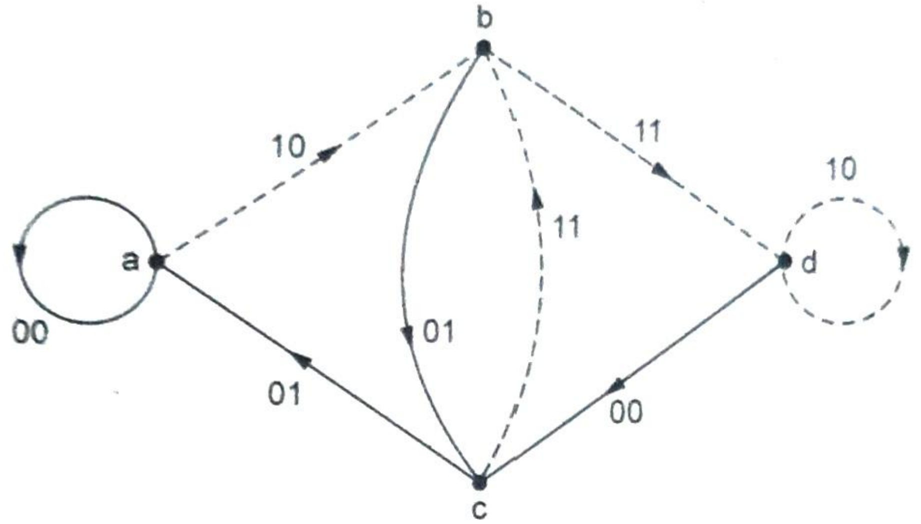


Fig. Q.16.3 : State diagram

Q.17 For the following convolutional encoder, find the coded output if input message is 10110000. ☞ [May-15, Marks 8]

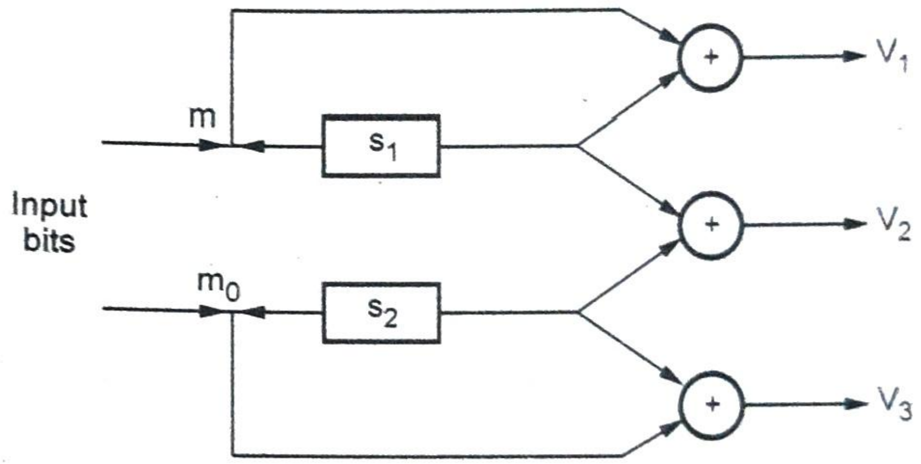


Fig. Q.17.1

Ans. : The convolutional encoder has two parallel inputs. $\therefore$ $R = 2$

It has three outputs $\therefore$ $n = 3$

$\therefore$ R(code rate) $= \frac{k}{n}$, $\therefore$ $R = \frac{2}{3}$

Now, there are two shift registers having constraint length of two each.

$\therefore$ $K = 2 + 2 = 4$

$\therefore$ The specification will be, $\left(\frac{2}{3}, 4\right)$

2) Logic table :

The encoder can be in any one of four states 00, 01, 10 and 11. From each state, there will be four outgoing paths.

The logic table is as shown in Table Q.17.1.

From encoder diagram,

$v_1 = m \oplus m_1, \quad v_2 = m_1 \oplus m_2, \quad v_3 = m_0 \oplus m_2$

3) Logic table :

Sr. No.	Current state m_2	m_1	Input m m_0	Output $v_1 = m_1$ $v_2 = m_1 \oplus m_2$ $v_3 = m_0 \oplus m_2$	Next state m m_o
1.	a = 0	0	0 0	0 0 0	0 0 i.e. a
		0	0 1	0 0 1	0 1 i.e. b
		0	1 0	1 0 0	1 0 i.e. c
		0	1 1	1 0 1	1 1 i.e. d
2.	b = 0	1	0 0	1 1 0	0 0 i.e. a
		1	0 1	1 1 1	0 1 i.e. b
		1	1 0	0 1 0	1 0 i.e. c
		1	1 1	0 1 1	1 1 i.e. d
3.	c = 1	0	0 0	0 1 1	0 0 i.e. a
		0	0 1	0 1 0	0 1 i.e. b
		0	1 0	1 1 1	1 0 i.e. c
		0	1 1	1 1 0	1 1 i.e. d
4.	d = 1	1	0 0	1 0 1	0 0 i.e. a
		1	0 1	1 0 0	0 1 i.e. b
		1	1 0	0 0 1	1 0 i.e. c
		1	1 1	0 0 0	1 1 i.e. d

Table Q.17.1 Logic table

4) The state diagram : It is a shown below :

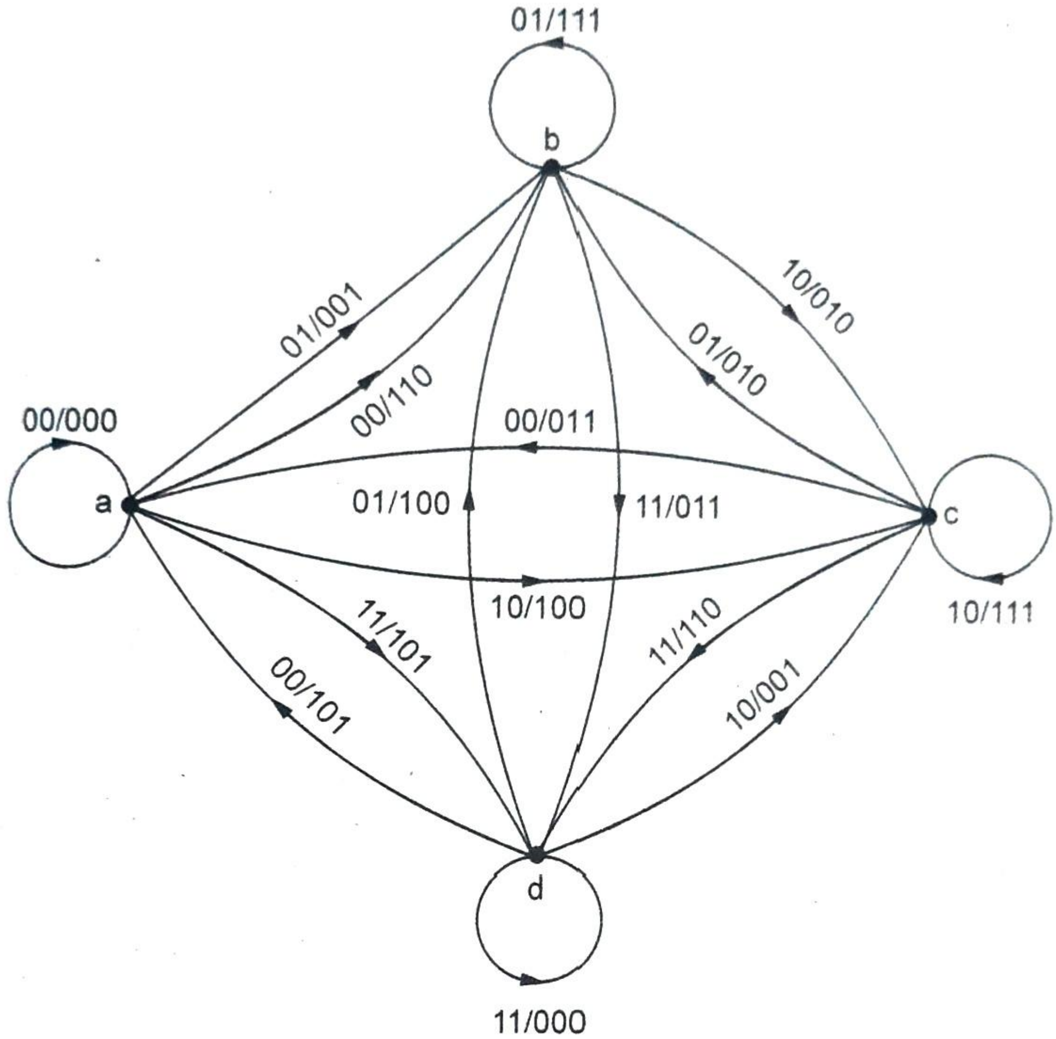


Fig. Q.17.2 State diagram

Note : 00/000 online Here 00 is input and 000 is output.

5) Encoding of sequence : Input ⇒ 1 0 1 1 0 0 0 0

Input	Output
1 0	1 0 0
1 1	1 1 0
0 0	1 0 1
0 0	0 0 0

∴ Output is 100 110 101 000

Q.18 Draw the trellis diagram for following encoder,

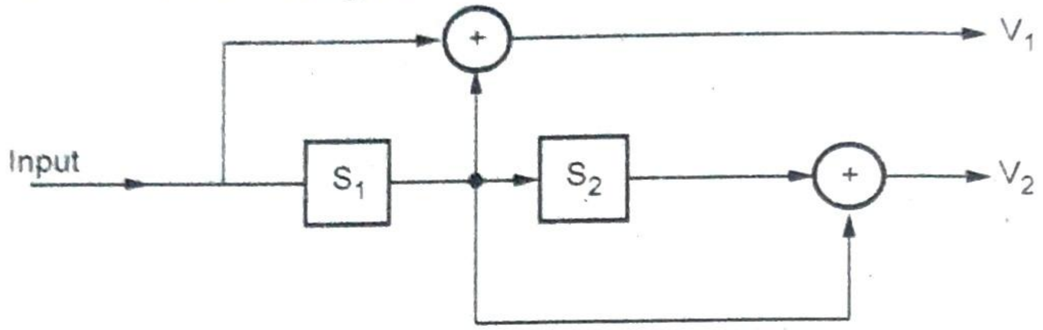


Fig. Q.18.1

[Dec.-15, Marks 8]

Ans. : Step 1 : Determine specifications :

Number of input message bits = 1, $\therefore K = 1$

For one input bit, there are two output bits, $\therefore n = 2$

$\therefore$ Code rate, $R = \frac{k}{n} = \frac{1}{2}$

Here output is influenced by three successive bits, $\therefore K = 3$

The encoder can be redrawn as in Fig. Q.18.2(a).

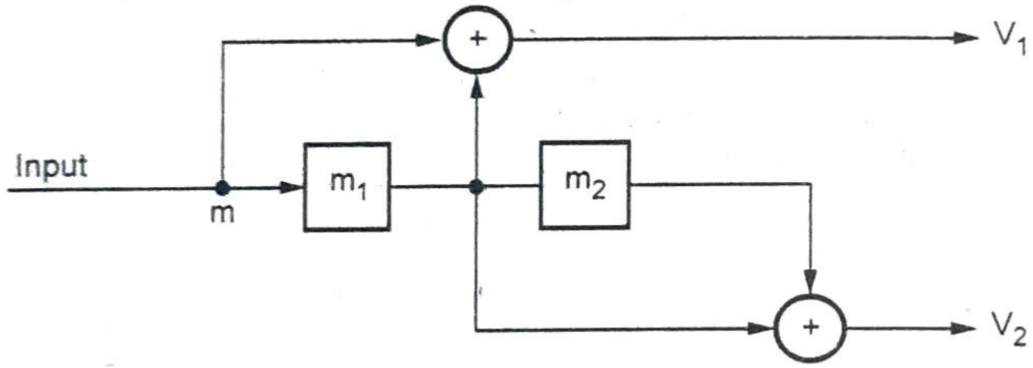


Fig. Q.18.2(a) Convolution encoder drawn alternately

Step 2 : Logic table :

Sr. No.	Current State m_2 m_1	Input m	Output $V_1 = m \oplus m_1$ $V_2 = m_1 \oplus m_2$	Next State m_1 m
1	a = 0 0	0	0 0	0 0 *i.e. a*
		1	1 0	0 1 *i.e. b*
2	$b = 0$ 1	0	1 1	1 0 *i.e. c*
		1	0 1	1 1 *i.e. d*

3	$c = 1\ 0$	0	0 1	0 0 *i.e. a*
		1	1 1	0 1 *i.e. b*
4	$d = 1\ 1$	0	1 0	1 0 *i.e. c*
		1	0 0	1 1 *i.e. d*

Table Q.18.1 Logic table of encoder in Fig. Q.18.2

Step 3 : To obtain Trellis diagram :

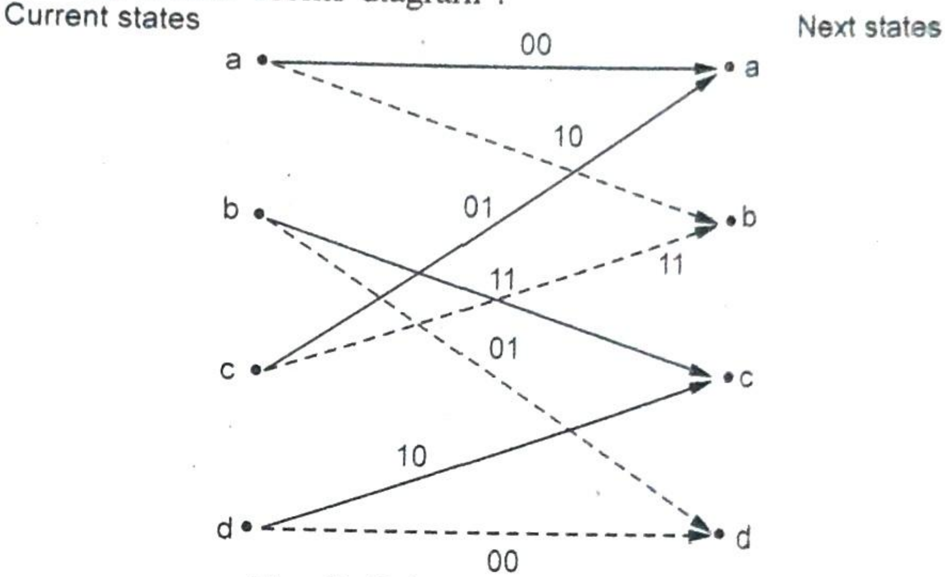


Fig. Q.18.3 : Trellis diagram

Q.19 A convolutional encoder shown in Fig. Q.19.1 Sketch the state diagram and Trellis diagram. Find out output data sequence 10011

☞ [SPPU : May-13, Marks 8, Dec.-17, Marks 10]

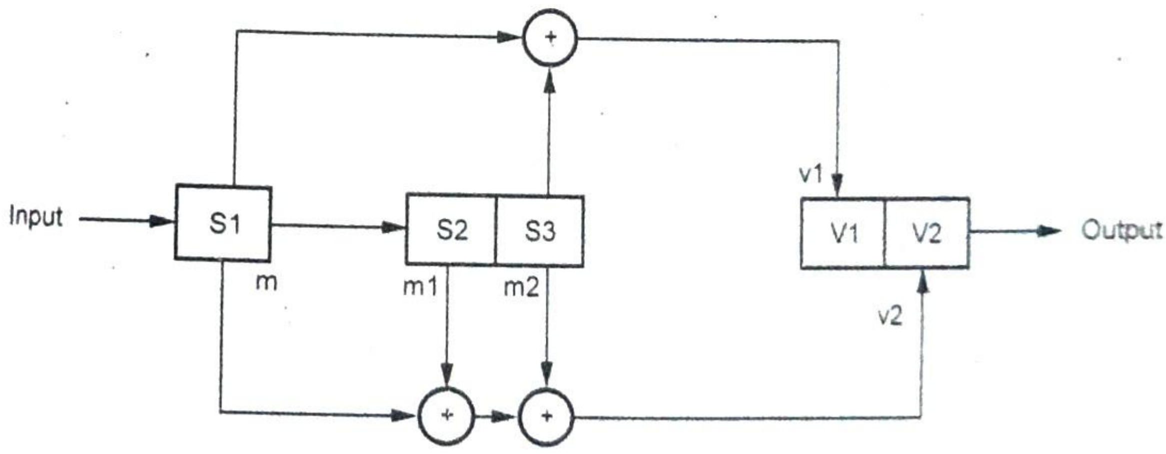


Fig. Q.19.1 : Convolution encoder

Ans. : i) To determine dimension of the code :

For every message bit $(k=1)$, two output bits $(n=2)$ are generated.

$\therefore$ Code rate (R) $= \frac{k}{n} = \frac{1}{2}$

Since there are three stages, output will be affected by the three successive bits.

$\therefore$ $$\boxed{K = 3}$$

$k = 1, n = 2$ and $k = 3$

Here $v_1 = m \oplus m_2$ and $v_2 = m \oplus m_1 \oplus m_2$

ii) Logic table : Ref. Q.1

iii) State diagram : Ref. Q.1

iv) Trellis diagram : Fig. Q.19.2 shows the trellis diagram.

v) Encoding of sequence (10011) : Same as Q.1.

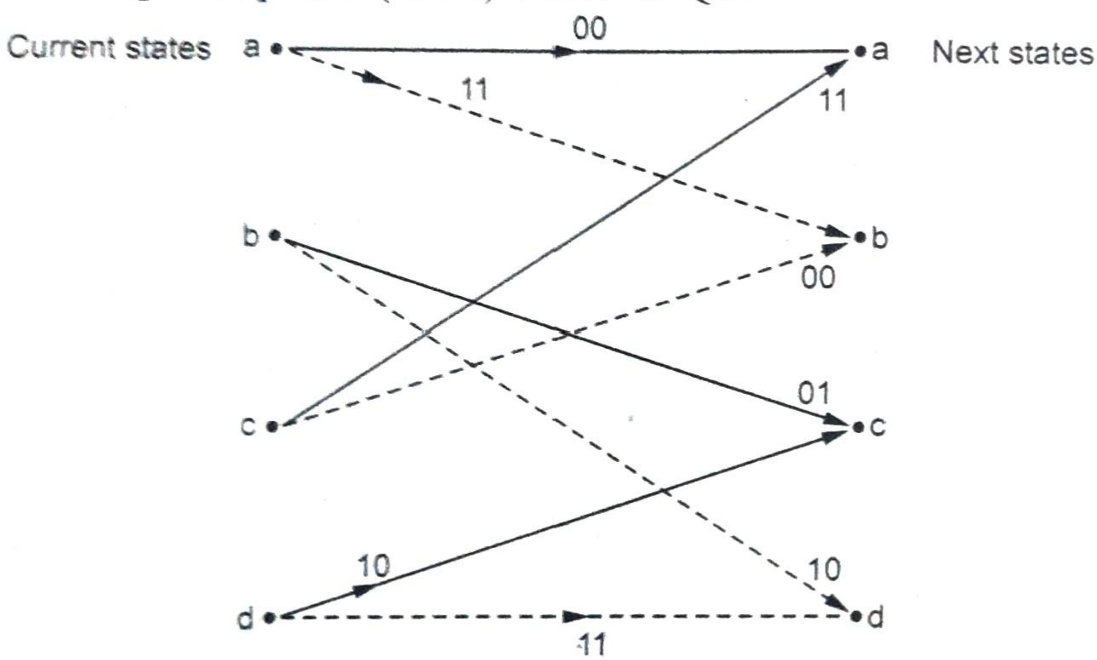


Fig. Q.19.2 Trellis diagram

Q.20 For the 1 3 convolution encoder has generating vectors as $g_1 = (111)$, $g_2 = (100)$ and $g_3 = (101)$. Sketch the encoder, state diagram and Trellis diagram code tree. Find output data sequence for the input data sequence 0101. **[May-12, Marks 10]**

Ans. : i) To determine dimensions of the code

This is rate 1/3, rate $(R) = \frac{k}{n} = \frac{1}{3}$ $\therefore$ $k = 1$ and $n = 3$

ii) To sketch encoder configuration :

Here, $k = 1$ and $n = 3$. This means each message bit generates three output bits. There will be three stage register. It will contain m, m_1, m_2

first output v_1. Since $g_1 = (111)$, $v_1 = m \oplus m_1 \oplus m_2$

second output v_2. Since $g_2 = (100)$, $v_2 = m$

third output v_3. Since $g_3 = (101)$, $v_3 = m \oplus m_2$

Fig. Q.20.1(a) shows the encoder based on above discussion.

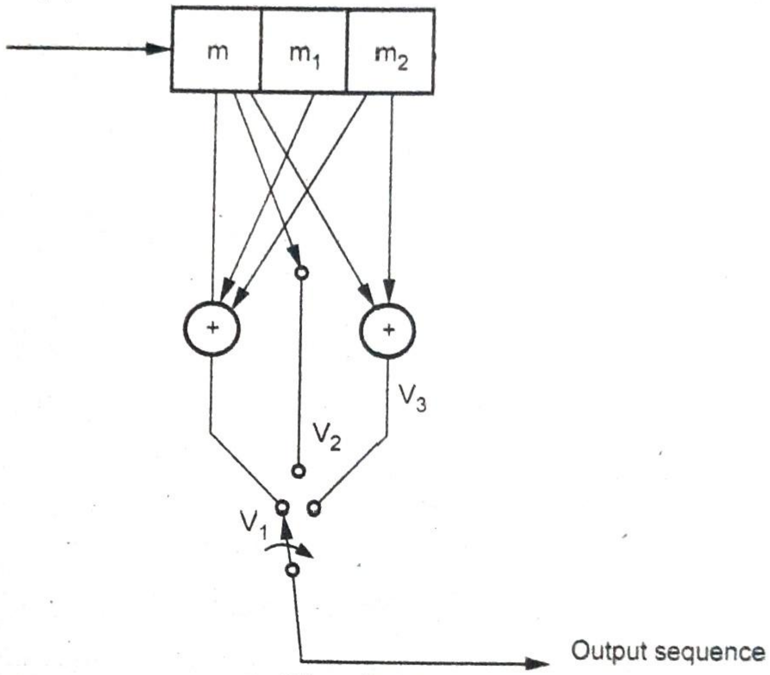


Fig. Q.20.1(a)

iii) To obtain logic table :

Sr. No.	Current State m_2 m_1	Input m	Output $V_1 = m \oplus m_1 \oplus m_2$ $V_2 = m$ $V_3 = m \oplus m_2$	Next State m_1 m
1	$a = 0$ 0	0	0 0 0	00, i.e. a
		1	1 1 1	01 i.e. b
2	$b = 0$ 1	0	1 0 0	10, i.e. c
		1	0 1 1	11 i.e. d
3	$c = 1$ 0	0	1 0 1	00, i.e. a
		1	0 1 0	01 i.e. b

4	$d = 1\ 1$	0	0 0 1	10, *i.e.* c
		1	1 1 0	11 *i.e.* d

Table Q.20.1 : Logic Table for Q.20.

iv) **To obtain state diagram :** Based on logic Table Q.20.1, state diagram is as shown in Fig. Q.20.2.

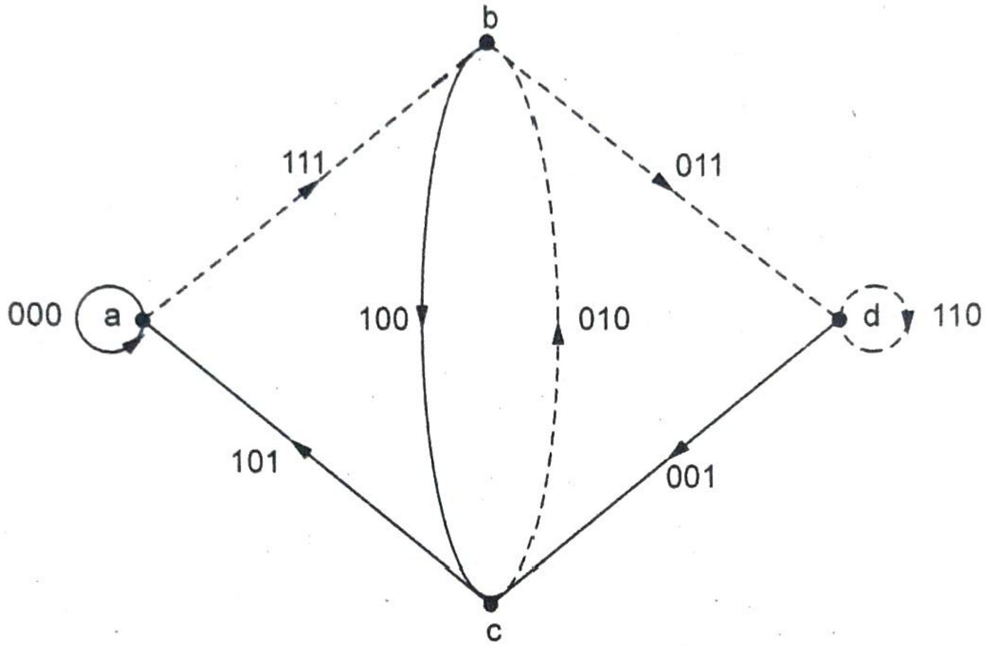


Fig. Q.20.2 : State diagram

v) **To obtain Trellis diagram :** Fig. Q.20.3 shows Trellis diagram.

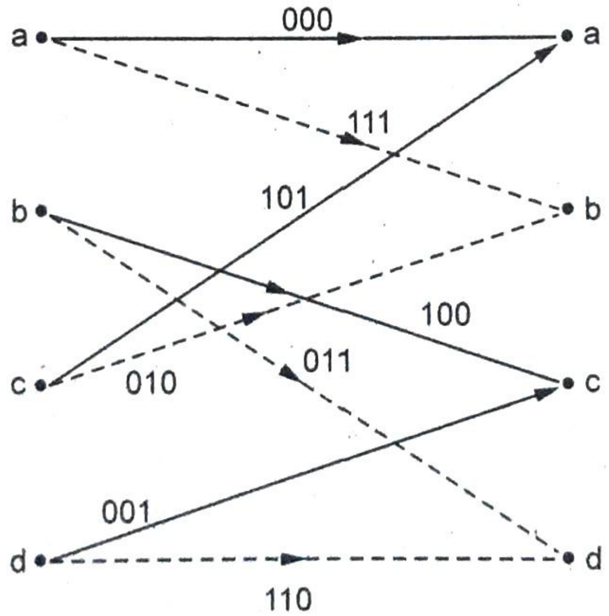


Fig. Q.20.3 Trellis diagram

vi) To obtain code tree and output for 0101 :

Assumption (i) Upward movement in code tree $m = 0$

(ii) Downward movement indicate $m = 1$

Based on above procedure, Fig. Q.20.4 is obtained

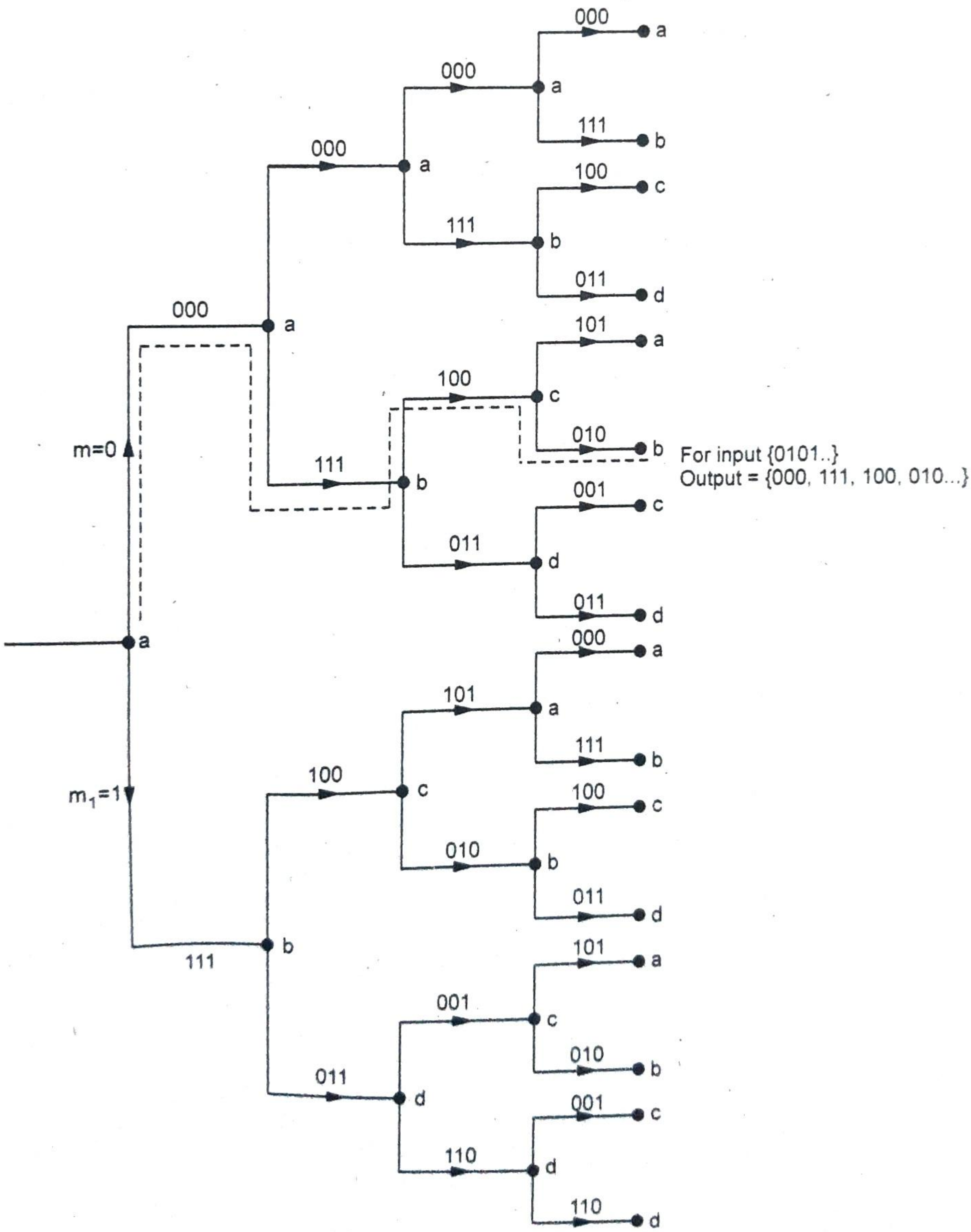


Fig. Q.20.4 : Code tree

Q.21 Compare state, tree and trellis representation of convolution codes with examples. **[SPPU : Dec.-14, Marks 9]**

Ans. : Comparison :

Sr. No.	Code tree	Trellis diagram	State diagram
1.	It indicates flow of encoded signal along nodes of the tree.	Trellis indicates transitions from current to next state.	It indicates transitions from current to next state.
2.	It is detailed way of representing coding process.	It is shorter or compact way of representing coding process.	It is shorter or compact way of representing coding process.
3.	Decoding is very simple using code tree.	Decoding is little complex using trellis diagram.	Decoding is complex using state diagram.
4.	Code tree repeats after number of stages used in encoder.	It repeats in every state. In steady state, it has only one state.	No repetition concept.
5.	It is complex to implement in programming.	It is simpler to implement in programming.	It is simplex to implement in programms.
6.	Refer Fig. Q.20.4	Fig. Q.20.3	Fig. Q.20.2

6.5 : Sequential and Viterbi Decoding

Important Points to Remember

- Viterbi decoding consists of succession of code trellis for each output symbol.
- Viterbi decoding evaluates the decoding path having lowest metric.
- Sequential decoding selects the path that have running metric within given threshold.

Q.22 Explain Viterbi's algorithm for decoding of convolutional codes.

☞ [SPPU : May-11, Marks 6, Dec.-15, Marks 8, May-17, Marks 3]

Ans. : Viterbi algorithm for decoding of convolutional codes (Maximum Likelihood Decoding)

Metric : It is the discrepancy between the received signal Y and the decoded signal at particular node. This metric can be added over few nodes for a particular path.

Surviving Path : This is the path of the decoded signal with minimum metric. In viterbi decoding a metric is assigned to each surviving path. Y is decoded as the surviving path with smallest metric.

Consider the following example of viterbi decoding. Let the first six received bits be

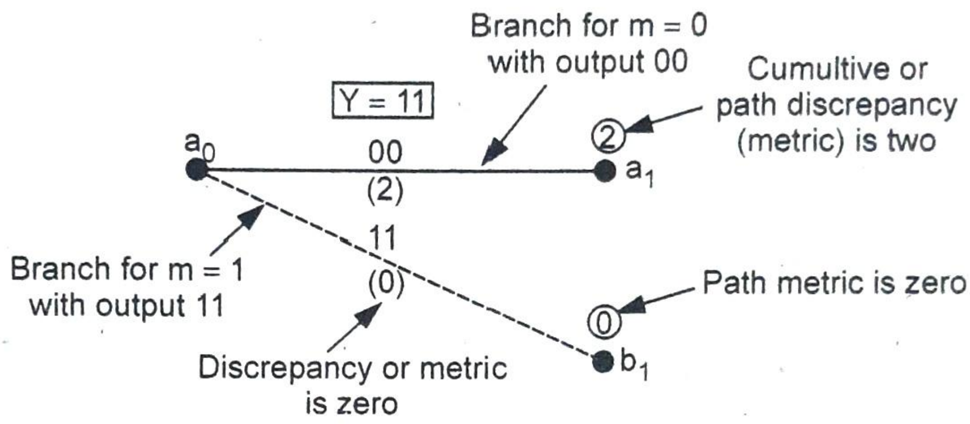


Fig. Q.22.1 Viterbi decoder results for first message bit

$$Y = 11 \quad 01 \quad 11$$

a) Decoding of first message bit for $Y = 11$

For single bit input the encoder transmits two bits $(v_1 v_2)$ outputs. These outputs are received at the decoder and represented by Y. As shown in Fig. Q.22.1. Two branches are shown from a_0. One branch is at next node a_1 representing decoded signal as 00 and other branch is at b_1 representing decoded signal as 11.

The branch from $a_0 b_1$ to represents decoded output as 11 which is same as received signal at that node i.e. 11. Thus there is no discrepancy between received signal and decoded signal. Hence 'Metric' of that branch is zero. This metric is shown in brackets along that branch. The metric of branch from a_0 to a_1 is two. The encoded number near a node shows path metric reaching to the node.

b) Decoding of second message bit Y = 01.

When the next part of bits $Y = 01$ is received at nodes a_1 and b_1, then from nodes a_1 and b_1 four possible next states a_2, b_2, c_2 and d_2 are possible. Fig. Q.22.2 shows all these branches, their decoded outputs and branch metrics corresponding to those decoded outputs. The encircled number near a_2, b_2, c_2 and d_2 indicate path metric emerging from a_0. For example the path metric of path a_0, a_1, a_2 is 'three'. The path metric of path $a_0 b_1 d_2$ is zero.

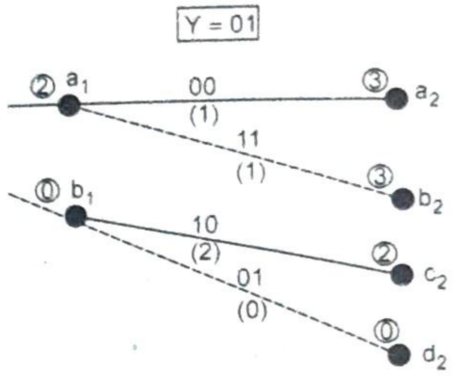


Fig. Q.22.2 : Viterbi decoder results for second message bit

c) Decoding of 3^{rd} message bit for Y = 11

Fig. Q.22.3 shows the trellis diagram for all the six bits of *Y*.

Fig. Q.22.3 shows the nodes with their path metrics on the right hand side at the end of sixth bit of *Y*. Thus two paths are common to node '*a*'. One path is $a_0 a_1 a_2 a_3$ with metric 5. The other path is $a_0 b_1 c_2 a_3$ with metric 2. Similarly there are two paths at other nodes also. According to viterbi decoding, only one path with lower metric should be retained at particular node. As shown in Fig. Q.22.3, the paths marked with × (cross) are cancelled because they have higher metrics than other path coming to that particular node. These four paths with lower metrics are stored in the decoder and the decoding continues to next received bits.

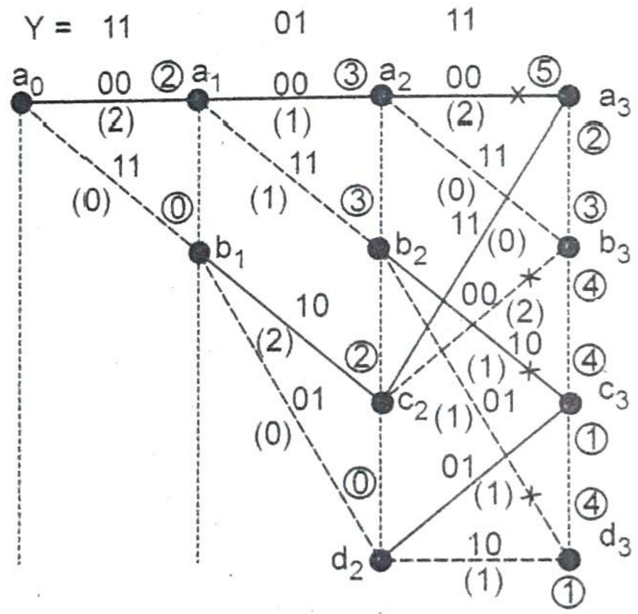


Fig. Q.22.3 Paths and their metrics for viterbi decoding

- Only one path of particular node is kept which is having lower metric. In case if there are two paths have same metric, then any one of them is continued.

- This is the complete explanation of viterbi decoding. The method of decoding used in viterbi decoding is called *maximum likelihood decoding.*

Q.23 What is sequential decoding ? Explain in brief.

[SPPU : May-15, Marks 6, May-17, Marks 3]

Ans. : Sequential decoding for convolutional codes

Sequential decoding uses metric divergence effect Fig. Q.23.1 (a) shows the code trellis for the convolutional encoder. Following are the important points about sequential decoding.

1) The decoding starts at a_0. It follows the single path by taking the branch with smallest metric. For example as shown in Fig. Q.23.1(a), the path for first three nodes is $a_0 b_1 d_2$ since its metric is the lowest.
2) If there are two or more branches from the same node with same metric, then decoder selects any one branch and continues decoding.
3) From (2) above if there are two branches from one nodes with equal metrics, then any one is selected at random. If the selected path is found to be unlikely with rapidly increasing merit, then decoder cancels that path and goes back to that node. It then selects other path emerging from that node. For example observe in the Fig. Q.23.1 (a) that two branches with same metric emerge from node d_2. One path is $d_2 d_3 c_4 a_5$ (or path marked 'B') with metric '3' at a_5. Therefore decoder drops this path and follows other path.
4) The decision about dropping a path is based on the expected value of running metric at a given node. Running metric at a particular j^{th} node is given as,

$$\text{Running metric} = jn\,\alpha \qquad \dots \text{(Q.23.1)}$$

where j is the node at which metric is to be calculated.

n is the number of encoded output bits for one message bit.

and α is the transmission error probability per bit.

The sequential decoder abandons a path whenever its running metric exceeds $(jn\alpha + \Delta)$. Here Δ is the should be above $jn\alpha$ at j^{th} node. Fig. Q.23.1 (b) shows the running metric at a particular node with respect to number of that node. The two dotted lines shows the range of threshold 'Δ' above $jn\alpha$ at a particular node. Observe that since metric of path 'B' exceeds the threshold at 5^{th} node, it is abandoned and decoder starts from node '2' again. Similarly path 'A' is also abandoned.

5) If the running metric of every path goes out of threshold limits then the value of threshold 'Δ' is increased and decoder tries back again. In Fig. Q.23.1(b) the value of $\alpha = 1/16$, for encoder of Fig. Q.23.1 we know that $n = 2$. Let's calculate $jn\,\alpha$ at 8^{th} node.

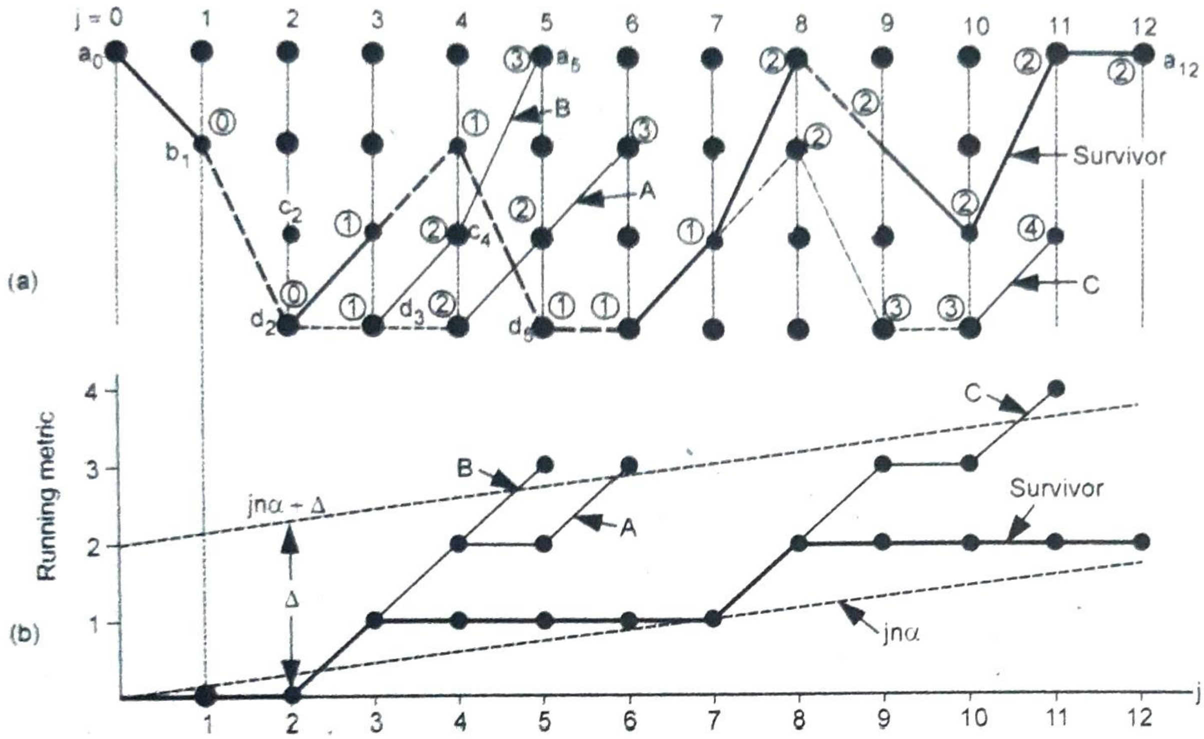


Fig. Q.23.1 Sequential decoding

At 8^{th} node $jn\,\alpha = 8 \times 2 \times 1/16 = 1$. The value of $\Delta = 2$. Therefore threshold will be,

Threshold $= jn\,\alpha + \Delta = 1 + 2 = 3$ at 8^{th} node. Similarly, the threshold at other nodes can be calculated.

Q.24 Diagram an convolution encoder of $O_1 = 110$ and $O_2 = 011$, draw state table and use viterbi Algorithm to decode the encoded sequence 10, 11, 11, 11, 01 **[Dec.-12, 14, Marks 10]**

Ans. : i) To obtain encoder diagram :

Rate $R = 1/2$, $\therefore$ $k = 1$ and $n = 2$

There will be three stages register. It will contain m, m_1 and m_2.

first output v_1, since $O_1 = 110$, $v_1 = m \oplus m_1$

second output v_2. since $O_2 = 011$, $v_2 = m_1 \oplus m_2$

Fig. Q.24.1(a) shows the encoder diagram based on above discussion.

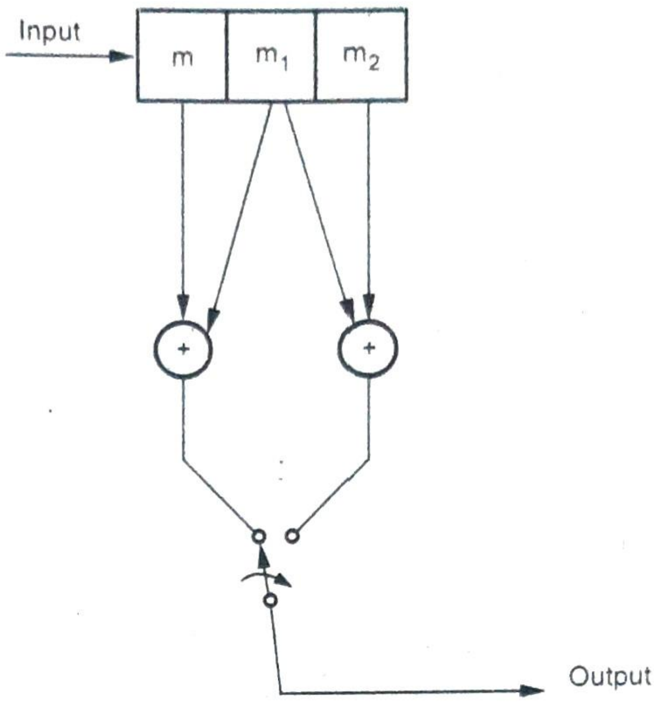


Fig. Q.24.1 : Encoder of Q.24

ii) State table :

Sr. No.	Current State m_2 m_1	Input m	Output $V_1 = m \oplus m_1$ $V_2 = m_1 \oplus m_2$	Next State m_1 m
1.	$a = 0\ 0$	0	0 0	00, *i.e.* *a*
		1	1 0	01 . *i.e.* *b*
2.	$b = 0\ 1$	0	1 1	10, *i.e.* *c*
		1	0 1	11 *i.e.* *d*
3.	$c = 1\ 0$	0	0 1	00, *i.e.* *a*
		1	1 1	01 *i.e.* *b*
4.	$d = 1\ 1$	0	1 0	10, *i.e.* *c*
		1	0 0	11 *i.e.* *d*

Table Q.24.1 Stable table of Q.24

iii) To obtain Decoded output :

Using viterbi algorithm, the decoded output is obtained as shown in Fig. Q.24.2.

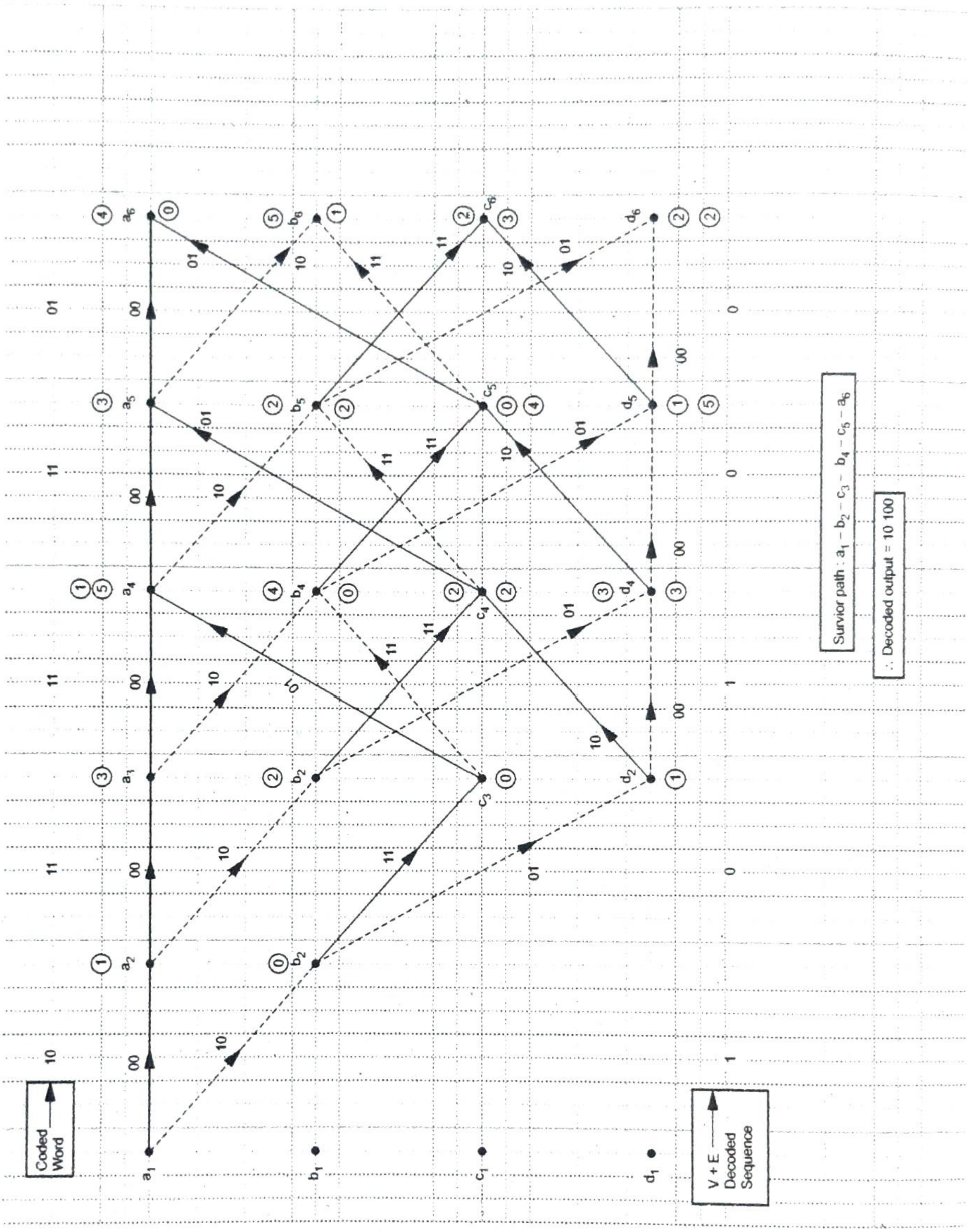

Coded Word
V + E Decoded Sequence
Survior path : a1 – b2 – c3 – b4 – c5 – a6
∴ Decoded output = 10 100

Fig. Q.24.2

Q.25 For the 1/3 convolution encoder has generating vectors as g_1 = (100), g_2 = (101) and (g_3) = (110). Sketch the encoder. Use viterbi algorithm to decode. **[SPPU : May-14, Marks 10]**

100 110 111 101 001 101 001 101 001 010

Ans. : i) To obtain encoder diagram

Rate $R = 1/3$, $\therefore$ $k = 1$ and $n = 3$

There will be three stage register. It will contain m, m_1 and m_2

first output v_1, since $g_1 = (100)$, $v_1 = m$

second output v_2, since $g_2 = (101)$, $v_2 = m \oplus m_2$

third output v_3, since $g_3 = (110)$, $v_3 = m \oplus m_1$

Fig. Q.25.1 shows encoder diagram based on above description.

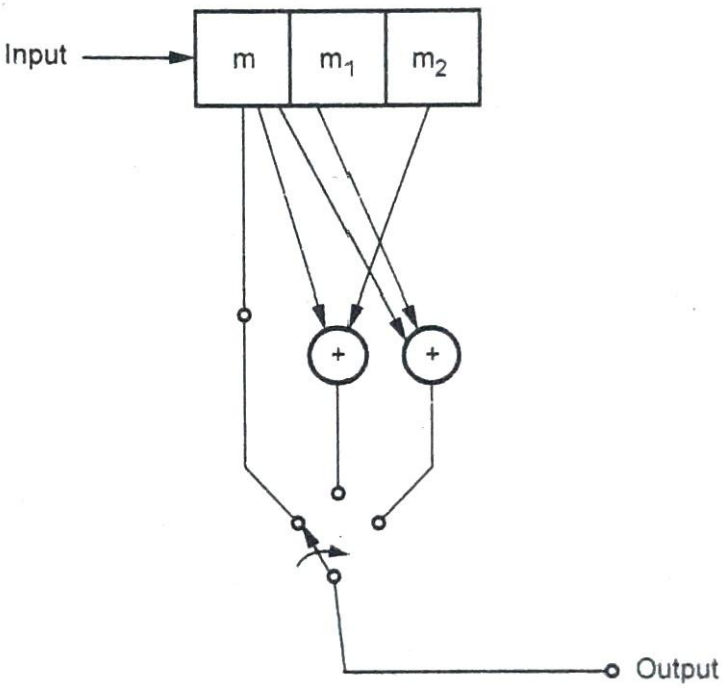


Fig. Q.25.1

ii) To obtain the state table :

Sr. No.	Current State m_2 m_1	Input m	Output $V_1 = m$, $V_2 = m \oplus m_2$, $V_3 = m \oplus m_1$	Next State m_1 m
1.	$a = 0$ 0	0	0 0 0	00, i.e. a
		1	1 1 1	01 i.e. b

2.	$b = 0\ 1$	0	0 0 1	10, i.e. c
		1	1 1 0	11 i.e. d
3.	$c = 1\ 0$	0	0 1 0	00, i.e. a
		1	1 0 1	01 i.e. b
4.	$d = 1\ 1$	0	0 1 1	10, i.e. c
		1	1 0 0	11 i.e. d

Table Q.25.1 Logic table

iii) To obtain decoded sequence using viterbi algorithm.

Viterbi algorithm to decode the given sequence

Fig. Q.25.2 shows the diagram based on viterbi decoding. It shows received sequence at the top. The decoded $(Y + E)$ sequence and decoded message sequence is shown at the bottom. (Refer Fig. Q.25.2 on next page)

The dark line shows maximum likelihood path. It has the lowest running metric, i.e. 3. Other path are also shown for reference. At any point only four paths are retained. The decoded message sequence is,

$$m = \{1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\}$$

Q.26 For the convolution encoder with constraint length and rate 1/2 as shown in Fig. Q.26.1. Draw the state diagram and trellis diagram. By using viterbi algorithm decode the sequence 010001000.

☞ **[SPPU : Dec.-13, Marks 10]**

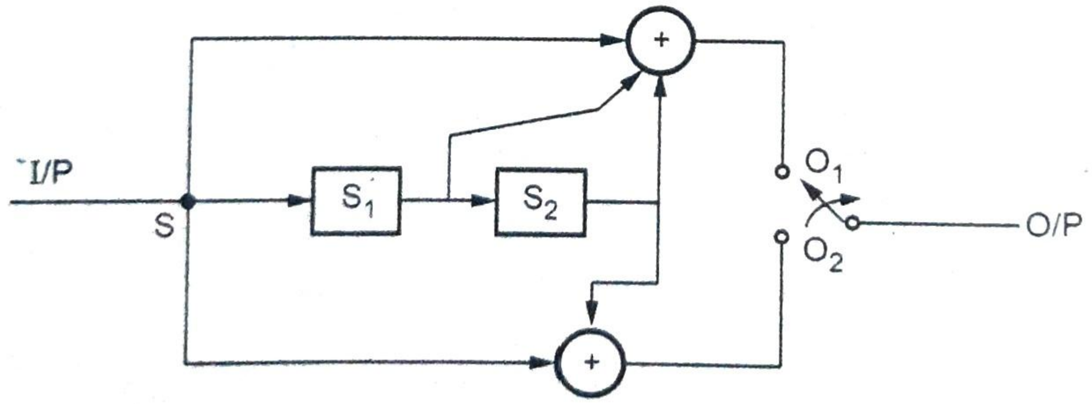


Fig. Q.26.1

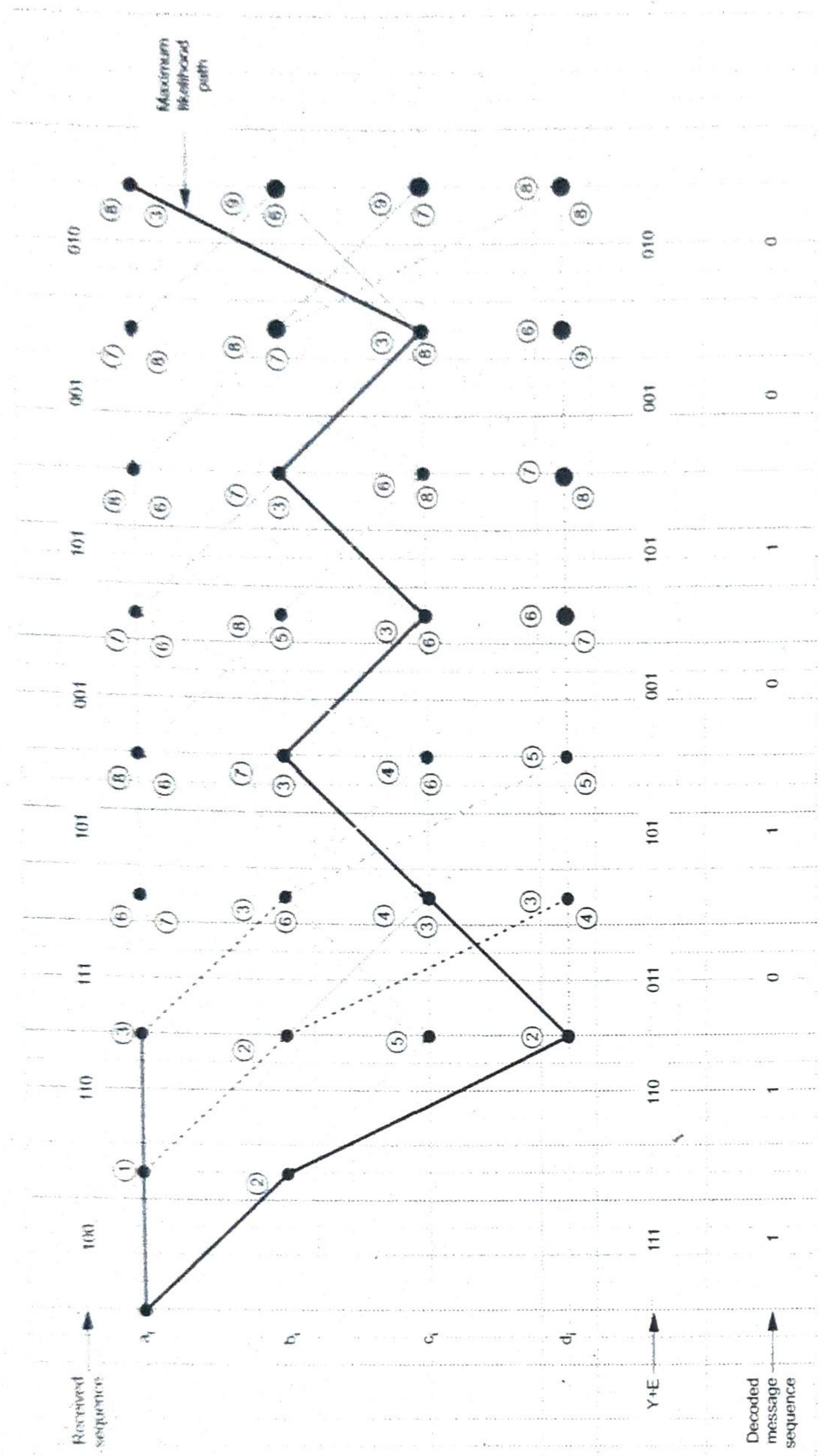


Fig. Q.25.2 Viterbi decoding for Q.25

Ans. : i) To determine dimensions of the code :

rate $= \frac{k}{n} = \frac{1}{2}$. Hence $k=1$ and $n=2$. For every message bit, there are two bits encoded at the output.

Constraint length $K=3$. Hence output is influenced by three shifts in the encoder.

State table :

Based on above table the trellis diagram and state diagrams are as shown below :

Sr. No.	Current state $s_2\ s_1$	Input s	Outputs $v_1 = s \oplus s_1 \oplus s_2$	$v_2 = s \oplus s_2$	Next state $s_1\ s$
1.	a = 0 0	0	0	0	0 0, i.e. a
		1	1	1	0 1, i.e. b
2.	b = 0 1	0	1	0	1 0, i.e. c
		1	0	1	1 1, i.e. d
3.	c = 1 0	0	1	1	0 0, i.e. a
		1	0	0	0 1, i.e. b
4.	d = 1 1	0	0	1	1 0, i.e. c
		1	1	0	1 1, i.e. d

Table Q.26.1 State transition table for encoder of Fig. Q.26.1

6.6 : Introduction to Turbo Codes

Important Points to Remember

- Turbo codes permutes the ordering of the input data sequence with the help of intervleaver. This operation provides high weight codewords.
- The decoder generates estimates of codewords in two stages of decoding and interleaving-deinter-leaving.

- This is like circulation of air in turbo engine for better performance. Hence these codes are called *Turbo codes*.

Q.27 With the help of block diagram explain turbo code encoder and decoder.

Ans. : Turbo Encoder

- Fig. Q.27.1 shows the block diagram of rate 1/3 turbo encoder. It is rate 1/3 encoder because it generates three output bits for one input bit. It generates x_k, y_k^1 and y_k^2 bits.

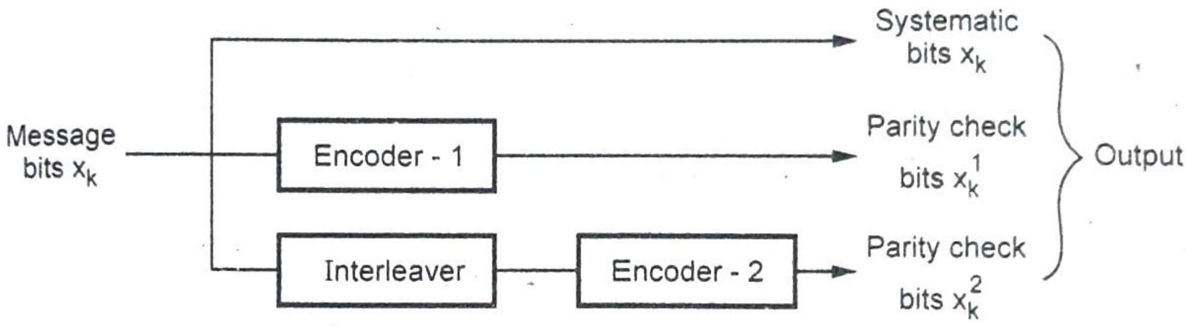


Fig. Q.27.1 Turbo encoder

- Encoder-1 and encoder-2 are convolutional encoders.
- The interleaver permutes input sequence x. Hence encoder-2 is high weight codeword.
- Like block codes, whole data block is present at the input before encoding begins. This is essential for interleaving operation.

Turbo Decoder

- Fig. Q.27.2 shows the block diagram of turbo decoder.

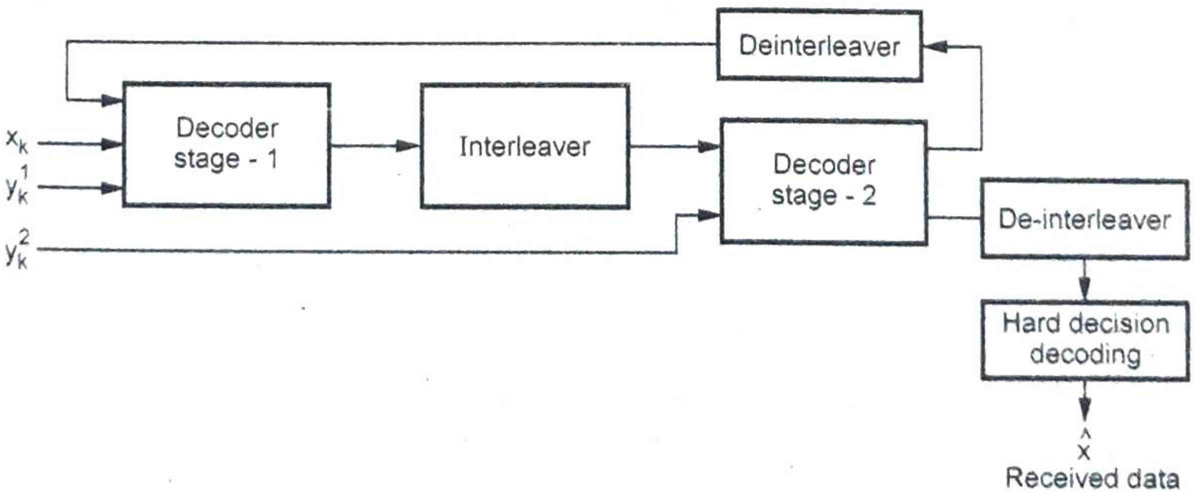


Fig. Q.27.2 Turbo decoder

- The systematic bits x_k and parity check bits y_k^1 are given to decoder stage - 1 . It generates the partial estimate of input data sequence.
- The output of decoder stage - 1 is passed through interleaver and given to decoder stage - 2. The decoder stage - 2 is also given parity check bits y_k^2. It generates the estimate.
- This second estimate is deinterleaved and looped back to first decoder.
- The decoder thus circulates the estimates of sent data till certain decoding criteria is met.
- The data is then decoded using hard decision decoding to get final estimate of transmitted sequence.

Q.28 State the advantages and disadvantages pf turbo codes.

Ans. : Advantages

1. Combination of block codes and convolutional codes.
2. High weight codes are produced without increasing data rate.
3. Iterative decoding is possible MAP algorithm.

Disadvantages

1. Encoding and decoding is complex due to interleaving.
2. Complete data sequence is required before encoding begins.

6.7 : Introduction to LDPC Codes

Important Points to Remember

- Turbo codes has achieved the performance close to theoretical bounds on the performance of error correction codes.
- The bit error rate of LDPC codes is almost same as that of turbo codes.
- The LDPC codes are known as Gallager codes after its invention.
- The LDPC codes are constructed by parity check matrix which has binary 1's at widely spaced intervals by 0's i.e. each row in H matrix has few 1's and many 0's.

Q.29 Explain the Generation of LDPC codes.

Ans. :

- Described by sparseness (low dense) parity check H matrix and graphs.
- Number of 1's in H grows linearly with block length.
- Number of edges in Tanner graph grows linearly with block length.
- Randomness of construction in placement of 1's in H matrix.
- There is a connectivity of variable and check nodes in tanner graphs.

The LDPC code can be denoted as $C\ (n,\ m,\ l)$

where n = Codeword length

m = Number of parity check equations involving each code bit

= (degree of each variable node in tanner graph)
(Number of 1's/column)

l = Number of code bits involved in each parity-check equation

= (degree of each check nodes in tanner graph)
number of 1's/row

Q.30 For a given specification for LDPC generate H matrix (n, m, l) (n, j, k) = (20, 3, 4).

Ans. : First $n/i = 5$ rows have $l = 4$ is each.

Then $(m - 1) = (3 - 1) = 2$ submatrices of size $n/l \times n = \frac{20}{4} \times 20 =$

$5 \times 20 = 100$ obtained by appling randomly chosen column permutation to first submatrix.

The result will be $m \times n/l \times n = 3 \times \frac{20}{4} \times 20$

$$= 3 \times 5 \times 20$$

$$= 15 \times 20$$

Parity check matrix for *a* $(n, m, l) = (20, 3, 4)$ LDPC code as shown below :

Can observe each column has only a single 1 and each row has equal number of 1's equal to 4 the binary 1 is sparse by 0's

S_1 S_2

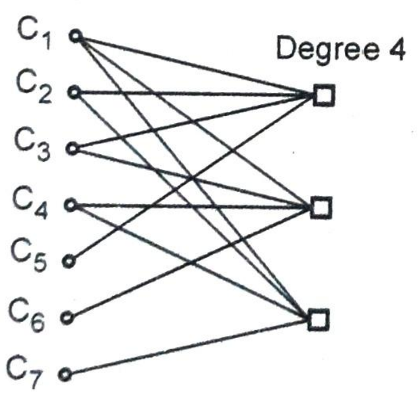
$$\left[\begin{array}{cccccccccccccccccccc}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
\hline
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
\hline
1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1
\end{array}\right]$$

Q.31 How to draw tanner graph ?

Ans. : Let's take an example of parity check matrix of Hamming code (7, 4) for understanding

$$\begin{array}{cc}
 & \begin{array}{ccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \end{array} \\
H = & \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}
\begin{array}{l}
C_1 + C_2 + C_3 + C_5 = 0 \\
C_1 + C_3 + C_4 + C_6 = 0 \\
C_1 + C_2 + C_4 + C_7 = 0
\end{array}$$

Bi - partite graph for above H matrix is



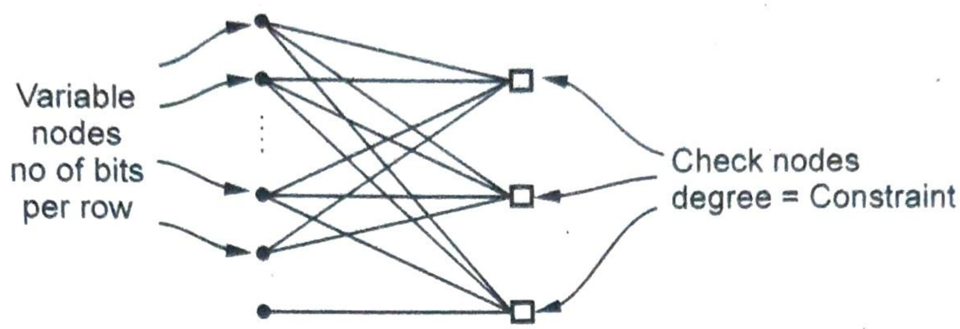
$C_1 + C_2 + C_3 + C_5 = 0$

$C_1 + C_3 + C_4 + C_6 = 0$

$C_1 + C_2 + C_4 + C_7 = 0$

On the similar line following terminology can be understood.

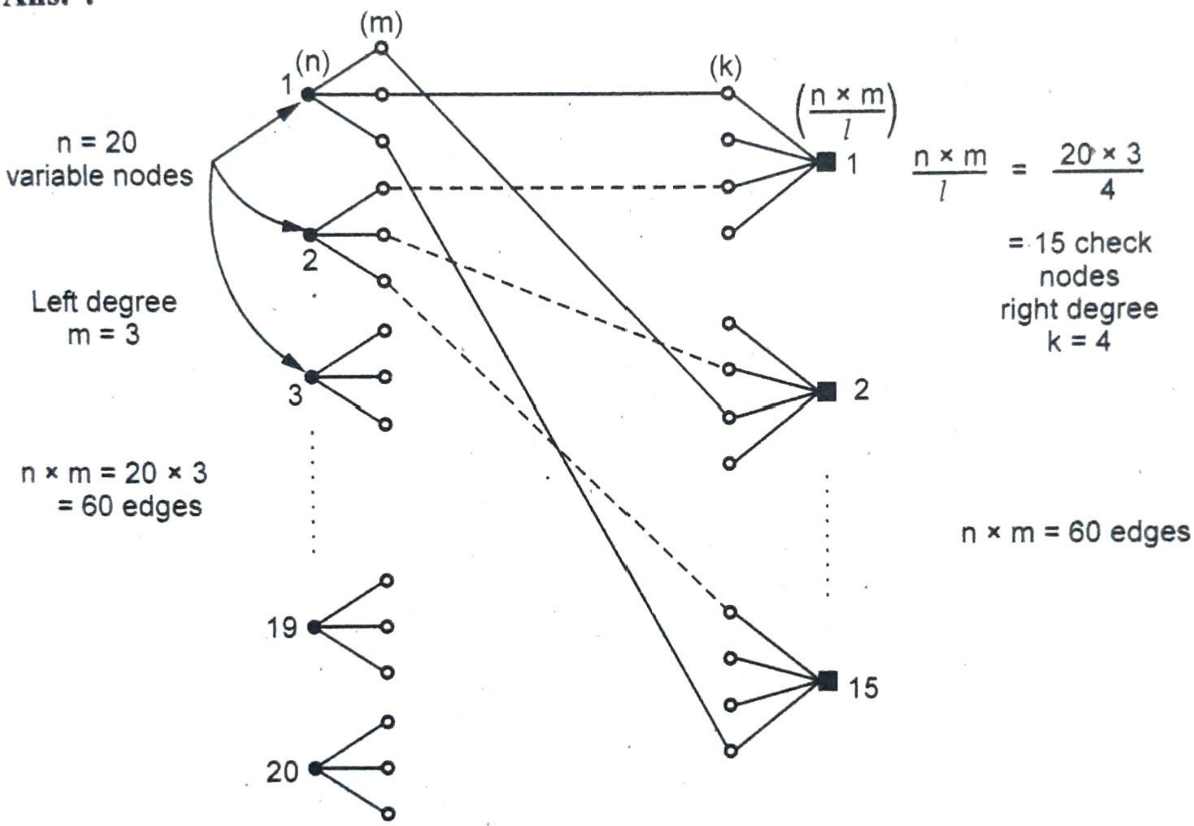
Tanner graph terminology can be understood by following graph.



The degree of a node is the number of edges connected to it.

Q.32 Explain the construction of tanner graph for C(20, 3, 4) LDPC Code.

Ans. :



Q.33 State the properties of LDPC codes.

Ans. :

1. The rate of LDPC code R (m, l) = $1 - m/l$.
   For example (20, 3, 4) with $R = 1 - 3/4 = 1/4$
2. The LDPC code normally has $m \geq 3$ the typical minimum distance of codes in the (m, l) ensemble grows linearly in the codeword length n.
3. The performance of LDPC under maximum likelihood decoding on binary symmetric channel is at least as good as the optimum code of a somewhat higher rate.

END...

SOLVED MODEL QUESTION PAPER (In Sem)

Digital Communication

T.E. (E&Tc) Semester - V [As Per 2019 Pattern]

Time : 1 Hour] **[Maximum Marks : 30**

N.B. : i) Attempt Q.1 or Q.2, Q.3 or Q.4.

ii) Neat diagrams must be drawn wherever necessary.

iii) Figures to the right side indicate full marks.

iv) Assume suitable data, if necessary.

Q.1 a) *Define mean, correlation and covariance of random process.* **(Refer Q.3 of Chapter - 1)** [5]

b) *Define and explain ergodic process.* **(Refer Q.9 of Chapter - 1)** [5]

c) *Define noise. What are the sources of noise ? Explain them in brief.* **(Refer Q.28 of Chapter - 1)** [5]

OR

Q.2 a) *Two random processes z (t) and y (t) are given by*

$z(t) = A \cos(\omega t + \theta)$, $y(t) = A \sin(\omega t + \theta)$

where A and ω *are constants and* θ *is a uniform random variable over* $(0, 2\pi)$. *Find the cross correlation of z (t) and y (t).*

(Refer Q.7 of Chapter - 1) [5]

b) *Define and explain Gaussian process.*

(Refer Q.10 of Chapter - 1) [5]

c) *Noise bandwidth of RC filter. Derive the expression for noise bandwidth of RC low pass filter.* **(Refer Q.41 of Chapter - 1)** **[5]**

Q.3 a) *State the advantages and disadvantages of passband transmission over baseband transmission.* **(Refer Q.13 of Chapter - 2) [5]**

b) *Binary data is transmitted using PSK at a rate 3 Mbps over RF link having bandwidth 10 MHz. Find signal power required at receiver input so that error probability is less than or equal to* 10^{-4}. *Assume noise PSD to be* 10^{-10} *watt/Hz. [Q (3.71)* $= 10^{-4}$*]***(Refer Q.31 of Chapter - 2)** **[7]**

c) *State the properties of matched filter.*
(Refer Q.3 of Chapter - 2) **[3]**

OR

Q.4 a) *Given the input binary sequence 1100100010, sketch the waveforms of the in-phase and quadrature components of a modulated wave obtained by using the QPSK scheme.* **(Refer Q.19 of Chapter - 2)** **[7]**

b) *Explain coherent BPSK transmitter and receiver. Derive the expression for receiver output. Derive and draw the spectrum of B PSK signal. Draw its constellation diagram. Mention the BW required.*
(Refer Q.14 of Chapter - 2) **[8]**

SOLVED MODEL QUESTION PAPER (End Sem)
Digital Communication
T.E. (E&Tc) Semester - V [As Per 2019 Pattern]

Time : $2\frac{1}{2}$ Hours] **[Maximum Marks : 70**

N.B. : *i) Attempt Q.1 or Q.2, Q.3 or Q.4, Q.5 or Q.6, Q.7 or Q.8.*

ii) Neat diagrams must be drawn wherever necessary.

iii) Figures to the right side indicate full marks.

iv) Assume suitable data, if necessary.

Q.1 a) *What is M-ary FSK ? Explain the transmitter and receiver of M-ary FSK.* **(Refer Q.3 of Chapter - 3)** **[7]**

b) *Binary data has to be transmitted over a telephone link that has a usable bandwidth of 3000 Hz and a maximum achievable signal-to-noise power ratio of 6 dB at its output.*

i) Determine the maximum signaling rate and probability of error of a coherent ASK scheme is used for transmitting binary data through this channel.

ii) If the data is maintained at 300 bits/sec calculate the error probability.

$$Q(3.4) = 0.0003,\ Q(6.4) = 10^{-10},\ Q(5.25) = 10^{-7}.$$

(Refer Q.15 of Chapter - 3) **[7]**

c) *What are the issues related to transmission and reception ?* **(Refer Q.20 of Chapter - 3)** **[4]**

OR

Q.2 a) *Explain GMSK in detail.* **(Refer Q.8 of Chapter - 3)** **[7]**

b) *Explain the use of eye diagram to measure ISI.* **(Refer Q.19 of Chapter - 3)** **[7]**

c) *Give the comparison of digital modulation techniques* **(Refer Q.22 of Chapter - 3)** [4]

Q.3 a) *What is the need for spread spectrum modulation technique ?* **(Refer Q.2 of Chapter - 4)** [4]

b) *Explain the application ranging using Direct Sequence Spread Spectrum.* **(Refer Q.22 of Chapter - 4)** [7]

c) *Explain the following terms : i) Slow frequency hopping ii) Fast frequency hopping.* **(Refer Q.13 of Chapter - 4)** [7]

OR

Q.4 a) *Explain DS-SS BPSK transmitter and receiver.* **(Refer Q.6 of Chapter - 4)** [7]

b) *What is PN sequence ? Explain the properties of maximum length sequences by giving the graphical representation of the autocorrelation function.* **(Refer Q.3 of Chapter - 4)** [7]

c) *Compare slow and fast FH-SS.* **(Refer Q.17 of Chapter - 4)** [4]

Q.5 a) *What do you mean by measure of information ?* **(Refer Q.2 of Chapter - 5)** [4]

b) *Design a Shannon-Fano code for a source generating 5 different messages with probabilities 0.45, 0.3, 0.15, 0.05, 0.05. Find the coding efficiency.* **(Refer Q.16 of Chapter - 5)** [6]

c) *A channel has Bandwidth of 5 kHz and a signal to noise power of 63. Determine the BW needed if the S/N power ratio is reduced to 31. What will be the signal power required if the channel BW is reduced to 3 kHz ?* **(Refer Q.38 of Chapter - 5)** [7]

OR

Q.6 a) *Why we need source coding techniques explain with example.* **(Refer Q.10 of Chapter - 5)** **[4]**

b) *Prove that the capacity of noisless channel is* $\log_2 M$ *where M is number of symbols generated by the source.*
(Refer Q.22 of Chapter - 5) **[6]**

c) *Explain differential entropy and mutual information for continuous ensembles.* **(Refer Q.32 of Chapter - 5)** **[7]**

Q.7 a) *What are hamming codes ? Explain with suitable example.* **(Refer Q.1 of Chapter - 6)** **[4]**

b) *With the help of block diagram explain turbo code encoder and decoder.* **(Refer Q.27 of Chapter - 6)** **[6]**

c) *Sketch the encoder and syndrome calculator for the generator polynomial* $g(x) = x^3 + x^2 + 1$, *(7,4) cyclic code and obtain the syndrome for the received code word 1001011.* **(Refer Q.13 of Chapter - 6)** **[7]**

OR

Q.8 a) *Compare state, tree and trellis representation of convolution codes with examples.* **(Refer Q.21 of Chapter - 6)** **[4]**

b) *What are cyclic hamming codes ? Give one example of cyclic hamming code.* **(Refer Q.6 of Chapter - 6)** **[6]**

c) *For a given specification for LDPC generate H matrix (n, m, l) (n, j, k) = (20, 3, 4).* **(Refer Q.30 of Chapter - 6)** **[7]**

END...