Unit III

3 Boundary Conditions

3.1 : Electric Dipole

Important Points to Remember

- The two point charges of equal magnitude but opposite sign, separated by a very small distance give rise to an electric dipole.

Q.1 Derive the expression for the potential and electric field intensity at a distance r from an electric dipole.

Ans. : • The two point charges of equal magnitude but opposite sign, seperated by a very small distance give rise to an **electric dipole.**

- Consider an electric dipole as shown in the Fig. Q.1.1. The two point charges + Q and − Q are separated by a very small distance d.

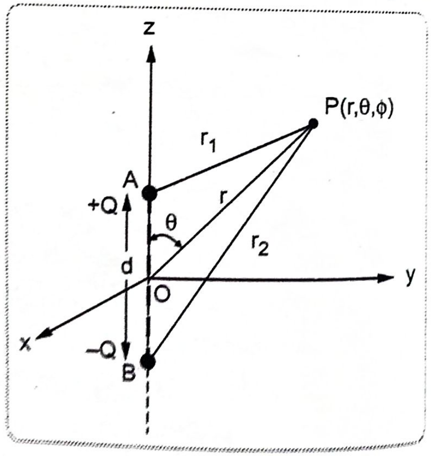


Fig. Q.1.1

- Consider a point P (r, θ, φ) in spherical coordinate system. It is required to find $\overline{E}$ due to an electric dipole at point P.
- Let O be the midpoint of AB. The distance of point P from A is r_1 while the distance of point P from B is r_2. The distance of point P from point O is r.
- The distance of separation of charges i.e. d is very small compared to the distances r_1, r_2 and r. The coordinates of A are $\left(0, 0, +\frac{d}{2}\right)$ and that of B are $\left(0, 0, -\frac{d}{2}\right)$.

- In spherical coordinates, the potential at point P due to the charge + Q is given by, $V_1 = \frac{+Q}{4\pi\varepsilon_0 r_1}$
- The potential at P due to the charge − Q is given by, $V_2 = \frac{-Q}{4\pi\varepsilon_0 r_2}$

The total potential at point P is the algebraic sum of V_1 and V_2.

$$\therefore \quad V = V_1 + V_2 = \frac{+Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \quad ...(1)$$

- Consider that P is located far away from the electric dipole. Thus r_1, r_2 and r can be assumed to be parallel to each other as shown in the Fig. Q.1.2.

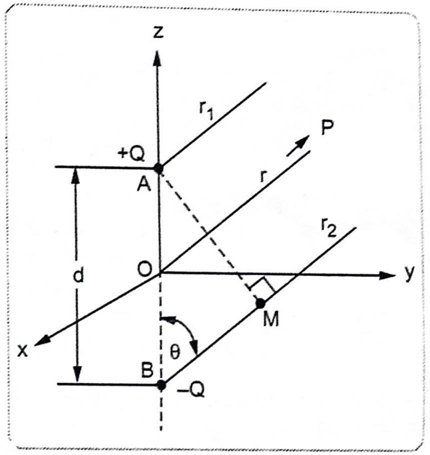


Fig. Q.1.2 Point P is too far away

- AM is drawn perpendicular from A on r_2. The angle made by r_1, r_2 and r with z axis is θ as all are parallel.

$$\therefore \quad BM = AB \cos\theta$$

$$= d \cos\theta \quad ...(2)$$

Now $PB = PM + BM$

and $PA = PM$ as AM is perpendicular.

and $PB = r_2, \quad PA = PM = r_1$

$$\therefore \quad BM = PB - PM = r_2 - PM = r_2 - r_1 \quad ...(3)$$

$$\therefore \quad r_2 - r_1 = d\cos\theta \quad ...(4)$$

As d is very small, $r_1 \approx r_2 \approx r$ hence $r_1 r_2 = r^2$ and use in (1),

$$\therefore \quad \boxed{V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta}{r^2}\right] \text{ V}} \quad ...(5)$$

Now $\quad \bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right]$ ...spherical

$$\therefore \quad \frac{\partial V}{\partial r} = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right] = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}(r^{-2})\right]$$

$$= \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}[-2r^{-3}] = \frac{-2Q\,d\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{Q\,d}{4\pi\varepsilon_0 r^2}[-\sin\theta] \quad \text{and} \quad \frac{\partial V}{\partial \phi} = 0$$

$$\therefore \quad \bar{E} = -\left[\frac{-2Q\,d\cos\theta}{4\pi\varepsilon_0 r^3}\bar{a}_r - \frac{Q\,d\sin\theta}{4\pi\varepsilon_0 r^3}\bar{a}_\theta\right]$$

$$\therefore \quad \boxed{\bar{E} = \frac{Q\,d}{4\pi\varepsilon_0 r^3}[2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta]} \quad \text{(Spherical System)} \quad \dots (6)$$

This is electric field $\bar{E}$ at point P due to an electric dipole.

3.2 : Conductors

Important Points to Remember

- Under the effect of applied electric field, the available free electrons start moving. The moving electrons strike the adjacent atoms and rebound in the random directions. This is called drifting of the electrons.
- After some time, the electrons attain the constant average velocity called **drift velocity** (v_d) and the corresponding current is called **drift current**.
- The drift velocity is directly proportional to the applied electric field and the constant of proportionality is called **mobility** of the electrons in a given material and denoted as μ_e.

$$\therefore \quad \boxed{\bar{v}_d = -\mu_e \bar{E}} \quad \dots (3.1)$$

$$\therefore \quad \boxed{\bar{J} = \rho_e \bar{v}_d \text{ where } \rho_e = \text{Charge density due to free electrons}} \quad \dots (3.2)$$

$$\therefore \quad \boxed{\bar{J} = -\rho_e \mu_e \bar{E}} \quad \dots (3.3)$$

- The charge density is given by ρ_e = ne where n is number of free electrons per m^3 and 'e' is the charge on one electron.

$$\therefore \quad \boxed{\bar{J} = -n\,e\,\mu_e \bar{E} = \sigma \bar{E} \text{ where } \sigma = \text{Conductivity of material}} \quad \dots (3.4)$$

The conductivity $\sigma = -n\,e\,\mu_e$ is measured in mhos per metre or Siemens per metre (S/m). The equation (4) is called **point form of Ohm's law.**

Q.2 State the properties of conductor.

[SPPU : May-01, 03, Dec.-02, Marks 4]

Ans. : It is seen that the charge inside the conductor is zero.

- If there exists a charge inside, there will be electric field and movement of charges under the influence of an electric field causing a current without any source. This is not possible hence charge and electric field are zero inside the conductor.
- The various properties of conductor are,
  1. Under static conditions, **no charge** and **no electric field** can exist at any point **within the conducting material.**
  2. The charge can exist on the surface of the conductor giving rise to surface charge density.
  3. Within a conductor, the charge density is always zero.
  4. The charge distribution on the surface depends on the shape of the surface.
  5. The conductivity of an ideal conductor is infinite.
  6. The conductor surface is an equipotential surface.

3.3 : Dielectric Polarization

Q.3 List the properties of dielectric materials. Define dielectric strength. **[SPPU : May-01, 02, 03, 05, Oct.-18, Marks 6]**

Ans. : The various properties of dielectric materials are,

1. The dielectrics do not contain any free charges but contain **bound charges.**
2. Bound charges are under the internal molecular and atomic forces and cannot contribute to the conduction.

3. When subjected to an external field $\overline{E}$, the bound charges shift their relative positions. Due to this, small electric dipoles get induced inside the dielectric. This is called **polarization.**
4. Due to the polarization, the dielectrics can store the energy.
5. Due to the polarization, the flux density of the dielectric increases by amount equal to the polarization.
6. The induced dipoles produce their own electric field and align in the direction of the applied electric field.
7. When polarization occurs, the volume charge density is formed inside the dielectric while the surface charge density is formed over the surface of the dielectric.
8. The electric field outside and inside the dielectric gets modified due to the induced electric dipoles.

- The minimum value of the applied electric field at which the dielectric breaks down is called **dielectric strength** of that dielectric. It is measures in v/m or kv/cm.

Q.4 Explain the phenomenon of polarization in dielectrics.

☞[SPPU : May-01,02,03,05,11,16,18, Dec.-07, 14 (In Sem), Oct.-18,19, Marks 6]

Ans. : To understand the polarization, consider an atom of a dielectric. This consists of a nucleus with positive charge and negative charges in the form of revolving electrons in the orbits.

- The negative charge is thus considered to be in the form of cloud of electrons. This is shown in the Fig. Q.4.1. Note that $\overline{E}$ applied is zero.
- The number of positive charges is same as negative charges and hence atom is electrically neutral.
- Due to symmetry, both positive and negative charges can be assumed to be point charges of equal amount, coinciding at the centre. Hence there cannot exist an electric dipole. This is called **unpolarized atom.**

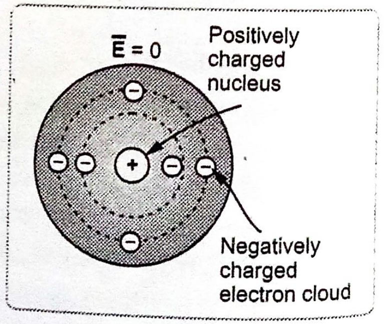


Fig. Q.4.1 Unpolarized atom of a dielectric

- When electric field $\overline{E}$ is applied, the symmetrical distribution of charges gets disturbed. The positive charges experience a force $\overline{F} = Q\overline{E}$ while the negative charges experience a force $\overline{F} = -Q\overline{E}$ in the opposite direction.
- Now there is separation between the nucleus and the centre of the electron cloud as shown in the Fig. Q.4.2(a). Such an atom is called **polarized atom.**

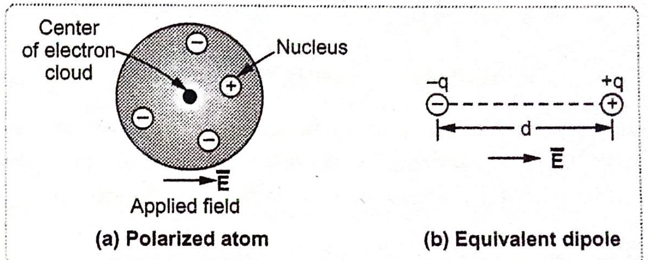


Fig. Q.4.2

- It can be seen that an electron cloud has a centre separated from the nucleus. This forms an electric dipole.
- The equivalent dipole formed is shown in the Fig. Q.4.2 (b). The dipole gets aligned with the applied field. This process is called **polarization** of dielectrics.
- When the dipole is formed due to the polarization, there exists an electric dipole moment $\overline{p}$ which is given by,

$$\boxed{\overline{p} = Q\,\overline{d}} \quad \text{... (1)}$$

where Q = Magnitude of one of the two charges

$\overline{d}$ = Distance vector from negative to positive charge

Let n = Number of dipoles per unit volume

Δv = Total volume of the dielectric, N = Total dipoles = $n\,\Delta v$

- Then the total dipole moment is to be obtained using superposition principle as,

$$\overline{p}_{total} = Q_1\,\overline{d}_1 + Q_2\,\overline{d}_2 + \ldots + Q_n\,\overline{d}_n = \sum_{i=1}^{n\Delta v} Q_i\,\overline{d}_i \quad \text{... (2)}$$

- The **polarization** $\bar{P}$ **is defined as the total dipole moment per unit volume.** Mathematically it is given by,

$$\bar{P} = \lim_{\Delta v \to 0} \frac{\sum_{i=1}^{n\Delta v} Q_i \bar{d}_i}{\Delta v} \quad \text{... (3)}$$

- It is measured in coulombs per square metre (C/m^2).

Important Points to Remember

- It can be seen that the units of polarization are same as that of flux density $\bar{D}$. Thus polarization increases the electric flux density in a dielectric medium. Hence we can write, flux density in a dielectric is,

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{P} \quad \text{and} \quad \bar{P} = \chi_e \varepsilon_0 \bar{E}$$

$$\therefore \quad \bar{D} = \varepsilon_0 \bar{E} + \chi_e \varepsilon_0 \bar{E} \quad \text{i.e.} \quad \bar{D} = (\chi_e + 1)\varepsilon_0 \bar{E} \quad \text{i.e.} \quad \bar{D} = \varepsilon \bar{E}$$

- But $\varepsilon = \varepsilon_R \varepsilon_0$ hence the quantity $\chi_e + 1$ is defined as **relative permittivity** or **dielectric constant** of the dielectric material.
- χ_e = Dimensionless quantity called **electric susceptibility** of the material. It tells us how sensitive is a given dielectric to the applied electric field $\bar{E}$.

$$\therefore \quad \varepsilon_R = \chi_e + 1$$

- The minimum value of the applied electric field at which the dielectric breaks down is called **dielectric strength** of that dielectric

3.4 : Bounary Conditions between Conductor and Free Space

Q.5 Derive the boundary conditions for an electric field at an interface between conductor and free space.

[SPPU : Dec.-2000,02,09,12,13,18,19, May-08,10,12, Insem-15, Oct.-16 (In sem), Marks 6]

Ans. : Consider the conductor free space boundary as shown in the Fig. Q.5.1.

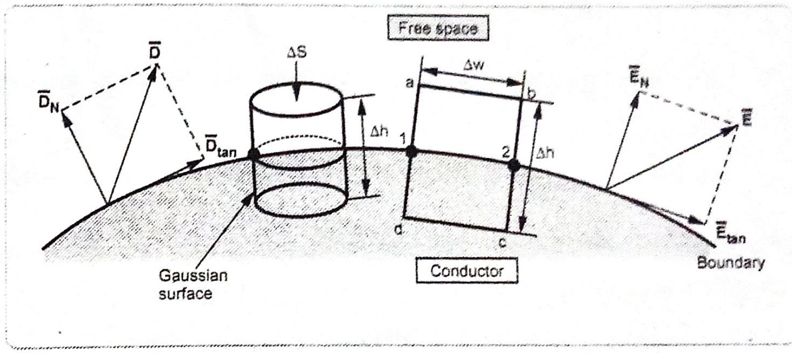


Fig. Q.5.1 Boundary between conductor and free space

- Let $\bar{E}$ be the electric field intensity, in the direction shown in the Fig. Q.5.1, making some angle with the boundary.
- This $\bar{E}$ can be resolved into two components :

1. The component tangential to the surface $(\bar{E}_{tan})$.
2. The component normal to the surface $(\bar{E}_N)$.

- It is known that, $\oint \bar{E} \bullet d\bar{L} = 0$ over a closed contour.
- Consider a rectangular closed path abcda as shown in the Fig. Q.5.1. It is traced in clockwise direction as a-b-c-d-a and hence,

$$\oint \bar{E} \bullet d\bar{L} = \int_a^b \bar{E} \bullet d\bar{L} + \int_b^c \bar{E} \bullet d\bar{L} + \int_c^d \bar{E} \bullet d\bar{L} + \int_d^a \bar{E} \bullet d\bar{L} = 0 \quad \text{... (1)}$$

- The closed contour is placed in such a way that its two sides a-b and c-d are parallel to the tangential direction to the surface while the other two are normal to the surface, at the boundary. And half of it is in conductor and other half in free space. As its height is Δh, $\Delta h/2$ is in the conductor and $\Delta h/2$ is in the free space.
- Now the portion c-d is in the conductor where $\bar{E} = 0$ hence the corresponding integral is zero.

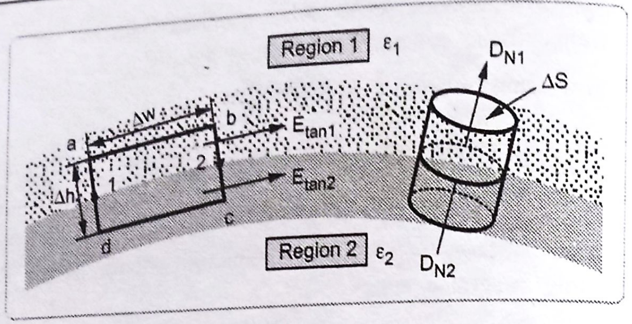


Fig. Q.6.1 Boundary between two perfect dielectrics

- Consider a closed path abcda rectangular in shape having elementary height Δh and elementary width Δw, as shown in the Fig. Q.6.1. It is placed in such a way that $\Delta h/2$ is in dielectric 1 while the remaining is in dielectric 2.
- Let us evaluate the integral of $\bar{E}\bullet d\bar{L}$ along this path.

$$\oint \bar{E}\bullet d\bar{L} = 0 \text{ i.e. } \int_a^b \bar{E}\bullet d\bar{L} + \int_b^c \bar{E}\bullet d\bar{L} + \int_c^d \bar{E}\bullet d\bar{L} + \int_d^a \bar{E}\bullet d\bar{L} = 0 \quad \dots (1)$$

- Both $\bar{E}_1$ and $\bar{E}_2$ in the respective dielectrics have both the components, normal and tangential.

Now $\quad \bar{E}_1 = \bar{E}_{1t} + \bar{E}_{1N}$ and $\bar{E}_2 = \bar{E}_{2t} + \bar{E}_{2N}$ $\quad \dots (2)$

- Let $\left|\bar{E}_{1t}\right| = E_{tan1}$, $\left|\bar{E}_{2t}\right| = E_{tan2}$, $\left|\bar{E}_{1N}\right| = E_{1N}$, $\left|\bar{E}_{2N}\right| = E_{2N}$
- For the rectangle to be reduced at the surface to analyse boundary conditions, $\Delta h \to 0$. As $\Delta h \to 0, \int_b^c$ and $\int_d^a$ become zero as these are line integrals along Δh and $\Delta h \to 0$. Hence equation (1) reduces to,

$$\int_a^b \bar{E}\bullet d\bar{L} + \int_c^d \bar{E}\bullet d\bar{L} = 0 \quad \dots (3)$$

- As a-b is in dielectric 1 component of $\bar{E}$ is E_{tan1}.

$$\therefore \quad \int_a^b \bar{E}\bullet d\bar{L} = E_{tan\,1}\int_a^b d\bar{L} = E_{tan\,1}\,\Delta w \quad \dots (4)$$

- While c-d is in dielectric 2 hence the component of $\bar{E}$ is E_{tan2}. But direction c-d is opposite to a-b hence corresponding integral is negative of the integral obtained for path a-b.

$$\therefore \quad \int_c^d \bar{E}\bullet d\bar{L} = -E_{tan\,2}\,\Delta w \quad \dots (5)$$

- Substituting in (3), $\quad E_{tan\,1}\,\Delta w - E_{tan\,2}\,\Delta w = 0 \quad \dots (6)$

$$\therefore \quad \boxed{E_{tan\,1} = E_{tan\,2}} \quad \dots (7)$$

- **Thus the tangential components of field intensity at the boundary in both the dielectrics remain same i.e. electric field intensity is continuous across the boundary.**

But $\quad D_{tan\,1} = \varepsilon_1 E_{tan\,1}$ and $D_{tan\,2} = \varepsilon_2 E_{tan\,2}$ $\quad \dots (8)$

$$\therefore \quad \frac{D_{tan\,1}}{\varepsilon_1} = \frac{D_{tan\,2}}{\varepsilon_2} \quad \dots (9)$$

$$\therefore \quad \boxed{\frac{D_{tan\,1}}{D_{tan\,2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}} \quad \dots (10)$$

- Thus **tangential components of $\bar{D}$ undergoes some change across the interface hence tangential $\bar{D}$ is said to be discontinuous across the boundary.**
- To find the **normal components**, let us use Gauss's law. Consider a Gaussian surface in the form of right circular cylinder, placed in such a way that half of it lies in dielectric 1 while the remaining half in dielectric 2. The height $\Delta h \to 0$ hence flux leaving from its **lateral surface is zero.** The surface area of its top and bottom is Δ S.

$$\therefore \quad \oint \bar{D}\bullet d\bar{S} = \left[\int_{top} + \int_{bottom} + \int_{lateral}\right] \bar{D}\bullet d\bar{S} = Q \quad \dots (11)$$

$$\therefore \quad \int_{top} \bar{D}\bullet d\bar{S} + \int_{bottom} \bar{D}\bullet d\bar{S} = Q \quad \dots (12)$$

- The flux leaving normal to the boundary is normal to the top and bottom surfaces. Hence $\left|\bar{D}\right| = D_{N1}$ for dielectric 1 and D_{N2} for dielectric 2.

$$\therefore \int_{top} \bar{D}\bullet d\bar{S} = D_{N1}\int_{top} d\bar{S} = D_{N1}\,\Delta S \quad \text{... (13)}$$

- For top surface, the direction of D_N is **entering** the boundary while for bottom surface, the direction of D_N is **leaving** the boundary. **Both are opposite in direction, at the boundary.**

$$\therefore \int_{bottom} \bar{D}\bullet d\bar{S} = -D_{N2}\int_{bottom} d\bar{S} = -D_{N2}\,\Delta S \quad \text{... (14)}$$

$$\therefore D_{N1}\,\Delta S - D_{N2}\,\Delta S = Q \quad \text{but } Q = \rho_S\,\Delta S \quad \text{... (15)}$$

$$\therefore D_{N1} - D_{N2} = \rho_S \quad \text{... (16)}$$

- **At the ideal diectric media boundary the surface charge density ρ_S can be assumed zero i.e. $\rho_S = 0$ i.e. $D_{N1} - D_{N2} = 0$.**

$$\therefore \boxed{D_{N1} = D_{N2}} \quad \text{... (17)}$$

- Hence the **normal component of flux density $\bar{D}$ is continuous at the boundary between the two perfect dielectrics.**
- Now $D_{N1} = \varepsilon_1 E_{N1}$ and $D_{N2} = \varepsilon_2 E_{N2}$ and equating,

$$\therefore \boxed{\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}} \quad \text{... (18)}$$

The normal components of the electric field intensity $\bar{E}$ are inversely proportional to the relative permitivities of the two media.

Q.7 Two extensive homogeneous isotropic dielectrics meet on plane z = 0. For z > 0, ε_{r1} = 4 and z < 0, ε_{r2} = 3. A uniform electric field $\bar{E}_1 = 5\hat{a}_x - 2\hat{a}_y + 3\hat{a}_z$ kV/m exists for $z \geq 0$.
Find : i) $\bar{E}_2$ for $z \leq 0$;
ii) The angle which E_1 makes with the interface
iii) The energy densitie (in J/m^3) for z > 0.

☞[SPPU : Aug.-17, Marks 4]

Ans. : The arrangement is shown in the Fig. Q.7.1.

$$\bar{E}_1 = 5\,\bar{a}_x - 2\,\bar{a}_y + 3\,\bar{a}_z \text{ kV/m}$$

i) The normal directions to the plane z = 0 are $\pm\,\bar{a}_z$.

$$\therefore \bar{E}_{N1} = 3\,\bar{a}_z,\ \bar{E}_{tan\,1} = 5\,\bar{a}_x - 2\,\bar{a}_y$$

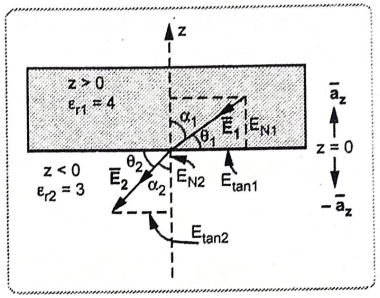


Fig. Q.7.1

- From boundary conditions,

$$\bar{E}_{tan\,1} = \bar{E}_{tan\,2} = 5\,\bar{a}_x - 2\,\bar{a}_y$$

$$\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} = \frac{3}{4} \quad \text{i.e.} \quad \bar{E}_{N2} = \frac{4}{3}\bar{E}_{N1} = 4\,\bar{a}_z$$

$$\therefore \bar{E}_2 = \bar{E}_{N2} + \bar{E}_{tan\,2}$$

$$= \mathbf{5\,\bar{a}_x - 2\,\bar{a}_y + 4\,\bar{a}_z \text{ kV/m}}$$

ii) $$\tan\theta_1 = \frac{E_{N1}}{E_{tan1}}$$

$$= \frac{3}{\sqrt{25+4}} \quad \text{i.e. } \theta_1 = \mathbf{29.12^\circ} \quad \text{...with interface}$$

$$\tan\theta_2 = \frac{E_{N2}}{E_{tan2}}$$

$$= \frac{4}{\sqrt{25+4}} \quad \text{i.e. } \theta_2 = \mathbf{36.6^\circ} \quad \text{...with interface}$$

iii) $$W_{E1} = \frac{1}{2}\varepsilon_1|\bar{E}_1|^2$$

$$= \frac{1}{2}\times 4\times 8.854\times 10^{-12}\times(\sqrt{25+4+9})^2\times 10^6$$

$$= \mathbf{672.904\ \mu J/m^3}$$

$$W_{E2} = \frac{1}{2}\varepsilon_2|\bar{E}_2|^2$$

$$= \frac{1}{2}\times 3\times 8.854\times 10^{-12}\times(\sqrt{25+4+16})^2\times 10^6$$

$$= \mathbf{597.645\ \mu J/m^3}$$

Q.8 A dielectric free space interface is defined by equation $3x + 2y + z = 12$ m. The origin side of interface has $\varepsilon_{r1} = 3$ and $\bar{E}_1 = 2\hat{a}_x + 5\hat{a}_z$ V/m. Find $\bar{E}_2$.

[SPPU : Oct.-18, Marks 8]

Ans. : The interface is shown in the Fig. Q.8.1 by its intersections with axes.

For $x = 0, y = 0, z = 12$ m

For $x = 0, z = 0, y = 6$ m

For $y = 0, z = 0, x = 4$ m

$\bar{E}_1 = 2\bar{a}_x + 5\bar{a}_z$

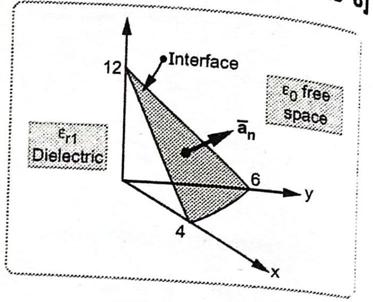


Fig. Q.8.1

$$\bar{a}_n = \frac{3\bar{a}_x + 2\bar{a}_y + \bar{a}_z}{\sqrt{9+4+1}} = \frac{3\bar{a}_x + 2\bar{a}_y + \bar{a}_z}{\sqrt{14}}$$

... Unit normal vector on free space side

- The projection of $\bar{E}_1$ on $\bar{a}_n$ is the normal component of $\bar{E}_1$ at the interface.

$$\therefore \bar{E}_1 \bullet \bar{a}_n = 2 \times \frac{3}{\sqrt{14}} + 0 + 5 \times \frac{1}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

$$\therefore \quad \bar{E}_{n1} = \frac{11}{\sqrt{14}} \bar{a}_n = 2.36\,\bar{a}_x + 1.57\bar{a}_y + 0.79\,\bar{a}_z$$

$$\therefore \quad \bar{E}_{tan1} = \bar{E}_1 - \bar{E}_{n1} = -0.36\,\bar{a}_x - 1.57\,\bar{a}_y + 4.21\,\bar{a}_z$$

But $\bar{E}_{tan1} = \bar{E}_{tan2}$ ... Boundary condition

While $\bar{D}_{n1} = \varepsilon_{r1}\varepsilon_0\,\bar{E}_{n1} = \varepsilon_0\left[7.08\,\bar{a}_x + 4.71\bar{a}_y + 2.37\bar{a}_z\right]$

But $\bar{D}_{n1} = \bar{D}_{n2}$ ... Boundary condition

$$\therefore \quad \bar{E}_{n2} = \frac{1}{\varepsilon_0}\bar{D}_{n2} = 7.08\,\bar{a}_x + 4.17\bar{a}_y + 2.37\bar{a}_z$$

$$\therefore \quad \bar{E}_2 = \bar{E}_{tan2} + \bar{E}_{n2} = 6.72\,\bar{a}_x + 3.14\,\bar{a}_y + 6.58\,\bar{a}_z \text{ V/m}$$

3.6 : Capacitance

Important Points to Remember

- A system which has two conducting surfaces carrying equal and opposite charges, separated by a dielectric is called **capacitive system** giving rise to a **capacitance.**
- The ratio of the magnitudes of the total charge on any one of the two conductors and potential difference between the conductors is called the **capacitance** of the two conductor system denoted as C.
- In general, $C = \frac{Q}{V}$ where Q = Charge in coulombs on any one plate

  and V = Potential difference in volts
- The capacitance is measured in **farads** (F).
- The capacitance can be expressed as,

$$C = \frac{Q}{V} = \frac{\oint_S \varepsilon\,\bar{E} \bullet d\bar{S}}{-\int_{-}^{+} \bar{E} \bullet d\bar{L}}\ \text{F}$$

3.7 : Parallel Plate Capacitor

Q.9 Derive the expression for capacitance of a parallel plate capacitor with single dielectric.

[SPPU : May-03,16,19, Dec.-07,14,15,17,19, In sem-15, Marks 6]

Ans. : • A parallel plate capacitor is shown in the Fig. Q.9.1.

- It consists of two parallel metallic plates separated by distance 'd'.
- The space between the plates is filled with a dielectric of permittivity ε.
- The lower plate, plate 1 carries the positive charge and is distributed over it with a charge density $+\rho_S$.

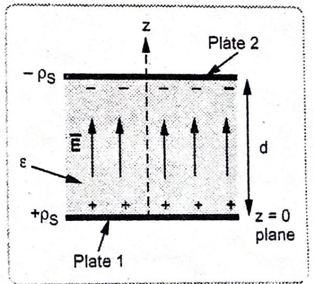


Fig. Q.9.1

- The upper plate, plate 2 carries the negative charge and is distributed over its surface with a charge density $-\rho_S$.
- The plate 1 is placed in z = 0 i.e. xy plane hence normal to it is z direction. While upper plate 2 is in z = d plane, parallel to xy plane.
- Let A = Area of cross section of the plates in m^2

$$\therefore \quad Q = \rho_S A \ \ C \qquad \dots (1)$$

- This is magnitude of charge on any one plate as charge carried by both is equal in magnitude.
- Assuming plate 1 to be infinite sheet of charge,

$$\bar{E}_1 = \frac{\rho_S}{2\varepsilon} \bar{a}_N = \frac{\rho_S}{2\varepsilon} \bar{a}_z \ \ V/m \qquad \dots (2)$$

- The $\bar{E}_1$ is **normal** at the boundary between conductor and dielectric without any tangential component.
- While for plate 2, we can write

$$\bar{E}_2 = \frac{-\rho_S}{2\varepsilon} \bar{a}_N = \frac{-\rho_S}{2\varepsilon} (-\bar{a}_z) \ \ V/m \qquad \dots (3)$$

- The direction of $\bar{E}_2$ is downwards i.e. in $-\bar{a}_z$ direction.
- In between the plates,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{\rho_S}{2\varepsilon} \bar{a}_z + \frac{\rho_S}{2\varepsilon} \bar{a}_z = \frac{\rho_S}{\varepsilon} \bar{a}_z \qquad \dots (4)$$

- The potential difference is given by,

$$V = -\int_{-}^{+} \bar{E} \bullet d\bar{L} = -\int_{upper}^{lower} \frac{\rho_S}{\varepsilon} \bar{a}_z \bullet d\bar{L}$$

$$\dots \ d\bar{L} = dx \ \bar{a}_x + dy \ \bar{a}_y + dz \ \bar{a}_z$$

$$\therefore \quad V = -\int_{z=d}^{z=0} \frac{\rho_S}{\varepsilon} dz = -\frac{\rho_S}{\varepsilon} [z]_d^0 = -\frac{\rho_S[-d]}{\varepsilon} = \frac{\rho_S d}{\varepsilon} \ \ V$$

- The capacitance is the ratio of charge Q to voltage V,

$$C = \frac{Q}{V} = \frac{\rho_S A}{\frac{\rho_S d}{\varepsilon}} = \frac{\varepsilon A}{d} \ \ F \qquad \dots (5)$$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r A}{d} \ \ F \quad \text{where } \varepsilon = \varepsilon_0 \varepsilon_r} \qquad \dots (6)$$

3.8 : Capacitance of a Co-axial Cable

Q.10 Derive the equation of a capacitance of a co-axial cable.

[SPPU : May-04, 05, Dec.-01, 08, Marks 8]

Ans. : Consider a co-axial cable or co-axial capacitor as shown in the Fig. Q.10.1.

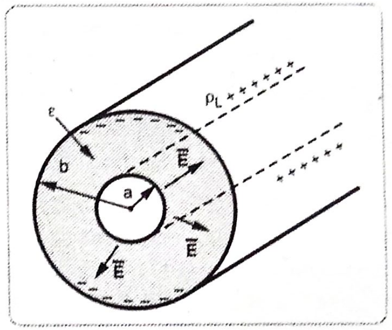


Fig. Q.10.1 Co-axial cable

- Let a = Inner radius

  b = Outer radius

- The two concentric conductors are separated by dielectric of permittivity ε.
- The length of the cable is L m.
- The inner conductor carries a charge density $+\rho_L$ C/m on its surface then equal and opposite charge density $-\rho_L$ C/m exists on the outer conductor.

$$\therefore \quad Q = \rho_L \times L \qquad \dots (1)$$

- Assuming cylindrical co-ordinate system, $\bar{E}$ will be radial from inner to outer conductor and for infinite line charge it is given by,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon r} \bar{a}_r \qquad \dots (2)$$

- $\bar{E}$ is directed from inner conductor to the outer conductor.
- The potential difference is work done in moving unit charge against $\bar{E}$ i.e. from r = b to r = a.
- To find potential difference, consider $d\bar{L}$ in radial direction which is $dr \ \bar{a}_r$

$$\therefore \quad V = -\int_{-}^{+} \bar{E} \bullet d\bar{L} = -\int_{r=b}^{r=a} \frac{\rho_L}{2\pi\varepsilon r} \bar{a}_r \bullet dr \ \bar{a}_r$$

$$= -\frac{\rho_L}{2\pi\varepsilon} [\ln r]_b^a = -\frac{\rho_L}{2\pi\varepsilon} \ln\left[\frac{a}{b}\right]$$

$$\therefore \quad V = \frac{\rho_L}{2\pi\varepsilon} \ln\left[\frac{b}{a}\right] V \qquad \dots (3)$$

$$\therefore \quad C = \frac{Q}{V} = \frac{\rho_L \times L}{\frac{\rho_L}{2\pi\varepsilon} \ln\left[\frac{b}{a}\right]} \quad \text{i.e} \quad \boxed{C = \frac{2\pi\varepsilon L}{\ln\left[\frac{b}{a}\right]} \ \text{F}}$$

- This is the capacitance of a **cylindrical capacitor** i.e. co-axial cable.

3.9 : Capacitance of a Spherical Capacitor

Q.11 Derive the expression for capacitance of a spherical capacitor.

[SPPU : May-12, 15, 18, Dec.-03, 09, 15, Oct.-16, 18, In Sem, Marks 6]

Ans. : • Consider a spherical capacitor formed of two concentric spherical conducting shells of radius a and b. The capacitor is shown in the Fig. Q.11.1.

- The radius of outer sphere is 'b' while that of inner sphere is 'a'. Thus $b > a$.
- The region between the two spheres is filled with a dielectric of permittivity ε.
- The inner sphere is given a positive charge $+Q$ while for the outer sphere it is $-Q$.

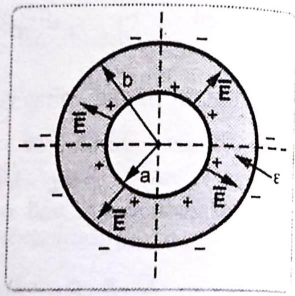


Fig. Q.11.1 Spherical capacitor

- Considering Gaussian surface as a sphere of radius r, it can be obtained that $\bar{E}$ is in radial direction and given by,

$$\bar{E} = \frac{Q}{4\pi\varepsilon r^2} \bar{a}_r \ \text{V/m} \quad \dots (1)$$

- The potential difference is work done in moving unit positive charge **against** the direction of $\bar{E}$ i.e. from $r = b$ to $r = a$.

$$\therefore \quad V = -\int_{-}^{+} \bar{E} \bullet d\bar{L} = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2} \bar{a}_r \bullet d\bar{L} \ (d\bar{L} = dr\ \bar{a}_r) \quad \dots (2)$$

$$\therefore \quad V = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2} dr = -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{r=b}^{r=a} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]_{r=b}^{r=a}$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] \ \text{V}$$

$$\therefore \quad C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]} \quad \text{i.e.} \quad \boxed{C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \ \text{F}} \quad \dots (3)$$

3.10 : Composite Parallel Plate Capacitor

Important Points to Remember

- The composite parallel plate capacitor is the one in which the space between the plates is filled with more than one dielectric.
- There are two cases,

i) Dielectric interface is parallel to the plates.

It is shown in the Fig. 3.1.

$$C_1 = \frac{\varepsilon_1 A}{d_1}, \quad C_2 = \frac{\varepsilon_2 A}{d_2}$$

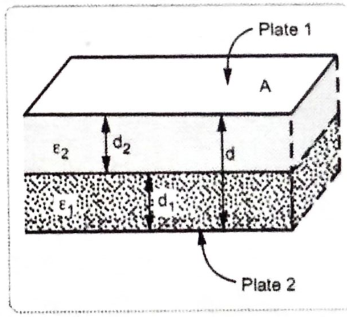


Fig. 3.1

Two are in series.

$$\therefore \quad C = \frac{C_1 C_2}{C_1 + C_2} = \frac{1}{\frac{d_1}{\varepsilon_1 A} + \frac{d_2}{\varepsilon_2 A}}$$

In general for ' n ' dielectrics,

$$C = \frac{A}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} + \dots + \frac{d_n}{\varepsilon_n}}$$

ii) Dielectric interface is normal to the plates. It is shown in Fig. 3.2.

$$C_1 = \frac{\varepsilon_1 A_1}{d}, \quad C_2 = \frac{\varepsilon_2 A_2}{d}$$

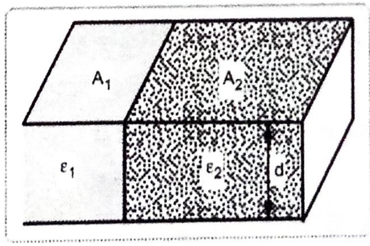


Fig. 3.2

Two are in parallel.

$$\therefore \quad C = C_1 + C_2 = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d}$$

In general for ' n ' dielectrics,

$$C = \frac{\varepsilon_1 A_1 + \varepsilon_2 A_2 + ... + \varepsilon_n A_n}{d}$$

Q.12 Let ε_{r1} = 2.5 for 0 < y < 1 mm, ε_{r2} = 4 for 1 < y < 3 mm, and ε_{r3} for 3 < y < 5 mm. Conducting surfaces are present at y = 0 and x = 5 mm. Calculate the capacitance per square meter of surface area if : i) ε_{r3} is that of air, ii) $\varepsilon_{r3} = \varepsilon_{r2}$, iii) $\varepsilon_{r3} = \varepsilon_{r2}$; iv) Region 3 is silver. **[SPPU : Aug.-17, Marks 4]**

Ans. : The three capacitors are in series.

$$\therefore \quad \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_1 = \frac{\varepsilon_0 \varepsilon_{r1} A}{d_1}, \quad C_2 = \frac{\varepsilon_0 \varepsilon_{r2} A}{d_2}, \quad C_3 = \frac{\varepsilon_0 \varepsilon_{r3} A}{d_3}$$

$$A = 1 m^2, \quad d_1 = 1 \text{ mm}, d_2 = 2 \text{ mm}, \quad d_3 = 2 \text{ mm}$$

$$\therefore \quad \frac{1}{C_{eq}} = \frac{1}{\varepsilon_0}\left[\frac{d_1}{\varepsilon_{r1}} + \frac{d_2}{\varepsilon_{r2}} + \frac{d_3}{\varepsilon_{r3}}\right]$$

$$= \frac{1 \times 10^{-3}}{8.854 \times 10^{-12}}\left[\frac{1}{2.5} + \frac{2}{4} + \frac{2}{\varepsilon_{r3}}\right]$$

$$C_{eq} = \frac{8.854 \times 10^{-9} \times \varepsilon_{r3}}{[2 + 0.9\, \varepsilon_{r3}]} \quad ...(1)$$

i) $\varepsilon_{r3} = 1$ as air

$\therefore$ C_{eq} = **3.05 nF** ... Using ε_{r3} = 1 in (1)

ii) $\varepsilon_{r3} = \varepsilon_{r1} = 2.5$

$\therefore$ C_{eq} = **5.21 nF**

iii) $\varepsilon_{r3} = \varepsilon_{r2} = 4$

$\therefore$ C_{eq} = **6.32 nF**

iv) Silver is perfect conductor hence $\varepsilon_{r3} \to \infty$

$$C_{eq} = \lim_{\varepsilon_r \to \infty} \frac{8.854 \times 10^{-9}\, \varepsilon_{r3}}{2 + 0.9\, \varepsilon_{r3}}$$

$$= \lim_{\varepsilon_r \to \infty} \frac{8.854 \times 10^{-9}}{\frac{2}{\varepsilon_{r3}} + 0.9} = \mathbf{9.83\ nF}$$

Q.13 Determine the capacitance of capacitor as shown in Fig. Q.13.1, if ε_{r1} = 4, ε_{r2} = 6, d = 5 mm, S = 30 cm^2.

[SPPU : Oct.-19, Marks 4]

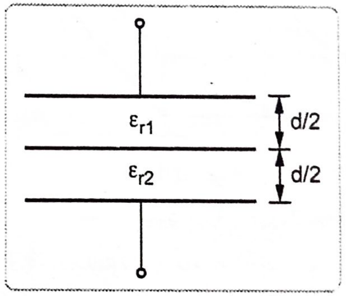


Fig. Q.13.1

Ans. : Two capacitors are in series

$$C_1 = \frac{\varepsilon_0\, \varepsilon_{r1}\, S}{\left(\frac{d}{2}\right)}$$

$$= \frac{8.854 \times 10^{-12} \times 4 \times 30 \times 10^{-4}}{2.5 \times 10^{-3}} = 42.5 \text{ pF}$$

$$C_2 = \frac{\varepsilon_0\, \varepsilon_{r2}\, S}{\left(\frac{d}{2}\right)}$$

$$= \frac{8.854 \times 10^{-12} \times 6 \times 30 \times 10^{-4}}{2.5 \times 10^{-3}} = 63.75 \text{ pF}$$

$$\therefore \quad C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \mathbf{25.5\ pF}$$

Q.14 Find the capacitance of parallel plate capacitor containing two dielectrics, ε_{r1} = 1.5 and ε_{r2} = 3.5, each comprising one half the volumes as shown in Fig. Q.14.1. Here area of plates A = 2 m² and d = 10^{-3} m.

[SPPU : May-17, Marks 6]

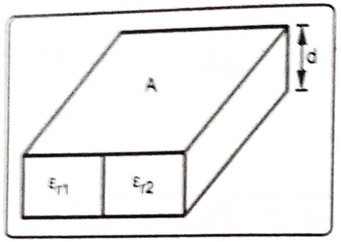


Fig. Q.14.1

Ans. : $\varepsilon_{r1} = 1.5$, $\varepsilon_{r2} = 3.5$, $A = 2\,m^2$, $d = 10^{-3}\,m$.

The dielectric interface is normal to the plates. Each occupies half the volume with constant ' d '.

$$\therefore \quad A_1 = A_2 = \frac{A}{2} = 1\ m^2$$

$$\therefore \quad C_1 = \frac{\varepsilon_0 \varepsilon_{r1} A_1}{d}, \quad C_2 = \frac{\varepsilon_0 \varepsilon_{r2} A_2}{d}$$

$$\therefore \quad C = C_1 + C_2 = \frac{\varepsilon_0}{d}[\varepsilon_{r1} A_1 + \varepsilon_{r2} A_2]$$

$$= \frac{8.854 \times 10^{-12}}{1 \times 10^{-3}}[1.5 \times 1 + 3.5 \times 1] = \mathbf{44.27\ nF}$$

3.11 : Energy Stored In a Capacitor and Energy Density

Q.15 Obtain an expression for an energy stored and energy density in a parallel plate capacitor.

[SPPU : May-03, Dec.-01, 06, Marks 6]

Ans. : • Consider a parallel plate capacitor as shown in the Fig. Q.15.1.

• It is supplied with the voltage V.

• Let $\bar{a}_N$ is the direction normal to the plates.

$$\therefore \quad \bar{E} = \frac{V}{d}\bar{a}_N \qquad \ldots (1)$$

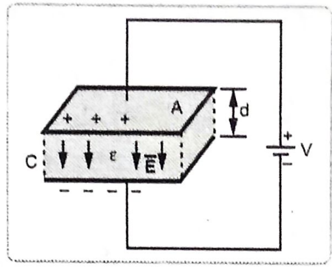


Fig. Q.15.1 Parallel plate capacitor

• The energy stored is given by,

$$W_E = \frac{1}{2}\int_{Vol} \bar{D} \bullet \bar{E}\, dv$$

$$= \frac{1}{2}\int_{Vol} \varepsilon \bar{E} \bullet \bar{E}\, dv$$

but $\bar{E} \bullet \bar{E} = |\bar{E}|^2$

$$= \frac{1}{2}\int_{Vol} \varepsilon |\bar{E}|^2\, dv \qquad \text{but } |\bar{E}| = \frac{V}{d}$$

$$= \frac{1}{2}\varepsilon \frac{V^2}{d^2}\int_{Vol} dv \qquad \text{but } \int_{Vol} dv = \text{Volume} = A \times d$$

$$= \frac{1}{2}\varepsilon \frac{V^2 Ad}{d^2} = \frac{1}{2}\frac{\varepsilon A}{d}V^2 \qquad \text{but } C = \frac{\varepsilon A}{d}$$

$$\therefore \quad \boxed{W_E = \frac{1}{2}CV^2\ J}$$

Energy Density :

• Energy density is the energy stored per unit volume as volume tends to zero.

$$\therefore W_E = \frac{1}{2}\varepsilon\int_{Vol}|\bar{E}|^2\, dv \text{ i.e. } \frac{W_E}{\text{volume}} = \frac{1}{2}\varepsilon|\bar{E}|^2 = \text{Energy density}$$

• Use $|\bar{D}| = \varepsilon|\bar{E}|$ hence energy density is expressed as,

$$\therefore \quad \boxed{\text{Energy density} = \frac{1}{2}\frac{|\bar{D}|^2}{\varepsilon} = \frac{1}{2}|\bar{D}||\bar{E}|\ \ J/m^3}$$

Q.16 For a parallel plate capacitor, area of plate A = 120 cm^2, spacing between plates d = 5 mm separated by dielectric of ε_r = 12, connected to 40 volt battery. Find : i) Capacitance ii) E iii) D iv) Energy stored in capacitor. **[SPPU : Oct.-16, In Sem, Marks 5]**

$A = 120 \text{ cm}^2, \; d = 5 \text{ mm}, \; \varepsilon_r = 12, \; V = 40 \text{ V}$

Ans. :

i) $C = \frac{\varepsilon A}{d} = \frac{\varepsilon_0 \varepsilon_r A}{d} = \mathbf{255 \; pF}$

ii) $E = \frac{V}{d} = \frac{40}{5 \times 10^{-3}} = \mathbf{8000 \; V/m}$

iii) $D = \varepsilon_0 \varepsilon_r E = 8.854 \times 10^{-12} \times 12 \times 8000 = \mathbf{0.85 \; \mu C/m^2}$

iv) $W_E = \frac{1}{2} C V^2 = \frac{1}{2} \times 255 \times 10^{-12} \times 40^2 = \mathbf{0.204 \; \mu J}$

3.12 : Poisson's and Laplace's Equations

Q.17 Derive the Poisson's and Laplace's equation from Gauss's law.

[SPPU : May- 04,05,11,15, Dec.-07,11,14,17 (In Sem), Marks 8]

Ans. : Consider the Gauss's law in the point form as,

$$\nabla \bullet \bar{D} = \rho_v \quad \text{... (1)}$$

where $\bar{D}$ = Flux density and ρ_v = Volume charge density

- It is known that for a homogeneous, isotropic and linear medium, flux density and electric field intensity are directly proportional. Thus,

$$\bar{D} = \varepsilon \bar{E} \quad \text{... (2)}$$

$$\therefore \quad \nabla \bullet \varepsilon \bar{E} = \rho_v \quad \text{... (3)}$$

- From the gradient relationship, $\bar{E} = -\nabla V$ ... (4)
- Substituting equation (4) in (3), $\nabla \bullet \varepsilon(-\nabla V) = \rho_v$ ... (5)
- Taking $-\varepsilon$ outside as constant, $-\varepsilon[\nabla \bullet \nabla V] = \rho_v$

$$\therefore \quad \nabla \bullet \nabla V = -\frac{\rho_v}{\varepsilon} \quad \text{... (6)}$$

- Now $\nabla \bullet \nabla$ operation is called **'del squared'** operation and denoted as ∇^2.

$$\therefore \quad \boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \quad \text{... (7)}$$

- This equation (7) is called **Poisson's equation.**

- If in a certain region, volume charge density is zero $(\rho_v = 0)$, which is true for dielectric medium then the Poisson's equation takes the form,

$$\boxed{\nabla^2 V = 0} \quad \text{(For charge free region)} \quad \text{... (8)}$$

- This is special case of Poisson's equation and is called **Laplace's equation.**
- The ∇^2 operation is called the **Laplacian of V.**
- The Laplace's equation in three co-ordinate systems is given by,

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(Cartesian)}}$$

$$\boxed{\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(Cylindrical)}}$$

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad \text{(Spherical)}}$$

Important Points to Remember

Steps to solve Laplace's equation

Step 1 : Solve the Laplace's equation using the method of integration. Assume constants of integration as per the requirement.

Step 2 : Determine the constants applying the boundary conditions given or known for the region. The solution obtained in step 1 with constants obtained using boundary conditions is an unique solution.

Step 3 : Then $\bar{E}$ can be obtained for the potential field V obtained, using gradient operation $-\nabla V$.

Step 4 : For homogeneous medium, $\bar{D}$ can be obtained as $\varepsilon \bar{E}$.

Step 5 : At the surface, $\rho_S = D_N$ hence once $\bar{D}$ is known, the normal component D_N to the surface is known. Hence the charge induced on the conductor surface can be obtained as $Q = -\int \rho_S dS$.

Step 6 : Once the charge induced Q is known and potential V is known then the capacitance C of the system can be obtained.

If $\rho_v \neq 0$ then similar procedure can be adopted to solve the Poisson's equation

3.13 : Magnetic Materials

Q.18 Classify different magnetic materials with suitable examples.

Ans.: • On the basis of the magnetic behaviour, the magnentic materials are classified as diamagnetic, paramagnetic, ferromagnetic, antiferromagnetic, ferrimagnetic and supermagnentic.

- The magnetic materials can also be classified on the basis of magnetic property namely relative permeability μ_r.
- In general, a material is said to be **non-magnetic** if the value of μ_r is less than 1 or approximately equal to 1.
- While a material is said to be **magnetic** if the value of μ_r is greater than or equal to 1.
- Generally the magnetic materials are classified into three main categories namely **diamagnetic, paramagnetic** and **ferromagnetic.**
- If a value of μ_r is slightly less than unity then it is a **diamagnetic material** ($\mu_r \leq 1$). is kept near either pole of a strong magnet gets repelled. The examples of diamagnetic materials are bismuth, lead, copper, silicon, diamond, graphite, sulphur, sodium chloride and inert gases.
- If the value of μ_r is slightly greater than unity, then it is **paramagnetic** material ($\mu_r \geq 1$). The common examples of paramagnetic materials are potassium, tungston, oxygen, rare earth metals.

- If the value of μ_r is very large than unity ($\mu_r >> 1$), then it is **ferromagnetic** marterial. Iron, nickel and cobalt are the examples of ferromagnetic materials.
- In general the value of μ_r is assumed to be unity for a paramagnetic and diamagnetic materials. Hence they are considered to be **linear** and **non-magnetic materials.**
- On the other hand, the ferromagnetic materials are always **non-linear and magnetic materials.**
- The materials in which the dipole moments of adjacent atoms line up in **antiparallel** fashion are called **antiferromagnetic** materials. The examples are oxides, chlorides and sulphides at low termperatures.
- The materials in which the magnetic dipole moments are lined up in antiparallel fashion and net magnetic moment is non-zero are called **ferrimagnetic** materials. The common examples of ferrites are nickel ferrite, nickel-zinc-ferrite and iron-oxide-magnetite.
- In **supermagnetic** materials, the ferromagnetic materials are suspended in the dielectric matrix. The common examples of supermagnetic materials are magnetic tapes used for audio, video and data recordings.

3.14 : Magnetization

Q.19 Write a note on magnetization.

Ans. : • When a magnetic material is subjected to magnetic field strength (H), the magnetic flux density (B) is produced.

- The magnetic field strength can be controlled by controlling the magnitude of current flowing through the material.
- When current is increased, the magnetic field strength increases. Due to this, magnetic field becomes stronger and stronger and the magnetic flux density increases. This is due to the fact that more number of magnetic dipoles align with the direction of magnetic flux density (B).
- If the behaviour of magnetic flux density is observed, initially it increases slowly and then it increases rapidly. And after one point, it flattens off and becomes constant. After this point though magnetic field strength is increased there is no change in the magnetic flux density. This is called saturation.

- This behaviour of magnetic flux density with magnetic field strength is called magnetization characteristics of the material.
- The Fig. Q.19.1 shows the magnetization characteristics.

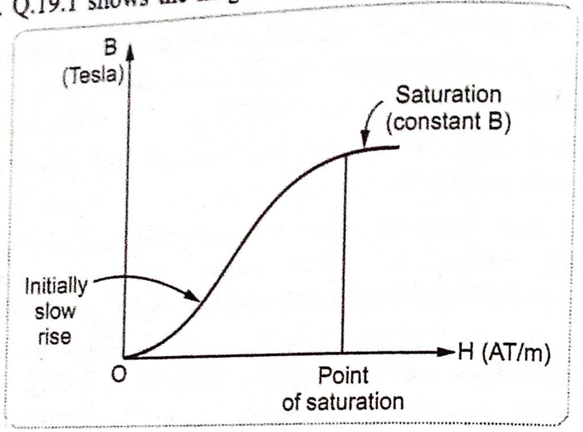


Fig. Q.19.1

- The **magnetization** is defined as the magnetic dipole moment per unit volume and is measured in units ampere/meter (A/m). Mathematically it is given by,

$$\overline{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{a=1}^{n\Delta v} \overline{m}_a \quad \text{... (1)}$$

3.15 : Magnetic Boundary Conditions

Q.20 Derive the boundary conditions at an interface between two magnetic media.

☞[SPPU : Aug-15 (In-Sem), Oct.-16 (In-Sem), May-19, Marks 5]

OR Derive the boundary condition for magnetic field at an interface between two magnetic medium having permeability μ_1 and μ_2.

☞[SPPU: May-10,14, Marks 9, Dec.-12, 13, 17, Oct.-18, Marks 8]

OR Define the magnetic boundary conditions as the conditions that H and B field must satisfy at the boundary between two different media.

☞[SPPU : May-13, Marks 8]

Ans. : • The conditions of the magnetic field $\overline{B}$ and $\overline{H}$ existing at the boundary of the two media when the magnetic field passes from one medium to other are called **boundary conditions** for magnetic fields or **magnetic boundary conditions.**

- To study conditions of $\overline{B}$ and $\overline{H}$ at the boundary, both the vectors are resolved into two components ;

a) Tangential to boundary and b) Normal to boundary.

- Consider a boundary between two isotropic, homogeneous linear materials with different permeabilities μ_1 and μ_2 as shown in the Fig. Q.20.1.

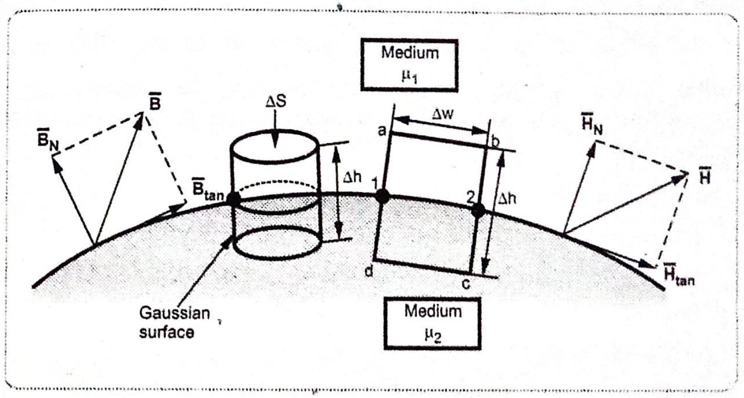


Fig. Q.20.1 Boundary between two magnetic materials of different permeabilities

- To determine the boundary conditions, we must use the closed path and the Gaussian surface.

Boundary Conditions for Normal Component

- To find the normal component of $\overline{B}$, select a closed Gaussian surface in the form of a right circular cylinder as shown in the Fig. Q.20.1.
- Let the height of the cylinder be Δh and be placed in such a way that $\Delta h/2$ is in medium 1 and remaining $\Delta h/2$ is in medium 2. Also the axis of the cylinder is in the normal direction to the surface.
- According to the Gauss's law for the magnetic field,

$$\oint_S \overline{B} \bullet d\overline{S} = 0 \quad \text{... (1)}$$

- The surface integral must be evaluated over three surfaces,

  i) Top ii) Bottom and iii) Lateral.

- Let the area of the top and bottom is same, equal to ΔS.

$$\therefore \oint_{top} \bar{B} \bullet d\bar{S} + \oint_{bottom} \bar{B} \bullet d\bar{S} + \oint_{lateral} \bar{B} \bullet d\bar{S} = 0 \quad \ldots (2)$$

- As we are very much interested in the boundary conditions, reduce Δh to zero. As $\Delta h \to 0$, the cylinder tends to boundary and only top and bottom surfaces contribute in the surface integral.
- Thus surface integrals are calculated for top and bottom surfaces only. These surfaces are very small.
- Let the magnitude of normal component of $\bar{B}$ be B_{N_1} and B_{N_2} in medium 1 and medium 2 respectively. As both the surfaces are very small, we can assume B_{N_1} and B_{N_2} constant over their surfaces. Hence we can write,

For top surfaces

$$\oint_{Top} \bar{B} \bullet d\bar{S} = B_{N_1} \oint_{Top} d\bar{S} = B_{N_1} \Delta S \quad \ldots (3)$$

For bottom surface

$$\oint_{Bottom} \bar{B} \bullet d\bar{S} = B_{N_2} \oint_{Bottom} d\bar{S} = B_{N_2} \Delta S \quad \ldots (4)$$

For lateral surface

$$\oint_{Lateral} \bar{B} \bullet d\bar{S} = 0 \quad \ldots (5)$$

Putting values of surface integrals in equation (2), we get

$$B_{N_1} \Delta S - B_{N_2} \Delta S = 0 \quad \ldots (6)$$

- Note that the negative sign is used for one of the surface integrals because normal component in medium 2 is entering the surface while in medium 1 the component is leaving the surface. Hence B_{N_1} and B_{N_2} are in opposite direction. From equation (6), we can write,

$$B_{N_1} \Delta S = B_{N_2} \Delta S$$

i.e.

$$\boxed{B_{N_1} = B_{N_2}} \quad \ldots (7)$$

- **Thus the normal component of $\bar{B}$ is continuous at the boundary.**

- But $\bar{B} = \mu \bar{H}$, so equation (7) can be written as,

$$\mu_1 H_{N_1} = \mu_2 H_{N_2}$$

$$\frac{H_{N_1}}{H_{N_2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_{r2}}{\mu_{r1}} \quad \ldots (8)$$

- **Hence the normal component of $\bar{H}$ is not continuous at the boundary. The field strengths in two media are inversely proportional to their relative permeabilities.**

Boundary Conditions for Tangential Component

- According to Ampere's circuital law,

$$\oint \bar{H} \bullet d\bar{L} = I \quad \ldots (9)$$

- Consider a rectangular closed path abcda as shown in the Fig. Q.20.1. It is traced in clockwise direction as a-b-c-d-a. This closed path is placed in a plane normal to the boundary surface. Hence $\oint \bar{H} \bullet d\bar{L}$ can be divided into six parts.

$$\oint \bar{H} \bullet d\bar{L} = \int_a^b \bar{H} \bullet d\bar{L} + \int_b^1 \bar{H} \bullet d\bar{L} + \int_1^c \bar{H} \bullet d\bar{L} + \int_c^d \bar{H} \bullet d\bar{L}$$

$$+ \int_d^2 \bar{H} \bullet d\bar{L} + \int_2^a \bar{H} \bullet d\bar{L} = I \quad \ldots (10)$$

- From the Fig. Q.20.1 it is clear that, the closed path is placed in such a way that its two sides a-b and e-d are parallel to the tangential direction to the surface while the other two sides are normal to the surface at the boundary.
- This closed path is placed in such a way that half of its portion is in medium 1 and the remaining is in medium 2.
- The rectangular path is an elementary rectangular path with elementary height Δh and elementary width Δw.
- Thus over small width Δw, $\bar{H}$ can be assume constant say H_{tan1} in medium 1 and H_{tan2} in medium 2. Similarly over a small height $\frac{\Delta h}{2}$, $\bar{H}$ can be assumed constant say H_{N1} in medium 1 and H_{N2} in medium 2.
- Now assume that $\bar{K}$ is the surface current normal to the path. Also from the Fig. Q.20.1 it is clear that the normal and tangential components in

medium 1 and medium 2 are in opposite direction. Thus equation (10) can be written as,

$$K \bullet dw = H_{tan1}(\Delta w) + H_{N1}\left(\frac{\Delta h}{2}\right) + H_{N2}\left(\frac{\Delta h}{2}\right) - H_{tan2}(\Delta w)$$

$$-H_{N2}\left(\frac{\Delta h}{2}\right) - H_{N1}\left(\frac{\Delta h}{2}\right) \quad \text{... (11)}$$

- To get conditions at boundary, $\Delta h \to 0$. Thus,

$$K \bullet dw = H_{tan1}(\Delta w) - H_{tan2}(\Delta w)$$

$$H_{tan1} - H_{tan2} = K \quad \text{... (12)}$$

- In vector form, we can express above relation by a cross product as

$$\overline{H}_{tan1} - \overline{H}_{tan2} = \bar{a}_{N12} \times \overline{K} \quad \text{... (13)}$$

where $\bar{a}_{N12}$ is the unit vector in the direction normal at the boundary from medium 1 to medium 2.

- For $\overline{B}$, the tangential components can be related with permeabilities of two media using equation (12),

$$\therefore \frac{B_{tan1}}{\mu_1} - \frac{B_{tan2}}{\mu_2} = K \quad \text{... (14)}$$

- Consider a special case that the boundary is free of current. In other words, media are not conductors; so K = 0. Then equation (12) becomes

$$H_{tan1} - H_{tan2} = 0$$

or $$H_{tan1} = H_{tan2} \quad \text{... (15)}$$

For tangential components of $\overline{B}$ we can write,

$$\frac{B_{tan1}}{\mu_1} - \frac{B_{tan2}}{\mu_2} = 0$$

$$\therefore \frac{B_{tan1}}{\mu_1} = \frac{B_{tan2}}{\mu_2}$$

$$\frac{B_{tan1}}{B_{tan2}} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}} \quad \text{... (16)}$$

From equations (15) and (16) it is clear that **tangential component of $\overline{H}$ are continuous, while tangential component of $\overline{B}$ are discontinuous at the boundary, with the condition that the boundary is current free.**

Q.21 The region x < 0 is medium 1 with μ_{r1} = 4.5 and $\overline{H}_1 = 4\hat{a}_x + 3\hat{a}_y - 6\hat{a}_z$ A/m. The region x > 0 is medium 2 with μ_{r2} = 6. Find $\overline{H}_2$ in medium 2 and angle made by $\overline{H}_2$ with normal to interface.

[SPPU : Aug.-15, Oct.-19, Marks 6]

Ans. : An interface is in y-z plane at x = 0. For x < 0, there is medium 1 ($\mu_{r1} = 4.5$) and for x > 0, there is medium 2 ($\mu_{r2} = 6$) as shown in the Fig. Q.21.1. Let us assume that the interface is current free.

From the Fig. Q.21.1 it is clear that, for y-z plane, x-axis is the normal to the interface. Thus the component of $\overline{H}_1$ along $\bar{a}_x$ is the normal component while the components along $\bar{a}_y$ and $\bar{a}_z$ are tangential components.

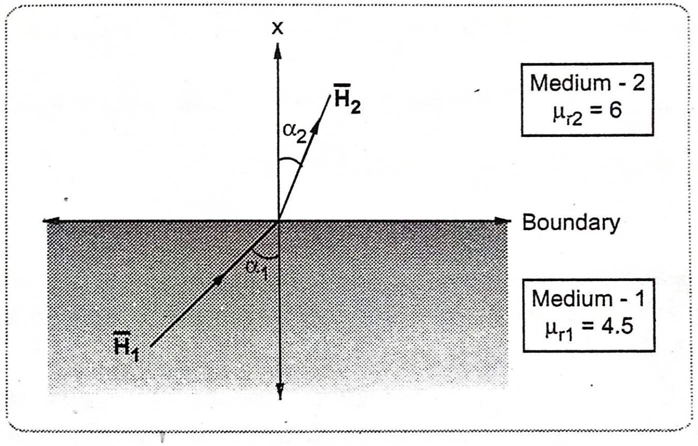


Fig. Q.21.1

$$\therefore \quad \overline{H}_{tan1} = 3\bar{a}_y - 6\bar{a}_z \ \text{A/m}$$

$$\text{and} \quad \overline{H}_{N1} = 4\,\bar{a}_x \ \text{A/m} \quad \text{... (1)}$$

From the boundary conditions, for current free boundary, the tangential component of $\overline{H}$ is continuous.

$$\therefore \quad \overline{H}_{tan1} = \overline{H}_{tan2} = 3\,\bar{a}_y - 6\bar{a}_z \ \text{A/m} \quad \text{...(2)}$$

At the boundary normal component of $\overline{H}$ is discontinuous.

$$\therefore \quad \frac{H_{N1}}{H_{N2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_0\mu_{r2}}{\mu_0\mu_{r1}} = \frac{6}{4.5}$$

$$\therefore \quad H_{N2} = \frac{(4.5)}{6} H_{N1} = (0.75)(4) = 3 \quad \text{... (3)}$$

As the normal direction is along x-axis,

$$\bar{H}_{N2} = 3\,\bar{a}_x \quad \text{A/m} \quad \text{... (4)}$$

From equations (2) and (4), the field intensity $\bar{H}_2$ in region 2 is given by,

$$\bar{H}_2 = \bar{H}_{tan2} + \bar{H}_{N2} = 3\,\bar{a}_x + 3\,\bar{a}_y - 6\,\bar{a}_z$$

Let the angle made by $\bar{H}_2$ with the normal to the interface be α_2. Then we can write

$$\bar{H}_2 \bullet \bar{a}_x = |\bar{H}_2|\,|\bar{a}_x| \cos\alpha_2$$

$$\therefore \left(3\bar{a}_x + 3\bar{a}_y - 6\bar{a}_z\right)\bullet(\bar{a}_x) = \left(\sqrt{(3)^2 + (3)^2 + (-6)^2}\right)(1)(\cos\alpha_2)$$

$$\therefore \quad 3 = (7.3484)\cos\alpha_2 \quad \text{.... } \bar{a}_x \bullet \bar{a}_y = \bar{a}_z \bullet \bar{a}_x = 0$$

$$\therefore \quad \cos\alpha_2 = \frac{3}{7.3484} = 0.4082$$

$$\therefore \quad \alpha_2 = \cos^{-1}(0.4082) = \mathbf{65.9^\circ}$$

Q.22 A current sheet $\bar{K} = 9\,\bar{a}_y$ A/m is located at z = 0. The region 1 which is at $z < 0$ has $\mu_{r1} = 4$ and region 2 which is at $z > 0$ has $\mu_{r2} = 3$. Given : $\bar{H}_2 = 14.5\,\bar{a}_x + 8\,\bar{a}_z$ A/m. Find $\bar{H}_1$.

☞[SPPU : Dec.-01, 15, May-15, Marks 6]

Ans. : From the given data, z-axis is normal to the boundary. The normal component of $\bar{H}_2$ is along $\bar{a}_z$, i.e. H_{N2} = 8 A/m. Similarly the tangential component of $\bar{H}_2$ is along $\bar{a}_x$, i.e. $H_{tan\,2}$ = 4.5 A/m.

As there is a current sheet at z = 0, tangential component of $\bar{H}$ is not continuous at the boundary. For the boundary containing current sheet we must write,

$$(\bar{H}_1 - \bar{H}_2) \times \bar{a}_{n12} = \bar{K}$$

where $\bar{a}_{n12}$ is unit vector normal to the boundary drawn from medium 1 to medium 2. In this case

$$\bar{a}_{n12} = \bar{a}_z$$

$$\therefore \quad (\bar{H}_1 - \bar{H}_2) \times \bar{a}_z = \bar{K} = 9\,\bar{a}_y \quad \text{... (1)}$$

For the boundary containing a current sheet, normal component of $\bar{H}$ is discontinuous. So we can write,

$$\frac{H_{n1}}{H_{n2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_0\mu_{r2}}{\mu_0\mu_{r1}} = \frac{\mu_{r2}}{\mu_{r1}}$$

$$\therefore \quad H_{n1} = (H_{n2})\frac{\mu_{r2}}{\mu_{r1}} = (8)\left(\frac{3}{4}\right) = \mathbf{6\ A/m}$$

Thus normal component of $\bar{H}_1$ is along $\bar{a}_z$ Similar to $\bar{H}_2$. Thus we can write $\bar{H}_1$ as,

$$\bar{H}_1 = H_{tan\,1}\,\bar{a}_x + 6\,\bar{a}_x \quad \text{A/m} \quad \text{... (2)}$$

$$\therefore \quad \bar{H}_1 - \bar{H}_2 = \left[H_{tan\,1}\,\bar{a}_x + 6\bar{a}_z\right] - \left[14.5\,\bar{a}_x + 2\,\bar{a}_z\right]$$

$$= (H_{tan1} - 14.5)\,\bar{a}_x + 2\,\bar{a}_z \quad \text{A/m}$$

Putting value of $(\bar{H}_1 - \bar{H}_2)$ in equation (1)

$$\left[(H_{tan1} - 14.5)\bar{a}_x + 2\bar{a}_z\right] \times \bar{a}_z = 9\,\bar{a}_y$$

$$\therefore \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ (H_{tan1} - 14.5) & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 9\,\bar{a}_y$$

$$\therefore -(H_{tan1} - 14.5)\bar{a}_y = 9\,\bar{a}_y$$

Comparing the components on both the sides,

$$\therefore \quad -H_{tan\,1} + 14.5 = 9$$

$$\therefore \quad H_{tan\,1} = \mathbf{5.5\ A/m} \quad \text{... (3)}$$

Hence $\bar{H}_1$ in medium is given by,

$$\therefore \quad \bar{H}_1 = \mathbf{5.5\ \bar{a}_x + 6\ \bar{a}_z \quad A/m}$$

3.16 : Magnetic Force

Q.23 Explain the magnetic force on a differential current element.

Ans. : • The force exerted on a differential element of charge dQ moving in a steady magnetic field is given by,

$$d\bar{F} = dQ\,\bar{v} \times \bar{B} \quad \text{N} \quad \text{... (1)}$$

- The current density $\bar{J}$ can be expressed interms of velocity of a volume charge density as,

$$\bar{J} = \rho_V \bar{V} \quad \text{... (2)}$$

- But the differential element of charge can be expressed in terms of the volume charge density as,

$$dQ = \rho_V \, dv \quad \text{... (3)}$$

- Substituting value of dQ in equation (1),

$$\therefore \quad d\bar{F} = \rho_V \, dv \, \bar{V} \times \bar{B}$$

- Expressing $d\bar{F}$ interms of $\bar{J}$ using equation (2), we can write,

$$d\bar{F} = \bar{J} \times \bar{B} \, dv \quad \text{... (4)}$$

- But we have already studied in previous chapters, the relationship as,

$$\bar{J} \, dv = \bar{K} \, dS = I \, d\bar{L}$$

- Then the force exerted on a surface current density is given by,

$$d\bar{F} = \bar{K} \times \bar{B} \, dS \quad \text{... (5)}$$

- Similarly the force exerted on a differential current element is given by,

$$d\bar{F} = \left(I \, d\bar{L} \times \bar{B}\right) \quad \text{... (6)}$$

- Integrating equation (4) over a volume, the force is given by,

$$\bar{F} = \int_{vol} \bar{J} \times \bar{B} \, dv \quad \text{... (7)}$$

- Integrating equation (5) over either open or closed surface, we get,

$$\bar{F} = \int_{S} \bar{K} \times \bar{B} \, dS \quad \text{... (8)}$$

- Similarly integrating equation (6) over a closed path, we get,

$$\bar{F} = \oint I \, d\bar{L} \times \bar{B} \quad \text{... (9)}$$

Q.24 Derive the expression for a force between two current carrying wires assuming the currents are in the same direction.

Ans. : • Let us consider two current elements $I_1 \, d\bar{L}_1$ and $I_2 d\bar{L}_2$ as shown in the Fig. Q.24.1.

- Note that the directions of I_1 and I_2 are same.
- Both the current elements produce their own magnetic fields.

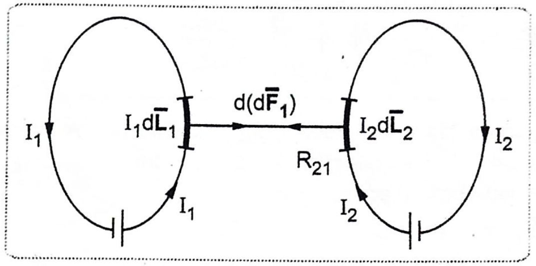


Fig. Q.24.1 Force between two current elements

- As the currents are flowing in the same direction through the elements, the force $d(d\bar{F}_1)$ exerted on element $I_1 d\bar{L}_1$ due to the magnetic field $d\bar{B}_2$ produced by other element $I_2 d\bar{L}_2$ is the force of attraction.

Derivation

- From the equation of force, the force exerted on a differential current element is given by,

$$d(d\bar{F}_1) = I_1 d\bar{L}_1 \times d\bar{B}_2 \quad \text{... (1)}$$

- According to Biot-Savart's law, the magnetic field produced by current element $I_2 d\bar{L}_2$ is given by, for free space,

$$d\bar{B}_2 = \mu_0 \, d\bar{H}_2 = \mu_0 \left[\frac{I_2 d\bar{L}_2 \times \bar{a}_{R21}}{4\pi R_{21}^2}\right] \quad \text{... (2)}$$

- Substituting value of $d\bar{B}_2$ in equation (1), we can write,

$$d(d\bar{F}_1) = \mu_0 \frac{I_1 d\bar{L}_1 \times (I_2 d\bar{L}_2 \times \bar{a}_{R21})}{4\pi R_{21}^2} \quad \text{... (3)}$$

- The equation (3) represents force between two current elements. It is very much similar to Coulomb's law.
- By integrating $d(d\,F_2)$ twice, the total force $\bar{F}_1$ on current element 1 due to current element 2 is given by,

$$\boxed{\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_1 \times (d\bar{L}_2 \times \bar{a}_{R21})}{R_{21}^2}} \quad \text{...(4)}$$

- Exactly following same steps, we can calculate the force $\bar{F}_2$ exerted on the current element 2 due to the magnetic field $\bar{B}_1$ produced by the current element 1. Thus,

$$\boxed{\bar{F}_2 = \frac{\mu_0 I_2 I_1}{4\pi} \oint_{L_2} \oint_{L_1} \frac{d\bar{L}_2 \times (d\bar{L}_1 \times \bar{a}_{R12})}{R_{12}^2}} \quad ...(5)$$

- Actually equation (5) is obtained from equation (4) by interchanging the subscripts 1 and 2. By using back-cab rule for expanding vector triple product, we can show that

$$\bar{F}_2 = -\bar{F}_1 \quad ...(6)$$

- Thus, above condition indicates that both the forces $\bar{F}_1$ and $\bar{F}_2$ obey Newton's third law that for every action there is equal and opposite reaction.

Q.25 Derive the force exerted between two parallel current carrying straight conductors, kept 'd' meters away.

Ans. : • Consider that two current carrying conductors are placed parallel to each other.

- Each of this conductor produces its own flux around it. So when such two conductors are placed close to each other, there exists a force due to the interaction of two fluxes.
- The force between such parallel current carrying conductors depends on the directions of the two currents.
- If the directions of both the currents are same, then the conductors experience a force of attraction as shown in the Fig. Q.25.1 (a). And if the directions of two currents are opposite to each other, then the conductors experience a force of repulsion as shown in the Fig. Q.25.1 (b).

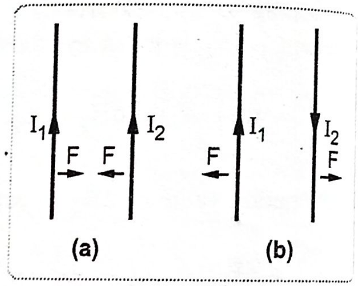


Fig. Q.25.1 Force between two long parallel conductors

- Consider two long parallel conductors of length l each placed in a medium. Assume that the conductors are separated by distance d as shown in the Fig. Q.25.2.

Derivation

- Assume that the conductors carry current in opposite direction as shown in the Fig. Q.25.2.

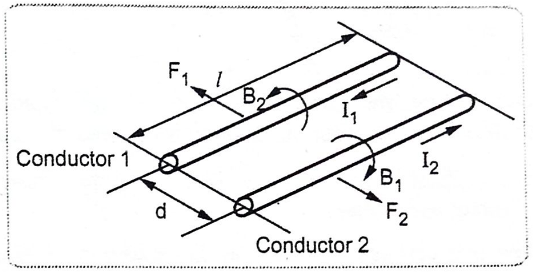


Fig. Q.25.2 Two parallel conductors of length l, separated by distance d and carrying current in opposite direction

- Now the magnitude of force exerted on conductor 1 i.e. $|\bar{F}_1|$ due to magnetic field $\bar{B}_2$ produced by conductor 2 is given by,

$$F_1 = |\bar{F}_1| = |I_1 l \times \bar{B}_2| = I_1\, l\, B_2 \sin 90 = I_1\, l B_2 \quad \text{... (Repulsive)} \quad ...(1)$$

- Now using Ampere circuital law, we can find magnitude of magnetic field intensity due to straight long conductor as

$$H_2 = \frac{I_2}{2\pi d} \quad ...(2)$$

- Hence by definition,

$$B_2 = \mu H_2 = \frac{\mu_0 \mu_r I_2}{2\pi d} \quad ...(3)$$

- Substituting value of B_2 in expression for F_1, we get,

$$F_1 = I_1\, l \left[\frac{\mu_0 \mu_r I_2}{2\pi d}\right]$$

$$F_1 = \frac{\mu_0 \mu_r I_1 I_2\, l}{2\pi d} \quad ...(4)$$

- Similarly the magnitude of force acting on conductor 2 due to magnetic field produced by conductor 1, we can write,

$$F_2 = \frac{\mu_0 \mu_r I_1 I_2\, l}{2\pi d} \quad ...(5)$$

- Thus in general, we can write that, for the two parallel conductors of length l carrying two same or different currents, the force exerted is given by,

$$F = \frac{\mu I_1 I_2 l}{2\pi d} \quad \ldots(6)$$

where I_1 and I_2 are the currents flowing through conductor 1 and conductor 2 and d is the distance of separation between two conductors.

- If the two currents flow in same directions, the current carrying conductors attract each other.
- While if, the two currents flow in opposite direction to each other, the current carrying conductors repel each other.

3.17 : Magnetic Torque

Q.26 Explain magnetic torque in brief.

Ans. :

- The **moment of a force or torque** about a specified point is defined as the vector product of the moment arm $\bar{R}$ and the force $\bar{F}$. It is measured in newton meter (Nm).

$$\bar{T} = \bar{R} \times \bar{F} \text{ Nm} \quad \ldots(1)$$

- Consider a point A at which force $\bar{F}$ is applied as shown in the Fig. Q.26.1.
- Let $\bar{R}$ be the arm from origin O to point A. Then the torque $\bar{T}$ about origin is nothing but a vector product of $\bar{R}$ and $\bar{F}$.
- The magnitude of the torque is equal to the product of the magnitudes of $\bar{R}$ and $\bar{F}$ and sine of the angle between $\bar{R}$ and $\bar{F}$, while the direction of the torque $\bar{T}$ is normal to both $\bar{R}$ and $\bar{F}$.

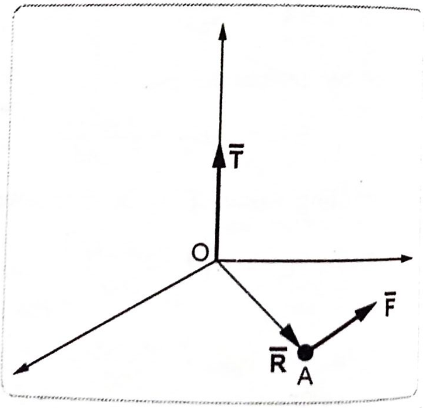


Fig. Q.26.1 Torque $\bar{T}$ about the origin

- Now consider that two forces namely $\bar{F}_1$ and $\bar{F}_2$ are applied at points A_1 and A_2 respectively. The arms for the two forces drawn from the origin be $\bar{R}_1$ and $\bar{R}_2$ respectively as shown in the Fig. Q.26.2.

- Assume that $\bar{F}_2 = -\bar{F}_1$. Then the total torque $\bar{T}$ about the origin due to the two forces is given by,

$$\bar{T} = \bar{R}_1 \times \bar{F}_1 + \bar{R}_2 \times \bar{F}_2$$

$$\therefore \quad \bar{T} = (\bar{R}_1 - \bar{R}_2) \times \bar{F}_1 \quad \ldots \bar{F}_2 = -\bar{F}_1$$

$$\therefore \quad \bar{T} = \bar{R}_{21} \times \bar{F}_1 \text{, where}$$

$\bar{R}_{21} = \bar{R}_1 - \bar{R}_2$ is a vector joining A_2 to A_1.

- From above expression it is clear that when total force is zero, the torque is independent of the choice of the origin.

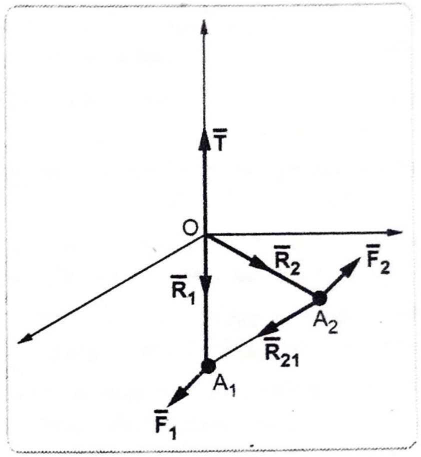


Fig. Q.26.2 Torque about the origin

Q.27 Derive an expression for a torque on a closed rectangular loop placed in uniform magnetic field.

Ans. : • Consider a differential current loop of rectangular shape with uniform magnetic field everywhere around it as shown in the Fig. Q.27.1.

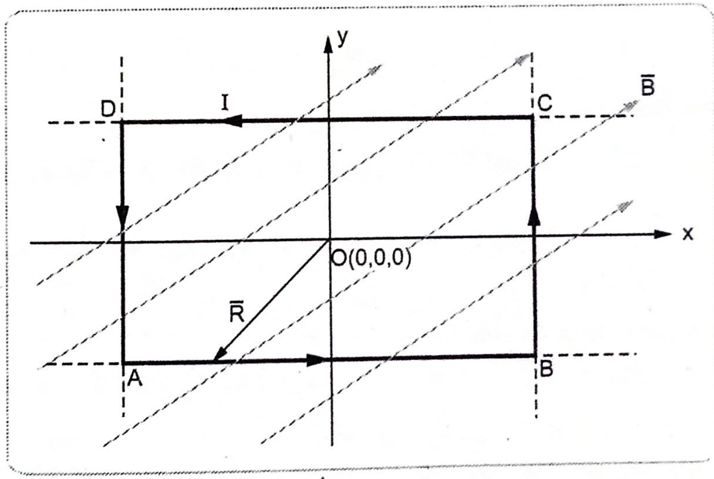


Fig. Q.27.1 A rectangular differential current loop in uniform magnetic field $\bar{B}$

- Assume that the loop is placed in xy-plane. Let the sides AB and CD be parallel to x-axis and sides BC and DA be parallel to y-axis.
- Let dx and dy be the lengths of the sides of the rectangular loop respectively.
- Considering origin of the rectangular co-ordinate system as the centre of the loop.
- The value of the magnetic field at the centre of the rectangular loop carrying current I in anticlockwise direction be B.
- As the rectangular loop is differential with the differential lengths dx and dy, the value of the magnetic field can be assumed B_o everywhere.
- The total force on the loop is zero and the origin for the torque can be selected as the centre of the loop.
- The force exerted on side AB is given by,

$$d\bar{F}_1 = I\,dx\bar{a}_x \times \bar{B}$$

$$\therefore \quad d\bar{F}_1 = I\,dx[\bar{a}_x \times (B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\bar{a}_z)]$$

$$\therefore \quad d\bar{F}_1 = I\,dx[B_y\,\bar{a}_z - B_z\,\bar{a}_y] \qquad ...(1)$$

- For the side AB, the arm extends from origin to the midpoint of the side AB. Thus the arm for side AB is given by,

$$\bar{R}_1 = \frac{1}{2}dy(-\bar{a}_y) = -\frac{1}{2}dy\,\bar{a}_y \qquad ...(2)$$

- Hence the torque on side 1 is given by,

$$d\bar{T}_1 = \bar{R}_1 \times d\bar{F}_1 = -\frac{1}{2}dy\,\bar{a}_y \times \{I\,dx[B_y\,\bar{a}_z - B_z\,\bar{a}_y]\}$$

$$= -\frac{1}{2}dx\,dy\,I\,B_y\bar{a}_x \qquad ...(\because \bar{a}_y \times \bar{a}_z = \bar{a}_x, \bar{a}_y \times \bar{a}_y = 0)$$

...(3)

- The force exerted on side BC is given by,

$$d\bar{F}_2 = I\,dy\,\bar{a}_y \times \bar{B} = I\,dy\,[\bar{a}_y \times (B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z)]$$

$$= I\,dy\,[B_z\,\bar{a}_x - B_x\,\bar{a}_z] \qquad ...(4)$$

- For the side BC, the arm extends from origin to the midpoint of the side BC. Thus the arm for side BC is given by,

$$\bar{R}_2 = \frac{1}{2}dx\,\bar{a}_x \qquad ...(5)$$

- Hence torque on side 2 is given by,

$$d\bar{T}_2 = \bar{R}_2 \times d\,\bar{F}_2 = \frac{1}{2}dx\,\bar{a}_x \times I\,dy\,[B_z\,\bar{a}_x - B_x\,\bar{a}_z]$$

$$= +\frac{1}{2}dxdy\,I\,B_x\,\bar{a}_y \qquad ...(6)$$

- For side CD, the torque contribution is exactly same as that by side AB. The torque on side 3 is given by,

$$d\bar{T}_3 = -\frac{1}{2}dx\,dy\,IB_y\,\bar{a}_x \qquad ...(7)$$

- Similarly for DA, the torque contribution is exactly same as that by side BC. The torque on side 4 is thus given by,

$$d\bar{T}_4 = +\frac{1}{2}dx\,dy\,I\,B_x\,\bar{a}_y \qquad ...(8)$$

- Hence the total torque is given by,

$$d\bar{T} = d\bar{T}_1 + d\bar{T}_2 + d\bar{T}_3 + d\bar{T}_4$$

$$\therefore \quad d\bar{T} = \left(-\frac{1}{2}dx\,dy\,I\,B_y\,\bar{a}_x\right) + \left(+\frac{1}{2}dx\,dy\,I\,B_x\,\bar{a}_y\right) + \left(-\frac{1}{2}dx\,dy\,I\,B_y\bar{a}_x\right) + \left(+\frac{1}{2}dx\,dy\,I\,B_x\,\bar{a}_y\right)$$

$$\therefore \quad d\bar{T} = -dx\,dy\,I\,B_y\,\bar{a}_x + dx\,dy\,I\,B_x\,\bar{a}_y$$

$$\therefore \quad d\bar{T} = I\,dx\,dy\,(B_x\,\bar{a}_y - B_y\,\bar{a}_x)$$

$$\therefore \quad d\bar{T} = I\,dx\,dy\,[\bar{a}_z \times (B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z)]$$

$$d\bar{T} = I\,dx\,dy\,(\bar{a}_z \times \bar{B}) \qquad ...(9)$$

- We can modify above equation by replacing the product term i.e. dx dy by vector area of the differential current loop i.e. $d\bar{S}$

$$\boxed{d\bar{T} = I\,d\bar{S} \times \bar{B}} \qquad ...(10)$$

- Above equation indicates that eventhough the total force exerted on the rectangular loop as a whole is zero, the torque exists along the axis of rotation. i.e. in the z-direction. The expression is valid for all the flat loops of any arbitrary shape.

3.18 : Application Case Study

Q.28 Explain the magnetic levitation.

Ans. : • Let us consider a permanent magnet which is moving at a speed of v m/s across a conducting ladder as shown in the Fig. Q.28.1.

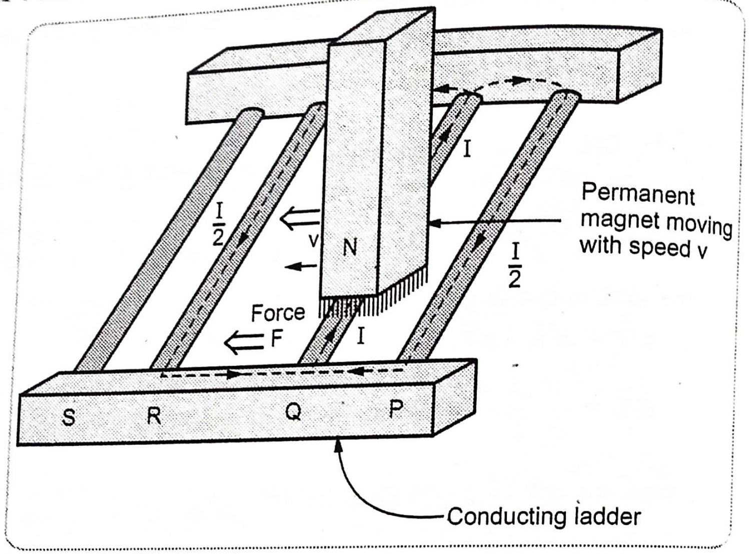


Fig. Q.28.1 Permanent magnet moving on conducting ladder

- This moving magnet tends to drag the conducting ladder along with it because of application of horizontal tractive force, $F = BIl$ where B is flux density in Wb/m^2, I is current flowing through the conductor and l is the active length of the conductor under the influence of magnetic field.
- The plane of motion of magnet and the plane of conducting ladder are perpendicular to each other due to which maximum force is exerted and its direction can be found using Fleming's left hand rule.
- In addition to this horizontal tractive force, a vertical force also exists between the moving magnet and conducting ladder which pushes the magnet away from the ladder in the upward direction. Let I be the current flowing through conductor Q. The front view of above figure is shown in the Fig. Q.28.2.

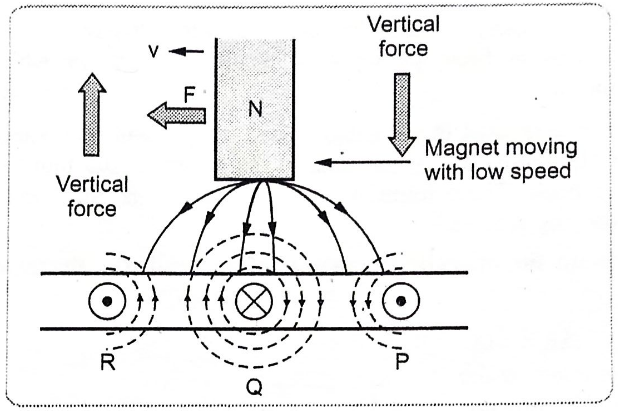


Fig. Q.28.2

- Now let us consider that the magnet is moving at a very high speed. This condition is represented in the Fig. Q.28.3.

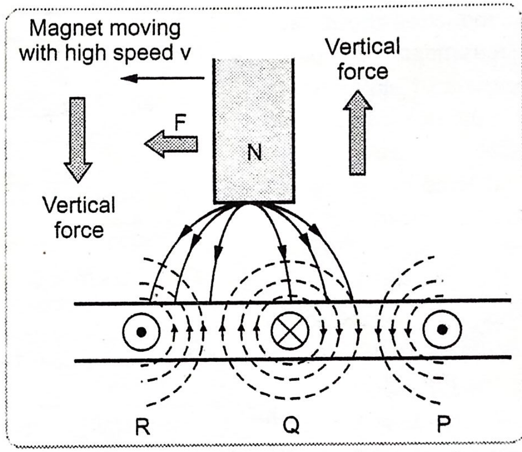


Fig. Q.28.3

- Due to inductance associated with the conductor, current in conductor Q reaches its maximum value a fraction of time, Δt, after voltage reaches its maximum value. This time interval Δt depends on L/R time constant of conductor circuit.
- This delay is very small at low speed such that voltage and current reach their maximum value virtually at the same time and place. At large speeds, this time dealy Δt is sufficient enough to produce a large shift in space between the points where the voltage and current achieve their maximum values.
- By the time current in Q reaches its maximum value, the centre of magnet is already ahead of conductor Q by a distance given as $v\Delta t$. The currents in conductors P, Q and R are established and their interaction with the field of magnet exerts a vertical force in such a way that the

front end of magnet is pushed downwards while the rear end is pulled upwards. This is basic principle of magnetic levitation which means floating in air.

- This principle is used in ultra high speed trains running at speeds in the range of 300 km/hr and which float in the air about 100 mm to 300 mm above the track. These trains do not need traditional steel rail and will not require any wheels.

Q.29 Explain the operating principle of a.c. and d.c. electromagnetic pump.

Ans. : - An a.c. electromagnetic pump works on the principle of induction. Similar to an induction motor, a.c. electromagnetic pump consists of stator which is divided into two parts. A three phase a.c. winding is placed on the stator. Between the two parts of stator, a liquid metal is placed which is required to be circulated under the developed motoring force. The construction is shown in the Fig. Q.29.1.

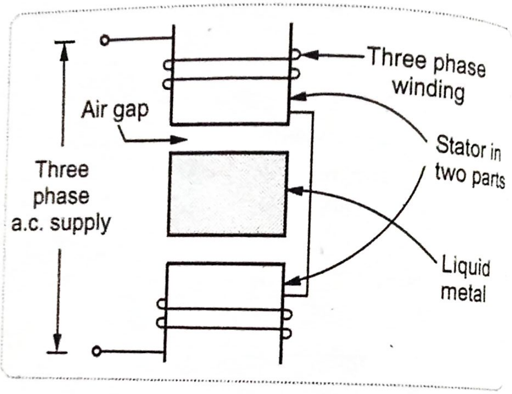


Fig. Q.29.1 A.C. Electromagnetic pump

- A liquid metal is highly electrically conductive. When three phase supply is given to the stator winding, it produces the rotating magnetic field. This rotating flux cut by the liquid metal to induce e.m.f. and circulating eddy currents through it according to Faraday's principle. These currents produce their own magnetic field. This field interacts with the stator magnetic field to produce Lorentz force on the liquid. This force on the liquid circulates the liquid through a required path.
- A pump which operates on the principle that a current carrying conductor placed in a magnetic field experiences a force is a d.c. electromagnetic pump. In a d.c. electromagnetic pump, a current is forced through a liquid metal. The liquid is placed in a magnetic field produced by permanent magnet or produced by the current flowing through the field coils. The current through a liquid metal produces its own flux which interacts with the main magnetic field producing force on the liquid. The direction of circulation of the liquid can be determined uing Fleming's left hand rule.

- The d.c. electromagnetic pump is shown in the Fig. Q.29.2.

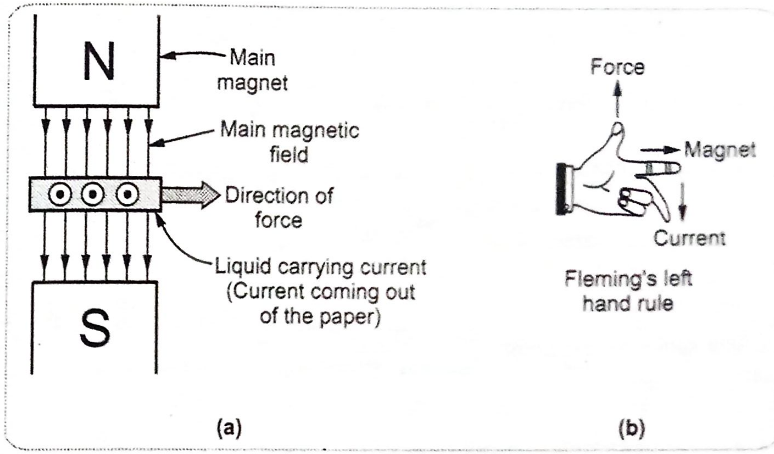


Fig. Q.29.2 D. C. Electromagnetic pump

- The various applications of electromagnetic pump are, use in liquid-metal-cooled reactor plants where liquid lithium, sodium, potassium, or sodium-potassium alloys are circulated, pouring and transportation of high temperature liquids in foundry, in wave soldering machines to circulate molten solder etc.

Formulae at a Glance

- The polarization $\bar{P} = \lim_{\Delta v \to 0} \dfrac{\sum_{i=1}^{n\Delta v} Q_i \bar{d}_i}{\Delta v} \quad (C/m^2)$
- Foe isotropic and linear medium,

$$\bar{P} = \chi_e \varepsilon_0 \bar{E}, \quad \bar{D} = \varepsilon_0 \bar{E} + \bar{P} = \varepsilon_0 \bar{E} + \chi_e \varepsilon_0 \bar{E} = (\chi_e + 1)\varepsilon_0 \bar{E}$$

$$\varepsilon_R = \chi_e + 1 = \text{Relative permittivity}$$

- Boundary conditions between conductor and dielectric,

$$E_{tan} = D_{tan} = 0, \quad D_N = \rho_S, \quad E_N = \frac{\rho_S}{\varepsilon} = \frac{\rho_S}{\varepsilon_0 \varepsilon_r}$$

For conductor-free space, use $\varepsilon_r = 1$ in above results.

- Boundary conditions at two perfect dielectrics are,

$$E_{\tan 1} = E_{\tan 2}, \quad \frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}},$$

$$D_{N1} = D_{N2}, \quad \frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}$$

- Law of refraction, $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\varepsilon_1}{\varepsilon_2} = \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}}$

- For parallel plate capacitor, $C = \dfrac{\varepsilon_0 \varepsilon_r A}{d}$ F where $\varepsilon = \varepsilon_0 \varepsilon_r$

- For co-axial cable, $C = \dfrac{2\pi\varepsilon L}{ln\left[\dfrac{b}{a}\right]}$ F

- For spherical capacitor, $C = \dfrac{4\pi\varepsilon}{\left[\dfrac{1}{a} - \dfrac{1}{b}\right]}$ F

- For composite parallel plate capacitor,

$$C = \frac{A}{\dfrac{d_1}{\varepsilon_1} + \dfrac{d_2}{\varepsilon_2} + \dfrac{d_3}{\varepsilon_3} + \ldots + \dfrac{d_n}{\varepsilon_n}}$$

- Energy stored in a capacitor, $W_E = \dfrac{1}{2} C V^2$ J

- Energy density $= \dfrac{1}{2}\dfrac{|\bar{D}|^2}{\varepsilon} = \dfrac{1}{2}|\bar{D}||\bar{E}|$ J/m^3

- The Laplace's equation in three co-ordinate systems is,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(Cartesian)}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(Cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

(Spherical)

- **Force on a differential current element**

$$\bar{F} = \oint I\, d\bar{L} \times \bar{B}$$

- **Force between differential current elements**

$$\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_1 \times (d\bar{L}_2 \times \bar{a}_{R21})}{R_{21}^2}$$

$$\bar{F}_2 = \frac{\mu_0 I_2 I_1}{4\pi} \oint_{L_2} \oint_{L_1} \frac{d\bar{L}_2 \times (d\bar{L}_1 \times \bar{a}_{R12})}{R_{12}^2}$$

- **Force exerted between current carrying parallel conductors**

$$F = \frac{\mu I_1 I_2 l}{2\pi d}$$

- **Magnetic boundary condition**

A) Boundary conditions for normal component

$$B_{N_1} = B_{N_2}$$

$$\frac{H_{N_1}}{H_{N_2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_{r2}}{\mu_{r1}}$$

B) Boundary conditions for tangential component

a) With current sheet

$$\bar{H}_{tan1} - \bar{H}_{tan2} = \bar{a}_{N12} \times \bar{K}$$

$$\frac{B_{tan1}}{\mu_1} - \frac{B_{tan2}}{\mu_2} = K$$

b) Without current sheet (current free region)

$$H_{tan1} - H_{tan2} = 0 \quad \text{or} \quad H_{tan1} = H_{tan2}$$

$$\frac{B_{tan1}}{B_{tan2}} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}$$

END...

Unit IV

4 Time Varying Electromagnetic Fields : Maxwell's Equations

4.1 : Scalar and Vector Magnetic Potentials

Q.1 State and explain in brief, scalar and vector magnetic potentials.

[SPPU : May-02,04,10,14, Dec.-03,05,07,09,12,14, (In Sem), In Sem-15, Marks 6]

Ans. : In case of magnetic fields there are two types of potentials which can be defined :

1. The scalar magnetic potential denoted as V_m

2. The vector magnetic potential denoted as $\bar{A}$.

- To define scalar and vector magnetic potentials, let us use two vector identities which are,

$$\nabla \times \nabla V = 0, \qquad V = \text{Scalar} \qquad \text{... (1)}$$

$$\nabla \bullet (\nabla \times \bar{A}) = 0, \qquad \bar{A} = \text{Vector} \qquad \text{... (2)}$$

- Every Scalar V_m and Vector $\bar{A}$ must satisfy these identities.

i) Scalar magnetic potential :

- If V_m is the scalar magnetic potential then it must satisfy the equation (1),

$$\therefore \quad \nabla \times \nabla V_m = 0 \qquad \text{... (3)}$$

- But the scalar magnetic potential is related to the magnetic field intensity $\bar{H}$ as,

$$\bar{H} = -\nabla V_m \qquad \text{... (4)}$$

- Using in equation (3),

$$\therefore \quad \nabla \times (-\bar{H}) = 0$$

$$\text{But} \quad \nabla \times \bar{H} = \bar{J} \qquad \text{i.e.} \quad \nabla \times \bar{H} = 0 \qquad \text{... (5)}$$

$$\text{i.e.} \quad \bar{J} = 0 \qquad \text{... (6)}$$

- **Thus scalar magnetic potential V_m can be defined for source free region where $\bar{J}$ i.e. current density is zero.**

$$\therefore \quad \bar{H} = -\nabla V_m \qquad \text{only for } \bar{J} = 0 \qquad \text{... (7)}$$

- Similar to the relation between $\bar{E}$ and electric scalar potential, magnetic scalar potential can be expressed interms of $\bar{H}$ as,

$$\boxed{V_{m\,a,b} = -\int_b^a \bar{H} \bullet d\bar{L}} \qquad \text{... Specified path} \qquad \text{... (8)}$$

ii) Vector magnetic potential :

- The vector magnetic potential is denoted as $\bar{A}$ and measured in Wb/m.
- It has to satisfy equation (2) that divergence of a curl of a vector is always zero.

$$\therefore \quad \nabla \bullet (\nabla \times \bar{A}) = 0 \qquad \text{... } \bar{A} = \text{Vector magnetic potential}$$

$$\text{But} \quad \nabla \bullet \bar{B} = 0 \qquad \text{... From Maxwell's equation}$$

$$\therefore \quad \boxed{\bar{B} = \nabla \times \bar{A}} \qquad \text{... (9)}$$

- Thus **curl of vector magnetic potential is the flux density.**
- Now $\nabla \times \bar{H} = \bar{J}$ i.e. $\nabla \times \dfrac{\bar{B}}{\mu_0} = \bar{J}$ $\qquad$ ... $\bar{B} = \mu_0 \bar{H}$

$$\therefore \quad \nabla \times \bar{B} = \mu_0 \bar{J} \qquad \text{... } \bar{B} = \nabla \times \bar{A}$$

$$\therefore \quad \nabla \times \nabla \times \bar{A} = \mu_0 \bar{J} \qquad \text{... (10)}$$

- Using vector identity to express left hand side we can write,

$$\nabla(\nabla \bullet \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$

$$\therefore \quad \bar{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \bar{A}\right] = \frac{1}{\mu_0}\left[\nabla(\nabla \bullet \bar{A}) - \nabla^2 \bar{A}\right] \qquad \text{... (11)}$$

- Thus if vector magnetic potential is known then current density $\bar{J}$ can be obtained. For defining $\bar{A}$ the current density need not be zero.

Q.2 Magnetic vector potential is given by $\bar{A} = -\dfrac{\rho^2}{4}\bar{a}_z$ Wb/m, calcualte total magnetic flux crossing the surface $\phi = \dfrac{\pi}{2}$, $1 \le \rho \le 2$ m, $0 < z < \le 5$ m.

[SPPU : May-14, Oct.-18, Marks 8]

Ans. : $\quad \bar{A} = -\dfrac{\rho^2}{4}\bar{a}_z, \quad A_\rho = 0, \quad A_\phi = 0, \; A_z = -\dfrac{\rho^2}{4}$

$$\therefore \bar{B} = \nabla \times \bar{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A\phi}{\partial z}\right]\bar{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\bar{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\bar{a}_z$$

$$= [0-0]\,\bar{a}_\rho + \left[0-\left(-\frac{2\rho}{4}\right)\right]\bar{a}_\phi + \frac{1}{\rho}[0-0]\bar{a}_z = \frac{\rho}{2}\bar{a}_\phi \text{ Wb/m}^2$$

$$\phi = \oint_S \bar{B} \bullet d\bar{S}, \; d\bar{S} = d\rho \, dz \, \bar{a}_\phi$$

$$= \int_{z=0}^{5}\int_{\rho=1}^{2} \frac{\rho}{2}\bar{a}_\phi \bullet d\rho \, dz \, \bar{a}_\phi = \int_{z=0}^{5}\int_{\rho=1}^{2} \frac{\rho}{2} d\rho \, dz$$

$$= \frac{1}{2}\left[\frac{\rho^2}{2}\right]_1^2 [z]_0^5 = \frac{1}{2}\times\frac{1}{2}\times[4-1]\times 5 = \mathbf{3.75 \ Wb}$$

Q.3 A current distribution gives rise to the vector magnetic potential $\bar{A} = x^2 y \, \hat{a}_x + y^2 x \hat{a}_y - 4xyz \, \hat{a}_z$ ub/m^2. Calculate -

i) $\bar{B}$ at (– 1, 2, 5)

ii) The flux through the surface defined by $z = 1, 0 \le x \le 1, -1 \le y \le 4$. **[SPPU : Oct.-19, Marks 5]**

Ans. : $\bar{A} = x^2 y \, \bar{a}_x + y^2 x \bar{a}_y - 4xyz \, \bar{a}_z$

i) $$\bar{B} = \nabla \times \bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & y^2 x & -4xyz \end{vmatrix}$$

$$= \bar{a}_x[-4xz-0] - \bar{a}_y[-4yz-0] + \bar{a}_z[y^2 - x^2] \quad ...(1)$$

At (–1, 2, 5) x = –1, y = 2, z = 5

$\therefore$ $$\bar{B} = 20\,\bar{a}_x + 40\,\bar{a}_y + 3\,\bar{a}_z \text{ Wb/m}^2$$

ii) $$\phi = \int \bar{B} \bullet d\bar{S} \quad \text{...Use equation (1) for } \bar{B}$$

The surface is at z = 1 hence $d\bar{S} = dxdy \, \bar{a}_z$

$$\phi = \int_{y=-1}^{4}\int_{x=0}^{1} (y^2 - x^2) \, dx \, dy \qquad \bar{a}_z \bullet \bar{a}_z = 1$$

$$= \int_{y=-1}^{4}\int_{x=0}^{1} y^2 \, dxdy - \int_{y=-1}^{4}\int_{x=0}^{1} x^2 \, dxdy$$

$$= \left[\frac{y^3}{3}\right]_{-1}^{4} [x]_0^1 - \left[\frac{x^3}{x}\right]_0^1 [y]_{-1}^{4}$$

$$= \frac{[64-(-17)]}{3} \times 1 - \frac{1}{3} \times [4-(-17)] = \mathbf{20 \ Wb}$$

4.2 : Poisson's and Laplace's Equations for Magnetic Fields

Q.4 Explain Poisson's and Laplace's equations for magnetic fields.

Ans. : Laplace's equation for scalar magnetic potential :

- It is known that as monopole of magnetic field is non existing,

$$\oint \bar{B} \bullet d\bar{S} = 0 \quad ... (1)$$

- Using Divergence theorem,

$$\oint \bar{B} \bullet d\bar{S} = \int_{vol} (\nabla \bullet \bar{B}) \, dv = 0 \quad ... (2)$$

$$\therefore \quad \nabla \bullet \bar{B} = 0 \quad ... (3)$$

$$\therefore \quad \nabla \bullet (\mu_0 \bar{H}) = 0 \quad \text{but } \mu_0 \ne 0 \quad ... (4)$$

$$\therefore \quad \nabla \bullet \bar{H} = 0 \quad ... (5)$$

$$\therefore \nabla \bullet (-\nabla V_m) = 0 \quad \text{... Using } \bar{H} = -\nabla V_m$$

$$\therefore \quad \boxed{\nabla^2 V_m = 0 \quad \text{for} \quad \bar{J} = 0} \quad ... (6)$$

- This is Laplace's equation for scalar magnetic potential. This is similar to the Laplace's equation for scalar electric potential $\nabla^2 V = 0$.

Poisson's equation for magnetic field :

- In vector magnetic potential $\bar{A}$ is known, then current density $\bar{J}$ can be defined as,

$$\bar{J} = \frac{1}{\mu_0}[\nabla(\nabla \bullet \bar{A}) - \nabla^2 \bar{A}]$$

- In a vector algebra, vector can be fully defined if its curl and divergence are defined.
- For $\bar{A}$, $\nabla \times \bar{A} = \bar{B}$ i.e. curl is defined. To define $\bar{A}$ completely let its divergence is known to bc zero.

As $\nabla \bullet \bar{A} = 0$, $\bar{J} = \frac{1}{\mu_0}[-\nabla^2 \bar{A}]$

$$\therefore \quad \boxed{\nabla^2 \overline{A} = -\mu_0 J}$$

- This is Poisson's equation for magnetic field.

4.3 : Faraday's Law and Lenz's Law

Q.5 State and explain Faraday's law and Lenz's Law.
[SPPU : Dec.-01,05,17, May-04,07,13, Marks 6]

Ans. : Faraday's Law :

- **Statement** : "The electromotive force (e.m.f.) induced in a closed path (or circuit) is proportional to rate of change of magnetic flux enclosed by the closed path (or linked with the circuit)."
- The statement of Faraday's law was based on the experiments performed by Faraday and the experiments are commonly known as **Faraday's experiment**.
- According to Faraday's law when a closed path moves in a magnetic field, current is generated and hence e.m.f. is generated.
- Also if closed path kept fixed and the magnetic field was varied the current and e.m.f. are generated. The effect is commonly called **electromagnetic induction.**
- **Faraday's law** can be stated as,

$$e = -N\frac{d\phi}{dt} \text{ volts} \qquad \dots (1)$$

where N = Number of turns in the circuit, e = Induced e.m.f.

- For single turn circuit i.e. N = 1, **Faraday's law** can be stated as,

$$e = -\frac{d\phi}{dt} \text{ volts} \qquad \dots (2)$$

- The minus sign in equations (1) and (2) indicates that the direction of the induced e.m.f. is such that to produce a current which will produce a magnetic field which will oppose the original field.
- **Lenz's law** : The direction of induced e.m.f. is such that it opposes the cause producing it i.e. change in magnetic flux.
- The induced e.m.f. is a scalar quantity measured in volts. Thus the induced e.m.f. is given by,

$$e = \oint \overline{E} \bullet d\overline{L} \qquad \dots (3)$$

- The induced e.m.f. in equation (3) indicates a voltage about a closed path such that if any part of the path is changed, the e.m.f. will also change.

- The magnetic flux ϕ passing through a specified area is given by

$$\phi = \int_S \overline{B} \bullet d\overline{S} \quad \text{Where B = Magnetic flux density}$$

- Using above result, equation (2) can be rewritten as

$$e = -\frac{d}{dt}\int_S \overline{B} \bullet d\overline{S} \qquad \dots (4)$$

$$e = \oint \overline{E} \bullet d\overline{L} = -\frac{d}{dt}\int_S \overline{B} \bullet d\overline{S} \qquad \dots (5)$$

Q.6 A circular loop lies in z = 0 plane has radius of 0.2 m and resistance of 10 ohm. Find the current flowing through the conductor due to field $\overline{B} = 0.2 \sin 10^3 t\ \hat{a}_z$. **[SPPU : May-19, Marks 8]**

Ans. : The loop is in z = 0 plane and $\overline{B}$ is in z direction i.e. perpendicular to the loop.

$$\phi = \int_S \overline{B} \bullet d\overline{S}$$

With cylindrical co-ordinates, $d\overline{S} = r\, dr\, d\phi\, \overline{a}_z$

$$\therefore \quad \phi = \int_{\phi=0}^{2\pi}\int_{r=0}^{0.2} [0.2 \sin 10^3 t\, \overline{a}_z] \bullet [r\, dr\, d\phi\, \overline{a}_z]$$

$$= 0.2 \sin 10^3 t\, [\phi]_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{0.2} = 0.0251 \sin 10^3 t \text{ Wb}$$

$$\therefore \quad e = -\frac{d\phi}{dt} = -\frac{d}{dt}[0.0251 \sin 10^3 t] = -25.1 \cos 10^3 t \text{ V}$$

$$\therefore \quad i = \frac{e}{R} = \frac{-25.1 \cos 10^3 t}{10} = \mathbf{-2.51 \cos 10^3 t\ A}$$

4.4 : Transformer and Mational E.M.F.

Q.7 Derive an expression for e.m.f. in moving conducting loop in a time varying field. **[SPPU : Dec.-10,17, May-11, Marks 8]**

Ans. : A) A stationary closed path in a time varying $\overline{B}$ field.

- When an e.m.f. is induced in a stationary closed path due to time varying $\overline{B}$ field, the e.m.f. is called **statically induced e.m.f.** or **transformer e.m.f.**
- The condition in which a closed path is stationary and the magnetic field $\overline{B}$ is varying with time is as shown in the Fig. Q.7.1.
- The closed circuit in which e.m.f. is induced is stationary and the magnetic flux is sinusoidally varying with time.

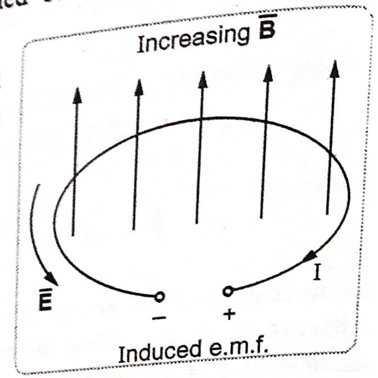


Fig. Q.7.1 Stationary closed path in time varying $\overline{B}$ field

- From equation (1) it is clear that the magnetic flux density is the only quantity varying with time.
- We can use partial derivative to define relationship as $\overline{B}$ may be changing with the co-ordinates as well as time. Hence we can write,

$$\oint \overline{E} \bullet d\overline{L} = -\int_S \frac{\partial \overline{B}}{\partial t} \bullet d\overline{S} \quad \dots (1)$$

- This is similar to **transformer action** and e.m.f. is called **transformer e.m.f.**.
- Using Stoke's theorem, a line integral can be converted to the surface integral as

$$\oint_S (\nabla \times \overline{E}) \bullet d\overline{S} = -\int_S \frac{\partial \overline{B}}{\partial t} \bullet d\overline{S} \quad \dots (2)$$

Assuming that both the surface integrals taken over identical surfaces.

$$\therefore (\nabla \times \overline{E}) \bullet d\overline{S} = -\frac{\partial \overline{B}}{\partial t} \bullet d\overline{S}$$

Hence finally, $\boxed{\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}}$ ... (3)

- Equation (3) represents one of the Maxwell's equations.

B) A time varying closed path in a static $\overline{B}$ field.

- When the e.m.f. is induced in a time varying closed path due to a static $\overline{B}$ field, the e.m.f. is called **dynamically induced** or **motional e.m.f.**
- The condition in which magnetic field $\overline{B}$ is stationary while a closed path is moving or revolving is as shown in the Fig. Q.7.2.

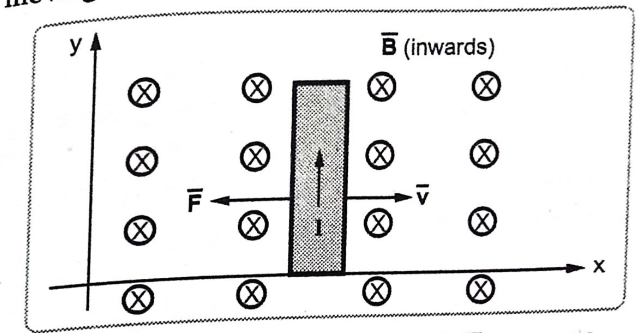


Fig. Q.7.2 A closed path moving in a static $\overline{B}$ field with velocity $\overline{v}$

- The magnetic field is stationary, constant not varying with time while the closed circuit is revolved to get the relative motion between them.
- This action is similar to **generator action**, hence the induced e.m.f. is called **motional** or **generator e.m.f.**
- Consider that a charge Q is moved in a magnetic field $\overline{B}$ at a velocity $\overline{v}$. Then the force on a charge is given by,

$$\overline{F} = Q\, \overline{v} \times \overline{B} \quad \dots (4)$$

But the motional electric field intensity is defined as the force per unit charge. It is given by,

$$\therefore \quad \overline{E}_m = \frac{\overline{F}}{Q} = \overline{v} \times \overline{B} \quad \dots (5)$$

Thus the induced e.m.f. is given by

$$\boxed{\oint \overline{E}_m \bullet d\overline{L} = \oint (\overline{v} \times \overline{B}) \bullet d\overline{L}} \quad \dots (6)$$

- Equation (6) represents total e.m.f. induced when a conductor is moved in a uniform constant magnetic field.

C) A time varying closed path in a time varying $\overline{B}$ field.

- A moving closed path in a time varying $\overline{B}$ field represents a general case in which both the e.m.f.s i.e. transformer e.m.f. and motional e.m.f. are present.

- Thus the induced e.m.f. is the combination of two e.m.f.s. Hence the induced e.m.f. for a moving closed path in a time varying field $\bar{B}$ can be expressed as,

$$\text{Total induced e.m.f.} = \text{Transformer e.m.f.} + \text{Motional e.m.f.}$$

$$\boxed{\oint \bar{E}\bullet d\bar{L} = -\int \frac{\partial \bar{B}}{\partial t}\bullet d\bar{S} + \oint (\bar{v}\times\bar{B})\bullet d\bar{L}} \quad \dots (7)$$

Q.8 A conducting bar can slide freely over two conducting rails as shown in Fig. Q.8.1 below. Calculate the induced voltage in the bar.

i) If the bar is stationed at y = 8 cm and $\bar{B} = 4\cos 10^6 t\, \bar{a}_z$ mWb/m²

ii) If the bar slides at a velocity $\bar{V} = 20\, \bar{a}_z$ m/s and $\bar{B} = 4\, \bar{a}_z$ mWb/m²

iii) If the bar slides at a velocity $\bar{V} = 20\, \bar{a}_y$ m/s and $\bar{B} = 4\cos(10^6 t - y)\, \bar{a}_z$ mWb/m²

[SPPU : May-18, Marks 8]

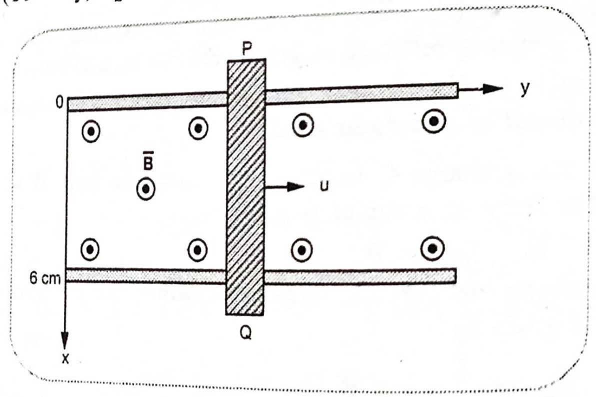


Fig. Q.8.1

Ans. :

a) $$e = -\int \frac{\partial \bar{B}}{\partial t}\bullet d\bar{S} = \int_{y=0}^{0.08}\int_{x=0}^{0.06} 4(10^{-3})(10^6)\sin 10^6 t \, dx\, dy$$

$$= 4\times10^3 \times 0.08\times 0.06\times \sin 10^6 t = \mathbf{19.2 \sin 10^6 t \ V}$$

b) $$e = \int (v\times\bar{B})\bullet d\bar{L} = \int_{x=l}^{0} (v\bar{a}_y \times B\bar{a}_z)\,(dx\bar{a}_x)$$

$$= -vBl - (20)\times(4\times10^{-3})\,(0.06) = \mathbf{-4.8\ mV}$$

c) $$e = -\int \frac{\partial \bar{B}}{\partial t}\bullet d\bar{S} + \int (v+\bar{B})\bullet d\bar{L}$$

$$= \int_{x=0}^{0.06}\int_{0}^{y} 4\times10^{-3}(10^6)\sin(10^6 t - y')\,dy'\,dx$$

$$+ \int_{0.06}^{0} [20\,\bar{a}_y + 4\times10^{-3}\cos(10^6 t - y)\bar{a}_z] \bullet (dx\bar{a}_x)$$

$$= \left[240\cos(10^6 t - y')\right]_0^y - 80(1\times10^{-3})(0.06)\cos(10^6 t - y)$$

$$= 240\cos(10^6 t - y) - 240\cos 10^6 t - 4.8\times10^3\cos(10^6 t - y)$$

$$= \mathbf{240\cos(10^6 t - y) - 240\cos 10^6 t \ \ V}$$

4.5 : Displacement Current Density and Displacement Current

Q.9 Define displacement current and displacement current density and hence show that $\nabla\times\bar{H} = \bar{J}_C + \bar{J}_D$, where $\bar{J}_C$ = conduction current density, $\bar{J}_D$ = displacement current density.

[SPPU : May-06,13,15,19, Dec.-15,18,19, Marks 8]

Ans. : • For static electromagnetic fields, according to Ampere's circuital law, we can write,

$$\nabla\times\bar{H} = \bar{J} \quad \dots (1)$$

- Taking divergence on both the sides,

$$\nabla\bullet(\nabla\times\bar{H}) = \nabla\bullet\bar{J}$$

- But according to vector identity, **'divergence of the curl of any vector field is zero'**. Hence we can write,

$$\nabla\bullet(\nabla\times\bar{H}) = \nabla\bullet\bar{J} = 0 \quad \dots (2)$$

- But the equation of continuity is given by,

$$\nabla\bullet\bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \dots (3)$$

- From equation (3) it is clear that when $\frac{\partial \rho_v}{\partial t} = 0$, then only equation (2) becomes true.

- Thus equations (2) and (3) are not compatible for time varying fields. We must modify equation (1) by adding one unknown term say $\bar{N}$. Then equation (1) becomes,

$$\nabla \times \bar{H} = \bar{J} + \bar{N} \quad \text{... (4)}$$

Again taking divergence on both the sides

$$\nabla \bullet (\nabla \times \bar{H}) = \nabla \bullet \bar{J} + \nabla \bullet \bar{N} = 0$$

As $\nabla \bullet \bar{J} = -\frac{\partial \rho_v}{\partial t}$, to get correct conditions we must write,

$$\nabla \bullet \bar{N} = \frac{\partial \rho_v}{\partial t}$$

But according to Gauss's law, $\rho_v = \nabla \bullet \bar{D}$

Thus replacing ρ_v by $\nabla \bullet \bar{D}$

$$\nabla \bullet \bar{N} = \frac{\partial}{\partial t}(\nabla \bullet \bar{D}) = \nabla \bullet \frac{\partial \bar{D}}{\partial t}$$

Comparing two sides of the equation,

$$\bar{N} = \frac{\partial \bar{D}}{\partial t} \quad \text{... (5)}$$

- Now we can write Ampere's circuital law in point form as,

$$\nabla \times \bar{H} = \bar{J}_C + \frac{\partial \bar{D}}{\partial t} \quad \text{.... (6)}$$

- The first term in equation (6) is **conduction current density** denoted by $\bar{J}_C$. Here attaching subscript C indicates that the current is due to the moving charges.
- The second term in equation (6) represents current density expressed in ampere per square meter.
- As this quantity is obtained from time varying electric flux density. This is also called **displacement density**. Thus this is called **displacement current density** denoted by $\bar{J}_D$.
- With these definitions we can write equation (6) as,

$$\nabla \times \bar{H} = \bar{J}_C + \bar{J}_D \quad \text{.... (7)}$$

- Thus in a given medium, both the types of the currents, namely the conduction current and the displacement current may flow. Hence the two current densities can be written as,

$$\left.\begin{aligned} \bar{J}_C &= \sigma \bar{E} && \text{....Conduction current density} \\ \bar{J}_D &= \frac{\partial \bar{D}}{\partial t} && \text{....Displacement current density} \end{aligned}\right\} \quad \text{... (8)}$$

- The total current density is given by,

$$\bar{J} = \bar{J}_C + \bar{J}_D \quad \text{... (9)}$$

Q.10 Show that the ratio of the amplitude of the conduction current and displacement current density is $\frac{\sigma}{\omega\varepsilon}$ for the applied field $E = E_{max}$ sin ωt V/m. **[SPPU : Dec.-13, May-16, Marks 4]**

Ans. : a) The conduction current density is given by

$$J_C = \sigma E = \sigma E_m \cos \omega t$$

The displacement current density is given by

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial \varepsilon E}{\partial t} = \varepsilon \frac{\partial}{\partial t}[E_m \cos \omega t]$$

$$J_D = -\omega \varepsilon E_m \sin \omega t$$

∴ The ratio of the amplitudes of the two densities is given by

$$\frac{|J_C|}{|J_D|} = \frac{\sigma E_m}{\omega \varepsilon E_m} = \frac{\sigma}{\omega \varepsilon}$$

Q.11 Show that the displacement current in dielectric of parallel plate capacitor is equal to the conduction current in the leads. **[SPPU : Dec.-02, Marks 8]**

Ans. : • Consider a parallel plate capacitor is connected to time varying voltage v as shown in the Fig. Q.11.1.

- Let distance of separation between parallel plates be d. Let the area of plates in parallel be A. Let the applied voltage be $v = V_m \sin \omega t$.
- The current flowing through the leads of the capacitor is conduction current and it flows when voltage is applied.

$$\therefore \quad i_C = C\frac{dv}{dt} = C(V_m \omega)\cos \omega t$$

$$\therefore \quad i_C = (C\omega) V_m \cos \omega t \quad \text{...(1)}$$

- The displacement current is the current flowing through the dielectric between parallel plates. By definition it is given by,

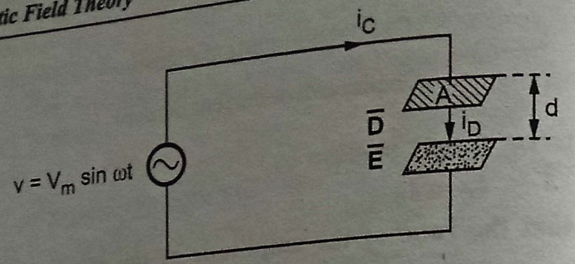


Fig. Q.11.1

$$i_D = \int_S \overline{J}_D \bullet d\overline{S} = \int_S \frac{\partial \overline{D}}{\partial t} \bullet d\overline{S} \quad ...(2)$$

- The flux density $\overline{D}$ between plates is normal to the plates. In other words, $\overline{D}$ and $d\overline{S}$ are in same direction. Hence the dot product changes to simple multiplication. So we can write,

$$i_D = \int_S \frac{\partial D}{\partial t} dS = \int_S \frac{\partial(\varepsilon E)}{\partial t} dS = \varepsilon \int_S \frac{\partial E}{\partial t} dS \quad ...(3)$$

- The electric field is related to potential through relation

$$E = \frac{v}{d}$$

$$\therefore \quad i_D = \varepsilon \int_S \frac{\partial}{\partial t}\left(\frac{v}{d}\right) dS = \frac{\varepsilon}{d} \int_S \frac{\partial v}{\partial t} dS = \frac{\varepsilon}{d} \frac{\partial v}{\partial t} \int_S dS \quad ...(4)$$

- Now the integral term represents area of the plates i.e. A, hence we get,

$$i_D = \frac{\varepsilon}{d} \frac{\partial v}{\partial t} A \quad ...(5)$$

- Now v is changing with respect to time we can change partial derivative to direct derivative and can modify equations as,

$$i_D = \frac{\varepsilon A}{d} \frac{\partial v}{\partial t}$$

But $\quad \frac{\varepsilon A}{d} = C = \text{Capacitance in farad}$

$$\therefore \quad i_D = C \frac{dv}{dt} = C \frac{d}{dt} (V_m \sin \omega t)$$

$$\therefore \quad i_D = (C\,\omega)\, V_m \cos \omega t \quad ...(6)$$

- Thus from equations (1) and (6) we can prove that the displacement current in the dielectric of parallel plate capacitor is equal to the conduction current in the leads.

Q.12 The electric field intensity $\overline{E}$ = 250 sin 10^{10}t V/m for a field propagating in a medium whose σ = 5.0 S/m and ε_r = 1.0 calculate J_D, J_C and frequency at which $J_C = J_D$.

[SPPU : Dec.-12,14,18, Marks 10]

Ans. : The conduction current density is given by,

$$J_C = \sigma E = 5\left(250 \sin 10^{10} t\right) = \mathbf{1250 \sin 10^{10}\ t\ \ A/m^2}$$

The displacement current density is given by

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon E) = \frac{\partial}{\partial t}[\varepsilon_0 \varepsilon_r E]$$

$$= \frac{\partial}{\partial t}\left[8.854 \times 10^{-12} \times 1 \times 250 \sin 10^{10} t\right]$$

$$= \left(8.854 \times 10^{-12} \times 250\right)\left(10^{10}\right)\left(\cos 10^{10} t\right)$$

$$= \mathbf{22.135 \cos 10^{10}\ t\ \ A/m^2}$$

For the two densities, the condition for magnitudes to be equal is,

$$\frac{|\overline{J}_C|}{|\overline{J}_D|} = \frac{\sigma}{\varepsilon \omega} = 1$$

$$\therefore \quad \omega = \frac{\sigma}{\varepsilon} = \frac{5}{8.854 \times 10^{-12} \times 1} = 5.6471 \times 10^{11}$$

But $\quad \omega = 2\pi f$

$$\therefore \quad f = \frac{\omega}{2\pi} = \frac{5.6471 \times 10^{11}}{2\pi} = \mathbf{89.87\ GHz}$$

Q.13 A parallel plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin 10^3t volts applied to it's plates. Calculate the displacement current, assuming $\varepsilon = 2\,\varepsilon_0$.

[SPPU : Dec.-06,19, Marks 8]

Ans. : $\quad D = \varepsilon E = \varepsilon \frac{V}{d}$

Hence the displacement current density is given by,

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon \frac{V}{d}\right) = \frac{\varepsilon}{d}\frac{dV}{dt}$$

Hence the displacement current is given by

$$i_D = J_D \cdot \text{Area} = \left(\frac{\varepsilon}{d}\frac{dV}{dt}\right)(A) \quad \text{... Plate area = A}$$

$$\therefore \quad i_D = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

This current is same as conduction current.

$$\therefore \quad i_C = \frac{dQ}{dt} = A\frac{dD}{dt} = \varepsilon A\frac{dE}{dt} = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

Hence the conduction current and displacement current is same. The displacement current is given by,

$$i_D = \frac{\varepsilon A}{d}\frac{dV}{dt} = \frac{(2\varepsilon_0)(A)}{d}\frac{dV}{dt}$$

$$= \frac{2\times 8.854\times10^{-12}\times5\times10^{-4}}{3\times10^{-3}}\frac{d}{dt}\left(50\sin 10^3 t\right)$$

$$= \frac{2\times 8.854\times10^{-12}\times5\times10^{-4}\times50\times10^{3}}{3\times10^{-3}}\cos 10^3 t$$

$$= \mathbf{0.1475\cos 10^3 t\ \mu A}$$

Important Points to Remember

- If $\frac{\sigma}{\omega\varepsilon} >> 1$, Medium is conductor
- If $\frac{\sigma}{\omega\varepsilon} << 1$, Medium is dielectric

Q.14 The magnetic field of an EM wave in free space in given by $\bar{H} = 0.5\ \varepsilon_0 \cos(\omega t - 100z)\hat{a}_y \frac{A}{m}$. Find the electric field intensity and displacement current density.

[SPPU : May-17, Marks 8]

Ans. : $\bar{H} = 0.5\varepsilon_0 \cos(\omega t - 100z)\bar{a}_y$ A/m

In free space, $\nabla\times\bar{H} = \frac{\partial\bar{D}}{\partial t} = \varepsilon_0\frac{\partial\bar{E}}{\partial t}$

$\bar{H}$ has no component in x and z direction hence $H_x = H_z = 0$.

$$\therefore \quad \nabla\times\bar{H} = -\frac{\partial H_y}{\partial z}\bar{a}_x + \frac{\partial H_y}{\partial x}\bar{a}_z \quad \dots \frac{\partial H_y}{\partial x} = 0$$

$$\therefore \quad \nabla\times\bar{H} = -0.5\varepsilon_0\frac{\partial}{\partial z}[\cos(\omega t - 100z)]\,\bar{a}_x$$

$$\therefore \quad \nabla\times\bar{H} = -0.5\varepsilon_0(-100)[-\sin(\omega t - 100z)]\,\bar{a}_x$$

$$\therefore \quad \varepsilon_0\frac{\partial\bar{E}}{\partial t} = -50\varepsilon_0\sin(\omega t - 100z)\,\bar{a}_x$$

$$\therefore \quad \frac{\partial\bar{E}}{\partial t} = -50\sin(\omega t - 100z)\,\bar{a}_x$$

Integrating, $\bar{E} = -50\int\sin(\omega t - 100z)\bar{a}_x\,dt$

$$\therefore \quad \bar{E} = -50\left[\frac{-\cos(\omega t - 100z)}{\omega}\right]\bar{a}_x$$

$$\therefore \quad \bar{E} = \mathbf{\frac{50}{\omega}\cos(\omega t - 100z)\,\bar{a}_x\ V/m}$$

$$\bar{J}_D = \nabla\times\bar{H} = \mathbf{0.5\varepsilon_0\sin(\omega t - 100z)\,\bar{a}_x\ A/m^2}$$

4.6 : Continuity Equation for Time Varying Fields

Q.15 Derive an expression for continuity equation for time varying fields.

[SPPU : Dec.-18, Marks 8]

Ans. : • The basic relation between an electric and magnetic field, starting from Faraday's law is given by,

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \quad \text{... (1)}$$

But we have $\bar{B} = \nabla\times\bar{A}$ where $\bar{A}$ is vector magnetic potential.

$$\therefore \quad \nabla\times\bar{E} = -\frac{\partial}{\partial t}(\nabla\times\bar{A}) \quad \text{... (2)}$$

- Interchanging operators at R.H.S. of above equation, we get,

$$\nabla\times\bar{E} = -\nabla\times\frac{\partial\bar{A}}{\partial t} \quad \text{i.e.} \quad \nabla\times\bar{E} + \nabla\times\frac{\partial\bar{A}}{\partial t} = 0$$

$$\therefore \quad \nabla\times\left(\bar{E} + \frac{\partial\bar{A}}{\partial t}\right) = 0 \quad \text{... (3)}$$

- But according to vector identity **'curl' of a gradient of a scalar is always zero**. Hence we can write,

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = \nabla v \quad \text{... (4)}$$

- As R.H.S. of the equation (3) including curl is zero, we can introduce negative sign at R.H.S. of the equation (4).

$$\therefore \quad \bar{E} = -\nabla v - \frac{\partial \bar{A}}{\partial t} \quad \text{... (5)}$$

- Now when the field is static, $\frac{\partial \bar{A}}{\partial t} = 0$, hence we get

$$\bar{E} = -\nabla v \quad \text{... (6)}$$

- Consider any closed surface. If the current is flowing out of the surface, we can write

$$I = \frac{dQ}{dt} \text{ A} \quad \text{i.e. C/sec}$$

- As current is flowing out of the surface, it indicates that positive charge is going out. So the positive charge is decreasing internally.
- Let Q_I be the internal charge,

$$\therefore \quad I = -\frac{dQ_I}{dt} \quad \text{... (7)}$$

If there is a volume charge ρ_V, then we can write,

$$Q_I = \int_V \rho_V \, dv \quad \text{... (8)}$$

$$\therefore \quad I = -\frac{d}{dt}\left[\int_V \rho_V \, dv\right]$$

Changing operations, we can write,

$$I = -\int_V \frac{d\rho_V}{dt} dv \quad \text{... (9)}$$

But current can be expressed as,

$$I = \int_V \bar{J} \bullet d\bar{S} \quad \text{... (10)}$$

Equating equations (9) and (10),

$$\int_V \bar{J} \bullet d\bar{S} = -\int_V \frac{d\rho_V}{dt} dv$$

- Using divergence theorem, converting surface integral to volume integral, assuming that the volume V is enclosed by the same surface S.

$$\int_V \nabla \bullet \bar{J} \, dv = -\int_V \frac{d\rho_V}{dt} dv$$

$$\therefore \quad \boxed{\nabla \bullet \bar{J} = -\frac{d\rho_V}{dt}} \quad \text{... (11)}$$

- Equation (11) is called **equation of continuity of current in point or differential form.**

4.7 : Time Varying Maxwell's Equations

Q.16 Explain Maxwell's equations for time varying electromagnetic fields. **☞[SPPU : May-13, 19, Marks 4, Dec.-11, Marks 8; Aug.-17, May-18, Dec.-18, 01, 02, Marks 6]**

OR State and explain Maxwell's equations for static electric and magnetic fields in both integral and point form. **☞[SPPU : May-16, 14, 11, 17, Dec.-14, 12, 17, Marks 8]**

OR Derive Maxwell's equations in both integral and differential form. **☞ [SPPU : Oct.-18, May-12, Marks 8]**

Ans. : • The basic Maxwell's equations for static fields are derived from Faraday's law, Ampere circuit law, Gauss's law for electrostatic field and Gauss's law for magnetostatic field.

Maxwell's Equations for Static Fields

A] Maxwell's Equation Derived from Faraday's Law

- According to the basic concept from an electrostatic field, the work done over a **closed path** or **closed contour** (i.e. starting point same as terminating point) is always zero. Mathematically it is represented as,

$$\oint \bar{E} \bullet d\bar{L} = 0 \quad \text{... (1)}$$

The equation (1) is called **integral form of Maxwell's equation derived from Faraday's law for static field.**

- Now using Stoke's theorem converting the closed line integral into the surface integral, we get,

$$\oint \bar{E} \bullet d\bar{L} = \int_S (\nabla \times \bar{E}) \bullet d\bar{S} = 0$$

$$\therefore \quad \int_S (\nabla \times \bar{E}) \bullet d\bar{S} = 0$$

But $d\bar{S}$ cannot be zero (i.e. $d\bar{S} \neq 0$). That means,

$$\nabla \times \bar{E} = 0 \qquad \dots (2)$$

The equation (2) is called **point or differential form of Maxwell's equation derived from Faraday's law for static fields.**

B] Maxwell's Equation Derived from Ampere's Circuit Law

- According to basic concept from magnetostatics an Ampere's circuital law states that the line integral of magnetic field intensity $\bar{H}$ around a closed path is exactly equal to the direct current enclosed by that path. Mathematically it is given as,

$$\oint \bar{H} \bullet d\bar{L} = I = \int_S \bar{J} \bullet d\bar{S} \quad \text{where} \quad \bar{J} = \text{Current density}$$

$$\oint \bar{H} \bullet d\bar{L} = \int_S \bar{J} \bullet d\bar{S} \qquad \dots (3)$$

The equation (3) is called **integral form of Maxwell's equation derived from Ampere's circuit law for static field.**

- Now to relate $\bar{H}$ with $\bar{J}$, converting closed line integral on L.H.S. of the equation (3) to surface integral using Stoke's theorem, we get

$$\oint \bar{H} \bullet d\bar{L} = \int_S (\nabla \times \bar{H}) \bullet d\bar{S} = \int_S \bar{J} \bullet d\bar{S} \qquad \dots (4)$$

The equation (4) is called **point or differential form of Maxwell's equation derived from Ampere's circuit law for static field.**

C] Maxwell's Equation Derived from Gauss's Law for Electrostatic Fields

- According to Gauss's law for electrostatic field, the electric flux passing through any closed surface is equal to the total charge enclosed by that surface. Mathematically we can write,

$$\psi = \oint \bar{D} \bullet d\bar{S} = Q_{enclosed}$$

The most common form to represent Gauss's law mathematically is with volume charge density ρ_V. Hence we can write,

$$\oint \bar{D} \bullet d\bar{S} = \int_V \rho_V \, Dv \qquad \dots (5)$$

The equation (5) is called **integral form of Maxwell's equation derived from Gauss's law for static electric field.**

Using Divergence theorem as,

$$\oint \bar{D} \bullet d\bar{S} = \int_V (\nabla \bullet \bar{D}) \, dv$$

Comparing above equation with equation (6), we can write,

$$\int_V (\nabla \bullet \bar{D}) \, dv = \int_V \rho_V \, dv$$

$$\nabla \bullet \bar{D} = \rho_V \qquad \dots (7)$$

The equation (7) is called **point or differential form of Maxwell's equation derived from Gauss's law for static electric field.**

D] Maxwell's Equation Derived from Gauss's Law for Magnetostatic Field

- According to the Gauss's law for the magnetostatic field, the magnetic flux cannot reside in a closed surface due to the non existance of single magnetic pole.

Mathematically we can write,

$$\oint_S \bar{B} \bullet d\bar{S} = 0 \qquad \dots (8)$$

The equation (8) is called **integral form of Maxwell's equation derived from Gauss's law for static magnetic field.**

Now using divergence theorem, we can write,

$$\oint_S \bar{B} \bullet d\bar{S} = \int_V (\nabla \bullet \bar{B}) \, dv = 0 \quad \text{i.e} \int_V (\nabla \bullet \bar{B}) \, dv = 0$$

Now dv cannot be zero, that means,

$$\nabla \bullet \bar{B} = 0 \quad \text{... (9)}$$

The equation (9) is called **point or differential form of Maxwell's equation derived from Gauss's law for static magnetic field.**

Maxwell's Equations for Time Varying Fields

A] Maxwell's Equation Derived from Faraday's Law

- From Faraday's law, we can write,

$$\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S} \quad \text{... (10)}$$

This is Maxwell's equation derived from Faraday's law expressed in **integral form**.

Statement : "The total electromotive force (e.m.f.) induced in a closed path is equal to the negative surface integral of the rate of change of flux density with respect to time over an entire surface bounded by the same closed path."

- Using Stoke's theorem, converting line integral of equation (10) to the surface integral,

$$\int_S (\nabla \times \bar{E}) \bullet d\bar{S} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S} \quad \text{... (11)}$$

Assuming that the integration is carried out over the same surface on both the sides, we get,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{... (12)}$$

This is **Maxwell's equation derived from Faraday's law** expressed in **point form** or **differential form.**

B] Maxwell's Equation Derived from Ampere's Circuit Law

- According to Ampere's circuit law, the line integral of magnetic field intensity $\bar{H}$ around a closed path is equal to the current enclosed by the path.

$$\therefore \quad \oint \bar{H} \bullet d\bar{L} = I_{enclosed}$$

Replacing current by the surface integral of conduction current density $\bar{J}$ over an area bounded by the path of integration of $\bar{H}$, we get more general relation as,

$$\oint \bar{H} \bullet d\bar{L} = \int_S \bar{J} \bullet d\bar{S} \quad \text{... (13)}$$

Above expression can be made further general by adding displacement current density to conduction current density as follows,

$$\oint \bar{H} \bullet d\bar{L} = \int_S \left[\bar{J} + \frac{\partial \bar{D}}{\partial t} \right] \bullet d\bar{S} \quad \text{... (14)}$$

Equation (14) is **Maxwell's equation derived from Ampere's circuit law**. This equation is in **integral form** in which line integral of $\bar{H}$ is carried over the closed path bounding the surface S over which the integration is carried out on R.H.S. In the circuit theory, closed path is called **Mesh**. Hence the equation considered above is also called **Mesh equation** or **Mesh relation.**

Statement : "The total magnetomotive force around any closed path is equal to the surface integral of the conduction and displacement current densities over the entire surface bounded by the same closed path."

Applying Stoke's theorem to L.H.S. of the equation (14), we get,

$$\int_S (\nabla \times \bar{H}) \bullet d\bar{S} = \int_S \left[\bar{J} + \frac{\partial \bar{D}}{\partial t} \right] \bullet d\bar{S}$$

Assuming that the surface considered for both the integrations is same, we can write,

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad \text{... (15)}$$

Above equation is the **Point form or differential form of Maxwell's equation derived from Ampere's circuit law.**

C] Maxwell's Equation Derived from Gauss's Law for Electric Field

According to Gauss's law, the total flux out of the closed surface is equal to the net charge within the surface. This can be written in integral form as,

$$\int_S \bar{D} \bullet d\bar{S} = Q_{enclosed} \quad \text{... (16)}$$

If we replace R.H.S. of above equation by the volume integral of volume charge density ρ_v through the volume enclosed by the surface S considered for integration at L.H.S. of equation (16), we get more general form of equation given by

$$\int_S \bar{D} \bullet d\bar{S} = \int_V \rho_v \, dv \quad \dots (17)$$

This equation is called **Maxwell's equation for electric fields derived from Gauss's law**, expressed in **integral form** and applied to a finite volume.

Statement : "The total flux leaving out of a closed surface is equal to the total charge enclosed by a finite volume."

Using divergence theorem, we can write

$$\int_V (\nabla \bullet \bar{D}) dv = \int_V \rho_v \, dv \text{........Assuming same volume for integration,}$$

$$\nabla \bullet \bar{D} = \rho_v \quad \dots (18)$$

This is Maxwell's equation for electric fields derived from Gauss's law which is expressed in point form or differential form.

D] Maxwell's Equation Derived from Gauss's Law Magnetic Fields

For magnetic fields, the surface integral of $\bar{B}$ over a closed surface S is always zero, due to non existance of monopole in the magnetic fields.

$$\int_S \bar{B} \bullet d\bar{S} = 0 \quad \dots (19)$$

This is **Maxwell's magnetic field equation** expressed in **integral form**. This is derived for Gauss's law applied to the magnetic fields.

Statement : "The surface integral of magnetic flux density over a closed surface is always equal to zero."

Using divergence theorem, the surface integral can be converted to volume integral as,

$$\int_S (\nabla \bullet \bar{B}) dv = 0$$

But being a finite volume, $dv \neq 0$,

$$\therefore \quad \nabla \bullet \bar{B} = 0 \quad \dots (20)$$

This is differential form or **point form of Maxwell's equation derived from Gauss's law applied to the magnetic fields.**

Differential form	Integral form	Significance
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$	Faraday's law
$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \bullet d\bar{L} = I + \int_S \frac{\partial \bar{D}}{\partial t} \bullet d\bar{S}$	Ampere's circuital law
$\nabla \bullet \bar{D} = \rho_v$	$\oint_S \bar{D} \bullet d\bar{S} = \int_V \rho_v \, dv$	Gauss's law
$\nabla \bullet \bar{B} = 0$	$\oint_S \bar{B} \bullet d\bar{S} = 0$	No isolated magnetic charges.

Table Q.16.1 Maxwell's equations

Q.17 State the Maxwell's equations in point form for static electric and steady magnetic fields. Explain how these are modified for the time varying fields and free space.

☞ [SPPU : Dec.-15,13,16,19, May-15,10, Marks 10]

Ans. : Maxwell'S Equations for Static and Time varying Fields

Refer answer of Q.16.

Maxwell's Equations for Free Space

- The Maxwell's equation, in the free space are as mentioned below.

Point form	Integral form
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$
$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \bullet d\bar{L} = \int_S \frac{\partial \bar{D}}{\partial t} \bullet d\bar{S}$
$\nabla \bullet \bar{D} = 0$	$\oint_S \bar{D} \bullet d\bar{S} = 0$
$\nabla \bullet \bar{B} = 0$	$\oint_S \bar{B} \bullet d\bar{S} = 0$

Maxwell's Equations for Harmonically Varying Fields

- Let us assume that the electric and magnetic fields are varying harmonically with time. The electric flux density can be written as,

$$\bar{D} = \bar{D}_0 e^{j\omega t}$$

- Similarly the magnetic flux density can be written as,

$$\bar{B} = \bar{B}_0 e^{j\omega t}$$

- Taking partial derivative with respect to time, we can write,

$$\frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}_0 e^{j\omega t} = j\omega\bar{D} \quad \text{and} \quad \frac{\partial \bar{B}}{\partial t} = j\omega \bar{B}_0 e^{j\omega t} = j\omega\bar{B}$$

A) Point Form :

i) $\nabla\times\bar{E} = -j\omega\bar{B} = -j\omega\mu\bar{H}$

ii) $\nabla\times\bar{H} = \bar{J} + j\omega\bar{D} = \sigma\bar{E} + j\omega(\varepsilon\bar{E}) = (\sigma + j\omega\varepsilon)\bar{E}$

iii) $\nabla\bullet\bar{D} = \rho_v$

iv) $\nabla\bullet\bar{B} = 0$

B) Integral Form :

i) $\oint \bar{E}\bullet d\bar{L} = -\int_S j\omega\bar{B}\bullet d\bar{S} = -\int_S j\omega\mu\,\bar{H}\bullet d\bar{S} = -j\omega\mu\int_S \bar{H}\bullet d\bar{S}$

ii) $\oint \bar{H}\bullet d\bar{L} = I + \int_S j\omega\bar{D}\bullet d\bar{S} = \int_S \bar{J}\bullet d\bar{S} + \int_S j\omega\varepsilon\,\bar{E}\bullet d\bar{S}$

$$= \int_S \sigma\bar{E}\bullet d\bar{S} + \int_S j\omega\varepsilon\,\bar{E}\bullet d\bar{S} = (\sigma + j\omega\varepsilon)\int_S \bar{E}\bullet d\bar{S}$$

iii) $\oint_S \bar{D}\bullet d\bar{S} = \oint_v \rho_v\, dv$

iv) $\oint_S \bar{B}\bullet d\bar{S} = 0$

Q.18 **Let $\mu = 10^{-5}$ H/m. $\varepsilon = 4\times10^{-9}$ F/m. $\sigma = 0$, $\rho_V = 0$.**

Find K (including units) so that each of the following pairs of field satisfies Maxwell's equations :

i) $D = 6\,\bar{a}_x - 2y\,\bar{a}_y + 2z\,\bar{a}_z$ nC/m², $H = Kx\,\bar{a}_x - 10y\,\bar{a}_y - 25z\,\bar{a}_z$ A/m

ii) $E = (20y - Kt)a_x$ V/m , $H = (y + 2\times10^6 t)\,a_z$ A/m.

[SPPU : Dec.-14,13, May-10,18, Marks 8]

Ans. : Given : $\mu = 10^{-5}$ H/m, $\varepsilon = 4\times10^{-9}$ F/m, $\sigma = 0$, $\rho_V = 0$

From above information, $\sigma = 0$ and $\rho_V = 0$ means given medium is free space.

i) For free space $\nabla\bullet\bar{B} = 0$ But $\bar{B} = \mu\bar{H}$

$$\therefore \quad (\nabla\bullet\mu\bar{H}) = 0$$

$$\therefore \quad \mu(\nabla\bullet\bar{H}) = 0$$

$$\therefore \quad 10^{-5}\left[\frac{\partial}{\partial x}H_x + \frac{\partial}{\partial y}H_y + \frac{\partial}{\partial z}H_z\right] = 0$$

$$\therefore 10^{-5}\left[\frac{\partial}{\partial x}(Kx) + \frac{\partial}{\partial y}(-10y) + \frac{\partial}{\partial z}(-25z)\right] = 0$$

$$10^{-5}[K - 10 - 25] = 0$$

$$\therefore \quad K = \mathbf{35\ A/m^2}$$

ii) For free space, $\nabla\times\bar{H} = +\frac{\partial\bar{D}}{\partial t}$

$$\text{L.H.S.} = \nabla\times\bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & (y + 2\times10^6 t) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(y + 2\times10^6 t) - 0\right]\bar{a}_x - [0-0]\bar{a}_y + [0-0]\bar{a}_z = (1)\,\bar{a}_x$$

$$\text{R.H.S.} = \frac{\partial\bar{D}}{\partial t}\ \frac{\partial}{\partial t}(\varepsilon\bar{E}) = \varepsilon\frac{\partial\bar{E}}{\partial t} = \varepsilon\frac{\partial}{\partial t}[(20y - Kt)]\,\bar{a}_x = \varepsilon[0-K]\bar{a}_x$$

$$= -K\varepsilon\bar{a}_x$$

Equating L.H.S. with R.H.S. we get,

$$\therefore \quad -K\varepsilon\bar{a}_x = 1\,\bar{a}_x$$

$$\therefore \quad K = -\frac{1}{\varepsilon} = \frac{-1}{4\times10^{-9}} = \mathbf{-2.5\times10^8\ V/ms}$$

Q.19 **In free space $\bar{E} = 20\cos(\omega t - 50x)\,\bar{a}_y$ V/m, calculate :**

i) $\bar{J}_d$ ii) $\bar{H}$ iii) ω. **[SPPU : May-13, Dec.-10,16,17, Marks 8]**

Ans.: Given : $\bar{E} = 20\cos(\omega t - 50x)\,\bar{a}_y$ V/m, $\mu_r = \varepsilon_r = 1$

... For free space

i) By definition,

$$\bar{D} = \varepsilon\bar{E} = (\varepsilon_0\varepsilon_r)\bar{E} = \varepsilon_0\bar{E}$$

$$\therefore \bar{D} = \varepsilon_0\left[20\cos(\omega t - 50x)\bar{a}_y\right] = 20\varepsilon_0\left[\cos(\omega t - 50x)\bar{a}_y\right] \quad ...(1)$$

The current density $\bar{J}_D$ is given by,

$$\bar{J}_D = \bar{J}_d = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}\left\{20\ \varepsilon_0 \cos(\omega t - 50x)\bar{a}_y\right\}$$

$$= 20\ \varepsilon_0\left[-\sin(\omega t - 50x)\right](\omega)\bar{a}_y$$

$$\therefore \quad \bar{J}_D = \bar{J}_d = -20\ \omega\ \varepsilon_0 \sin(\omega t - 50x)\ \bar{a}_y \quad A/m^2 \quad ...(2)$$

ii) By Maxwell's equation, for free space,

$$\nabla \times \bar{E} = \frac{-\partial \bar{B}}{\partial t}$$

$$\therefore \quad \frac{-\partial \bar{B}}{\partial t} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 20\cos(\omega t - 50x) & 0 \end{vmatrix}$$

$$= \left\{0 - \frac{\partial}{\partial z}20\cos(\omega t - 50x)\right\}\bar{a}_x - \{0-0\}\bar{a}_y + \left\{\frac{\partial}{\partial x}20\cos(\omega t - 50x) - 0\right\}\bar{a}_z$$

$$\therefore \quad -\frac{\partial \bar{B}}{\partial t} = 20\left[-\sin(\omega t - 50x)(-50)\right]\bar{a}_z = 1000\sin(\omega t - 50x)\bar{a}_z$$

$$\therefore \quad -\frac{\partial \bar{B}}{\partial t} = -1000\sin(\omega t - 50x)\bar{a}_z$$

Separating variables,

$$\partial \bar{B} = \left[-1000\sin(\omega t - 50x)\bar{a}_z\right]dt$$

Integrating both sides

$$\bar{B} = -1000\left[-\frac{\cos(\omega t - 50x)}{\omega}\right]\bar{a}_z = \frac{1000}{\omega}\cos(\omega t - 50x)\bar{a}_z \ T \quad ...(3)$$

By definition,

$$\bar{H} = \frac{\bar{B}}{\mu} = \frac{\bar{B}}{\mu_0\mu_r} = \frac{\bar{B}}{\mu_0}$$

$$= \frac{1000}{\omega\mu_0}\cos(\omega t - 50x)\ \bar{a}_z\ A/m \quad ...(4)$$

Now to find value of ω, let us use Maxwell's equation as follows,

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} = \bar{J}_D$$

$$\nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \frac{1000}{\omega\mu_0}\cos(\omega t - 50x) \end{vmatrix} = \frac{\partial \bar{D}}{\partial t}$$

$$\therefore \left[\frac{\partial}{\partial y}\frac{1000}{\omega\mu_0}\cos(\omega t - 50x) - 0\right]\bar{a}_x - \left[\frac{\partial}{\partial x}\left\{\frac{1000}{\omega\mu_0}\cos(\omega t - 50x)\right\} - 0\right]\bar{a}_y$$

$$+ [0-0]\bar{a}_z = \frac{\partial \bar{D}}{\partial t}$$

$$\therefore [0-0]\bar{a}_x - \left[\frac{1000}{\omega\mu_0}\{-\sin(\omega t - 50x)\}(-50) - 0\right]\bar{a}_y + [0]\bar{a}_z = \frac{\partial \bar{D}}{\partial t}$$

$$\therefore \quad \frac{-50000}{\omega\mu_0}\sin(\omega t - 50x)\ \bar{a}_y = \frac{\partial \bar{D}}{\partial t} \quad ...(5)$$

Comparing equations (2) and (5) as both are of same form and representing same quantity, we can write,

$$-20\omega\ \varepsilon_0 = \frac{-50000}{\omega\ \mu_0}$$

$$\therefore \quad \omega\ \mu_0 = \frac{2500}{\omega\ \varepsilon_0} \quad ...(6)$$

Putting equation (6) in equation (4), we can represent $\bar{H}$ in another form as,

$$\bar{H} = \frac{1000}{\left(\frac{2500}{\omega\varepsilon_0}\right)}\cos(\omega t - 50x)\ \bar{a}_z = 0.4\ \omega\ \varepsilon_0 \cos(\omega t - 50x)\ \bar{a}_z\ A/m \quad ...(7)$$

Now rearranging equation (6)

$$\omega^2 = \frac{2500}{\mu_0\varepsilon_0} = \frac{2500}{(4\times\pi\times10^{-7})(8.854\times10^{-12})} = 224694\times10^{20}$$

$$\therefore \quad \omega = 1.4989\times10^{10}\ rad/sec \approx 1.5\times10^{10}\ rad/sec$$

Thus representing values of $\bar{J}_d$ and $\bar{H}$ by putting values of ε_0 and ω using equations (2) and (7) as follows

$$\bar{J}_d = 20(8.854\times10^{-12})\cos(1.5\times10^{10}t - 50x)\bar{a}_y$$

$$\therefore \quad \bar{J}_d = 1.7708\times10^{10} \cos\left(1.5\times10^{10}t-50x\right)\bar{a}_y \quad A/m^2$$

Similarly,

$$\bar{H} = 0.4\left(1.5\times10^{10}\right)\left(8.854\times10^{-12}\right)\cos(\omega t-50x)\bar{a}_z$$

$$= 0.053124\cos(\omega t-50x)\bar{a}_z \quad A/m$$

Q.20 Determine value of k such that following pairs of fields satisfies Maxwell's equations in the region where $\sigma = 0$, $\sigma_v = 0$

a) $\bar{E} = [kx-100\,t]\,\bar{a}_y$ V/m, $\bar{H} = [x+20\,t]\,\bar{a}_z$ A/m and $\mu = 0.25$ H/m, $\varepsilon = 0.01$ F/m

b) $\bar{D} = 5x\,\bar{a}_x - 2y\,\bar{a}_y + kz\,\bar{a}_z$ $\mu C/m^2$, $\bar{B} = 2\,\bar{a}_y$ mT and $\mu = \mu_0$, $\varepsilon = \varepsilon_0$.

[SPPU : Dec.-15,12,19, May-15,09,2000, Marks 8]

Ans.: a) For time varying fields, we can write Maxwell's equation as,

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

We can write,

$$\nabla\times\bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & [kx-100\,t] & 0 \end{vmatrix}$$

$$= -\frac{\partial}{\partial z}[kx-100\,t]\bar{a}_x + \frac{\partial}{\partial x}[kx-100\,t]\,\bar{a}_z$$

Again $\bar{E}$ is varying with respect to x and not with z.

$$\therefore \quad \nabla\times\bar{E} = \frac{\partial}{\partial x}[kx-100\,t]\,\bar{a}_z = -\frac{\partial\bar{B}}{\partial t}$$

$$\therefore \quad k\,\bar{a}_z = -\frac{\partial}{\partial t}\left(\mu\,\bar{H}\right) = -\mu\frac{\partial}{\partial t}[x+20\,t]\,\bar{a}_z = -20\,\mu\,\bar{a}_z$$

... $\bar{B} = \mu\,\bar{H}$

Comparing, $k = -20\mu = -20\,(0.5) = \mathbf{-5}$ **V/m²**

b) Consider Maxwell's equation derived from Gauss's law for electric fields,

$$\nabla\bullet\bar{D} = \rho_v$$

$$\therefore \quad \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v = 0 \qquad \text{... Given}$$

From given expressions of $\bar{D}$,

$D_x = 5x, \quad D_y = -2y, \quad D_z = kx$

Putting values of D_x, D_y and D_z, we get,

$$\frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(kx) = 0$$

$$\therefore \quad 5-2+k = 0$$

$$\therefore \quad k = -3 \quad \mu C/m^3$$

Note that in part (a), k is unknown in the expression of $\bar{E}$ which is expressed in V/m. In the expression k is multiplied with x which is expressed in metres (m). Hence accordingly k is expressed in V/m². While in part (b), k is the part of expression of $\bar{D}$ which is expressed in μ C/m². k is multiplied by z which is expressed in m, in expression of $\bar{D}$. Hence k is expressed in μC/m².

4.8 : Retarded Potential

Important Points to Remember

- Retarded Electric Scalar Potential $V = \int_V \frac{[\rho_v]}{4\pi\varepsilon R}dV$
- Retarded Magnetic Vector Potential $\bar{A} = \int_V \mu\frac{[\bar{J}]}{4\pi R}dV$

4.9 : Phasor Representation of Vector

Q.21 The electric field and magnetic field in free space are given by :

$$\bar{E} = \frac{50}{\rho}\cdot\cos\left(10^6 t+\beta\tau\right)\bar{a}_\phi \text{ V/m}, \quad \bar{H} = \frac{H_0}{\rho}\cdot\cos\left(10^6 t+\beta\tau\right)\bar{a}_\rho \text{ A/m}$$

Express these in phasor form and determine the constants H_0 and β such that the fields satisfy Maxwell's equation.

☞[SPPU : May-13, Dec.-10, Marks 8]

Ans. : The instantaneous value of $\bar{E}$ can be written as,

$$\bar{E} = \text{Re}\left[E_S \cdot e^{j\omega t}\right] \quad \text{where } \omega = 10^6 \text{ rad/sec}$$

The phasor form is given by,

$$\bar{E}_S = \frac{50}{\rho} e^{j\beta Z} \bar{a}_\phi \quad ...(1)$$

The instantaneous values of $\bar{H}$ can be written as,

$$\bar{H} = \text{Re}\left[H_S \cdot e^{j\omega t}\right] \quad \text{where } \omega = 10^6 \text{ rad/sec}$$

The phasor form is given by,

$$\bar{H}_S = \frac{H_0}{\rho} e^{j\beta Z} \bar{a}_\rho \quad ...(2)$$

For free space, the Maxwell's equations are

i) $$\nabla \bullet \bar{D} = \nabla \bullet (\varepsilon_0 \bar{E}) = \varepsilon_0 (\nabla \bullet \bar{E}_S) = 0$$

$$\therefore \nabla \bullet \bar{E}_S = 0 \quad ...(3)$$

ii) $$\nabla \bullet \bar{B} = \nabla \bullet (\mu_0 \bar{H}) = \mu_0 (\nabla \bullet \bar{H}_S) = 0$$

$$\therefore \nabla \bullet \bar{H}_S = 0 \quad ...(4)$$

iii) $$\nabla \times \bar{H} = \bar{J} + \frac{\partial (\varepsilon_0 \bar{E})}{\partial t}$$

$$\therefore \nabla \times \bar{H} = \sigma \bar{E} + \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

But $\sigma = 0$ for free space, hence

$$\nabla \times \bar{H} = \varepsilon_0 [j\omega \bar{E}] = j\,\omega\varepsilon_0 [\bar{E}] = j\,\omega\,\varepsilon_0 [\bar{E}_S] \quad ...(5)$$

iv) $$\nabla \times \bar{E} = \frac{-\partial \bar{B}}{\partial t} = \frac{-\partial}{\partial t}(\mu_0 \bar{H}) = -\mu_0 [j\omega \bar{H}] = -j\,\omega\mu_0\,\bar{H}$$

$$= -j\,\omega\,\mu_0\,\bar{H}_S \quad ...(6)$$

Substituting equation (1) in equation (3), we get,

$$\nabla \times \bar{E}_S = \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{50}{\rho} e^{j\beta z}\right) = 0 \quad ...(7)$$

Substituting equation (2) in equation (4), we get

$$\nabla \times \bar{H}_S = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho\left(\frac{H_0}{\rho} e^{j\beta z}\right)\right] = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[H_0 e^{j\beta z}\right] = 0 \quad ...(8)$$

So from equations (7) and (8) it is clear that given $\bar{E}$ and $\bar{H}$ satisfy two maxwell's equations represented by equations (3) and (4).

$$\text{Now,} \quad \nabla \times \bar{H}_S = \nabla \times \left(\frac{H_0}{\rho} e^{j\beta z} \bar{a}_\rho\right) = \frac{jH_0\beta}{\rho} e^{j\beta z} \bar{a}_\phi \quad ...(9)$$

Substituting equations (1), (2), (9) in equation (5), we get,

$$\frac{jH_0\beta}{\rho} e^{j\beta z} \bar{a}_\phi = j\,\omega\,\varepsilon_0 \frac{50}{\rho} e^{j\beta z} \bar{a}_\phi$$

Comparing,

$$H_0\beta = 50\omega\varepsilon_0 \quad ...(10)$$

Similarly substituting equations (1) and (2) in equations (6), we get,

$$\nabla \times \bar{E}_S = \nabla \times \left(\frac{50}{\rho} e^{j\beta z} \bar{a}_\phi\right) = -j\omega\mu_0\left(\frac{H_0}{\rho} e^{j\beta z} \bar{a}_\rho\right)$$

$$\therefore -j\beta\frac{50}{\rho} e^{j\beta z} \bar{a}_\rho = -j\omega\mu_0\left(\frac{H_0}{\rho} e^{j\beta z} \bar{a}_\rho\right)$$

Comparing,

$$\frac{H_0}{\beta} = \frac{50}{\omega\mu_0} \quad ...(11)$$

Multiplying equations (10) and (11)

$$H_0^2 = 2500\frac{\varepsilon_0}{\mu_0}$$

$$\therefore H_0 = \sqrt{2500\frac{\varepsilon_0}{\mu_0}} = \sqrt{2500\left[\frac{8.854\times10^{-12}}{4\times\pi\times10^{-7}}\right]}$$

$$= \pm 0.1327 \text{ A/m}$$

Dividing equation (10) by (11), we get,

$$\beta^2 = \omega^2\mu_0\varepsilon_0$$

$$\therefore \quad \beta = \pm\omega\sqrt{\mu_0\varepsilon_0}$$

$$\therefore \quad \beta = \pm 10^6\sqrt{(4\times\pi\times10^{-7})(8.854\times10^{-12})}$$

$\therefore \quad \beta = \pm 3.3356\times10^{-3}$ rad

Thus $H_0 = +0.1327$ and $\beta = \pm 3.3356\times10^{-3}$ or $H_0 = -0.1327$ and $\beta = \pm 3.3356\times10^{-3}$ are the only values which can satisfy all four Maxwell's equations.

4.10 : Power and Poynting Theorem

Q.22 State and prove Poynting theorem. Interprete each term.

[SPPU : May-14,16,17,18,19, Dec.-13,14,16,17,18,19, Marks 8]

Ans.: • By the means of electromagnetic (EM) waves, an energy can be transported from transmitter to receiver.

- The energy stored in an electric field and magnetic field is transmitted at a certain rate of energy flow which can be calculated with the help of Poynting theorem.
- $\overline{E}$ is electric field expressed in V/m; while $\overline{H}$ is magnetic field measured in A/m. So if we take product of two fields, dimensionally we get a unit $V \cdot A/m^2$ or $watt/m^2$.
- Thus the product of $\overline{E}$ and $\overline{H}$ gives a quantity called **power density** which is expressed as watt per unit area.

Statement of poynting theorem

Poynting theorem states that **the vector product of electric field intensity $\overline{E}$ and magnetic field intensity $\overline{H}$ at any point is a measure of the rate of energy flow per unit area at that point and the direction of power flow is perpendicular to $\overline{E}$ and $\overline{H}$ both along the direction of $\overline{E}\times\overline{H}$.**

- To calculate a power density, we must carry out a **cross product** of $\overline{E}$ and $\overline{H}$. The power density is given by

$$\overline{P} = \overline{E}\times\overline{H} \quad \text{where } \overline{P} \text{ is } \textbf{Poynting Vector} \qquad \ldots (1)$$

- Poynting. $\overline{P}$ is the instantaneous power density vector associated with the electromagnetic (EM) field at a given point.
- The direction of $\overline{P}$ indicates instantaneous power flow at the point. To get a net power flowing out of any surface, $\overline{P}$ is integrated over same closed surface.

- The Poynting theorem is based on law of conservation of energy in electromangetism. Poynting theorem can be stated as follows :
- The net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within volume v minus the ohmic power dissipated. This can be well illustrated by the Fig. Q.22.1.

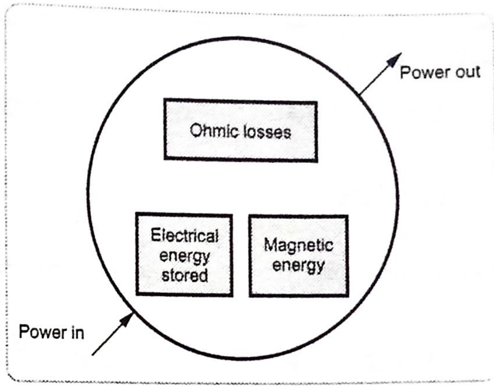


Fig. Q.22.1 Power balance representation in electromagnetic fields

- The power density measured in watt/m^2 is given by,

$$\overline{P} = \frac{E_m^2}{\eta_0}\cos^2(\omega t - \beta z)\,\overline{a}_z \quad W/m^2 \qquad \ldots (2)$$

Integral and Point Forms of Poynting Theorem

Consider Maxwell's equations as given below :

$$\nabla\times\overline{E} = -\frac{\partial\overline{B}}{\partial t} = -\mu\frac{\partial\overline{H}}{\partial t} \qquad \ldots (3)$$

$$\nabla\times\overline{H} = \overline{J}+\frac{\partial\overline{D}}{\partial t} = \sigma\overline{E}+\varepsilon\frac{\partial\overline{E}}{\partial t} \qquad \ldots (4)$$

Dotting both the sides of equation (4) with $\overline{E}$, we get,

$$\overline{E}\bullet\left(\nabla\times\overline{H}\right) = \overline{E}\bullet\left(\sigma\overline{E}\right)+\overline{E}\bullet\left(\varepsilon\frac{\partial\overline{E}}{\partial t}\right)$$

$$\nabla\bullet\left(\overline{E}\times\overline{H}\right) = \overline{E}\bullet(\sigma\overline{E})+\overline{E}\bullet\left(\varepsilon\frac{\partial\overline{E}}{\partial t}\right) \qquad \ldots (5)$$

Let us make use of a vector identity as given below,

$$\nabla\bullet\left(\overline{A}\times\overline{B}\right) = \overline{B}\bullet\left(\nabla\times\overline{A}\right)-\overline{A}\bullet\left(\nabla\times\overline{B}\right)$$

Applying above vector identity to L.H.S. of the equation (5) with $\bar{A} = \bar{E}$ and $\bar{B} = \bar{H}$,

$$\bar{H}\bullet(\nabla\times\bar{E})-\nabla\bullet(\bar{E}\times\bar{H}) = \bar{E}\bullet(\sigma\bar{E})+\bar{E}\bullet\left(\varepsilon\frac{\partial\bar{E}}{\partial t}\right)$$

$$\therefore \quad \bar{H}\bullet(\nabla\times\bar{E})-\nabla\bullet(\bar{E}\times\bar{H}) = \sigma E^2+\bar{E}\bullet\left(\varepsilon\frac{\partial\bar{E}}{\partial t}\right) \qquad \dots (6)$$

Consider first term on left of equation (6). Putting value of $\nabla\times\bar{E}$ from equation (3) we can write,

$$\bar{H}\bullet(\nabla\times\bar{E}) = \bar{H}\bullet\left(-\mu\frac{\partial\bar{H}}{\partial t}\right) = -\mu\,\bar{H}\bullet\frac{\partial\bar{H}}{\partial t} \qquad \dots (7)$$

Now consider term,

$$\frac{\partial}{\partial t}(\bar{H}\bullet\bar{H}) = \bar{H}\bullet\frac{\partial\bar{H}}{\partial t}+\bar{H}\bullet\frac{\partial\bar{H}}{\partial t}$$

$$\therefore \quad \frac{\partial}{\partial t}H^2 = 2\bar{H}\bullet\frac{\partial\bar{H}}{\partial t}$$

$$\therefore \quad \frac{1}{2}\frac{\partial}{\partial t}(H^2) = \bar{H}\bullet\frac{\partial\bar{H}}{\partial t} \qquad \dots (8)$$

Similarly we can write,

$$\frac{1}{2}\frac{\partial}{\partial t}(E^2) = \bar{E}\bullet\frac{\partial\bar{E}}{\partial t} \qquad \dots (9)$$

Using results obtained in equations (7), (8) and (9) in equation (6),

$$-\frac{\mu}{2}\frac{\partial}{\partial t}(H^2)-\nabla\bullet(\bar{E}\times\bar{H}) = \sigma E^2+\frac{\varepsilon}{2}\frac{\partial}{\partial t}(E^2)$$

$$\therefore \quad -\nabla\bullet(\bar{E}\times\bar{H}) = \sigma E^2+\frac{1}{2}\frac{\partial}{\partial t}\left[\mu H^2+\varepsilon E^2\right]$$

$$\therefore \quad \nabla\bullet(\bar{E}\times\bar{H}) = -\sigma E^2-\frac{1}{2}\frac{\partial}{\partial t}\left[\mu H^2+\varepsilon E^2\right]$$

But $\bar{E}\times\bar{H}$ is nothing but Poynting vector; $\bar{P}$, rewriting equation,

$$\nabla\bullet\bar{P} = -\sigma E^2-\frac{1}{2}\frac{\partial}{\partial t}\left[\mu H^2+\varepsilon E^2\right] \qquad \dots (10)$$

Equation (10) represents **Poynting theorem in point form.** If we integrate this power over a volume, we get energy distribution as,

$$\int_V \nabla\bullet\bar{P}\,dv = -\int_V \sigma E^2\,dv-\frac{\partial}{\partial t}\int_V \frac{1}{2}\left[\mu H^2+\varepsilon E^2\right]dv$$

Applying divergence theorem to left of above equation, we get,

$$\oint_S \bar{P}\bullet d\bar{S} = -\int_V \sigma E^2\,dv-\frac{\partial}{\partial t}\int_V \frac{1}{2}\left[\mu H^2+\varepsilon E^2\right]dv \qquad \dots (11)$$

Equation (11) represents **Poynting theorem in integral form.**

Interpretation of the terms in the equation

1. The term on the left-hand side indicates the net power flowing out of the surface. The term $\bar{P}$ is poynting vector and is equal to $\bar{E}\times\bar{H}$. It is the instantaneous power density vector associated with the electromagnetic field at any point.
2. The first term on the right-hand side represents power dissipated in the medium where $\sigma \neq 0$. It is actually the total ohmic power loss within the volume.
3. The second term on the left hand side represents the rate of decrease in energy stored in the electric and magnetic fields.
4. According to the law of conservation of energy, the sum of two terms on the right-hand side is equal to the total power flowing out of the volume. This can be interpreted as,

Total power leaving volume	Total ohmic power loss ($\sigma \neq 0$)	Rate of decrease in energy stored in the electric and magnetic fields.
↓	↓	↓
$\oint_S \bar{P}\bullet d\bar{S}$	$= -\int_V \sigma E^2\,dv$	$-\ \frac{\partial}{\partial t}\int_V \frac{1}{2}[\mu H^2+\varepsilon E^2]\,dv$

Q.23 Derive the expression for average power density.

[SPPU : Dec.-17, Marks 4]

Ans. : • To find average power density, let us integrate power density in z-direction over one cycle and divide by the period T of one cycle.

$$\therefore \quad P_{avg} = \frac{1}{T}\int_0^T \frac{E_m^2}{\eta}\cos(\omega t-\beta z)\,dt = \frac{E_m^2}{T\eta}\int_0^T \frac{1+\cos 2(\omega t-\beta z)}{2}\,dt$$

$$= \frac{E_m^2}{T\eta}\left[\frac{t}{2}+\frac{\sin 2(\omega t-\beta z)}{2(2\omega)}\right]_0^T = \frac{E_m^2}{T\eta}\left[\frac{t}{2}+\frac{\sin 2(\omega t-\beta z)}{4\omega}\right]_0^T$$

$$= \frac{E_m^2}{T\eta}\left[\frac{t}{2}+\frac{\sin(2\omega t-2\beta z)}{4\omega}\right]_0^T$$

$$= \frac{E_m^2}{T\eta}\left[\frac{T}{2}+\frac{\sin(2\omega T-2\beta z)}{4\omega}-\frac{\sin(-2\beta z)}{4\omega}\right]$$

But $\omega T = 2\pi$,

$$\therefore \quad P_{avg} = \frac{E_m^2}{T\eta}\left[\frac{T}{2}+\frac{\sin(4\pi-2\beta z)}{4\omega}+\frac{\sin 2\beta z}{4\omega}\right]$$

$$= \frac{E_m^2}{T\eta}\left[\frac{T}{2}-\frac{\sin 2\beta z}{4\omega}+\frac{\sin 2\beta z}{4\omega}\right] = \frac{E_m^2}{2\eta}$$

• Hence the average power is given by

$$\boxed{P_{avg} = \frac{1}{2}\frac{E_m^2}{\eta}\ \text{W/m}^2} \qquad \dots (1)$$

• The average power flowing through any area S normal to the direction of power flow is given by,

$$\boxed{P_{avg} = \frac{E_m^2}{2\eta}\,S\ \text{watts}} \qquad \dots (2)$$

4.11 : Application Case Study

Q.24 Write a note on memresistor.

Ans. : • A two terminal electronic component that shows a nonlinear characteristics and which relates the electric charge with the magnetic field produced by the circuit is called a **memresistor**. It has the dynamic relation between the voltage and current and with a memory which consists of the history of the voltages and currents applied in the past. The relation between the charge and the magnetic flux is governed by the quasi-static expansion of Maxwell's equations. It is measured in ohms.

The basic construction of memresistor is shown in the Fig. Q.24.1 (a) while its symbol is shown in the Fig. Q.24.1 (b).

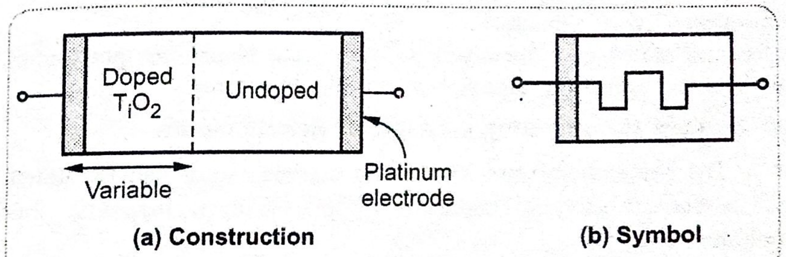


Fig. Q.24.1 Memresistor

• In memresistor, an element like titanium oxide (T_iO_2) having resistive properties is placed between two platinum electrodes. But in the section in between the electrodes, one part is doped with titanium oxide while the other part is kept undoped. Depending upon the polarity of the voltage applied, the doped side becomes thinner or thicker and produces a variable resistance. When voltage applied is removed, it stores a past value hence called resistance with memory.

If R_M is the memresistance then according to ohm's law we can write,

$$v(t) = R_M i(t) \quad \text{while} \quad v(t) = \frac{d\phi}{dt},\ i(t) = \frac{dq}{dt}$$

$$\therefore \quad R_M = \frac{d\phi/dt}{dq/dt} = \frac{d\phi}{dq}$$

Thus if shows a nonlinear relationship between magnetic flux and the charge. When current through it is zero, $\frac{dq}{dt} = 0$ at $t = t_0$ where

$q = q(t_0)$ and it gives a past value of resistance at $t = t_0$ and acts as a nonvolatile memory.

- For ordinary resistors, there exists a linear relation between voltage and current. But memresistors show the characteristics similar to hysteresis curve of a magnetic material. This is shown in the Fig. Q.24.2. The two straight lines are within the curve indicating two separate resistance states of a memresistor. While the remaining portion of the curve is the transition regions between the two states.

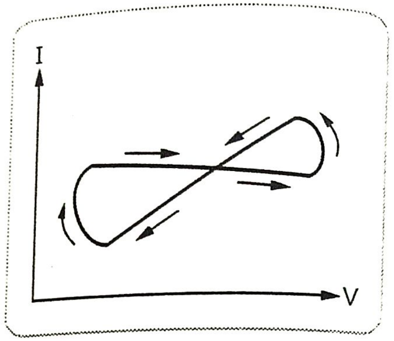
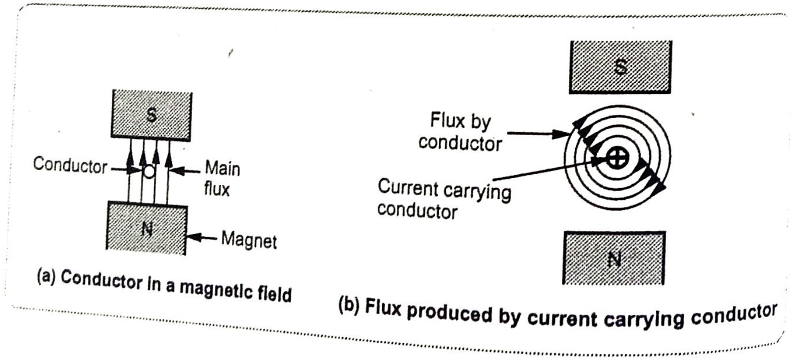


Fig. Q.24.2

Q.25 Explain the operating principle of electric motor.

Ans. : The principle of operation of an electric motor can be stated as ' when a current carrying conductor is placed in a magnetic field, it experiences a force'.

- Consider a single conductor placed in a magnetic field as shown in the Fig. Q.25.1 (a). The magnetic field is produced by a permanent magnet but in a practical d.c. motor it is produced by the field winding when it carries a current.

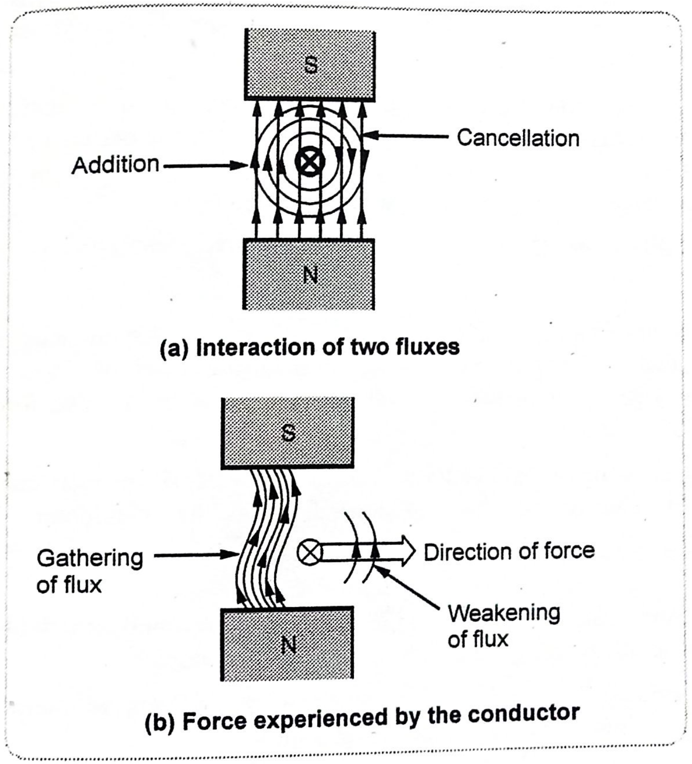


(a) Conductor in a magnetic field (b) Flux produced by current carrying conductor

Fig. Q.25.1

- Now this conductor is excited by a separate supply so that it carries a current in a particular direction. Consider that it carries a current away from an observer as shown in the Fig. Q.25.1 (b). Any current carrying

conductor produces its own magnetic field around it, hence this conductor also produces its own flux, around. The direction of this flux can be determined by right hand thumb rule. For direction of current considered, the direction of flux around a conductor is clockwise. For simplicity of understanding, the main flux produced by the permanent magnet is not shown in the Fig. Q.25.1 (b).



(a) Interaction of two fluxes

(b) Force experienced by the conductor

Fig. Q.25.2

- Now there are two fluxes present,
  1. The flux produced by the permanent magnet called main flux.
  2. The flux produced by the current carrying conductor.
- These are shown in the Fig. Q.25.2 (a). From this, it is clear that on one side of the conductor, both the fluxes are in the same direction. In this case, on the left of the conductor there is gathering of the flux lines as two fluxes help each other. As against this, on the right of the conductor, the two fluxes are in opposite direction and hence try to cancel each other. Due to this, the density of the flux lines in this area

gets weakened. So on the left, there exists high flux density area while on the right of the conductor there exists low flux density area as shown in the Fig. Q.25.2 (b).

- This flux distribution around the conductor acts like a stretched rubber band under tension. This exerts a mechanical force on the conductor which acts from high flux density area towards low flux density area, i.e. from left to right for the case considered as shown in the Fig. Q.25.2 (b).
- The armature conductors are placed in the slots on the periphery of a circular armature. Thus the individual force experienced by the conductors acts as a twisting force on the armature which is called torque. Hence overall armature starts rotating.

Q.26 Explain the operating principle of electric generator.

Ans. :

- All generators work on the principle of dynamically induced e.m.f. It states that, 'whenever the number of magnetic lines of force i.e. flux linking with a conductor or a coil changes, an electromotive force is set up in that conductor or coil.'
- The magnitude of induced e.m.f. in a conductor is proportional to the rate of change of flux associated with the conductor. This is mathematically given by, e (magnitude) $\propto \frac{d\phi}{dt}$.
- The relative motion can be achieved by rotating conductor with respect to flux or by rotating flux with respect to a conductor.
- So a voltage gets generated in a conductor, as long as there exists a relative motion between conductor and the flux.
- Such an induced e.m.f. which is due to physical movement of coil or conductor with respect to flux or movement of flux with respect to coil or conductor is called **dynamically induced e.m.f.**
- So a generating action requires following basic components to exist, i) The conductor or a coil ii) The flux iii) The relative motion between conductor and flux.
- To have a large voltage as the output, the number of conductors are connected together in a specific manner, to form a winding. This winding is called **armature winding** and the part on which this winding is kept is called **armature** of a d.c. machine.

- To have the rotation of conductors, the conductors placed on the armature are rotated with the help of some external device. Such an external device is called a **prime mover**. The commonly used prime movers are diesel engines, steam engines, steam turbines, water turbines etc.
- The necessary magnetic flux is produced by current carrying winding which is called **field winding.**
- The direction of the induced e.m.f. can be obtained by using Fleming's right hand rule.
- If angle between the plane of rotation and the plane of the flux is ' θ ' as measured from the axis of the plane of flux then the induced e.m.f. is given by,

$$\boxed{E = B\, l\, (v \sin\theta) \text{ volts}}$$

where v sin θ is the component of velocity which is perpendicular to the plane of flux and hence responsible for the induced e.m.f.

Formulae at a Glance

- **Faraday's Law** $e = \oint \bar{E} \bullet d\bar{L} = -\frac{d}{dt}\int_S \bar{B} \bullet d\bar{S}$

I) Transformer E.M.F. (Statically induced E.M.F.)

Integral Form $\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$

Point Form $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

II) Motional/Generator E.M.F. (Dynamically induced E.M.F.)

$$\oint \bar{E}_m \bullet d\bar{L} = \oint (\bar{v} \times \bar{B}) \bullet d\bar{L}$$

III) Closed Path and magnetic field both varying with time

$$\oint \bar{E} \bullet d\bar{L} = \int \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S} + \oint (\bar{v} \times \bar{B}) \bullet d\bar{L}$$

• Displacement and conduction current density

$\bar{J}_C = \sigma \bar{E}$ Conduction current density

$\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$ Displacement current density

• General Maxwell's Equations

Differential form	Integral form	Significance
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$	Faraday's law
$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \bullet d\bar{L} = I + \int_S \frac{\partial \bar{D}}{\partial t} \bullet d\bar{S}$	Ampere's circuital law
$\nabla \bullet \bar{D} = \rho_v$	$\oint_S \bar{D} \bullet d\bar{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \bullet \bar{B} = 0$	$\oint_S \bar{B} \bullet d\bar{S} = 0$	No isolated magnetic charges.

• Maxwell's equation for free space

Point form	Integral form
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$
$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \bullet d\bar{L} = \int_S \frac{\partial \bar{D}}{\partial t} \bullet d\bar{S}$
$\nabla \bullet \bar{D} = 0$	$\oint_S \bar{D} \bullet d\bar{S} = 0$
$\nabla \bullet \bar{B} = 0$	$\oint_S \bar{B} \bullet d\bar{S} = 0$

• Maxwell's equations for good conductor medium

Point form	Integral form
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$
$\nabla \times \bar{H} = \bar{J}$	$\oint \bar{H} \bullet d\bar{L} = I = \int_S \bar{J} \bullet d\bar{S}$
$\nabla \bullet \bar{D} = 0$	$\oint_S \bar{D} \bullet d\bar{S} = 0$
$\nabla \bullet \bar{B} = 0$	$\oint_S \bar{B} \bullet d\bar{S} = 0$

• Maxwell's equations for harmonically varying fields

A) Point form :

i) $\nabla \times \bar{E} = -j\omega \bar{B} = -j\omega\mu \bar{H}$

ii) $\nabla \times \bar{H} = \bar{J} + j\omega \bar{D} = \sigma \bar{E} + j\omega(\varepsilon \bar{E}) = (\sigma + j\omega\varepsilon) \bar{E}$

iii) $\nabla \bullet \bar{D} = \rho_v$

iv) $\nabla \bullet \bar{B} = 0$

B) Integral form :

i) $\oint \bar{E} \bullet d\bar{L} = -\int_S j\omega \bar{B} \bullet d\bar{S} = -\int_S j\omega\mu \bar{H} \bullet d\bar{S} = -j\omega\mu \int_S \bar{H} \bullet d\bar{S}$

ii) $\oint \bar{H} \bullet d\bar{L} = I + \int_S j\omega \bar{D} \bullet d\bar{S} = (\sigma + j\omega\varepsilon) \int_S \bar{E} \bullet d\bar{S}$

iii) $\oint_S \bar{D} \bullet d\bar{S} = \oint_v \rho_v \, dv$

iv) $\oint_S \bar{B} \bullet d\bar{S} = 0$

• Retarded Potenetials

Retarded Electric Scalar Potential $V = \int_V \frac{[\rho_v]}{4\pi\varepsilon R} dV$

Retarded Magnetic Vector Potential $\bar{A} = \int_V \mu \frac{[J]}{4\pi R} dV$

- **Poynting Vector**

Definition $\bar{P} = \bar{E} \times \bar{H}$

Point Form $\nabla \bullet \bar{P} = -\sigma E^2 - \frac{1}{2}\frac{\partial}{\partial t}\left[\mu H^2 + \varepsilon E^2\right]$

Integral Form $\oint_S \bar{P} \bullet d\bar{S} = -\int_V \sigma E^2 dv - \frac{\partial}{\partial t}\int_V \frac{1}{2}\left[\mu H^2 + \varepsilon E^2\right] dv$

- **Power Density** $\bar{P} = \frac{E_m^2}{\eta_0} \cos^2(\omega t - \beta z)\, \bar{a}_z \quad W/m^2$

- **Average Power Density** $P_{avg} = \frac{1}{2}\frac{E_m^2}{\eta} \quad W/m^2$

END...

Unit V

5 Uniform Plane Waves

5.1 : Maxwell's Equation in Phasor Form

Q.1 Obtain Maxwell's equation in pahsor form.

Ans. : Maxwells Equations in Phasor Form :

Let the applied electric field be $E = E_m \cos(\omega t + \phi)$

Let $\theta = \omega t + \phi$

According to Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$

The real and imaginary parts of $E_m e^{j\theta}$ are given by,

$$\mathrm{Re}(E_m e^{j\theta}) = E_m \cos(\omega t + \phi)$$

$$\mathrm{Im}(E_m e^{j\theta}) = E_m \sin(\omega t + \phi)$$

Here $E = \mathrm{Re}(E_m e^{j\theta}) = \mathrm{Re}(E_m e^{j\omega t} e^{j\phi})$

$E_m e^{j\omega t} = E_S$ is the phasor representation of electric field.

The time harmonic Maxwell's equations assuming time factor $e^{j\omega t}$:

Let $\bar{D} = \bar{D}_o e^{j\omega t}$

$$\bar{B} = \bar{B}_o e^{j\omega t}$$

$$\therefore \quad \frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}_o e^{j\omega t} = j\omega \bar{D}$$

$$\frac{\partial B}{\partial t} = j\omega \bar{B}_o e^{j\omega t} = j\omega \bar{B}$$

The Maxwell's equations are written as,

In differential form

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \times \bar{E} = -j\omega \mu \bar{H} = -j\omega \bar{B}$$

$$\nabla \times \bar{H} = \bar{J} + j\omega\bar{D} = \sigma\bar{E} + j\omega\varepsilon\bar{E} = (\sigma + j\omega\varepsilon)\bar{E}$$

In Integral form,

$$\oint_S \bar{B}\cdot d\bar{S} = 0$$

$$\oint_S \bar{D}\cdot d\bar{S} = \int_v \rho_v \, dv$$

$$\oint_L \bar{E}\cdot d\bar{l} = -j\omega\int_S \bar{B}\cdot d\bar{S} = -j\omega\mu\int_S \bar{H}\cdot d\bar{S}$$

$$\oint_L \bar{H}\cdot d\bar{l} = \int_S (\bar{J} + j\omega\bar{D})\cdot d\bar{S} = \int_S (\sigma\bar{E} + j\omega\,\varepsilon\,\bar{E})\cdot d\bar{S}$$

$$= \int_S (\sigma + j\omega\,\varepsilon)\bar{E}\cdot d\bar{S} = (\sigma + j\omega\,\varepsilon)\int_S \bar{E}\cdot d\bar{S}$$

5.2 : Introduction to Uniform Plane Wave

Important Points to Remember

- The **waves** are the means of transporting energy or information from source to destination. The waves consisting electric and magnetic fields are called **electromagnetic waves.**
- Important properties of wave are as follows :
  1. The waves are function of time and space.
  2. They assume properties of waves while travelling.
  3. They travel with high velocity.
  4. They radiate outwards from source in all directions.
- General wave equations interms of electric and magnetic fields are as follows :

$$\nabla^2\bar{E} = \mu\,\sigma\frac{\partial\bar{E}}{\partial t} + \mu\,\varepsilon\frac{\partial^2\bar{E}}{\partial t^2}$$

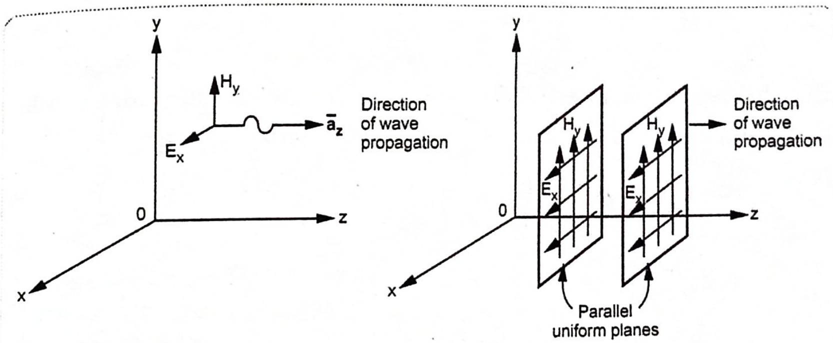
$$\nabla^2\bar{H} = \mu\,\sigma\frac{\partial\bar{H}}{\partial t} + \mu\,\varepsilon\frac{\partial^2\bar{H}}{\partial t}$$

Q.2 Explain the term uniform plane wave and its transverse nature.

[SPPU : Dec.-13, May-18,19, Marks 4]

Ans. : • Consider an electromagnetic wave propagating through the free space. For free space, $\sigma = 0$.

- Consider that the electric field in the wave is in x-direction only while the magnetic field is in y-direction only. Both the fields, i.e. electric field and magnetic field do not vary with x and y but vary only with z.
- The fields also vary with time as wave propagates in the free space.
- Basically **plane waves** means, the electric field vector $\bar{E}$ and the magnetic field vector $\bar{H}$ lie in the same plane. Also the different planes **along the direction** consisting $\bar{E}$ and $\bar{H}$ vectors are parallel to each other along the direction of propagation of wave.
- The **uniform plane wave** means the $\bar{E}$ and $\bar{H}$ field vector are in same plane. Moreover the amplitude and phase of field vectors $\bar{E}$ and $\bar{H}$ is constant over the planes parallel to each other. A uniform plane with field vectors $\bar{E}$ and $\bar{H}$ is illustrated in the Fig. Q.2.1 (a) and (b).



(a) Illustration of direction of propagation

(b) Illustration of uniform plane waves

Fig. Q.2.1

- Electric field vector is in $\bar{a}_x$ direction while magnetic field is in $\bar{a}_y$ direction. That means $\bar{E}$ and $\bar{H}$ lie in x-y plane.
- So in any of the planes in the wave, the vectors $\bar{E}$ and $\bar{H}$ are independet of x and y.
- Thus $\bar{E}$ and $\bar{H}$ are function of z and t only. Moreover as $\bar{E}$ and $\bar{H}$ are mutually perpendicular to each other, the electromagnetic waves are also called **transverse unform plane wave.**
- **As the wave travels in the z-direction. It is clear that direction of the electromagnetic wave i.e. uniform plane wave is perpendicular or orthogonal to the plane consisting the $\bar{E}$ and $\bar{H}$ field vectors.**

5.3 : Electromagnetic Wave Equation (Helmholtz's Equation)

Q.3 What is meant by uniform plane wave ? Obtain the wave equation travelling in free space in terms of E. What are Helmholtz's equations ? **[SPPU : Dec.-15, May-15,19, Dec.-16,19, Marks 8]**

Ans. : • Assume that electric field vector is in $\bar{a}_x$ direction while magnetic field is in $\bar{a}_y$ direction that means $\overline{E}$ and $\overline{H}$ lie in x-y plane.

• Let us consider wave equations for the $\overline{E}$ and $\overline{H}$ fields given by,

$$\nabla^2\overline{E} = \mu\sigma\frac{\partial\overline{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} \quad ...(1)$$

$$\nabla^2\overline{H} = \mu\sigma\frac{\partial\overline{H}}{\partial t}+\mu\varepsilon\frac{\partial^2\overline{H}}{\partial t^2} \quad ...(2)$$

• But for free space, $\sigma = 0$ and $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ substituting these values in equations (1) and (2), the wave equations are modified as,

$$\therefore \quad \nabla^2\overline{E} = \mu_0\varepsilon_0\frac{\partial^2\overline{E}}{\partial t^2} \quad ...(3)$$

$$\nabla^2\overline{H} = \mu_0\varepsilon_0\frac{\partial^2\overline{H}}{\partial t^2} \quad ...(4)$$

• Consider equation (3), we can write,

$$\nabla^2\overline{E} = \frac{\partial^2\overline{E}}{\partial x^2}+\frac{\partial^2\overline{E}}{\partial y^2}+\frac{\partial^2\overline{E}}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2\overline{E}}{\partial t^2} \quad ...(5)$$

• But the wave travels in the z-direction, hence $\overline{E}$ is independent of x and y. Hence first two differential terms in above equation are zero. Hence we can write,

$$\frac{\partial^2\overline{E}}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2\overline{E}}{\partial t^2} \quad ...(6)$$

Modifying equation (6) by rearranging terms, we get,

$$\frac{\partial^2\overline{E}}{\partial t^2} = \frac{1}{\mu_0\varepsilon_0}\frac{\partial^2\overline{E}}{\partial z^2} \quad ...(7)$$

Now according to the results in physics,

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c \quad \text{i.e.} \quad v^2 = \frac{1}{\mu\varepsilon} = c^2,$$

where $\quad c = 3\times10^8$ m/s = Velocity of light

Substituting in equation (7), we get,

$$\boxed{\frac{\partial^2\overline{E}}{\partial t^2} = v^2\frac{\partial^2\overline{E}}{\partial z^2}} \quad ... (8)$$

Similar to this we can also write,

$$\boxed{\frac{\partial^2\overline{H}}{\partial t^2} = v^2\frac{\partial^2\overline{H}}{\partial z^2}} \quad ... (9)$$

The equations, 8 and 9 are called **Helmholtz's equations.**

Let us consider equation (6), given as,

$$\frac{\partial^2\overline{E}}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2\overline{E}}{\partial t^2}$$

For the wave propagating in z-direction, $\overline{E}$ may have E_x and E_y component but definately not E_z. According to assumption, $\overline{E}$ is in $\bar{a}_x$ direction, so let us consider that only E_x is present. Then we can rewrite above equation as,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2 E_x}{\partial t^2} \quad ...(10)$$

Let $\quad E_x = E_m e^{j\omega t}$

where $\quad E_m$ = Amplitude of the electric field, ω = Angular frequency

Partially differentiating E_x twice with respect to t, we get,

$$\frac{\partial^2 E_x}{\partial t^2} = E_m(j\omega)(j\omega)e^{j\omega t} = -\omega^2 E_m e^{j\omega t}$$

But $\quad E_m e^{j\omega t} = E_x$, thus we can write,

$$\frac{\partial^2 E_x}{\partial t^2} = -\omega^2 E_x$$

Substituting in equation (10), we get,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 [-\omega^2 E_x] = -\omega^2 \mu_0 \varepsilon_0 E_x \quad ...(11)$$

Let $\frac{\partial}{\partial z} = D$ i.e. $\frac{\partial^2}{\partial z^2} = D^2$, then equation (11) becomes,

$$D^2 E_x = -\omega^2 \mu_0 \varepsilon_0 E_x$$

$$\therefore \quad D^2 E_x + \omega^2 \mu_0 \varepsilon_0 E_x = 0$$

Thus auxiliary equation becomes,

$$(D^2 + \omega^2 \mu_0 \varepsilon_0) E_x = 0$$

Hence equating bracket term to zero, we get,

$$D^2 + \omega^2 \mu_0 \varepsilon_0 = 0$$

or

$$D^2 = -\omega^2 \mu_0 \varepsilon_0 = \pm j\omega\sqrt{\mu_0 \varepsilon_0} = \pm j\beta \quad ...(12)$$

Where $\beta = \omega\sqrt{\mu_0 \varepsilon_0}$ which is called **phase shift constant measured in rad/m.**

Hence the solution of equation (11) can be written as,

$$E_x = K_1 e^{-j\omega\sqrt{\mu_0\varepsilon_0}\, z} + K_2 e^{+j\omega\sqrt{\mu_0\varepsilon_0}\, z}$$

i.e.

$$E_x = K_1 . e^{-j\beta z} + K_2 e^{+j\beta z} \quad ...(13)$$

Let K_1 and K_2 be the constants with respect to z but are functions of t. Let us assume K_1 and K_2 as,

$$K_1 = E_m^+ e^{+j\omega t} \text{ and } K_2 = E_m^- e^{+j\omega t}$$

Substituting values of K_1 and K_2 in equation (13) we get,

$$E_x = E_m^+ e^{+j\omega t} . e^{-j\beta z} + E_m^- e^{j\omega t} . e^{j\beta z}$$

$$\boxed{E_x = E_m^+ e^{j(\omega t+\beta z)} + E_m^- e^{j(\omega t+\beta z)}} \quad ...(14)$$

To find the electric field in the time domain, taking real part of equation (14), we get,

$$E_x = R_e[E_m^+ e^{j(\omega t-\beta z)} + E_m^- e^{j(\omega t+\beta z)}]$$

$$\therefore \quad \boxed{E_x = E_m^+ \cos(\omega t-\beta z) + E_m^- \cos(\omega t+\beta z)\, V/m} \quad ...(15)$$

- Above equation (15) is the sinusoidal function consisting two components of an electric field; one in **forward direction** and other in **backward direction**. The wave thus consists **one component of the field travelling in positive z-direction having amplitude E_m^+; while other component of the field travelling in negative z-direction having amplitude E_m^-.**
- The equation for H_y can be obtained in the similar way by simplifying equation (4) and the equation is given by,

$$H_y = H_m^+ \cos(\omega t-\beta z) - H_m^- \cos(\omega t+\beta z)\, A/m \quad ... (16)$$

- This equation is very much similar to equation (15) representing two components of a magnetic field, one in **forward direction,** while other in **backward direction.**
- Thus from equations (15) and (16) it is clear that $\bar{E}$ is in x-direction while $\bar{H}$ is in y-direction.
- Both $\bar{E}$ and $\bar{H}$ are in time phase and are mutually perpendicular to each other. Both these fields lie in the plane which is mutually perpendicular to the direction of wave propagation.
- Thus $\bar{E}$ and $\bar{H}$ together form transverse electromagnetic wave (TEM wave) and are only functions of time and direction of travel.

Q.4 Derive the general wave equation.

[SPPU : May-02, 06, Dec.-01, Marks 6]

Ans. : • Assume that the electric and magnetic fields exist in a linear, homogeneous and isotropic medium with the parameters μ, ε and σ.

- Also assume that the medium is source free and the medium obeys the ohm's law i.e. $\bar{J} = \sigma \bar{E}$. Then the Maxwell's equations are given by,

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \quad ...(1)$$

$$\nabla \times \bar{H} = \sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t} \quad ...(2)$$

$$\nabla \bullet \bar{B} = 0 \quad \text{i.e.} \quad \nabla \bullet \bar{H} = 0 \quad ...(3)$$

$$\nabla \bullet \bar{D} = 0 \quad \text{i.e.} \quad \nabla \bullet \bar{E} = 0 \quad ...(4)$$

Derivation

To eliminate $\bar{H}$ from equation (1), taking curl on both the sides of equation (1), we get,

$$\nabla\times(\nabla\times\overline{E}) = -\mu\left(\nabla\times\frac{\partial\overline{H}}{\partial t}\right) \quad \ldots(5)$$

∇ operator indicates differentiation with respect to space while $\frac{\partial}{\partial t}$ operates differentiation with respect to time. Both are independent of each other, the operators can be interchanged.

So we get,

$$\nabla\times\nabla\times\overline{E} = -\mu\frac{\partial}{\partial t}(\nabla\times\overline{H}) \quad \ldots(6)$$

Substituting value of $\nabla\times\overline{H}$ from equation (2), we get,

$$\nabla\times\nabla\times\overline{E} = -\mu\frac{\partial}{\partial t}\left[\sigma\overline{E}+\varepsilon\frac{\partial\overline{E}}{\partial t}\right]$$

$$\nabla\times\nabla\times\overline{E} = -\mu\sigma\frac{\partial\overline{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} \quad \ldots(7)$$

Now according to the vector identity,

$$\nabla\times\nabla\times\overline{E} = \nabla(\nabla\bullet\overline{E})-\nabla^2\overline{E} \quad \ldots(8)$$

Substituting $\nabla\bullet\overline{E} = 0$ from equation (4), we can modify equation (8) as,

$$\nabla\times\nabla\times\overline{E} = -\nabla^2\overline{E} \quad \ldots(9)$$

Substituting value of $\nabla\times\nabla\times\overline{E}$ from equation (9) in equation (10) we get,

$$-\nabla^2\overline{E} = -\mu\sigma\frac{\partial\overline{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2}$$

$$\therefore \quad \nabla^2\overline{E} = \mu\sigma\frac{\partial\overline{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} \quad \ldots(10)$$

By multiplying both the sides of equation (10) by ε,

$$\nabla^2(\varepsilon\overline{E}) = \mu\sigma\frac{\partial\varepsilon\overline{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\varepsilon\overline{E}}{\partial t^2}$$

i.e. $$\nabla^2\overline{D} = \mu\sigma\frac{\partial\overline{D}}{\partial t}+\mu\varepsilon\frac{\partial^2\overline{D}}{\partial t^2} \quad \ldots(11)$$

The wave equation for $\overline{H}$ can be obtained by taking curl on both the sides of equation (2), we get,

$$\nabla\times(\nabla\times\overline{H}) = \nabla\times(\sigma\overline{E})+\varepsilon\nabla\times\frac{\partial\overline{E}}{\partial t} \quad \ldots(12)$$

As ∇ operator and $\frac{\partial}{\partial t}$ represent independent relationship between the two, we can interchange them as follows.

$$\nabla\times\nabla\times\overline{H} = \sigma(\nabla\times\overline{E})+\varepsilon\frac{\partial}{\partial t}(\nabla\times\overline{E}) \quad \ldots(13)$$

Substituting $\nabla\times\overline{E} = -\mu\frac{\partial\overline{H}}{\partial t}$ in equation (12), we get,

$$\nabla\times\nabla\times\overline{H} = \sigma\left(-\mu\frac{\partial\overline{H}}{\partial t}\right)+\varepsilon\left(-\mu\frac{\partial\overline{H}}{\partial t}\right)$$

$$\therefore \quad \nabla\times\nabla\times\overline{H} = -\mu\sigma\frac{\partial\overline{H}}{\partial t}-\mu\varepsilon\frac{\partial\overline{H}}{\partial t} \quad \ldots(14)$$

From the vector indentity,

$$\nabla\times\nabla\times\overline{H} = \nabla(\nabla\bullet\overline{H})-\nabla^2\overline{H} \quad \ldots(15)$$

Substituting $\nabla\bullet\overline{H} = 0$ from equation (4) in equation (15), we get

$$\nabla\times\nabla\times\overline{H} = -\nabla^2\overline{H} \quad \ldots(16)$$

Substituting value of $\nabla\times\nabla\times\overline{H}$ in equation (14) we get,

$$-\nabla^2\overline{H} = -\mu\sigma\frac{\partial\overline{H}}{\partial t}-\mu\varepsilon\frac{\partial\overline{H}}{\partial t}$$

i.e. $$\nabla^2\overline{H} = \mu\sigma\frac{\partial\overline{H}}{\partial t}+\mu\varepsilon\frac{\partial^2\overline{H}}{\partial t} \quad \ldots(17)$$

Now multiplying both the sides by μ, we get,

$$\nabla^2(\mu\overline{H}) = \mu\sigma\frac{\partial\mu\overline{H}}{\partial t}+\mu\varepsilon\frac{\partial^2(\mu\overline{H})}{\partial t}$$

i.e. $$\nabla^2\overline{B} = \mu\sigma\frac{\partial\overline{B}}{\partial t}+\mu\varepsilon\frac{\partial^2\overline{B}}{\partial t} \quad \ldots(18)$$

Hence in general we can write,

$$\nabla^2\begin{bmatrix}\overline{E}\\\overline{D}\\\overline{H}\\\overline{B}\end{bmatrix} = \mu\sigma\frac{\partial}{\partial t}\begin{bmatrix}\overline{E}\\\overline{D}\\\overline{H}\\\overline{B}\end{bmatrix}+\mu\varepsilon\frac{\partial^3}{\partial t^2}\begin{bmatrix}\overline{E}\\\overline{D}\\\overline{H}\\\overline{B}\end{bmatrix} \quad \ldots(19)$$

5.4 : Relation Between $\bar{E}$ and $\bar{H}$

Q.5 Prove that intrinsic impedance of free space is 377 Ω.

[SPPU : May-03, 04, 2000, Dec.-08, Marks 6]

OR Derive the relation between $\bar{E}$ and $\bar{H}$ in free space.

Ans. : • Consider Maxwell's equation derived from Faraday's law,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t} \qquad ...(1)$$

$$\therefore \quad \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = -\mu \frac{\partial}{\partial t}[H_x \bar{a}_x + H_y \bar{a}_y + H_z \bar{a}_z]$$

• Assume that uniform plane wave is propagating in z-direction.

$$\therefore \quad \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & \partial/\partial z \\ E_x & E_y & 0 \end{vmatrix} = -\mu \frac{\partial}{\partial t}[H_x \bar{a}_x + H_y \bar{a}_y]$$

• Also assume that $\bar{E}$ and $\bar{H}$ are mutually perpendicular to each other and the direction of propagation. Then $E_y = 0$ and $H_x = 0$. Thus above equation gets simplified as,

$$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = -\mu \frac{\partial}{\partial t}[H_y \bar{a}_y]$$

$$\therefore \quad \frac{\partial E_x}{\partial z}\bar{a}_y = -\mu \frac{\partial}{\partial t} H_y \bar{a}_y$$

i.e.

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad ...(2)$$

But from equation (15) from Q.3.

$$E_x = E_m^+ \cos(\omega t - \beta z) + E_m^- \cos(\omega t + \beta z)$$

Differentiating with respect to z, we get,

$$\frac{\partial E_x}{\partial z} = \beta[E_m^+ \sin(\omega t - \beta z) - E_m^- \sin(\omega t + \beta z)]$$

Substituting $\frac{\partial E_x}{\partial z}$ in equation (2) we get,

$$\beta[E_m^+ \sin(\omega t - \beta z) - E_m^- \sin(\omega t + \beta z)] = -\mu \frac{\partial H_y}{\partial t}$$

$$\therefore \quad \frac{\partial H_y}{\partial t} = -\frac{\beta}{\mu}\left[E_m^+ \sin(\omega t - \beta z) - E_m^- \sin(\omega t + \beta z)\right] \qquad ...(3)$$

Integrating equation (3) with respect to t, we get,

$$H_y = \frac{-\beta}{\mu}\left[-\frac{E_m^+ \cos(\omega t - \beta z)}{\omega} - \left\{-\frac{E_m^- \cos(\omega t - \beta z)}{\omega}\right\}\right]$$

$$\therefore \quad H_y = \frac{\beta}{\omega\mu}[E_m^+ \cos(\omega t - \beta z) - E_m^- \cos(\omega t + \beta z)]$$

$$H_y = \frac{\beta}{\omega\mu} E_m^+ \cos(\omega t - \beta z) - \frac{\beta}{\omega\mu} E_m^- \cos(\omega t + \beta z) \qquad ...(4)$$

- Comparing equation (4) with equation (16) from Q.3 we get,

$$H_m^+ = \frac{\beta}{\mu\,\omega} E_m^+ \quad \text{and} \quad H_m^- = -\frac{\beta}{\mu\,\omega} E_m^-$$

Hence

$$H_m^+ = \frac{E_m^+}{\frac{\mu\,\omega}{\beta}} = \frac{E_m^+}{\frac{\mu\,\omega}{\omega\sqrt{\mu\,\varepsilon}}} = \frac{E_m^+}{\sqrt{\frac{\mu}{\varepsilon}}} \qquad ...(5)$$

Also

$$H_m^- = -\frac{E_m^-}{\frac{\mu\,\omega}{\beta}} = -\frac{E_m^-}{\frac{\mu\,\omega}{\omega\sqrt{\mu\,\varepsilon}}} = -\frac{E_m^-}{\sqrt{\frac{\mu}{\varepsilon}}} \qquad ...(6)$$

- The radical term is nothing but the impedance with unit ohm.
- Such impedance is expressed interms of μ and ε which are properties of the medium.
- Hence the impedance is called **intrinsic impedance of the medium** and is denoted by η.
- Hence in general,

$$\eta = \frac{E_m^+}{H_m^+} = -\frac{E_m^-}{H_m^-} = \sqrt{\frac{\mu}{\varepsilon}}\ \Omega \qquad ...(7)$$

- But for free space, $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$ Hence for free space, the intrinsic impedance is denoted by η_0 and is given by,

$$\eta_0 = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}} = 377\ \Omega \text{ (resistive)} \quad \text{... (8)}$$

- The general expression for η interms of η_0 is given as,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \sqrt{\frac{\mu_r}{\varepsilon_r}} = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} \quad \text{... (9)}$$

Q.6 Obtain the propagation constant and wavelength for uniform plane wave in free space.

Ans. : Propagation Constant (γ)

- Using the simplified equation, we can write,

$$\nabla^2 \bar{E} = \mu \sigma \frac{\partial \bar{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2}$$

- Rewriting above equation in phasor from, we get,

$$\nabla^2 \bar{E} = \mu \sigma (j\omega \bar{E}) + \mu \varepsilon (j\omega)^2 \bar{E}$$

$$\nabla^2 \bar{E} = [j\omega\mu (\sigma + j\omega\varepsilon)]\bar{E} \quad \text{... (1)}$$

- Similarly for the magnetic field we can write,

$$\nabla^2 \bar{H} = [j\omega\mu (\sigma + j\omega\varepsilon)]\bar{H} \quad \text{... (2)}$$

- Equation (1) and (2) are called **wave equations in phasor form.**
- In equation (1) and (2), the term inside the bracket is same. This term represents the properties of the medium through which the wave is travelling.
- It is square of the propagation constant (γ). Hence wave equations can be rewritten as,

$$\nabla^2 \bar{E} = \gamma^2 \bar{E} \quad \text{...(3)}$$

and

$$\nabla^2 \bar{H} = \gamma^2 \bar{H} \quad \text{...(4)}$$

- Hence, in general, the propagation constant can be expressed interms of the properties of the medium as,

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \quad \text{...(5)}$$

- But the propagation constant γ is the complex quantity made up of real and imaginary term. Thus

$$\boxed{\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} \quad \text{...(6)}$$

- In general, when wave travels through medium it gets attenuated. That means the amplitude of the medium reduces. It is represented by the real part of the propagation constant. It is called **attenuation constant** (α). It is measured in **neper per metor (Np/m).** But practically α is expressed in **decibel (dB).**
- Similarly when a wave travels through the medium, phase change occurs. Such a phase change is expressed by an imaginary part of the propagation constant. It is called **phase shift constant** or simply **phase constant** (β). It is measured in **radian per meter (rad/m).** Thus imaginary part of the propagation constant i.e. β is given by,
- Now for free space, $\sigma = 0$, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, then,

$$\gamma = \alpha + j\beta = 0 + j\omega\sqrt{\mu_0 \varepsilon_0}$$

- Hence for free space, $\alpha = 0$ and $\beta = \omega\sqrt{\mu_0 \varepsilon_0}$
- Thus **for free space, the propagation constant is purely imaginary.**

4. Wavelength (λ)

- In general, the wave repeats itself after 2π radian.
- Thus the distance that must be travelled by the wave to change phase by 2π radian is called **wavelength** and is denoted by λ.

$$\text{Wavelength} = \lambda = \frac{2\pi}{\beta} \text{ mi} \quad \text{... (7)}$$

Putting $\beta = \omega\sqrt{\mu\,\varepsilon}$, we get,

$$\boxed{\lambda = \frac{2\pi}{\omega\sqrt{\mu\,\varepsilon}} = \frac{\frac{1}{\sqrt{\mu\,\varepsilon}}}{\left(\frac{\omega}{2\pi}\right)} = \frac{v}{f} \text{ m}} \quad \text{...(8)}$$

- For free space, $\lambda = \frac{v_0}{f} = \frac{c}{f}$

Q.7 A uniform plane wave $\bar{E}_y = 10 \sin (2 \pi \times 10^8 t - \beta x) \bar{a}_y$ is travelling in x-direction in free space. Find i) Phase constant, ii) Phase velocity and iii) Expression for $\bar{H}_z$. Assume $\bar{E}_z = 0 = \bar{H}_y$.

Ans. : A wave travelling in x-direction with the electric field in y-direction can be expressed as,

$$\bar{E}_y = E_{y0} \sin (\omega t - \beta z) \bar{a}_y \quad \text{V/m} \qquad \cdots (1)$$

where E_{y0} is the magnitude of the electric field in y-direction.

By comparing the given expression of $\bar{E}_y$ with equation (1), we can write,

$$\omega = 2\pi \times 10^8 \text{ rad/sec.} \quad \text{and} \quad E_{y0} = 10 \text{ V/m}$$

For free space $\varepsilon_r = 1 = \mu_r$. For a uniform plane wave in free space, $v = \frac{\omega}{\beta}$

In free space, $v = c = 3 \times 10^8$ m/sec. Hence phase constant is given by,

$$\therefore \quad \beta = \frac{\omega}{v} = \frac{2 \times \pi \times 10^8}{3 \times 10^8} = \mathbf{2.09435 \text{ rad/m}}$$

For free space, the intrinsic impedance is given by,

$$\eta_0 = 120 \pi = 377 \ \Omega$$

The magnitude of the magnetic field is given by,

$$H_{z0} = \frac{E_{y0}}{\eta_0} = \frac{10}{377} = 0.026525 \text{ A/m} = 26.5251 \text{ mA/m}$$

As wave is travelling in positive x-direction and the electric field is in y-direction, the magnetic field must be in positive z-direction. Hence the magnetic field expression can be given as,

$$\bar{H}_z = H_{y0} \sin (\omega t - \beta x) \bar{a}_z \text{ A/m}$$

Substituting the values of H_{y0}, ω and β, we can write,

$$\bar{H}_z = \mathbf{26.5251 \sin (2 \times \pi \times 10^8 t - 2.09435x) \bar{a}_z \text{ mA/m}}$$

Q.8 A 10 GHz plane wave travelling in free space as an amplitude $E_x = 10$ V/m. Find v, λ, β, η and amplitude and direction of $\bar{H}$.

[SPPU : Dec.-18, Marks 8]

Ans. : In free space,

i) $$v = 3 \times 10^8 \text{ m/s}$$

ii) $$\lambda = \frac{v}{f} = \frac{3 \times 10^8}{10 \times 10^9} = \mathbf{0.03 \text{ m}}$$

iii) $$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.03} = \mathbf{209.44 \text{ rad/m}}$$

iv) $$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}} = 376.7 \approx \mathbf{377 \ \Omega}$$

v) $$H = \frac{E}{\eta_0} = \frac{10}{377} = \mathbf{0.0265 \text{ A/m}}$$

$\bar{H}$ is in **+ z direction** and H_y = **0.0265 A/m** is amplitude.

Q.9 A plane wave in a nonmagnetic medium has $\bar{E} = 50 \sin (10^8 + 2z) \hat{a}_y$ V/m. Find : i) Direction of wave propagation ii) Wavelength, frequency, iii) Magnetic field H

[SPPU : May-19, Marks 8]

Ans. : i) Due to +ve sign of 2z in $\bar{E}$, the direction of wave propagation is in **– z direction.**

ii) From equation, $\omega = 10^8$ rad/sec

$$\therefore \quad f = \frac{\omega}{2\pi} = \mathbf{15.915 \text{ MHz}}$$

From equation, $\beta = 2$

$$\therefore \quad \lambda = \frac{2\pi}{\beta} = \mathbf{3.142 \text{ m}}$$

iii) For nonmagnetic material $\mu_r = 1$

$$\lambda = \frac{v}{f} \quad \text{i.e. } v = \lambda f = 50 \times 10^6 \text{ m/s}$$

But $$v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon_r}}$$

and $$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}}$$

$$\therefore \quad 50 \times 10^6 = \frac{1}{\sqrt{4 \pi \times 10^{-7} \times 8.854 \times 10^{-12} \varepsilon_r}} \quad \text{i.e. } \varepsilon_r = 35.95$$

$$\therefore \quad \eta = \sqrt{\frac{4 \pi \times 10^{-7}}{8.854 \times 10^{-12} \times 35.95}} = 62.71$$

$$\therefore \quad H_m = \frac{E_m}{\eta} = \frac{50}{62.71} = 0.796$$

$$\therefore \quad \bar{H} = 0.796 \sin(10^8 + 2z)\,\bar{a}_x \ \text{A/m}$$

5.5 : Electromagnetic Wave Equation in Phasor Form

Q.10 Derive the electromagnetic wave equation in phasor form.

[SPPU : May-05, 13,18, Dec.-01,18, Marks 8]

Ans. : • Consider Maxwell's equation derived from Faraday's law,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t} \quad \dots (1)$$

• Taking curl on both the sides of the equation,

$$\therefore \quad \nabla \times \nabla \times \bar{E} = -\mu\left[\nabla \times \frac{\partial \bar{H}}{\partial t}\right] = -\mu\left[\frac{\partial}{\partial t}(\nabla \times \bar{H})\right] \quad \dots (2)$$

• Using vector identity to the left of equation (2),

$$\therefore \quad \nabla(\nabla \bullet \bar{E}) - \nabla^2 \bar{E} = -\mu\left[\frac{\partial}{\partial t}(\nabla \times \bar{H})\right] \quad \dots (3)$$

• But according to another Maxwell's equation,

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

• Putting value of $\nabla \times \bar{H}$ in equation (3),

$$\therefore \quad \nabla(\nabla \bullet \bar{E}) - \nabla^2 \bar{E} = -\mu\left[\frac{\partial}{\partial t}\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\right] \quad \dots (4)$$

• Since most of the regions are source or charge free,

$$\therefore \quad \nabla \bullet \bar{E} = 0$$

$$\therefore \quad \nabla(\nabla \bullet \bar{E}) = 0$$

• Putting value of $\nabla(\nabla \bullet \bar{E})$ in equation (4), assuming charge free medium,

$$-\nabla^2 \bar{E} = -\mu\left[\frac{\partial}{\partial t}\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\right]$$

• Making both sides positive,

$$\nabla^2 \bar{E} = \mu\left[\frac{\partial}{\partial t}\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\right] \quad \dots (5)$$

• Consider a general electromagnetic wave with both the fields, $\bar{E}$ and $\bar{H}$ varying with respect to time. When any field varies with respect to time, its partial derivative taken with respect to time can be replaced by $j\omega$. Rewriting equation (5) in phasor form,

$$\therefore \quad \nabla^2 \bar{E} = \mu\left[j\omega\left(\bar{J} + j\omega\bar{D}\right)\right] = j\omega\mu\left[(\sigma\bar{E}) + j\omega(\varepsilon\bar{E})\right]$$

$$\therefore \quad \nabla^2 \bar{E} = \left[j\omega\sigma\mu\bar{E} + (j\omega)\varepsilon\mu\bar{E}\right] = \left[j\omega\mu(\sigma + j\omega\varepsilon)\right]\bar{E} \quad \dots (6)$$

• In similar way, we can write another phasor equation as,

$$\nabla^2 \bar{H} = \left[j\omega\mu(\sigma + j\omega\varepsilon)\right]\bar{H} \quad \dots (7)$$

The terms inside the bracket in equations (6) and (7) are exactly similar and represent the properties of the medium in which wave is propagating.

Q.11 Derive expression of electromagnetic wave equation in phasor form. Also derive expression of α and β from it.

[SPPU : May-18, Marks 8]

Ans. : Refer Q.10 of Unit - VI

$$\nabla^2 \bar{E} = \gamma^2 \bar{E} \quad \text{and} \quad \nabla^2 \bar{H} = \gamma^2 \bar{H}$$

• So the propagation constant γ can be expressed interms of properties of the medium as,

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \quad \dots (1)$$

• The real and imaginary parts of γ are attenuation constant (α) and phase constant (β) and both can be expressed interms of the properties of the medium,

$$\therefore \quad \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)} \quad \dots (2)$$

$$\text{and} \quad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)} \quad \dots (3)$$

5.6 : Uniform Plane Wave in Lossless Dielectric

Q.12 Explain in detail the behaviour of uniform plane wave in lossless dielectric. **[SPPU : May-01, 06, Dec.-01, 04, 08, Marks 8]**

Ans. : • Consider that the uniform plane wave is propagating through a perfect dielectric. If the medium is perfect dielectric, then its properties are given by, $\sigma = 0$, $\mu = \mu_r \mu_0$ and $\varepsilon = \varepsilon_r \varepsilon_0$. For the perfect dielectric as conductivity is zero (i.e. $\sigma = 0$), the medium is also called **lossless medium.**

• The analysis of the uniform plane waves propagating through the perfect dielectric is very much similar to that for the wave propagating through the free space as in both cases $\sigma = 0$. But the expressions are different as the values of permeability and permittivity are different. For the free space, $\mu = \mu_0$ and $\varepsilon = \varepsilon_r$. Let the values of the permittivity and permeability for the perfect dielectric be $\varepsilon = \varepsilon_0 \varepsilon_r$ and $\mu = \mu_0 \mu_r$ respectively.

• The velocity of propagation is given by,

$$v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{(\mu_0 \mu_r)(\varepsilon_0 \varepsilon_r)}}$$

... (For perfect dielectric, $\varepsilon = \varepsilon_0 \varepsilon_r$ and $\mu = \mu_0 \mu_r$)

$$\therefore \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}\sqrt{\mu_r \varepsilon_r}} = \frac{1/\sqrt{\mu_0 \varepsilon_0}}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \text{ m/s} \quad ...(1)$$

Also

$$\boxed{v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{\omega}{\omega\sqrt{\mu \varepsilon}} = \frac{\omega}{\beta} \text{ m/s}} \quad ...(2)$$

• The **propagation constant** is given by,

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \text{ m}^{-1}$$

• For the perfect dielectric, substituting $\sigma = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$ and $\mu = \mu_0 \mu_r$ in above expression we get,

$$\gamma = \sqrt{j\omega\mu(0 + j\omega\varepsilon)}$$

$$\therefore \quad \boxed{\gamma = \pm j\omega\sqrt{\mu \varepsilon} \text{ m}^{-1}} \quad ...(3)$$

Also $\gamma = \alpha + j\beta$

• Hence the **attenuation constant** for the perfect dielectric is given by,

$$\boxed{\alpha = 0} \quad ...(4)$$

• The **phase constant** for the perfect dielectric is given by,

$$\boxed{\beta = \omega\sqrt{\mu \varepsilon} \text{ rad/m}} \quad ...(5)$$

• The **intrinsic impedance** is given by,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

• Putting $\sigma = 0$ for perfect dielectric, we get,

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$$\therefore \quad \boxed{\eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{\mu_r}{\varepsilon_r}} \ \Omega} \quad ...(6)$$

5.7 : Uniform Plane Wave in Lossy Dielectric

Q.13 Explain in detail the behaviour of uniform plane wave in lossy dielectric. **[SPPU : May-01, 02, Dec.-05, 07, Marks 8]**

Ans. : • Practically all the dielectric materials exhibit some conductivity ($\sigma \neq 0$). So during analysis of the uniform plane waves through dielectric with some amount of conductivity, we cannot neglect σ by assuming it to be zero. Due to certain conductivity, certain amount of loss in the medium takes place. Hence the wave travelling through such medium gets attenuated ($\alpha \neq 0$). Such dielectric is called **lossy dielectric**. The results obtained for the lossy dielectric are different as compared with the results for perfect dielectric medium.

• Let us consider that a uniform plane wave travels in z-direction through the lossy dielectric ($\alpha \neq 0$). The wave equation for the electric field vector $\bar{E}$ is given by,

$$\nabla^2 \bar{E} = \mu\sigma\frac{\partial^2 \bar{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 \bar{E}}{\partial t^2} \quad ...(1)$$

- Using vector indentity on L.H.S. of equation (1), we get,

$$\frac{\partial^2 \bar{E}}{\partial x^2}+\frac{\partial^2 \bar{E}}{\partial y^2}+\frac{\partial^2 \bar{E}}{\partial z^2} = \mu\sigma\frac{\partial \bar{E}}{\partial t}+\mu\varepsilon\frac{\partial^2 \bar{E}}{\partial t^2} \qquad ...(2)$$

- Now the wave travels in z-direction, then $\bar{E}$ is the function of z and t only, hence equation (2) becomes,

$$\frac{\partial^2 \bar{E}}{\partial z^2} = \mu\sigma\frac{\partial \bar{E}}{\partial t}+\mu\varepsilon\frac{\partial^2 \bar{E}}{\partial t^2} \qquad ...(3)$$

- Let $\bar{E}$ has only one component i.e. in x-direction (E_x). Then equation (3) can be modified as,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\sigma\frac{\partial E_x}{\partial t}+\mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} \qquad ...(4)$$

- Expressing E_x in phasor form as,

$$E_x = E_m e^{j\omega t}$$

Then $$\frac{\partial E_x}{\partial t} = (j\omega)\, E_m e^{j\omega t} = (j\omega) E_x \qquad ...(5)$$

Also $$\frac{\partial^2 E_x}{\partial t^2} = (j\omega)(j\omega)\, E_m e^{j\omega t} = -(j\omega)^2 E_x \qquad ...(6)$$

- Substituting values of $\frac{\partial E_x}{\partial t}$ and $\frac{\partial^2 E_x}{\partial t^2}$ in equation (4) we get,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\sigma(j\omega)E_x + \mu\varepsilon(j\omega)^2 E_x$$

$$\therefore \quad \frac{\partial^2 E_x}{\partial z^2} = [j\omega\mu(\sigma + j\omega\varepsilon)]E_x \qquad ...(7)$$

- By definition of propagation constant, $\gamma = \sqrt{j\omega\mu(\sigma + j\omega t)}$
- Hence substituting bracket value as γ^2 in equation (7), we get

$$\frac{\partial^2 E_x}{\partial z^2} = \gamma^2 E_x \qquad ...(8)$$

where $$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

- Equation (8) can be written as,

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0 \qquad ...(9)$$

Let $$\frac{\partial}{\partial z} = D \text{ and } \frac{\partial^2}{\partial z^2} = D^2, \text{ then we get,}$$

$$D^2 E_x - \gamma^2 E_x = 0$$

$$\therefore \quad (D^2 - \gamma^2)E_x = 0 \qquad ...(10)$$

- Hence equating term inside bracket to zero, we get,

$$D^2 - \gamma^2 = 0$$

$$\therefore \quad D^2 = \gamma^2$$

$$\therefore \quad D = \pm\gamma$$

- Hence solution of equation (9) is given by,

$$E_x = K_1 e^{-\gamma z} + K_2 e^{+\gamma z} \qquad ...(11)$$

- where K_1 and K_2 are constants with respect to z but be the functions of time t.

$$\therefore \quad K_1 = E_m^+ e^{j\omega t}$$

$$K_2 = E_m^- e^{j\omega t}$$

- Substituting values of K_1 and K_2 in equation (11), we get,

$$E_x = E_m^+ e^{j\omega t} e^{-\gamma z} + E_m^- e^{j\omega t} e^{-\gamma z}$$

$$\therefore \quad E_x = E_m^+ e^{j\omega t} e^{-(\alpha + jB)z} + E_m^- e^{j\omega t} e^{(\alpha + jB)z}$$

$$\therefore \quad \boxed{E_x = E_m^+ e^{-\alpha z} \cdot e^{j(\omega t - \beta z)} + E_m^- e^{\alpha z} \cdot e^{j(\omega t + \beta z)}} \qquad ...(12)$$

- Above equation indicates the equation of electric field in the uniform plane wave travelling through lossy dielectric in the z-direction. Assuming that wave travels in +z-direction, the wave equation becomes,

$$E_x = E_m^+ e^{-\alpha z} e^{j(\omega t - Bz)} \qquad ...(13)$$

- Taking real part of the term on R.H.S., we get,

$$E_x = R\cdot E[E_m^+ e^{-\alpha z} e^{j(\omega t - Bz)}]$$

$$= E_m^+ e^{-\alpha z} \cos(\omega t - Bz) \qquad ...(14)$$

- Thus from equations (12), (13) and (14), the wave travels in positive z-direction with velocity $\frac{\omega}{\beta}$ m/s. As the wave progresses, it gets attenuated by the factor $e^{-\alpha}$.

5.8 : Depth of Penetration

Q.14 Explain the skin effect and depth of penetration.

[SPPU : Dec.-17,18,19, May-18,19, Marks 4]

Ans. : • Consider only the component of the electric field E_x travelling in positive z-direction. When it travels in good conductor, the conductivity is very high and attenuation constant α is also high. Thus we can write such a component in phasor form as

$$E_x = E_m^+ e^{-\alpha z} e^{-j\beta z} \quad \text{... (1)}$$

- When such a wave propagates in good conductor, there is a large attenuation of the amplitude as shown in the Fig. Q.14.1
- At z = 0, amplitude of the component E_x is E_m ; while at z = d, amplitude is $E_m e^{-\alpha d}$.

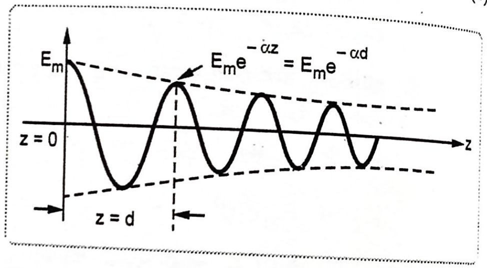


Fig. Q.14.1 Effect of attenuation constant (α) on amplitude of E_x

- In distance z = d, the amplitude gets reduced by the factor $e^{-\alpha d}$. If we select $d = \frac{1}{\alpha}$, then the factor becomes e^{-1} = 0.368.
- So over a distance $d = \frac{1}{\alpha}$ the amplitude of the wave decreases to approximately 37 % of its original value.
- The distance through which the amplitude of the travelling wave decreases to 37 % of the original amplitude is called **skin depth** or **depth of penetration**. It is denoted by δ.

∴

$$\boxed{\text{Skin depth} = \delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu \sigma}} \text{ m}} \quad \text{... (2)}$$

- From the expression of the skin depth, it is clear that δ is inversely proportional to the square root of frequency.
- So for the frequencies in the microwave range, the skin depth or depth of penetration is very small for good conductors. And all the fields and currents may be considered as confined to a very thin layer near the surface of the conductor. This thin layer is nothing but the skin of the conductor, hence this effect is called **skin effect.**
- The intrinsic impedance of a good conductor interms of skin depth δ is given by,

$$\boxed{\eta = \left(\frac{1}{\sigma\delta} + j\frac{1}{\sigma\delta}\right) = \frac{\sqrt{2}}{\sigma\delta} \angle 45^\circ \ \Omega} \quad \text{... (3)}$$

Q.15 Calculate skin depth, propagation constant and wave velocity v at a frequency of 1.6 MHz in Aluminium Where σ = 38.2 MS/m and μ_r = 1

[SPPU : Dec.-17, 18, Marks 7]

Ans. : For a given metal i.e. aluminium, σ = 38.2 MS/m which is very high. Thus it can be used as good conductor material. For good conductor, the propagation constant is given by,

$$\gamma = \sqrt{j\omega\mu\sigma}$$

$$\therefore \quad \gamma = \sqrt{j(2\pi f)(\mu_0\mu_r)\sigma}$$

$$\therefore \quad \gamma = \sqrt{j\left(2\times\pi\times1.6\times10^6\right)\left(4\times\pi\times10^{-7}\times1\right)\left(38.2\times10^6\right)}$$

$$\therefore \quad \gamma = \sqrt{j\left(4.8258\times10^8\right)} = \sqrt{4.8258\times10^8} \angle 45^\circ$$

$$\therefore \quad \gamma = 21967.7 \angle 45^\circ \text{ m}^{-1}$$

$$\therefore \quad \gamma = \alpha + j\beta = \mathbf{15.533\times10^3 + j\,15.533\times10^3 \ m^{-1}}$$

Hence,

Attenuation constant α = 15.533×10^3 Np/m,

Phase constant β = 15.533×10^3 rad/m

The skin depth for aluminium at 1.6 MHz is given by

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} = \frac{1}{15.533\times10^3} = \mathbf{64.38\ \mu m}$$

The velocity of propagation is given by

$$v = \frac{\omega}{\beta} = \frac{2\times\pi\times1.6\times10^6}{15.533\times10^3} = \mathbf{647.2 \quad m/s}$$

Q.16 Calculate the skin depth and wave velocity at 2 MHz in aluminium with conductivity 40 MS/m and $\mu_r = 1$.

☞ [SPPU : Dec.-14, Marks 6]

Ans. : For aluminium with very high conductivity, the propagation constant is given by,

$$\gamma = \sqrt{j\omega\mu\sigma}$$

$$= \sqrt{j(2\pi f)(\mu_0\mu_r)\sigma}$$

$$= \sqrt{j(2\times\pi\times2\times10^6)(4\times\pi\times10^{-7}\times1)(40\times10^6)}$$

$$= \sqrt{j\,6.3165\times10^8}$$

$$= \sqrt{6.3165\times10^8 \angle 90^\circ}$$

$$= 25.1326\times10^3 \angle 45^\circ \ m^{-1}$$

$$= 17.771\times10^3 + j\,17.771\times10^3 \ m^{-1}$$

But $\quad \gamma = \alpha + j\beta$

$$= 17.771\times10^3 + j\,17.771\times10^3$$

By comparing real and imaginary terms, we get,

$$\alpha = 17.771\times10^3 \ Np/m$$

and $\quad \beta = 17.771\times10^3 \ rad/m$

Hence skin depth for aluminium is given by,

$$\delta = \frac{1}{\alpha} = \frac{1}{17.771\times10^3} = \mathbf{56.2714 \ \mu m}$$

Similarly the velocity of propagation is given by,

$$v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\times\pi\times2\times10^6}{17.771\times10^3} = \mathbf{707.12 \ m/s}$$

Q.17 Find the depth of penetration for Cu at 1 MHz, $\sigma = 6\times10^6$ S/m $\mu_r = 1$.

☞ [SPPU : Dec.-19, Marks 6]

Ans. : $\quad \gamma = \sqrt{j\omega\mu\sigma} = \sqrt{j2\pi f\mu_0\mu_r\sigma}$

$$= \sqrt{j\times2\pi\times1\times10^6\times4\pi\times10^{-7}\times1\times6\times10^6}$$

$$= \sqrt{47.874\angle 90^\circ} = 6.882\times10^3 \angle 45^\circ$$

$$= 4866.31 + j\,4866.31 = \alpha + j\beta$$

$$\therefore \quad \alpha = 4866.31 \ Np/m$$

$$\therefore \quad \delta = \frac{1}{\alpha} = \mathbf{205.494 \ \mu m}$$

5.9 : Polarization of Uniform Plane Waves

Q.18 What is meant by polarization of the wave. State its types and explain any one in detail.

☞ [SPPU : May-18,19, Dec.-17, 19, Marks 8]

Ans. : The **polarization of uniform plane waves** is defined as time varying behaviour of the electric field intensity vector $\bar{E}$ at some fixed point in space, along the direction of propagation.

- Consider that the electric field $\bar{E}$ has only x-component and y-component of $\bar{E}$ is zero. Then looking from the direction of propagation, the wave is said to be **linearly polarized in x-direction.** Similarly if only the y-component in $\bar{E}$ is present and x-component of $\bar{E}$ is zero then the wave is said to be **linearly polarized in y-direction.**
- Let us assume that both the components of $\bar{E}$ are present denoted by $\bar{E}_x$ and $\bar{E}_y$. Both these components are in phase having different amplitudes. As $\bar{E}_x$ and $\bar{E}_y$ are in phase they will have their amplitudes reaching maximum or minimum value simultaneously. Also if the amplitude of $\bar{E}_x$ increases or decreases the amplitude of $\bar{E}_y$ also increases or decreases. In other words, at any point along positive z-axis the ratio of amplitudes of both the components is constant as both of them are in phase having same wavelength.
- The electric field $\bar{E}$ is the resultant of $\bar{E}_x$ and $\bar{E}_y$ and the direction of it depends on the relative magnitude of $\bar{E}_x$ and $\bar{E}_y$. Thus the angle made by $\bar{E}$ with x-axis is given by,

$$\theta = \tan^{-1}\frac{E_y}{E_x}$$

- where E_x and E_y are the magnitudes of $\overline{E}_x$ and $\overline{E}_y$ respectively.
- This angle is constant with respect to time. In other words, the resultant vector $\overline{E}$ is oriented in a direction which is constant with time, thus the wave is said to be **linearly polarized** as shown in the Fig. Q.18.1.

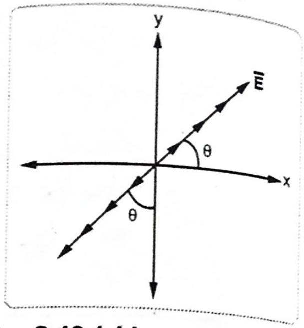


Fig. Q.18.1 Linear polarization

- When both the components have same amplitudes we get a polarization of $\overline{E}$ as linear polarization with a constant angle of 45°.
- From Fig. Q.18.1 it is clear that when E_x increases or decreases, E_y also increases and decreases as both are in phase. And the resultant $\overline{E}$ changes along the straight line at an angle $\theta°$ with respect to x-axis.
- **Thus when $\overline{E}_x$ and $\overline{E}_y$ components are in phase with either equal or unequal amplitudes, for a uniform plane wave travelling in z-direction, the polarization is linear.**
- The various types of polarization are,

1) Linear 2) Elliptical

3) Circular 4) Linear polarization of uniform plane wave

5.10 : Reflection of Uniform Plane Waves

Q.19 Explain the normal incidence at plane dielectric boundary. Derive the expressions for transmission and reflection coefficients.

Ans. : • Consider a uniform plane wave striking the interface between the two dielectrics at right angles as shown in the Fig. Q.19.1.

- Assume that a uniform plane wave travels along +z direction and incidence at right angles at the boundary between two dielectric media i.e. at z = 0.
- Below z = 0, let the properties of medium 1 be $\varepsilon_1, \mu_1, \sigma_1, \eta_1$ and above z = 0, the properties of medium 2 be $\varepsilon_2, \mu_2, \sigma_2, \eta_2$.

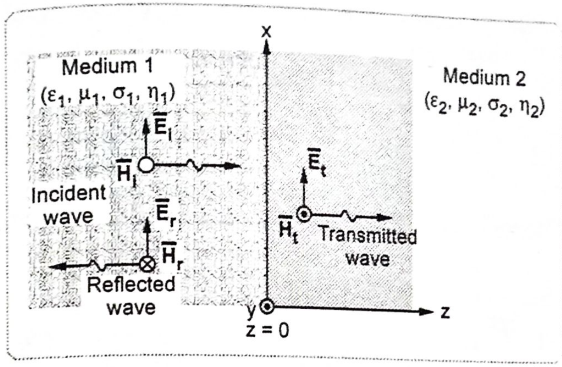


Fig. Q.19.1 Normal incidence at plane dielectric boundary

- So depending upon the properties of two media, part of the wave will be transmitted in medium 2 while other part will be reflected back in medium 1.
- Let E_i and H_i be the field strengths of the incident wave striking the boundary. Let E_t and H_t be the field strengths of the transmitted wave in the medium 2. And let E_r and H_r be the field strengths of the reflected wave in the medium 1 returning back from the interface.
- From the Fig. Q.19.1 it is clear that in medium 1, the total field comprises both the incident and reflected fields. But in medium 2, only the transmitted field gives the total field.
- So the conditions for the total fields in medium 1 are given by,

$$\overline{E}_1 = \overline{E}_i + \overline{E}_r \quad \text{and} \quad \overline{H}_1 = \overline{H}_i + \overline{H}_r$$

- Similarly the conditions for the total fields in medium 2 are given by,

$$\overline{E}_2 = \overline{E}_t \quad \text{and} \quad \overline{H}_2 = \overline{H}_t$$

- According to the boundary conditions, the tangential components of $\overline{E}$ and $\overline{H}$ must be continuous at the interface z = 0.
- As the waves are transverse in nature, at the boundary the fields $\overline{E}$ and $\overline{H}$ both are tangential to the interface.

$$\therefore \quad \overline{E}_{1\,tan} = \overline{E}_{2\,tan} \quad \text{And} \quad \overline{H}_{1\,tan} = \overline{H}_{2\,tan}$$

- Thus at the interface z = 0, we can write

$$\overline{E}_i + \overline{E}_r = \overline{E}_t \quad \text{And} \quad \overline{H}_i + \overline{H}_r = \overline{H}_t$$

- The relationships between the magnitudes of $\bar{E}$ and $\bar{H}$ at $z = 0$ are given by following expressions.

$$E_i = \eta_1 H_i,$$

$$E_r = -\eta_1 H_r \quad \text{... As direction of reflected wave is opposite}$$

$$E_t = \eta_2 H_t$$

- Interms of magnitudes of the fields $\bar{E}$ and $\bar{H}$ at the interface, we can write,

$$E_i + E_r = E_t \quad \text{... (1)}$$

and $$H_i + H_r = H_t \quad \text{... (2)}$$

- In equation (2), putting values of H_i, H_r and H_t interms of E_i, E_r and E_t respectively,

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

$$\therefore \quad E_i - E_r = \frac{\eta_1}{\eta_2} E_t \quad \text{... (3)}$$

- Adding equations (1) and (3),

$$2E_i = \left(1 + \frac{\eta_1}{\eta_2}\right) E_t$$

$$\therefore \quad 2E_i = \frac{\eta_1 + \eta_2}{\eta_2} E_t$$

$$\therefore \quad E_t = \frac{2\eta_2}{\eta_1 + \eta_2} E_i \quad \text{... (4)}$$

- The **transmission coefficient** is denoted by τ and it is given by,

$$\boxed{\tau = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}} \quad \text{... (5)}$$

- Eliminating E_t from equations (1) and (3), we get,

$$E_i + E_r = \frac{\eta_2}{\eta_1} E_i - E_r$$

$$\therefore \eta_1 (E_i + E_r) = \eta_2 (E_i - E_r)$$

$$\therefore E_i (\eta_1 - \eta_2) = - E_r (\eta_1 + \eta_2)$$

$$\therefore \quad E_r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_i \quad \text{... (6)}$$

- The **reflection coefficient** is denoted by Γ and it is given by,

$$\boxed{\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \quad \text{... (7)}$$

- Thus we can draw important conclusions as,

A) $1 + \Gamma = \tau$

B) $0 \leq |\Gamma| \leq 1$

C) Both Γ and τ are dimensionless and may be complex in nature.

Q.20 Briefly explain about the wave incident normally on a perfect conductor. **[SPPU : Dec.-17, Marks 8]**

Ans. : • Consider a uniform plane wave striking the interface between two media ; where medium 1 is perfect dielectric ($\sigma = 0$, lossless) and medium 2 is perfect conductor ($\sigma = \infty$). For medium 2, intrinsic impedance $\eta_2 = 0$ being a perfect conductor.

- Thus the transmission coefficient is given by,

$$\tau = \frac{2\eta_2}{\eta_1 + \eta_2} = 0 \quad \text{... (1)}$$

Also the reflection coefficient is given by,

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1 \quad \text{... (2)}$$

- From the values of the reflection and transmission coefficients it is clear that the wave is totally reflected and there is no transmitted wave in medium 2.
- It is obvious from the fact that no field can exists in the perfect conductor.
- As totally reflected wave combines with the incident wave, standing waves are formed.
- A standing wave does not travel and it consists of two travelling waves ; one incident $(\bar{E}_i)$ and other reflected $(\bar{E}_r)$. Both the waves have same amplitudes but the directions in which they propagate are different.

- Let the standing wave in medium be denoted by $\bar{E}_{1s}$. Then in phasor form we can write,

$$\bar{E}_{1s} = \bar{E}_i + \bar{E}_r = \left(E_i e^{-\gamma_1 z} + E_r e^{\gamma_1 z}\right) \bar{a}_x$$

- But as the reflection coefficient is

$$\Gamma = \frac{E_r}{E_i} = -1$$

- Also for medium 1, $\sigma = 0$ indicates perfect dielectric medium

$$\gamma_1 = \alpha_1 + j\beta_1 \quad \text{where} \quad \alpha = 0 \quad \text{for} \quad \sigma = 0$$

$$\gamma_1 = j\beta_1$$

$$\bar{E}_{1s} = \left(E_i e^{-j\beta_1 z} - E_i e^{j\beta_1 z}\right) \bar{a}_x$$

$$\bar{E}_{1s} = -E_i \left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right) \bar{a}_x \quad \ldots (4)$$

Simplifying,

$$\bar{E}_{1s} = -2j E_0 \sin\beta_1 z \, \bar{a}_x \quad \ldots (5)$$

Hence the field in medium 1 is given by,

$$\bar{E}_1 = R_e\left(\bar{E}_{1s} e^{j\omega t}\right) \quad \ldots (6)$$

or

$$\bar{E}_1 = 2E_i \sin\beta_1 z \sin\omega t \, \bar{a}_x \quad \ldots (7)$$

- Exactly similar to above derivation, we can obtain magnetic field H_1 in medium 1 as

$$\bar{H}_1 = \frac{2E_i}{\eta_1} \cos\beta_1 z \cos\omega t \, \bar{a}_y \quad \ldots (8)$$

- From equations (7) and (8) it is clear that the total fields in the medium 1 are not travelling eventhough these fields are obtained by adding two waves of equal amplitude but travelling in opposite directions. Also in these two equations, the factors with time and distance are represented by separate trigonometric terms.
- Consider the equation for total electric field in medium 1 given by equation (7). This field is zero everywhere whenever $\omega t = n\pi$. Also this field is zero for all time at the planes for which $\beta_1 z = n\pi$.
- The planes at which the electric field is zero are located where,

$$\beta_1 z = n\pi \quad \ldots\ldots\ldots \quad n = 0, \pm 1, \pm 2, \ldots\ldots$$

$$\therefore \quad z = \frac{n\pi}{\beta_1} = \frac{n\pi}{\left(\frac{2\pi}{\lambda_1}\right)} = n\frac{\lambda_1}{2}$$

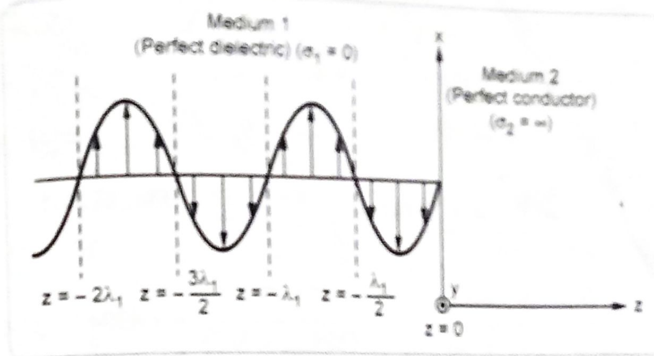


Fig. Q.20.1 Representation of instantaneous values of the total electric field at $t = \frac{\pi}{2}$, in medium 1

- The above condition clearly indicates that the electric field is zero at $z = 0$ and it repeats after every half wavelength from the boundary in the medium 1 where $z < 0$.
- The Fig. Q.20.1 shows the instantaneous values of the electric field in medium 1 at $t = \pi/2$.
- Now consider equation (Q.20.8) representing a standing wave. The magnitude of the magnetic field is maximum at the positions where the electric field is zero. Furthermore it is 90° out of time phase with the electric field everywhere.

Q.21 Determine the amplitude of the reflected and transmitted E and H at the interface of two media with the following properties. Medium 1 : $\xi_r = 8.5, \mu_r = 1, \sigma = 0$, Medium 2 : Free Space. Assume normal incidence and the amplitude of E in medium 1 at the interface is 1.5 mV/m. **[SPPU : May-18, Marks 8]**

Ans. : The interface is between perfect dielectric (region 1) and free space (region 2).

For region 1,

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$= \sqrt{\frac{4\times\pi\times10^{-7}\times1}{8.8542\times10^{-12}\times8.5}}$$

$$= 129.22\ \Omega$$

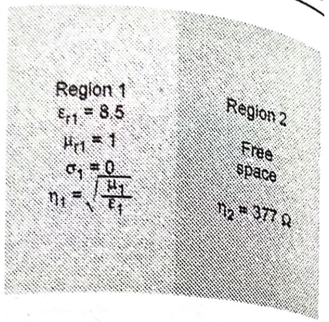


Fig. Q.21.1

For region 2,

$$\eta_2 = 120\ \pi = 377\ \Omega$$

By definitions,

$$\Gamma = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1+\eta_2} = \frac{2(377)}{(129.22+377)} = 1.4894$$

But $\quad E_i = 1.5$ mV/m

$$\therefore \quad E_t = \tau(E_i) = (1.4894)\left(1.5\times10^{-3}\right)$$

$$= \mathbf{2.2341\ mV/m}$$

$$\Gamma = \frac{E_r}{E_i} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{(377-129.22)}{(377+129.22)} = 0.4894$$

But $\quad E_i = 1.5$ mV/m

$$\therefore \quad E_r = \Gamma(E_i) = (0.4894)\left(1.5\times10^{-3}\right)$$

$$= \mathbf{0.7341\ mV/m}$$

As we know, $H_t = \frac{E_t}{\eta_2}$ and $H_r = \frac{H_r}{-\eta_1}$ we can write,

$$H_t = \frac{2.2341\times10^{-3}}{377} = \mathbf{5.9259\ \mu A/m} \quad \text{and}$$

$$H_r = \frac{0.7341\times10^{-3}}{-129.22} = \mathbf{-5.681\ \mu A/m}$$

Q.22 Given the intrinsic impedance : $\eta_1 = 100\ \Omega$ and $\eta_2 = 300\ \Omega$, the normal incident eletric field $E_i = 100$ m V/m, calculate
i) Reflection and transmission coefficient
ii) Reflected and transmitted eletric field $\overline{E}$
iii) Reflected and transmitted magnetic field $\overline{H}$

[SPPU : May-19, Marks 8]

Ans. : $\quad \eta_1 = 100,\ \eta_2 = 300,\ E_i = 100$ mV/m

i)
$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \mathbf{0.5}$$

$$\tau = \frac{2\eta_2}{\eta_1+\eta_2} = \mathbf{1.5}$$

ii)
$$E_r = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}E_i = \Gamma E_i = 0.5\times100 = \mathbf{50\ mV/m}$$

$$E_t = \tau E_i = 1.5\times100 = \mathbf{150\ mV/m}$$

iii)
$$H_t = \frac{E_t}{\eta_2} \quad \text{and} \quad H_r = \frac{E_r}{-\eta_1}$$

$$\therefore \quad H_t = \frac{150}{300} = \mathbf{0.5\ mA/m}$$

$$\therefore \quad H_r = \frac{50}{-100} = \mathbf{-0.5\ mA/m}$$

5.11 : Snell's Law

Q.23 Explain oblique incidence at a plane dielectric boundary. State Snell's law. **[SPPU : Dec.-17,19, Marks 4]**

Ans. : • Consider that a uniform plane wave is incident obliquely at the interface between the two dielectrics. The two dielectric media are assumed to be lossless with different constants. Such as μ_1, ε_1 for medium 1 while μ_2, ε_2 for medium 2. As the two media are having different constitutive parameters, at the boundary, thus there is as if discontinuity. Because of this, at the interface, a part of the incident wave is reflected; while a part is transmitted as shown in the Fig. Q.23.1.

• Let the lines OA, O′A′ and O′B be the intersections of the plane of incidence with the equiphase surfaces of the incident, reflected and

For region 1,

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$= \sqrt{\frac{4\times\pi\times10^{-7}\times1}{8.8542\times10^{-12}\times8.5}}$$

$$= 129.22\ \Omega$$

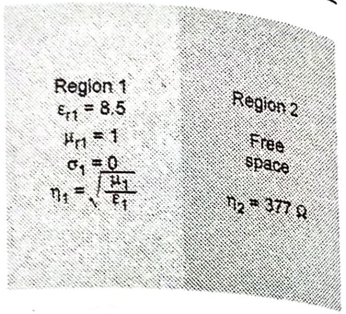


Fig. Q.21.1

For region 2,

$$\eta_2 = 120\,\pi = 377\ \Omega$$

By definitions,

$$\Gamma = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1+\eta_2} = \frac{2(377)}{(129.22+377)} = 1.4894$$

But $\quad E_i = 1.5$ mV/m

$$\therefore \quad E_t = \tau(E_i) = (1.4894)\left(1.5\times10^{-3}\right)$$

$$= \mathbf{2.2341\ mV/m}$$

$$\Gamma = \frac{E_r}{E_i} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{(377-129.22)}{(377+129.22)} = 0.4894$$

But $\quad E_i = 1.5$ mV/m

$$\therefore \quad E_r = \Gamma(E_i) = (0.4894)\left(1.5\times10^{-3}\right)$$

$$= \mathbf{0.7341\ mV/m}$$

As we know, $H_t = \frac{E_t}{\eta_2}$ and $H_r = \frac{H_r}{-\eta_1}$ we can write,

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Q.22 Given the intrinsic impedance : $\eta_1 = 100\ \Omega$ and $\eta_2 = 300\ \Omega$, the normal incident eletric field $E_i = 100$ m V/m, calculate
i) Reflection and transmission coefficient
ii) Reflected and transmitted eletric field $\overline{E}$
iii) Reflected and transmitted magnetic field $\overline{H}$

☞ [SPPU : May-19, Marks 8]

Ans. : $\quad \eta_1 = 100, \quad \eta_2 = 300, \quad E_i = 100$ mV/m

i) $$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \mathbf{0.5}$$

$$\tau = \frac{2\eta_2}{\eta_1+\eta_2} = \mathbf{1.5}$$

ii) $$E_r = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}E_i = \Gamma E_i = 0.5\times100 = \mathbf{50\ mV/m}$$

$$E_t = \tau E_i = 1.5\times100 = \mathbf{150\ mV/m}$$

iii) $$H_t = \frac{E_t}{\eta_2} \quad \text{and} \quad H_r = \frac{E_r}{-\eta_1}$$

$$\therefore \quad H_t = \frac{150}{300} = \mathbf{0.5\ mA/m}$$

$$\therefore \quad H_r = \frac{50}{-100} = \mathbf{-0.5\ mA/m}$$

5.11 : Snell's Law

Q.23 Explain oblique incidence at a plane dielectric boundary. State Snell's law. **☞ [SPPU : Dec.-17,19, Marks 4]**

Ans. : • Consider that a uniform plane wave is incident obliquely at the interface between the two dielectrics. The two dielectric media are assumed to be lossless with different constants. Such as μ_1, ε_1 for medium 1 while μ_2, ε_2 for medium 2. As the two media are having different constitutive parameters, at the boundary, thus there is as if discontinuity. Because of this, at the interface, a part of the incident wave is reflected; while a part is transmitted as shown in the Fig. Q.23.1.

• Let the lines OA, O′A′ and O′B be the intersections of the plane of incidence with the equiphase surfaces of the incident, reflected and

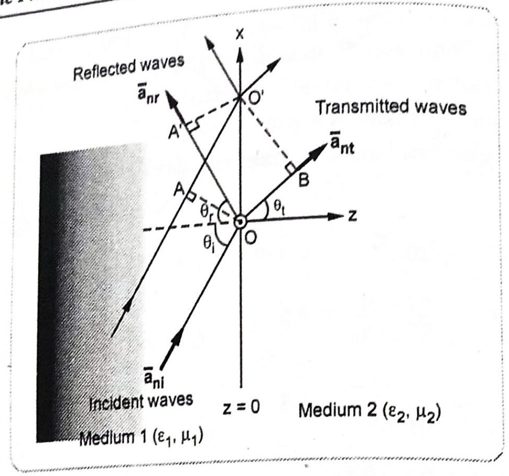

Fig. Q.23.1 Oblique incidence at a plane dielectric boundary

transmitted waves respectively. The incident and reflected waves travel in medium 1, while transmitted wave in medium 2. The velocity for the waves in medium 1 is same say v_1. Then distances $\overline{OA'}$ and $\overline{AO'}$ must be same.

$$\therefore \quad \overline{OO'} \sin\theta_i = \overline{OO'} \sin\theta_i$$

i.e. $$\boxed{\theta_r = \theta_i} \quad \ldots (1)$$

- From equation (1) it is clear that the angle of incidence and angle of reflection are equal. This is nothing but **Snell's law of reflection.**
- It is also observed the time taken by the incident wave to travel from A to O′ in medium 1 is equal to that taken by the transmitted wave to travel from O to B in medium 2. Thus we can write,

$$\frac{\overline{OB}}{v_2} = \frac{\overline{AO'}}{v_1}$$

where v_1 is the velocity of the wave in medium 1 while v_2 is velocity of the wave in medium 2.

$$\therefore \quad \frac{\overline{OB}}{\overline{AO'}} = \frac{\overline{OO'} \sin\theta_t}{\overline{OO'} \sin\theta_i} = \frac{v_2}{v_1}$$

Simplifying above equation,

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{v_2}{v_1} = \frac{(\omega/\beta_2)}{(\omega/\beta_1)} = \frac{\beta_1}{\beta_2} \quad \ldots (2)$$

- But for any medium, the ratio of the speed of light in free space to that in medium is called **index of refraction**. It is denoted by n and it is given by,

$$n = \frac{c}{v}$$

where c = Velocity of light in free space

v = Velocity of light in medium

- Then for medium 1, index of refraction is given by,

$$n_1 = \frac{c}{v_1}$$

- Similarly, for medium 2, index of refraction is given by,

$$n_2 = \frac{c}{v_2}$$

- Substituting values of v_1 and v_2 interms of n_1 and n_2 in equation (2), we get,

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2}} \quad \ldots (3)$$

Equation (3) represents **Snell's law of refraction.**

5.12 : Application Case Study

Q.24 Give the comparison between circuit theory at low frequency and field theory at high frequency.

Ans. : • Circuit theory is applicable at low frequencies to analyse and synthesis of systems. The systems may be linear or nonlinear, time dependent or time independent. The basic variables in circuit theory are analysis to determine energy distribution across different components in circuit. Network theorems are applicable for systems with lumped parameters. Electrical circuits and electronic circuits can be represented as two port networks for which parameters can be calculated. In circuit theory a linear circuit can be modeled using differential equations, transfer functions, state equation etc. The limitations of circuit theory are it cannot be applied in free space, it is applicable at low frequency only it cannot analyze radiation phenomenon.

- The electromagnetic field theory is dependent on the vector fields $\bar{E}$, $\bar{H}$, $\bar{B}$ and $\bar{D}$. Various vector fields satisfy the Maxwells equation in static fields as well as in time vaying conditions. The propagation of electromagnetic energy takes place in the direction of vector called poynting vector. The field vectors are function of space and time. The electric and magnetic scalar potentials satisfy Poisson's and Laplace's equations. The propagation of electromagnetic wave can be represented by Helmholtz equation. The advantages of field theory are it is applicable in free space, applicable at all frequencies. It is used to explain the radiation phenomenon.

Q.25 Explain the antenna radiation mechanism.

Ans. : • An antenna is a metallic device in the form of either wire or rod used for radiating electromagnetic wave or transmitting energy. Antenna is also a transducer which acts as impedance marking device between transmitter and free space or free space and receiver. It can also be considered as matching device between the transmission line or waveguide and the medium.

It directs radiated energy in desired direction.

- Consider two wire antenna driven by transmission line driven by voltage source.
- When the source is applied, electric field get developed between conductors.
- The electric lines of force are tangent to electric field at each point with strength dependent on electric field intensity. The electric lines of force out on the free electrons easily detachable from atoms in conductors. It results in displacement of charges which causes current giving rise to magnetic field intensity.

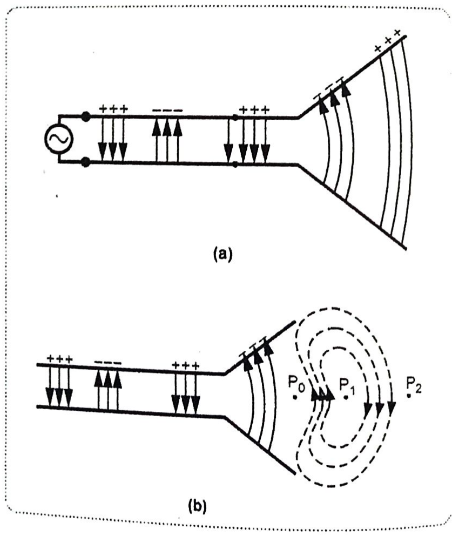

Fig. Q.25.1 Radiation from two wire antennas

- The electric field lines travel from positive charges to negative charges. They can also start on positive charge and end at infinity or travel from infinity to negative charge or form closed loops.
- Magnetic field lines form closed loops and uncircle current carrying conductor.
- The time varying electric and magnetic fields between the conductors form electromagnetic waves and are detected from antenna.

Formulae at a Glance

- **Uniform Plane wave in any General Medium**

Velocity (v)	$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{\omega}{\beta} = f.\lambda\, m/s$
Propagation constant (γ)	$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}\ m^{-1}$
Attenuation constant (α)	$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}\ Np/m$
Phase constant (β)	$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}\ rad/m$
Intrinsic Impedance (η)	$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}\ \Omega$
Wavelength (λ)	$\lambda = \frac{2\pi}{\beta} = \frac{v}{f}\ m$

- **Uniform Plane wave in Free Space**

For free space, $\sigma = 0$, $\mu = \mu_0$, $\varepsilon = \varepsilon_0$

Velocity (v)	$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8\ m/s = c$
Propagation constant (γ)	$\gamma = \alpha + j\beta = 0 + j\omega\sqrt{\mu_0\varepsilon_0}\ m^{-1}$
Attenuation constant (α)	$\alpha = 0$

Phase constant (β)	$\beta = \omega\sqrt{\mu_0 \varepsilon_0}$ rad/m
Intrinsic Impedance (η_0)	$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$
Wavelength (λ)	$\lambda = \frac{2\pi}{\beta} = \frac{c}{f}$ m

- **Uniform Plane wave in Perfect Dielectric**

Velocity (v)	$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$ m/s
Propagation constant (γ)	$\gamma = \alpha + j\beta = 0 + j\omega\sqrt{\mu\varepsilon}\ m^{-1}$
Attenuation constant (α)	$\alpha = 0$
Phase constant (β)	$\beta = \omega\sqrt{\mu\varepsilon}$ rad/m
Intrinsic impedance (η)	$\eta = 377\sqrt{\frac{\mu_r}{\varepsilon_r}}\ \Omega$
Wavelength (λ)	$\lambda = \frac{2\pi}{\beta}$ m

- **Uniform Plane wave in Lossy Dielectric**

Velocity (v)	$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$ m/s
Propagation constant (γ)	$\gamma = \alpha + j\beta = 0 + j\omega\sqrt{\mu\varepsilon}\ m^{-1}$
Attenuation constant (α)	$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$ Np/m
Phase constant (β)	$\beta = \omega\sqrt{\mu\varepsilon}$ rad/m
Intrinsic impedance (η)	$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{j\omega\varepsilon}\right)\Omega$
Wavelength (λ)	$\lambda = \frac{2\pi}{\beta}$ m
Loss Tangent	$\tan\theta = \frac{\sigma}{\omega\varepsilon}$

- **Uniform Plane wave in Good Conductor**

Velocity (v)	$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$ m/s
Propagation constant (γ)	$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$
Attenuation constant (α)	$\alpha = \sqrt{\pi f \mu \sigma}$ Np/m
Phase constant (β)	$\beta = \sqrt{\pi f \mu \sigma}$ Rad/m
Intrinsic impedance (η)	$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\pi f \mu}{\sigma}}\ (1 + j1)$
Wavelength (λ)	$\lambda = \frac{2\pi}{\beta}$ m
Depth of penetration	$\delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu \sigma}}$

Normal Incidence

At Plane Dielectric Boundary

Transmission coefficient $\tau = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}$

Reflection coefficient $\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

At Plane Conducting Boundary

Reflection coefficient $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Standing Wave Ratio $S = \frac{E_{1s\,max}}{E_{1s\,min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$

Reflection coefficient Interms of SWR $|\Gamma| = \frac{S-1}{S+1}$

END...

Unit VI

6 Transmission Line Theory

6.1 : Transmission Line Parameters

Q.1 State and explain the primary constants of a transmission line.

☞ [SPPU : May-07, Dec.-16, 18, Marks 8]

Ans. : 1. Resistance : The resistance is uniformly distributed all along the length of the transmission line. Its total value depends on the overall length of the transmission line. Hence its value is given per unit length of the transmission line. It is denoted as R and given in ohms per unit length.

2. Inductance : When the conductors carry the current, the magnetic flux is produced around the conductors. It depends on the magnitude of the current flowing through the conductors. This gives rise to the effect called inductance of the transmission line. It is also distributed all along the length of the transmission line. It is denoted as L and measured in henry per unit length.

3. Capacitance : The transmission line consists of two parallel conductors, separated by a dielectric like air. Such parallel conductors separated by an insulating dielectric produces a capacitive effect. So there exists a capacitance which is distributed along the length of the conductor. It is denoted as C and measured in farads per unit length.

4. Conductance : The dielectric in between the conductors is not perfect. Hence a very small amount of current flows through the dielectric called **displacement current**. This leakage current gives rise to a leakage conductance. It exists between the conductors and distributed along the length of the transmission line. It is denoted as G and measured in mho per unit length.

Q.2 Write expressions for inductance and capacitance of open wire transmission line and coaxial line.

Ans. : For Open Wire Line

$$L = 10^{-7}\left[\frac{\mu}{\mu_0} + 4\,ln\frac{d}{a}\right] \text{H/m} = 10^{-7}\left[\mu_r + 9.210\log_{10}\frac{d}{a}\right] \text{H/m}$$

$$C = \frac{\pi\varepsilon}{ln\frac{d}{a}} \text{ F/m} = \frac{12.07\,\varepsilon_r}{\log_{10}\frac{d}{a}} \text{ p F/m}$$

For Co-axial Line

$$L = \frac{\mu_0}{2\pi}\,ln\frac{b}{a} = 2\times10^{-7}\,ln\frac{b}{a} \text{ H/m} = 4.6\times10^{-7}\log\frac{b}{a} \text{ H/m}$$

$$C = \frac{2\pi\varepsilon}{ln\frac{b}{a}} \text{ F/m} = \frac{24.14\,\varepsilon_r}{\log_{10}\frac{b}{a}} \text{ pF/m}$$

6.2 : The Infinite Line

Important Points to Remember

- The input impedance of the infinite line is called **characteristic impedance** of the transmission line and is denoted by Z_0.
- As the line has an infinite length, no waves will ever reach the receiving end and hence there is no possibility of any reflected wave. Thus complete power applied at the sending end is absorbed by the line.
- As the reflected waves are absent, the characteristic impedance Z_0 at the sending end will decide the current flowing, with a voltage is at the sending end. The current will not be affected by the terminating impedance Z_R at the receiving end which is fulfilled by the long lines in practice.

6.3 : Short Line

Important Points to Remember

- The short line means a practical line of finite length. As it is a practical line with finite length, it is also called **finite line**.
- **A finite line which is terminated in its characteristic impedance behaves as an infinite line.** This means that its input impedance will be Z_0 and there will be no reflection.

Q.3 Derive the expression for Z_0 for finite line terminated in Z_0.

Ans. : Consider a short line terminted in its characteristic impedance Z_0 as shown in the Fig. Q.3.1 (a). The short line is a symmetrical network and hence can be represented by the equivalent T-section as shown in the Fig. Q.3.1 (b). Such a representation is the property of a symmetrical network.

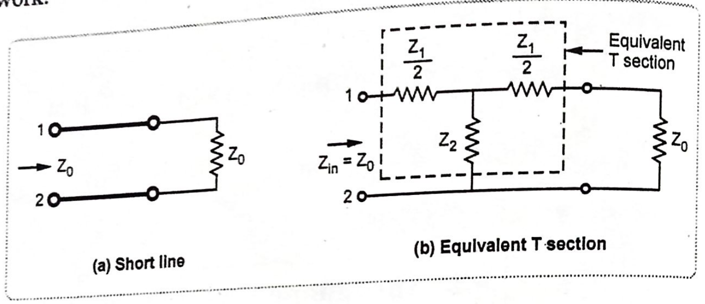

Fig. Q.3.1

It is known that finite line terminated in Z_0 behaves as an infinite line hence the input impedance Z_{in} of the equivalent T-section network also must be Z_0.

The input impedance Z_{in} of the equivalent T section network is given by,

$$Z_{in} = \frac{Z_1}{2} + \left\{ Z_2 \,||\, \left(\frac{Z_1}{2} + Z_0 \right) \right\} \quad \text{... (1)}$$

But $Z_{in} = Z_0$. Hence by simplifying above equation with $Z_{in} = Z_0$, the Z_0 for the equivalent T section of the finite line is given by,

$$\therefore \quad \boxed{Z_0 = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}} \quad \text{... (2)}$$

Q.4 Prove that $Z_0 = \sqrt{Z_{OC} Z_{SC}}$. **[SPPU : Dec.-16, Marks 6]**

Ans. : For a finite line terminated in Z_0,

$$Z_0 = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$$

To obtain Z_1 and Z_2 practically, two measurements are done called open circuit and short circuit.

In open circuit, the line is kept open and input impedance is measured which is denoted as Z_{OC}. This is shown in the Fig. Q.4.1.

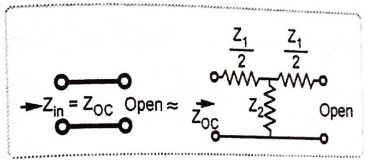

Fig. Q.4.1

From the Fig. Q.4.1 we can write,

$$Z_{OC} = \frac{Z_1}{2} + Z_2 \quad \text{... (1)}$$

In short circuit case, the second end of the line is shorted and the input impedance is measured. It is denoted as Z_{SC}. This is shown in the Fig. Q.4.2.

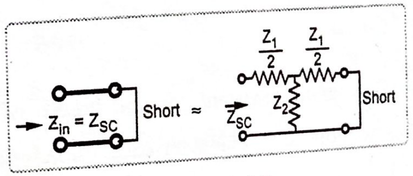

Fig. Q.4.2

From Fig. Q.4.2, We can write,

$$Z_{SC} = \frac{Z_1}{2} + \left[Z_2 \,||\, \frac{Z_1}{2} \right] = \frac{Z_1}{2} + \left[\frac{\frac{Z_1 Z_2}{2}}{\frac{Z_1}{2} + Z_2} \right]$$

$$= \frac{Z_1}{2} + \frac{Z_1 Z_2}{Z_1 + 2Z_2} = \frac{Z_1^2 + 4Z_1 Z_2}{2(Z_1 + 2Z_2)}$$

$$= \frac{Z_1^2 + 4Z_1 Z_2}{4\left(\frac{Z_1}{2} + Z_2 \right)} = \frac{Z_0^2}{Z_{OC}}$$

$$\therefore \quad Z_0^2 = Z_{OC} Z_{SC} \quad \text{i.e. } Z_0 = \sqrt{Z_{OC} Z_{SC}} \quad \text{... Proved}$$

6.4 : Line of Cascaded Sections

Important Points to Remember

- A line terminated in Z_0 behaves as an infinite line, divided into number of identical sections of unit length as shown in the Fig. 6.1.

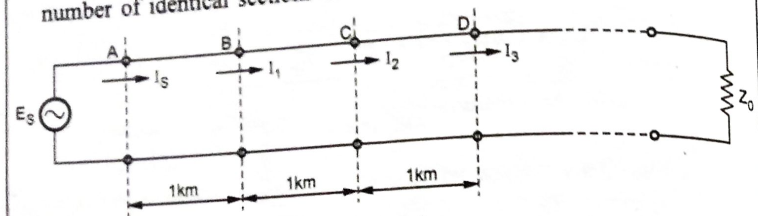

Fig. Q.6.1

- The voltage E_S is applied at the sending end of the line at A.
- At point B, current is I_1. There is phase shift between I_S and I_1.

$$\therefore \quad \frac{I_S}{I_1} = e^{\gamma}$$

Where γ = Propagation constant per unit length of line.

- Each section is terminated in characteristics impedance Z_0. Thus in general,

$$\frac{I_S}{I_n} = \frac{I_S}{I_1} \times \frac{I_1}{I_2} \times \ldots \times \frac{I_{n-1}}{I_n} = e^{n\gamma}$$

$$\therefore \quad \boxed{I_n = I_S e^{-n\gamma}}$$

- Similarly,

$$\boxed{E_n = E_S e^{-n\gamma}.}$$

$$\gamma = \alpha + j\beta$$

Where α = Attenuation constant in nepers / km.

β = Phase or wavelength constant in radians / km.

$$\therefore \quad \boxed{\frac{I_S}{I_n} = e^{n\alpha} \angle n\beta}$$

- Practically the attenuation is measured in decibels (dB). The conversion is,

$$\therefore \quad \boxed{1 \text{ Neper} = 8.686 \text{ dB}}$$

- Also the phase shift is measured in degrees where

$$\therefore \quad \boxed{1 \text{ Radian} = 57.3 \text{ degrees}}$$

Q.5 A cable has an attenuation of 3.5 dB/km and a phase constant of 0.28 rad/km. If 3 V is applied to the sending end then what will be the voltage at point 10 km down the line when line is terminated with Z_0. **[SPPU : Dec.-15, May- 15, Marks 8]**

Ans. : Given x = 10 km,

α = 3.5 dB/km = (3.5)(0.115) Np/m = 0.40285 Np/km,

β = 0.28 rad/km, E_s = 3 V Now1 Neper=8.686 dB

$$\therefore \quad 3.5 \text{ dB/km} = \frac{3.5}{8.686} \text{ Neper/km} = 0.4029 \text{ Neper/km}$$

Now $E_x = E_S e^{-\gamma x} = E_S e^{-\alpha x} \angle -\beta x$ ... Use α in Neper/km

$$\therefore \quad E_x = 3 e^{-0.4029 \times 10} \angle -0.28 \times 10 \text{ rad}$$

$= 0.05337 \angle -2.8$ rad ... 1 rad = 57.3°

$= \mathbf{0.05337 \angle -160.44° \ V}$

This is the voltage at a point 10 km down the line.

6.5 : Wavelength and Velocity

Q.6 Define wavelength and velocity of a transmission line.

Ans. : Wavelength (λ)

- The distance between two points along the line at which currents or voltages differ in phase by 2π radians is called **wavelength**. It is denoted by λ.
- It can also be defined as the distance between any point and the next point along the line at which the current or voltage is in the same phase.

- Mathematically it is expressed as,

$$\boxed{\lambda = \frac{2\pi}{\beta} \text{ dB meter}}$$

Velocity (v)

- The wave travels distance of λ in one cycle, in time equal to $1/f$ sec. Hence the **velocity of propagation** (v) is defined as,

$$\boxed{v = \frac{\text{Distance travelled}}{\text{Time taken}} = \frac{\lambda}{\left(\frac{1}{f}\right)} = f\lambda = \frac{2\pi f}{\beta} = \frac{\omega}{\beta}}$$

- It is measured in km/sec if β is in rad/km and in m/sec if β is in rad/m and so on. As it is related to phase constant of the line, the velocity is called **phase velocity.**
- When β is a function of ω then velocity is produced by introduction of a group of frequencies travelling through the system, then velocity is called **group velocity** and can be obtained as,

$$v_g = \frac{d\omega}{d\beta}$$

6.6 : Relationship between Primary and Secondary Constants

Q.7 Derive the expression for characteristics impedance (Z_0) and propagation constant (γ) in terms of primary constants of transmission line. **☞[SPPU : Dec.-15, 19, May-15,17, Marks 8]**

OR State primary and secondary constants of a transmission line and hence derive relationship between primary and secondary constants of a transmission line.

☞ [SPPU : Dec.-14, 17, 18, May-16, 18, 19, Marks 8]

Ans. : • Any practical line has following constants namely, R(Resistance per unit length), G (Conductance per unit length), L (Inductance per unit length), C (Capacitance per unit length).

- All these constants are assumed to be independent of frequency and are called **primary constants** of the transmission line. All these constants are measured considering both the wires of transmission line.

- Apart from R, G, L and C few other constants related to the transmission line are **characteristic impedance** Z_0, the **propagation constant** γ (which in turn gives attenuation constant α and phase constant β).
- All these constants are fixed at one particular frequency but change their values as the frequency changes. Hence these constants are called **secondary constants.**

Relationship Between Primary and Secondary Constants of a Line

- Consider a short length of line 'l' km.is shown in the Fig. Q.7.1 (a). This section will have resistance of $Rl\ \Omega$, conductance of Gl mho, inductance of Ll H and capacitance of Cl F. Its characteristic impedance is Z_0.

This short line can be represented by symmetrical T network as shown in the Fig. Q.7.1 (b).

- If the length of the line is small, then the total series impedance of the section represents Z_1 and the total parallel impedance of the section represents Z_2.

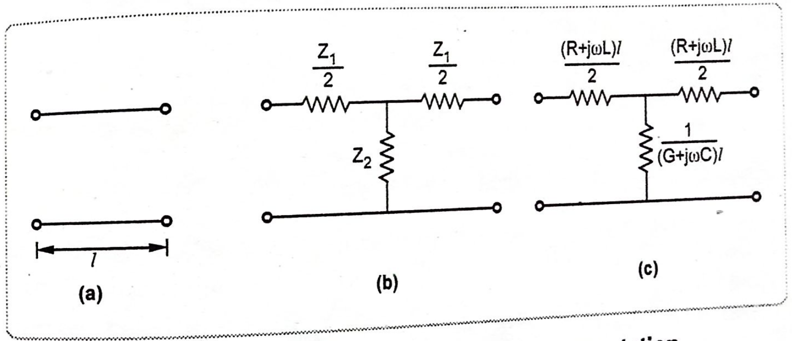

Fig. Q.7.1 Short transmission line representation

$$\therefore \quad Z_1 = (R + j\omega L)\,l \quad \text{and} \quad Z_2 = \frac{1}{(G + j\omega C)\,l}$$

- The total series impedance per unit length is denoted as Z while the total parallel impedance per unit length is denoted as Y.

$$\therefore \quad Z = R + j\omega L \qquad \text{... per unit length}$$

$$Y = Z_{shunt} = G + j\omega C \qquad \text{... per unit length}$$

- Hence the equivalent T network can be shown as in the Fig. Q.7.1 (c).

- Note that this assumption is valid only when length of the line is small i.e. $l \to 0$.

Determination of Z_0 Interms of Primary Constants

- For the T section we can write,

$$Z_0 = \sqrt{\frac{Z_1^2}{4}+Z_1 Z_2} = \sqrt{\frac{(R+j\omega L)^2 l^2}{4}+\frac{(R+j\omega L)l}{(G+j\omega C)l}}$$

$$= \sqrt{\frac{R+j\omega L}{G+j\omega C}+\frac{(R+j\omega L)^2 l^2}{4}}$$

But as $l \to 0$, $l^2 \to 0$ and can be neglected.

$$\therefore \quad Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{Z}{Y}} \quad \text{... (1)}$$

Representing in polar form we can write,

$$Z_0 = \sqrt{\frac{\sqrt{R^2+\omega^2 L^2}}{\sqrt{G^2+\omega^2 C^2}} \angle \tan^{-1}\frac{\omega L}{R}-\tan^{-1}\frac{\omega C}{G}}$$

$$\therefore \quad Z_0 = \sqrt[4]{\frac{R^2+\omega^2 L^2}{G^2+\omega^2 C^2}} \angle \frac{1}{2}\left[\tan^{-1}\frac{\omega L}{R}-\tan^{-1}\frac{\omega C}{G}\right] \quad \text{... (2)}$$

- When frequency is very small, $\omega \to 0$ hence

$$Z_0 = \sqrt{\frac{R}{G}}$$

- When frequency is very large $R^2 << \omega^2 L^2$ and $G^2 << \omega^2 C^2$

$$Z_0 = \sqrt{\frac{L}{C}}$$

Determination of γ Interms of Primary Constants

- If γ is the propagation constant per unit length defined for a line then its value for a line of length l is γl. Hence we can write,

$$e^{\gamma l} = 1+\frac{Z_1}{2Z_2}+\frac{Z_0}{Z_2} \text{.......for the T section}$$

Substituting values of Z_1 and Z_2 for a short line of length l,

$$e^{\gamma l} = 1+\frac{(R+j\omega L)l}{2\left[\frac{1}{(G+j\omega C)l}\right]}+\frac{Z_0}{\left[\frac{1}{(G+j\omega C)l}\right]}$$

$$\therefore \quad e^{\gamma l} = 1+\frac{l^2(R+j\omega L)(G+j\omega C)}{2}+Z_0\, l(G+j\omega C)$$

$$= 1+\frac{l^2(R+j\omega L)(G+j\omega C)}{2}+\sqrt{\frac{R+j\omega L}{G+j\omega C}}\times l(G+j\omega C)$$

$$\therefore \quad e^{\gamma l} = 1+\frac{l^2(R+j\omega L)(G+j\omega C)}{2}+l\sqrt{(R+j\omega L)(G+j\omega C)} \quad \text{... (3)}$$

But mathematically exponential term $e^{\gamma l}$ can be expanded in a series as,

$$e^{\gamma l} = 1+\gamma l+\frac{\gamma^2 l^2}{2!}+\ldots$$

As $l \to 0$, neglecting higher order terms,

$$e^{\gamma l} = 1+\gamma l+\frac{\gamma^2 l^2}{2} \quad \text{... (4)}$$

Comparing equations (Q.7.3) and (Q.7.4),

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{ZY}$$

Q.8 The characteristic impedance of the uniform transmission line is 2040 Ω at a frequency of 800 Hz. At this frequency the propagation constant is 0.054 ∠873.9°. Determine. R,L,G, v and λ.

[SPPU : Dec.-17, Marks 10]

Ans. : $\omega = 2\pi f = 2\pi \times 800 = 5026.5482$ rad/sec

$$\therefore \quad Z_0 = 2040\ \Omega \quad \text{and} \quad \gamma = 0.054\ \angle 87.9^\circ$$

$$R+j\omega L = Z_0\gamma = 2040\ \angle 0^\circ \times 0.054\ \angle 87.9^\circ$$

$$= 110.16\ \angle 87.9^\circ = 4.0366+j\,110.086$$

$$\therefore \quad R = \mathbf{4.0366\ \Omega/km}$$

and $$\omega L = 110.086$$

$$\therefore \quad L = \frac{110.086}{5026.5482} = \mathbf{0.0219\ \ H/km}$$

and $G + j\omega C = \frac{\gamma}{Z_0} = \frac{0.054 \angle 87.9^\circ}{2040 \angle 0^\circ} = 2.647 \times 10^{-5} \angle 87.9^\circ$

$= 9.7 \times 10^{-7} + j\, 2.6452 \times 10^{-5}$

$\therefore \quad G = 9.7 \times 10^{-7}$ mho/km $=$ **0.97 μmho/km**

and $\quad \omega C = 2.6452 \times 10^{-5}$

$\therefore \quad C = \frac{2.6452 \times 10^{-5}}{5026.5482} =$ **5.2626×10^{-9} F/km**

$\gamma = 0.054 \angle 87.9^\circ = 1.978 \times 10^{-3} + j0.05396 = \alpha + j\beta$

$\therefore \quad \beta = 0.05396$ rad/km

$\therefore \quad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.05396} =$ **116.44 km**

$v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\pi \times 800}{0.05396} =$ **93.153×10^3 km/sec**

Q.9 Calculate the primary constants of a transmission line having $Z_0 = 692 \angle -12^\circ \Omega$ and $\gamma = 0.0363 \angle 78^\circ$ at f = 1 kHz. Also calculate velocity and wavelength. **☞[SPPU : May-02, 07, Dec.-19, Marks 8]**

Ans. : It is known that,

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

and

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$\therefore \quad Z_0 \gamma = R + j\omega L$

$\therefore \quad R + j\omega L = 692 \angle -12^\circ \times 0.0363 \angle 78^\circ = 25.1196 \angle 66^\circ$

$= 10.217 + j\, 22.9478$

$\therefore \quad R =$ **10.217 Ω/km**

$\therefore \quad \omega L = 22.9478$

$\therefore \quad L = \frac{22.9478}{2\pi f} = \frac{22.9478}{2\pi \times 1000} =$ **3.6522 mH/km**

... f = 1 kHz

and $\quad \frac{\gamma}{Z_0} = G + j\omega C$

$\therefore \quad G + j\omega C = \frac{0.0363 \angle 78^\circ}{692 \angle -12^\circ} = 0.000052456 \angle 90^\circ$

$= 0 + j\, 0.000052456$

$\therefore \quad G =$ **0 mho/km**

$\omega C = 0.000052456$

$\therefore \quad C = \frac{0.000052456}{2\pi \times 1000} = 8.3486 \times 10^{-9}$ F/km

$\gamma = \alpha + j\beta = 0.007547 + j\, 0.0355$

$\therefore \quad \beta = 0.0355$ rad/km

$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.0355} =$ **176.957 km**

$v = \frac{\omega}{\beta} = \frac{2\pi \times 1000}{0.0355} =$ **176957.45 km/sec**

Q.10 A transmission line cable has following primary constants R = 11 Ω/km, G = 0.8 μ mho/km L = 0.00367 H/km C = 8.35 nF/km At a signal of 1 kHz calculate i) Characteristic impedance Z_0 ii) Attenuation constant (α) in Np/km iii) Phase constant (β) in radians/km iv) Wavelength (λ) in km v) Velocity of signal in km/sec.

☞[SPPU : Dec.-15, 18, May-15, Marks 10]

Ans. : $Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{Z}{Y}}$ where $\omega = 2 \times \pi \times f = 2 \times \pi \times 1 \times 10^3$ rad/sec

$Z = R + j\omega L = 11 + j(2 \times \pi \times 1 \times 10^3)(0.00367) = 11 + j23.06 = 25.549 \angle 64.5^\circ$

$Y = G + j\omega C = 0.8 \times 10^{-6} + j\,(2 \times \pi \times 10^3)(8.35 \times 10^{-9})$

$= 0.8 \times 10^{-6} + j52.46 \times 10^{-6} = 5.246 \times 10^{-5} \angle 89.12^\circ$

$\therefore \quad Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{25.549 \angle 64.5^\circ}{5.246 \times 10^{-5} \angle 89.12^\circ}} = \sqrt{487.0186 \times 10^3 \angle -24.62}$

$=$ **$697.86 \angle -12.31^\circ \Omega$**

$$\gamma = \sqrt{ZY} = \sqrt{(25.549\angle 64.5^\circ)(5.246\times10^{-5}\angle 89.12^\circ)}$$

$$= \sqrt{1.3403\times10^{-3}\angle 153.62^\circ}$$

$$\therefore \quad \gamma = 0.0366\angle 76.81^\circ = 0.0083 + j0.03563 \text{ km}^{-1} = \alpha + j\beta$$

$$\therefore \quad \alpha = \mathbf{0.0083\ N/km} \quad \text{and} \quad \beta = \mathbf{0.03563\ rad/km}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.03563} = \mathbf{176.345\ km}$$

$$\therefore \quad v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\times\pi\times10^3}{0.03563} = \mathbf{176.3453 \times 10^3 km/sec}$$

$$Z_0 = \frac{E_S}{I_S} \quad \therefore I_S = \frac{E_S}{Z_0} = \frac{1\angle 0^\circ}{697.86\angle -12.31^\circ} = \mathbf{1.433 \times 10^{-3} \angle 12.31^\circ A}$$

$$I_R = I_S \cdot e^{-(\alpha+j\beta)l} = I_S . e^{-\alpha l}\angle -\beta l$$

$$\therefore \quad I_R = 1.433\times10^{-3} \cdot e^{-0.0083\times50}\angle -0.03563\times50$$

$$\therefore I_R = 9.4626\times10^{-4}\angle -1.7815 \text{ rad} = \mathbf{9.4626\times10^{-4}\angle -102.07^\circ\ A}$$

Now $\dfrac{E_R}{I_R} = Z_0$

$$\therefore \quad E_R = I_R \cdot Z_0 = (9.4626\times10^{-4}\angle -102.07^\circ)(697.86\angle -12.31^\circ)$$

$$\therefore \quad E_R = \mathbf{0.6603\angle -114.38^\circ V}$$

Q.11 A generator of 1 V, 1 kHz supplies power to a 100 km open wire transmission line terminated in Z_0. The line parameters are,
R = 10.4 Ω/km, **L** = 0.00367 H/km , **G** = 0.8×10^{-6} mho/km
C = 0.00835×10^{-6} F/km
Calculate $Z_0, \alpha, \beta, \lambda$, velocity (v), received current, voltage and power.

☞[SPPU : May-15, 18, Dec.-15, Marks 8]

Ans. : The Z_0 is given by,

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad \text{where} \quad \omega = 2\pi f = 2\pi\times1\times10^3 \text{ rad/sec}$$

$$Z_0 = \sqrt{\frac{10.4 + j(2\pi\times10^3\times0.00367)}{0.8\times10^{-6} + j(2\pi\times10^3\times0.00835\times10^{-6})}}$$

$$= \sqrt{\frac{10.4 + j\,23.059}{0.8\times10^{-6} + j\,5.246\times10^{-5}}}$$

$$= \sqrt{\frac{25.29\angle 65.72^\circ}{5.246\times10^{-5}\angle 89.126^\circ}} = \sqrt{4.8208\times10^5\angle 65.72^\circ - 89.126^\circ}$$

$$\therefore \quad Z_0 = \mathbf{694.32\ \angle -11.703^\circ\Omega}$$

$$\text{Now} \quad \gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{25.29\angle 65.72^\circ \times 5.246\times10^{-5}\angle 89.126^\circ}$$

$$= \sqrt{0.001326\angle 154.846^\circ} = 0.03641\angle 77.423^\circ$$

$$= 0.007928 + j\,0.03553 = \alpha + j\beta$$

$$\therefore \quad \alpha = \mathbf{0.007928\ Nepers/km} \quad \text{and} \quad \beta = \mathbf{0.03553\ radians/km}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.03553} = \mathbf{176.841\ km}$$

$$v = \frac{\omega}{\beta} = \frac{2\pi\times1\times10^3}{0.03553} = \mathbf{1.95\times10^4\ km/sec}$$

$$\text{Now} \quad Z_0 = \frac{E_S}{I_S} \quad \ldots E_S = 1\text{ V}$$

$$\therefore \quad I_S = \frac{E_S}{Z_0} = \frac{1\angle 0^\circ}{694.32\angle -11.703^\circ}$$

$$= 1.4402\times10^{-3}\angle +11.703^\circ \text{ A}$$

$$\text{Now} \quad I_R = I_S\, e^{-\gamma x} \quad \text{where } x = l = 100 \text{ km}$$

$$\therefore \quad I_R = I_S\, e^{-(\alpha+j\beta)l} = I_S\, e^{-\alpha l}\angle -\beta l$$

$$= 1.4402\times10^{-3}\times e^{-0.007928\times100}\angle -0.03553\times100 \text{ rad}$$

$$\therefore \quad I_R = 6.518\times10^{-4}\angle -3.553 \text{ rad}$$

$$= \mathbf{6.518\times10^{-4}\angle -203.58^\circ\ A}$$

Now $\frac{E_R}{I_R} = Z_0$

∴ $E_R = I_R Z_0$

$= 6.518\times10^{-4} \angle -203.58^\circ \times 694.32 \angle -11.703^\circ$

∴ $E_R = \mathbf{0.4525 \angle -215.283^\circ\ V}$

Thus I_R and E_R are receiving end current and voltage respectively.

The power received is given by,

$$P_R = E_R\, I_R \cos\left(E_R \wedge I_R\right)$$

where $E_R \wedge I_R$ = angle between E_R and I_R

$$\theta = E_R \wedge I_R = 215.28^\circ - 203.58^\circ = +11.703^\circ$$

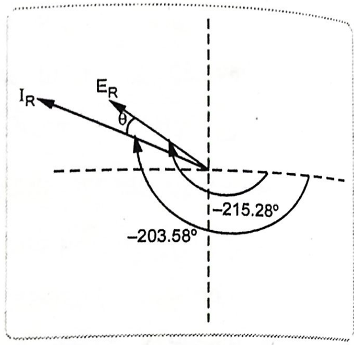

Fig. Q.11.1

This is nothing but angle of Z_0

∴ $P_R = 6.518\times10^{-4} \times 0.4525 \times \cos(+11.703^\circ)$

$= \mathbf{288\times10^{-6}}$ **watts = 288 μW**

Q.12 The characteristic impedance of a uniform transmission line is 2040 Ω at a frequency of 800 Hz. At this frequency, the propagation constant is 0.054 ∠ 87.9°. Determine the values of R, L, G and C.

[SPPU : Dec.-99, 09, May-17, Marks 8]

Ans. : $\omega = 2\pi f = 2\pi \times 800 = 5026.5482$ rad/sec

∴ $Z_0 = 2040\ \Omega$ and $\gamma = 0.054 \angle 87.9^\circ$

$R + j\omega L = Z_0\,\gamma = 2040 \angle 0^\circ \times 0.054 \angle 87.9^\circ$

$= 110.16 \angle 87.9^\circ = 4.0366 + j\,110.086$

∴ $R = \mathbf{4.0366\ \Omega/km}$

and $\omega L = 110.086$

∴ $L = \frac{110.086}{5026.5482} = \mathbf{0.0219\ \ H/km}$

and $G + j\omega C = \frac{\gamma}{Z_0} = \frac{0.054 \angle 87.9^\circ}{2040 \angle 0^\circ} = 2.647\times10^{-5} \angle 87.9^\circ$

$= 9.7\times10^{-7} + j\,2.6452\times10^{-5}$

∴ $G = 9.7\times10^{-7}$ mho/km = **0.97 μmho/km**

and $\omega C = 2.6452\times10^{-5}$

∴ $C = \frac{2.6452\times10^{-5}}{5026.5482} = \mathbf{5.2626\times10^{-9}\ \ F/km}$

6.7 : General Solution of a Transmission Line and its Physical Significance

Q.13 Write the equations for voltage and current at any point along the length of transmission line and hence explain physical significance of general solution of transmission line.

[SPPU : Dec.-14, 16, May-16, 18, Marks 8]

Ans. : • The general solution of a transmission line includes the expressions for current and voltage at any point along a line of any length having uniformly distributed constants.

• The expression for voltage and current at any point along the length of transmission line are given by,

$$\therefore\ E = \frac{E_R(Z_R + Z_0)}{2Z_R}\left[e^{\sqrt{ZY}\,s} + \frac{(Z_R - Z_0)}{(Z_R + Z_0)}e^{-\sqrt{ZY}\,s}\right] \quad \text{... (1)}$$

$$\therefore\ I = \frac{I_R(Z_R + Z_0)}{2Z_0}\left[e^{\sqrt{ZY}\,s} - \frac{(Z_R - Z_0)}{(Z_R + Z_0)}e^{-\sqrt{ZY}\,s}\right] \quad \text{... (2)}$$

• The equations given above represent is the **general solution** of a transmission line.

• Another way of representing the equations is,

$$\therefore\ E = E_R \cosh\left(\sqrt{ZY}\,s\right) + I_R Z_0 \sinh\left(\sqrt{ZY}\,s\right) \quad \text{... (3)}$$

and

$$I = I_R \cosh\left(\sqrt{ZY}\,s\right) + \frac{E_R}{Z_0}\sinh\left(\sqrt{ZY}\,s\right) \quad \text{... (4)}$$

- The similar equations can be obtained interms of sending end voltage E_S and I_S. If x is the distance measured down the line **from the sending end** then, $x = l - s$, then the equations (3) and (4) get transferred interms of E_S and I_S as,

$$E = E_S \cosh(\sqrt{ZY}\, x) - I_S Z_0 \sinh(\sqrt{ZY}\, x) \quad \dots (5)$$

$$I = I_S \cosh(\sqrt{ZY}\, x) - \frac{E_S}{Z_0} \sinh(\sqrt{ZY}\, x) \quad \dots (6)$$

- If receiving end parameters are known and s is distance measured from the receiving end then,

$$E = E_R \cosh(\gamma s) + I_R Z_0 \sinh(\gamma s)$$

$$I = I_R \cosh(\gamma s) + \frac{E_R}{Z_0} \sinh(\gamma s)$$

- And if sending end parameters are known and x is distance measured from the sending end then,

$$E = E_S \cosh(\gamma x) - I_S Z_0 \sinh(\gamma x)$$

$$I = I_S \cosh(\gamma x) - \frac{E_S}{Z_0} \sinh(\gamma x)$$

Physical Significance of General Solution

- From the equations, the sending end current can be obtained by substituting $s = l$ measured from the receiving end.

$$E_S = E_R \cosh(\gamma l) + I_R Z_0 \sinh(\gamma l) \quad \dots (7)$$

and $$I_S = I_R \cosh(\gamma l) + \frac{E_R}{Z_0} \sinh(\gamma l) \quad \dots (8)$$

Now $$Z_R = \frac{E_R}{I_R}$$

$$\therefore \quad I_S = I_R \cosh(\gamma l) + \frac{Z_R}{Z_0} I_R \sinh(\gamma l)$$

$$\therefore \quad I_S = I_R \left[\cosh(\gamma l) + \frac{Z_R}{Z_0} \sinh(\gamma l)\right] \quad \dots (9)$$

- Now if the line is terminated in its characteristic impedance Z_0 then,

$$I_S = I_R [\cosh(\gamma l) + \sinh(\gamma l)] \quad \dots \text{as } Z_R = Z_0$$

$$\therefore \quad \frac{I_S}{I_R} = [\cosh(\gamma l) + \sinh(\gamma l)] = e^{\gamma l} \quad \dots (10)$$

- This is the equation which is already derived for the line terminated in Z_0. Using $E_R = I_R Z_R$ in equation (7),

$$E_S = Z_R I_R \cosh(\gamma l) + I_R Z_0 \sinh(\gamma l)$$

$$\therefore \quad E_S = I_R [Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)] \quad \dots (11)$$

Dividing equation (11) by equation (9),

$$\frac{E_S}{I_S} = \frac{I_R [Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)]}{I_R \left[\cosh(\gamma l) + \frac{Z_R}{Z_0} \sinh(\gamma l)\right]}$$

But $$\frac{E_S}{I_S} = Z_S$$

$$\therefore \quad Z_S = \frac{Z_0 [Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)]}{[Z_0 \cosh(\gamma l) + Z_R \sinh(\gamma l)]} \quad \dots (12)$$

- When the line is terminated in Z_0 then $Z_R = Z_0$. So substituting in equation (12) we get,

$$Z_S = Z_0 \quad \dots (13)$$

- This shows that for a line terminated in its characteristic impedance, its input impedance is also its characteristic impedance.
- Now consider an infinite line with $l \to \infty$. Using this in equation (12) we get,

$$Z_S = \frac{Z_0 [Z_R + Z_0 \tanh(\gamma l)]}{[Z_0 + Z_R \tanh(\gamma l)]} \quad \text{and } \tanh(\gamma l) \to 1 \text{ as } l \to \infty, \quad \dots (14)$$

$$\therefore \quad Z_S = Z_0$$

- This shows that finite line terminated in its characteristic impedance behaves as an infinite line, to the sending end generator.

Q.14 Derive the expression for input impedance of transmission line. Hence state the effect of open circuit and short circuit of line on impedance. **[SPPU : Dec.-15, Marks 8]**

Ans. : Input Impedance of Transmission Line : Refer answer of Q.13.

Open and Short Circuit conditions

The current and voltage at any point, if the distance s is measured from the receiving end then $s = l - x$ and the equations of E and I are,

$$E = E_R \cosh[\gamma(l-x)] + I_R Z_0 \sinh[\gamma(l-x)] \quad \text{... (1)}$$

$$I = I_R \cosh\left[\gamma(l-x) + \frac{E_R}{Z_0}\sinh[\gamma(l-x)]\right] \quad \text{... (2)}$$

1. Finite Line Terminated in Z_0

The case is equally applicable for an infinite line.

At $x = l$, $E = E_R$ and $I = I_R$

$$\therefore \quad E_R = E_S \cosh(\gamma l) - I_S Z_0 \sinh(\gamma l)$$

and $$I_R = I_S \cosh(\gamma l) - \frac{E_S}{Z_0}\sinh(\gamma l)$$

Using $E_R / Z_R = Z_R$ we get,

$$Z_R = \frac{E_R}{I_R} = \frac{Z_0 [E_s \cosh\gamma l - I_S Z_0 \sinh\gamma l]}{[I_S Z_0 \cosh\gamma l - E_S \sinh\gamma l]}$$

But $Z_R = Z_0$

$$Z_0 = \frac{Z_0 [E_S \cosh\gamma l - I_S Z_0 \sinh\gamma l]}{[I_S Z_0 \cosh\gamma l - E_S \sinh l]} \quad \text{... (3)}$$

Solving we get,

$$\frac{E_S}{I_S} = Z_0 = Z_S \quad \text{... (4)}$$

Thus **input impedance for a finite line terminated in Z_0 is also Z_0.**

2. Finite Line Open Circuited at Distant End

The line of length 'l' is open circuited at a distance $x = l$ i.e. $I_R = 0$.

$$\therefore \quad 0 = I_S \cosh(\gamma l) - \frac{E_S}{Z_0}\sinh(\gamma l)$$

$$\therefore \quad \frac{E_S}{I_S} = Z_0 \frac{\cosh(\gamma l)}{\sinh(\gamma l)} = Z_0 \coth(\gamma l)$$

But E_S / I_S is the input impedance so let us call this input impedance of open circuited line as Z_{OC}.

$$\therefore \quad \boxed{Z_{in} = Z_{OC} = Z_0 \coth(\gamma l)} \quad \text{... (5)}$$

If $l \to \infty$, then $\coth(\gamma l) \to 1$ and thus input impedance Z_{OC} approaches to Z_0 for an infinite line, on open circuit.

3. Finite Line Short Circuited at Distant End

The line of length 'l' is short circuited at a distance $x = l$ i.e. $E_R = 0$.

$$0 = E_S \cosh(\gamma l) - I_S Z_0 \sinh(\gamma l)$$

$$\therefore \quad \frac{E_S}{I_S} = Z_0 \tanh(\gamma l)$$

But E_S / I_S is the input impedance, so let us call this input impedance of short circuited line as Z_{SC}.

$$\therefore \quad \boxed{Z_{in} = Z_{SC} = Z_0 \tanh(\gamma l)} \quad \text{... (6)}$$

6.8 : Waveform Distortion

Q.15 What are the various types of distortions in the transmission line ? **[SPPU : Dec.-16, Marks 5]**

Ans. : • When the received signal is not the exact replica of the transmitted signal then the signal is said to be distorted.

• There are three types of distortions present in the transmitted wave along the transmission line.

1. Due to variation of characteristic impedance Z_0 with frequency.
2. Frequency distortion due to the variation of α with frequency.
3. Phase distortion due to the variation of β with frequency.

1. Distortion due to Z_0 Varying with Frequency

• The characteristic impedance Z_0 of the line varies with the frequency while the line is terminated in an impedance which does not vary with frequency in similar fashion as that of Z_0.

• This causes the distortion. The power is absorbed at certain frequencies while it gets reflected for certain frequencies. So there exists the **selective power absorption**, due to this type of distortion.

2. Frequency Distortion

• The attenuation constant α is a function of frequency. Hence the different frequencies transmitted along the line will be attenuated to the different extent.

- For example a voice signal consisting many frequencies will not be attenuated equally along the transmission line.
- Hence received signal will not be exact replica of the input signal at the sending end. Such a distortion is called a **frequency distortion.** Such a distortion is very serious and important for audio signals.

3. Phase Distortion

- The phase constant β also varies with frequency. The velocity of propagation of waves also varies with frequency.
- Hence some waves will reach receiving end very fast while some waves will get delayed than the others.
- Hence all frequencies will not have same transmission time.
- Thus the output wave at the receiving end will not be exact replica of the input wave at the sending end. This type of distortion is called **phase distortion** or **delay distortion.**
- It is not much important for the audio signals due to the characteristics of the human ears. But such a distortion is very serious in case of video and picture transmission.

6.9 : Distortionless Line

Q.16 What are the various types of distortions in the transmission line ? Derive condition for distortionless line.

[SPPU : May-16, Dec.-18, 19, Marks 8]

Ans. : Types of Distortion : Refer Q.15.

- A line in which there is no phase or frequency distortion and also it is correctly terminated, is called a **distortionless line.**

Derivation of the condition for distortionless line

Consider expression for propagation constant given by,

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\therefore \quad \gamma^2 = (R+j\omega L)(G+j\omega C) = \left(RG-\omega^2 LC\right)+j\omega(RC+LG) \quad ...(1)$$

- It is known that for minimum attenuation $L = \frac{RC}{G}$ i.e. $LG = RC$.

- Substituting this condition in equation (1) we get,

$$\gamma^2 = RG-\omega^2 LC+j\,2\,\omega RC$$

But $\quad RC = LG = \sqrt{RCLG}$

$$\therefore \quad \gamma^2 = RG-\omega^2 LC+j2\omega\sqrt{RCLG} = \left(\sqrt{RG}+j\omega\sqrt{LC}\right)^2$$

$$\therefore \quad \gamma = \sqrt{RG}+j\omega\sqrt{LC} \quad ...(2)$$

But $\quad \gamma = \alpha+j\beta$

$$\therefore \quad \boxed{\alpha = \sqrt{RG}} \quad ...(3)$$

$$\text{and} \quad \boxed{\beta = \omega\sqrt{LC}} \quad ...(4)$$

- It can be seen from the equation (3) that α does not vary with frequency which eliminates the frequency distortion.

Now, $\quad \beta = \omega\sqrt{LC} \quad$ ... for the condition LG = CR

Now, $\quad v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}}$ km/sec ... (5)

- Thus for the condition LG = CR, the velocity becomes independent of frequency. This eliminates the phase distortion.
- It is already proved that for RC = LG, the Z_0 becomes resistive and line can be correctly terminated to eliminate distortion due to Z_0 varying with frequency. Thus all the distortions are eliminated for a condition,

$$\boxed{RC = LG \quad \text{i.e.} \quad \frac{R}{G} = \frac{L}{C}} \quad ...(6)$$

This is the required condition for a **distortionless line.**

Q.17 A distortionless line has $Z_0 = 60\ \Omega$, $\alpha = 20$ m Np/m, velocity of propagation = 0.6 c, where c is the speed of light in a vaccum. Find R, L, G, and λ at 100 MHz. **[SPPU : May-19, Marks 10]**

Ans. : $\quad Z_0 = 60\ \Omega, \quad \alpha = 20\times10^{-3}$ Np/m

For distortionless line, $\frac{R}{G} = \frac{L}{C}$ and $Z_0 = \sqrt{\frac{L}{C}}$

$$\frac{L}{C} = Z_0^2 = 60^2 = \frac{R}{G} \text{ and } \alpha = \sqrt{RG}$$

$$\therefore \quad \alpha = \sqrt{R} \times \sqrt{\frac{RC}{L}} = \frac{R}{Z_0} \qquad \ldots G = \frac{RC}{L}$$

$$\therefore \quad 20 \times 10^{-3} = \frac{R}{60} \quad \text{i.e. } R = \mathbf{1.2\ \Omega/m}$$

$$\therefore \quad G = \frac{R}{60^2} = \mathbf{333.333\ \mu mho/m}$$

$$\therefore \quad v = 0.6\ c = 0.6 \times 3 \times 10^8 \text{ m/s}$$

But $$v = \frac{1}{\sqrt{LC}} \quad \text{i.e. } LC = 3.0864 \times 10^{-17}$$

$$\therefore \quad 60^2\ C \times C = 3.0864 \times 10^{-17}$$

$$\therefore \quad C = \mathbf{92.59\ pF/m}$$

$$\therefore \quad L = \mathbf{333.33\ nH/m}$$

$$\lambda = \frac{v}{f} = \frac{0.6 \times 3 \times 10^8}{100 \times 10^6} = \mathbf{1.8\ m}$$

6.10 : Reflection on a Line Not Teminated in Z_0

Q.18 Explain the phenomenon of reflection on the transmission line and hence define reflection coefficient.

[SPPU : May-15, 16, 17, 19, Dec.-14, 15, Marks 8]

OR What is reflection on transmission line ? What are disadvantages of same ? Explain the terms reflection coefficient.

[SPPU : Dec.-09, 06, May-09, 08, Marks 8]

Ans. : • If a line is not terminated in Z_0 or it is joined to some impedance having value other than Z_0 then part of the wave is reflected back from the distant end or from the point of discontinuity.

• Thus **reflection phenomenon** exists for a line which is not terminated in Z_0.

• Such a reflection is maximum when the line is on open circuit i.e. $Z_R = \infty$ or short circuit i.e. $Z_R = 0$.

• The reflection is zero when $Z_R = Z_0$.

• From the general solution of a line we can write,

$$E = \frac{E_R(Z_R + Z_0)}{2 Z_R}\left[e^{\sqrt{ZY}\, s} + \frac{Z_R - Z_0}{Z_R + Z_0} e^{-\sqrt{ZY}\, s}\right] \qquad \ldots (1)$$

and $$I = \frac{I_R(Z_R + Z_0)}{2 Z_0}\left[e^{\sqrt{ZY}\, s} - \frac{Z_R - Z_0}{Z_R + Z_0} e^{-\sqrt{ZY}\, s}\right] \qquad \ldots (2)$$

where the value of Z_R is not equal to Z_0 of the line.

• When Z_R is not equal to Z_0, each E and I consists of 2 parts,

1. One part varying exponentially with positive s
2. One part varying exponentially with negative s

$$E = \frac{E_R(Z_R + Z_0)}{2 Z_R} e^{\gamma s} + \frac{E_R(Z_R - Z_0)}{2 Z_R} e^{-\gamma s} \qquad \ldots (3)$$

and $$I = \frac{I_R(Z_R + Z_0)}{2 Z_0} e^{\gamma s} - \frac{I_R(Z_R - Z_0)}{2 Z_0} e^{-\gamma s} \qquad \ldots (4)$$

• The first component of E or I which varies exponentially with + s is called **incident wave** which flows **from the sending end to the receiving end.**

$$E_1 = \frac{E_R(Z_R + Z_0)}{2 Z_R} e^{\gamma s} = \text{Incident voltage wave}$$

and $$I_1 = +\frac{I_R(Z_R + Z_0)}{2 Z_0} e^{\gamma s} = \text{Incident current wave}$$

• The second component of E or I varies exponentially with – s. It flows from the receiving end towards the sending end. When s is minimum (s = 0) at the receiving end, its amplitude is maximum while due to negative index, when s is maximum $(s = l)$ at the sending end, its amplitude is minimum. Thus such a wave which flows from the receiving end towards the sending end, with decreasing amplitude is called **reflected wave.**

$$\therefore \quad E_2 = \frac{E_R (Z_R - Z_0)}{2 Z_R} e^{-\gamma s} = \text{Reflected voltage wave}$$

and
$$I_2 = -\frac{I_R (Z_R - Z_0)}{2 Z_0} e^{-\gamma s} = \text{Reflected current wave}$$

- Thus the total instantaneous voltage or current at any point on line is the phasor sum of voltage or current of incident and reflected waves.
- The term $\frac{Z_R - Z_0}{Z_R + Z_0}$ decides the relative phase angles between incident and reflected waves. Thus magnitudes and phase angles of Z_R and Z_0 are important to determine the phase angle between the two waves.

Reflection Coefficient

- The ratio of the amplitudes of the reflected and incident voltage waves at the receiving end of the line is called the **reflection coefficient**. It is denoted by K

$$K = \frac{\text{Reflected voltage at load}}{\text{Incident voltage at load}} = \frac{Z_R - Z_0}{Z_R + Z_0}$$

Disadvantages of Reflection

1. If the attenuation is not large then the reflected wave appears as echo at the sending end.
2. There is reduction in efficiency.
3. The part of the received energy is rejected by the load hence output reduces.
4. If the generator impedance at the sending end is not Z_0 then reflected wave is reflected again from the sending end and becomes a new incident wave. The energy is transmitted back and forth till all the energy get dissipated in the line losses.

Q.19 A transmission line has characteristic impedance of 50 ohm. Find the reflection coefficient if line is terminated with 1) 50 Ohm 2) 0 Ohm 3) 75 + j75 Ohm 4) 75 + j40 Ohm.

☞[SPPU : Dec.-14, Marks 8]

Ans. : The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} \quad \text{where } Z_0 = 50\ \Omega.$$

1) $Z_R = 50\ \Omega$: $K = \frac{50-50}{50+50} = 0$ Perfect impedance matching hence no reflection.

2) $Z_R = 0\ \Omega$: $K = \frac{0-50}{50-50} = -1 = 1\angle 180°$

3) $Z_R = 75 + j\,75\ \Omega$:

$$K = \frac{(75+j75)-50}{(75+j75)+50} = \frac{25+j75}{125+j75} = \frac{79.056\angle 71.56°}{145.7738\angle 30.96°} = 0.5423\angle 40.6°$$

4) $Z_R = 75 + j\,40\ \Omega$:

$$K = \frac{(75+j40)-50}{(75+j40)+50} = \frac{25+j40}{125+j40} = \frac{47.1699\angle 57.99°}{131.244\angle 17.74°} = 0.3594\angle 40.25°$$

6.11 : Reflection Loss, Reflection Factor and Return Loss

Q.20 Define reflection factor, reflections loss and return loss.

Ans. : • The ratio which indicates the change in current in the load due to reflection at the mismatched junction is called **reflection factor**. It is denoted by K and defined by,

$$K = \text{Reflection factor} = \left|\frac{2\sqrt{Z_R Z_0}}{Z_R + Z_0}\right|$$

- The **reflection loss** is defined as the number of nepers or decibels by which the current in the load under image matched conditions would exceed the current actually flowing in the load. It is inversely proportional to the reflection factor.

$$\text{Reflection loss} = 20 \log \frac{|Z_R + Z_0|}{|2\sqrt{Z_R Z_0}|} \text{ dB} = 20 \log \frac{1}{|K|}$$

- **Return loss** is defined as,

$$\text{Return loss} = 10 \log \frac{P_1}{P_3} \text{ dB} = 20 \log \left|\frac{Z_R + Z_0}{Z_R - Z_0}\right| \text{ dB}$$

- The return loss is also called 'Singing point.

6.12 : Line Constants for Dissipationless Line

Important Points To Remember

For the radio frequency line, the standard assumptions made for the analysis of the performance of the line are as follows.

1) At very high frequency, the **skin effect** is considerable. Hence it is assumed that the currents may flow on the surface of conductor. Then the internal inductance becomes zero.
2) It is observed that due to the skin effect, resistance R increases with $\sqrt{f}$. But the line reactance ωL increases directly with frequency f. Hence the second assumption is $\omega L >> R$.
3) The third assumption is that the line at radio frequency is constructed such that the leakage conductance G may be considered zero. Hence the third assumption is $\omega C >> G$.

Q.21 What do you mean by dissipationless line ? Derive the expressions for characteristics impedance and propagation constant for dissipationless line.

[SPPU : Dec.-14, 16, 17, May-15, 16, 17, Marks 8]

Ans.: • There are two considerations for the analysis of the line performances. I) R is slightly small with respect to ωL and II) R is completely negligible as compared with ωL.

- If R is neglected completely, then such a line is termed as **zero dissipation line or dissipationless line..** This concept is useful when the line is used for transmission of power at a high frequency and the losses are neglected completely.
- While if R is small, then such a line is termed as **small dissipation line.**

Characteristics Impedance and Propagation Constant for Dissipationless Line :

- In general, the **characteristic impedance** (Z_0) and **propagation constant** (γ) of a line are given by,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}} \quad \text{... (1)}$$

and

$$\gamma = \sqrt{ZY} = \sqrt{(R+j\omega L)(G+j\omega C)} \quad \text{... (2)}$$

- According to the standard assumptions for line at a high frequency, $j\omega L >> R$ and $j\omega C >> G$. Hence characteristics impedance is given by,

$$\therefore \quad Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} \quad \text{... (3)}$$

- As the value of characteristic impedance is real and resistive, it is represented by symbol R_0,

$$\therefore \quad \boxed{Z_0 = R_0 = \sqrt{\frac{L}{C}}} \quad \text{... (4)}$$

- Similarly the propagation constant γ is given by,

$$\gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC} = 0 + j\,\omega\sqrt{LC} \quad \text{... (5)}$$

But $\quad \gamma = \alpha + j\beta$

Hence α = **Attenuation constant** = 0

β = **Phase constant** = $\omega\sqrt{LC}$ radian/m

The velocity of propagation is,

$$\therefore \quad \boxed{v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \text{ m/s}}$$

6.13 : Expressions for Voltages and Currents on Dissipationless Line

Q.22 Derive the expressions for voltage and current at any point on the radio frequency line terminated in Z_R.

Ans.: • Consider a transmission line of length l and terminated in Z_R as shown in the Fig. Q.22.1.

- In general for any line the voltage E and current I at a distance x from sending end are expressed in terms of receiving end voltage E_R and receiving end current I_R.

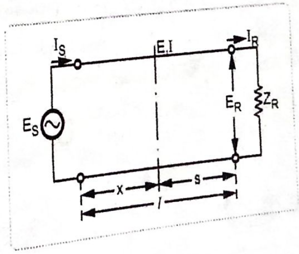

Fig. Q.22.1

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and

$$\gamma = \sqrt{ZY} = \sqrt{(R+j\omega L)(G+j\omega C)} \quad \text{... (2)}$$

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$$\therefore \quad Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} \quad \text{... (3)}$$

- As the value of characteristic impedance is real and resistive, it is represented by symbol R_0,

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- Similarly the propagation constant γ is given by,

$$\gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC} = 0 + j\,\omega\sqrt{LC} \quad \text{... (5)}$$

But $\quad \gamma = \alpha + j\beta$

Hence α = **Attenuation constant** = 0

β = **Phase constant** = $\omega\sqrt{LC}$ radian/m

The velocity of propagation is,

$$\therefore \quad \boxed{v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \text{ m/s}}$$

6.13 : Expressions for Voltages and Currents on Dissipationless Line

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Ans.: • Consider a transmission line of length l and terminated in Z_R as shown in the Fig. Q.22.1.

- In general for any line the voltage E and current I at a distance x from sending end are expressed in terms of receiving end voltage E_R and receiving end current I_R.

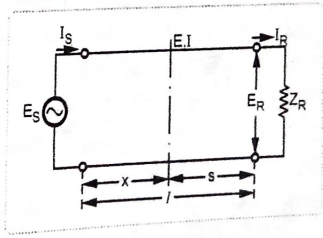


Fig. Q.22.1

- The voltage E at a distance x from the sending end is given by,

$$E = E_R \cdot \cosh \gamma(l-x) + I_R \cdot Z_0 \sinh \gamma(l-x) \quad \text{... (1)}$$

Putting $(l-x) = s$, equation (1) reduces to,

$$E = E_R \cosh \gamma s + I_R \cdot Z_0 \sinh \gamma s \quad \text{... (2)}$$

But at very high frequencies,

$$Z_0 = R_0 \quad \text{and} \quad \gamma = j\beta$$

Hence equation (2) can be rewritten as,

$$E = E_R \cosh(j\beta s) + I_R \; R_0 \sinh(j\beta s)$$

$$\therefore \quad E = E_R \left[\frac{e^{j\beta s} + e^{-j\beta s}}{2}\right] + j \, I_R \; R_0 \left[\frac{e^{j\beta s} - e^{-j\beta s}}{2}\right]$$

$$\therefore \quad E = E_R \cos(\beta s) + j \, I_R \; R_0 \sin(\beta s) \quad \text{... (3)}$$

- Above equation represents a voltage in terms of receiving end voltage and current, at a point distance 's' away from receiving end.
- Similarly for current at a point distance 's' away from receiving end is given by,

$$I = I_R \cos(\beta s) + j \frac{E_R}{R_0} \sin(\beta s) \quad \text{... (4)}$$

But $\beta = \frac{2\pi}{\lambda}$

Hence equations (3) and (4) reduce to,

$$\boxed{E = E_R \cos \frac{2\pi s}{\lambda} + j \; I_R \cdot R_0 \sin \frac{2\pi s}{\lambda}} \quad \text{... (5)}$$

and

$$\boxed{I = I_R \cos \frac{2\pi s}{\lambda} + j \frac{E_R}{R_0} \sin \frac{2\pi s}{\lambda}} \quad \text{... (6)}$$

6.14 : Standing Waves

Q.23 What are the standing waves ? Define node and antinode.

Ans. : • When a line is not matched with its load impedance i.e. load impedance is not equal to the characteristics impedance, then the energy delivered to the load is reflected back to the source in the form of reflected wave.

- The combination of incident and reflected waves gives rise to standing waves as shown in the Fig. Q.23.1.

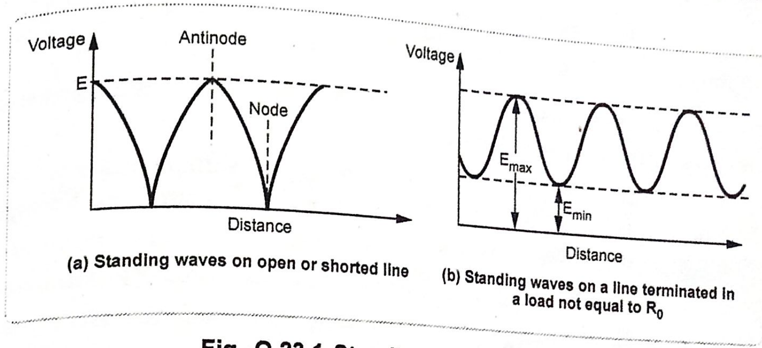


Fig. Q.23.1 Standing waves on a line

- If a line is terminated in a load other than R_0, the distribution of voltage at a point along the length of the line consists maximum and minimum values of voltage as shown in the Fig. Q.23.1 (b).
- The variations of voltage and current are except for a $\lambda/4$ shift in a position of maxima and minima.
- The standing waves with the nodes and antinodes for a line either open or short circuited at load side are as shown in the Fig. Q.23.1 (a).
- The points along the line where magnitude of voltage or current is zero are called **Nodes** while the points along the lines where magnitude of voltage or current is maximum are called **Antinodes** or **Loops.**
- When a line is terminated in R_0, the standing waves are absent, such a line is called **smooth line.**

6.15 : Standing Wave Ratio

Q.24 Explain standing waves and why they generate? Derive the relation between the SWR and magnitude of reflection coefficient.

☞[SPPU : Dec.-15, May-15, Marks 9]

Ans. : Standing Waves : Refer answer of Q.23.

- The ratio of the maximum to minimum magnitudes of voltages or currents on a line having standing waves is called **standing wave ratio** and it is denoted by S.

- The standing wave ratio (S) is given by,

$$S = \frac{|E_{max}|}{|E_{min}|} = \frac{|I_{max}|}{|I_{min}|}$$

- When line is not terminated properly, standing waves are produced because total power is not absorbed.
- The standing wave ratio S is measured using **RF voltmeter across** the line at a point. Then the ratio of E_{max} to E_{min} is referred as **Voltage Standing Wave Ratio (VSWR).**
- Similarly the ratio of I_{max} to I_{min} can be measured using **RF Ammeter in series** with the line at a point. Then such ratio is referred as **current standing wave ratio (ISWR).**
- But in practice, ISWR calculation is very impractical because for this one has to cut the line, insert RF ammeter and then rejoin the line. Hence practically only VSWR measurement is done.
- Theoretically the value of S lies between 1 and ∞. The relation between standing wave ratio and reflection coefficient is given by,

$$S = \frac{1+|K|}{1-|K|} \quad \text{or} \quad |K| = \frac{S-1}{S+1} = \frac{|E_{max}|-|E_{min}|}{|E_{max}|+|E_{min}|}$$

Relation between the SWR and magnitude of reflection coefficient

- The voltage at a point distance s away from the receiving end is given by,

$$E = E_R \cdot \cosh(j\beta s) + I_R \cdot Z_0 \sinh(j\beta s) \quad \text{... (1)}$$

$$\therefore \quad E = E_R \frac{e^{j\beta s}+e^{-j\beta s}}{2} + I_R \cdot Z_0 \frac{e^{j\beta s}-e^{-j\beta s}}{2}$$

$$\therefore \quad E = \frac{e^{j\beta s}}{2}[E_R + I_R \cdot Z_0] + \frac{e^{-j\beta s}}{2}[E_R - I_R \cdot Z_0]$$

$$\therefore \quad E = \frac{e^{j\beta s}}{2}[I_R \cdot Z_R + I_R \cdot Z_0] + \frac{e^{-j\beta s}}{2}[I_R \cdot Z_R - I_R \cdot Z_0]$$

$$\therefore \quad E = I_R \cdot \frac{e^{j\beta s}}{2}\left[(Z_R+Z_0)+e^{-j2\beta s}(Z_R-Z_0)\right]$$

$$\therefore \quad E = I_R \frac{(Z_R+Z_0)}{2} e^{j\beta s}\left[1+\frac{Z_R-Z_0}{Z_R+Z_0} e^{-j2\beta s}\right] \quad \text{... (2)}$$

But $\quad K = \frac{Z_R - Z_0}{Z_R + Z_0} = |K| \angle \phi$ say as k is complex,

$$\therefore \quad E = I_R \frac{Z_R+Z_0}{2} e^{j\beta s}\left[1+|K| e^{j\phi} \cdot e^{-j2\beta s}\right]$$

$$\therefore \quad E = I_R \frac{Z_R+Z_0}{2} e^{j\beta s}\left[1+|K| e^{j(\phi-2\beta s)}\right]$$

$$\therefore \quad E = I_R \cdot \frac{Z_R+Z_0}{2} e^{j\beta s}\left[1\angle 0+|K| \angle \phi-2\beta s\right] \quad \text{... (3)}$$

- In above equation, the first term represents voltage in the incident wave while the second term represents voltage in the reflected wave.
- Thus the voltage E at any point is the vector sum of voltages in incident and reflected wave.
- This voltage will be maximum when both, the incident and reflected waves, are in phase. When both waves are in phase, their phase angles will be same. Thus for E_{max},

$$0 = \phi - 2\beta s \quad \text{... (4)}$$

- Then equation (3) is modified as,

$$E_{max} = I_R \frac{Z_R+Z_0}{2} e^{j\beta s}\left[1\angle 0+|K|\angle 0\right]$$

$$\therefore \quad E_{max} = I_R \frac{Z_R+Z_0}{2} e^{j\beta s}\left[1+|K|\right] \quad \text{... (5)}$$

- When the incident wave and reflected wave are out of phase, we get minimum voltage. Then the difference of angles of the two waves is π. Thus for E_{min},

$$0+\pi = \phi - 2\beta s \quad \text{... (6)}$$

- Then equation (3) is modified as,

$$E_{min} = I_R \frac{Z_R+Z_0}{2} e^{j\beta s}\left[1\angle 0+|K|\angle \pi\right]$$

$$\therefore \quad E_{min} = I_R \frac{Z_R+Z_0}{2} e^{j\beta s}\left[1\angle 0-|K|\right]$$

$$\therefore \quad E_{min} = I_R \frac{Z_R+Z_0}{2} e^{j\beta s}\left[1-|K|\right] \quad \text{... (7)}$$

- Hence from equations (4) and (7), the standing wave ratio S can be determined as,

$$\boxed{S = \frac{E_{max}}{E_{min}} = \frac{1+|K|}{1-|K|}}$$

Q.25 A line with characteristic impedance of 692 ∠−12° is terminated in 200 Ω resistor. Determine K and S.

[SPPU : May-02, Dec.-99, 16, Marks 4]

Ans. : The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{200-(678.878-j\,143.87)}{200+(678.878-j\,143.87)} = \frac{-467.878+j\,143.87}{878.878-j\,143.87}$$

$$\therefore \quad = \frac{498.1\angle 163.21}{890.57\angle -9.29} = 0.559\angle 172.5^\circ$$

Then the standing wave ratio is given by,

$$S = \frac{1+|K|}{1-|K|} = \frac{1+0.559}{1-0.559} = 3.535$$

6.16 : Expression for Input Impedance of the Dissipationless Line Interms of Reflection Coefficient

Q.26 Explain what do you understand by standing waves and standing wave voltage ratio and hence derive the expression for input impedance of line in terms of characteristics impedance and propagation constant. [SPPU : Dec.-14, May-15, May-16, Marks 10]

Ans. : Standing waves : Refer answer of Q.23.

Standing wave ratio : Refer answer of Q.24.

Input impedance of line in terms of characteristics impedance and propagation constant :

- The expressions for sending end voltage and current at a distance s from receiving end for a line of length s are given by,

$$E_S = E_R \cos\beta s + j\, I_R\, R_0 \sin\beta s$$

and

$$I_S = I_R \cos\beta s + j\frac{E_R}{R_0}\sin\beta s$$

- Consider a line of length s and terminated in Z_R as shown in the Fig. Q.26.1.
- The input impedance of such a line is given by,

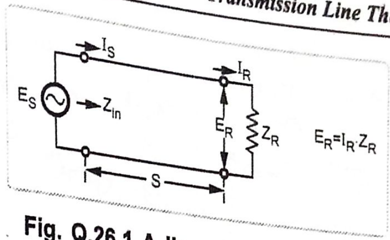


Fig. Q.26.1 A line of length s and terminated in Z_R

$$Z_{in} = \frac{E_S}{I_S} = \frac{E_R\cos\beta s + j\,I_R\cdot R_0\sin\beta s}{I_R\cos\beta s + j\frac{E_R}{R_0}\sin\beta s} = R_0\left[\frac{E_R\cos\beta s + j\,I_R\cdot R_0\sin\beta s}{I_R\,R_0\cos\beta s + j\,E_R\sin\beta s}\right]$$

$$= R_0\left[\frac{E_R + j\,I_R\cdot R_0\tan\beta s}{I_R\,R_0 + j\,E_R\tan\beta s}\right] = R_0\left[\frac{\frac{E_R}{I_R} + j\,R_0\tan\beta s}{R_0 + j\frac{E_R}{I_R}\tan\beta s}\right]$$

$$\therefore \quad Z_{in} = R_0\left[\frac{Z_R + j\,R_0\tan\beta s}{R_0 + j\,Z_R\tan\beta s}\right] \quad \ldots (1)$$

- From equation (1), it is clear that the input impedance Z_{in} is complex.
- **Another convenient form of the input impedance** is obtained as follows.

$$Z_{in} = R_0\left[\frac{Z_R + j\,R_0\tan\beta s}{R_0 + j\,Z_R\tan\beta s}\right] = R_0\left[\frac{Z_R\cos\beta s + j\,R_0\sin\beta s}{R_0\cos\beta s + j\,Z_R\sin\beta s}\right]$$

Writing numerator and denominator in exponential forms and rearranging,

$$Z_{in} = R_0\left[\frac{1+\frac{Z_R - Z_0}{Z_R + Z_0}\,e^{-j2\beta s}}{1-\frac{Z_R - Z_0}{Z_R + Z_0}\,e^{-j2\beta s}}\right] \quad \ldots (2)$$

But as we know,

$K = \dfrac{Z_R - Z_0}{Z_R + Z_0}$ where K is reflection coefficient and writing in modulus-angle form,

$$\boxed{Z_{in} = R_0\left[\frac{1+|K|\ \phi\cdot e^{-2\beta s}}{1-|K|\ \phi\cdot e^{-j2\beta s}}\right] = R_0\left[\frac{1+|K|\ \angle\phi-2\beta s}{1-|K|\ \angle\phi-2\beta s}\right]}$$

6.17 : Input Impedance of Disspationless Lines with Open and Short Circuit Conditions

Important Points to Remember

- The input impedance of a line either open circuited or short circuited is pure reactive in nature and its value is repeated after every $s = \frac{\lambda}{2}$ period. For first quarter wavelength, short circuited line acts as an inductance while the open circuited line acts as a capacitance. After each quarter wavelength, the nature of reactances reverses.
- When transmission line is open circuited at its end, the current is always zero and voltage is maximum at the open end which repeates after every $(\lambda/2)$ distance. At a voltage maxima, there is current minima and at voltage minima we get current maxima.
- When a line is shorted at terminating end, the voltage is zero and current is maximum at that end which repeat themselves at half wavelength intervals back from the short circuit.

6.18 : Power and Impedance Measurement on Dissipationless Line

Important Points to Remember

- This impedance in voltage loop represented as R_{max} and that in current loop represented by R_{min} respectively given by ,

$$\frac{E_{max}}{I_{min}} = S\ R_{max} = R_0 \quad \text{and} \quad \frac{E_{min}}{I_{max}} = R_{min} = \frac{R_0}{S}$$

- The effective power flowing into a resistance R_{max} is the power passing voltage loop at voltage E_{max} and is given by

$$P = \frac{E^2_{max}}{R_{max}} \qquad \text{... (1)}$$

- Since there is dissipation in the line same power must flow into the resistance R_{min} in the current loop at voltage E_{min} given by,

$$P = \frac{E^2_{min}}{R_{min}} \qquad \text{... (2)}$$

Multiplying above expressions $P^2 = \frac{E^2_{max} \cdot E^2_{min}}{R_{max} \cdot R_{min}}$

- $P = \frac{(|E_{max}| \cdot |E_{min}|)}{R_0} = (|I_{max}| \cdot |I_{min}|) \cdot R_0$

- The **impedance is minimum at a point where voltage is minimum** and the **impedance is maximum at a point where voltage is maximum.**

6.19 : Impedance Matching and Impedance Transformation

Q.27 What is impedance matching ? Explain necessity of it. Name the techniques used. **[SPPU : May-17, Marks 3]**

Ans. : • **Impedance matching** is a technique in which two resistive loads or a resistive load and a transimission line or two transmission lines with different characteristic impedance are mathed perfectly.

Necessity of Impedance Matching

- When a line is terminated into its characteristic impedance, then at any point along the line the impedance is same and equal to characteristic impedance of line. Also in such case there are no reflections along line. The maximum power is transfered to load.
- But if line is not matched with its characteristic impedance, then maximum power transfer to load is not possible. Also there are reflected waves alog line causing standing waves.
- Hence to avoid reflections, impedance matching is necessary.
- The transmission line with different wavelengths can be used for impedance matching and impedance transformation. The most commonly used i) Quarter wave transformer, ii) Single stub matching technique, iii) Double stub matching technique

6.20 : Impedance Matching Using Single Stub Technique

Q.28 What is impedance matching ? Explain necessity of it. What is stub matching ? Explain single stub matching with its merits and demerits. **[SPPU : Dec.-15, 16, May-15, 17, Marks 8]**

Ans.: Impedance Matching and its necessity : Refer Answer of Q.27.

Single Stub Matching

- A stub is a section of transmission line connected in shunt with the main transmission line.
- A stub with suitable length is placed at a certain distance from load so that the impedance seen beyond the stub connected is nothing but the characteristic impedance of the main line. Use of stub is as shown in the Fig. Q.28.1.

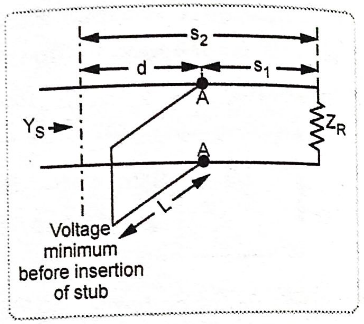


Fig. Q.28.1 Use of single stub for impedance matching

- By using such stub, antiresonance is achieved providing impedance at resonance equal to R_0.
- Because of paralleling of the element, it is convenient to work with admittances. Then the input admittance Y_S, looking towards the load from any point on line is given by ... (1)

$$Y_S = G_0 \pm jB$$

- This may be the admittance at point A before stub was connected. The point A is located such that at the point A, $G_0 = \frac{1}{R_0}$.
- Then at the point A, a short stub line is connected. This line is selected such that its input succeptance is $\mp B$. This stub is connected across the transmission line.
- Then the total admittance at input is given by,

$$Y_S = (G_0 \pm jB) + (\mp jB) = G_0 = \frac{1}{R_0} \quad ... (2)$$

$$\therefore \quad Z_S = R_0 \quad ... (3)$$

- Thus, the line from the source to the point A is then terminated in R_0. It acts as a smooth line.
- For the impedance matching using single stub it is very much essential to know the exact point at which the stub is to be connected to the line and also the length of the stub.

Merits

1. It is better technique than quarter wave transformer.
2. It is capable of matching any complex load impedance to characteristic impedance.
3. It is more suitable of matching fixed impedance at fixed frequency at high frequency.
4. The reflection losses are cosiderably reduced.

Demerits

1. It is suitable for fixed frequency only. For different frequencies location of stub must be changed everytime.
2. It is not suitable for variable load impedances.
3. Repositioning of stub at a final position is difficult in case of transmission line of coaxial type.

6.21 : The Smith Chart

Important Points to Remember

- **P.H. Smith of Bell Laboratories,** developed a new chart, similar to the impedance of circle diagram chart, in which the drawbacks of the circle diagram were removed.
- The modified chart is named as the **Smith chart** after P.H. Smith and it is extensively used for the analysis of the transmission line problems.
- The **Smith Chart** is a valuable graphical tool for solving radio frequency transmission line problems.
- In almost all the transmission line problems, the main objective is to match the impedances of line to that with load.

Properties of the Smith Chart

1. The Smith chart may be used for impedances as well as for admittances.
2. The Smith chart consists constant r_i - circles and constant x_i - circles superpositioned on one chart. The constant r_i - circles have their centres on the horizontal axis i.e. u-axis and constant x_i - circles have their centers on the vertical axis i.e v-axis.
3. The Smith chart is based on the assumption that,

$$|K| \angle \phi - 2\beta s = u + jv$$

The maximum magnitude of u + jv is the maximum value of $|K|$ i.e. unity. Thus, in the chart, it is possible to locate all possible values of impedances inside the outer circle of unit radius.

4. The impedance of a line of any length can be read at any point on the S-circle. For properly terminated line , the impedance is represented by the point (1, 0) which acts as the centre of the Smith chart.
5. The horizontal line passing through the centre of the smith chart for represents real axis or r_i-axis for impedance plot or g_i-axis for admittance plot.

To the extreme left of the chart, there is **zero impedances at load point** or the **short circuit condition**. Similarly, to the extreme right of the chart, there is **infinite impedance at load point** or the **open circuit condition.**

6. The outer rim of the chart is scaled into either degrees or wavelengths with an arrow. This arrow indicates the direction of travel along the line. The outer circle is called **β s scale** of the chart which indicates the electrical length of the line.
7. A complete revolution of 360° around the center of the chart represents a distance of $\frac{\lambda}{2}$ on the line. The **clockwise movement** along the outer rim indicates the **travel towards the generator from load** along the line. Vice a versa, **anticlockwise movements** along outer rim indicates the **travel towards the load from the generator**. Both these directions are indicated by the arrows G and L.

The distance $\frac{\lambda}{2}$ on the line corresponds to a movement of 360° on the chart. Thus the distance λ corresponds to a 720° movement on the chart.

8. On the periphery of the Smith chart, three scales are provided, which are useful in determining the distance from the load or generator in degrees or wavelengths.
9. If the Smith chart is used for impedances, the inductive reactance are above r_i-axis or u-axis while the capacitive reactance are below u-axis. Similarly if the Smith chart is used for admittances, the r_i axis becomes g_i axis while x_i axis becomes b_i axis. Then the extreme left of g_i axis represents **zero conductances** or **open circuit,** while the extreme right of g_i - axis represents **infinite conductance** or **short circuit.**

Q.29 A loss less transmission line with characteristic impedance 50 ohm is 30 m long and operates at 2 MHz. The line is terminated with a load of (60 + j40). If phase velocity is 0.6 C. where c is speed of light then find using Smith chart.
1) Reflection coefficient 2) The standing wave ratio
3) The input impedance.

[SPPU : May-04, 09, 16, Dec.-02, 09, 14, 17, Marks 8]

Ans. : The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{Z_R - R_0}{Z_R + R_0} = \frac{(60 + j\,40) - 50}{60 + j\,40 + 50} = \frac{10 + j40}{110 + j40}$$

$$= \frac{41.231 \angle 75.96^\circ}{117.0469 \angle 9.98^\circ}$$

$$\therefore \quad K = 0.3523 \angle 56^\circ$$

The standing wave ratio is given by,

$$S = \frac{1 + |K|}{1 - |K|} = \frac{1 + 0.3523}{1 - 0.3523} = \frac{1.3523}{0.6477} = 2.088$$

The velocity on the line given by

$$v = \frac{\omega}{\beta} \quad \text{i.e.} \quad \beta = \frac{\omega}{v}$$

But the electrical length of the line is βs where s = 30 m.

$$\therefore\ \beta s = \frac{\omega}{v}\,s = \frac{2\times\pi\times f}{v}.s = \frac{2\times\pi\times 2\times 10^6\times 30}{0.6\times 3\times 10^8} = (0.6666)\pi = \mathbf{120^\circ}$$

Then the input impedance is given by,

$$Z_{in} = R_0\left[\frac{Z_R + jR_0\tan\beta s}{R_0 - jZ_R\tan\beta s}\right]$$

$$\therefore\quad Z_{in} = 50\left[\frac{(60+j40)+j\,50\tan 120^\circ}{50-j(60+j\,240)\cdot\tan 120^\circ}\right]$$

$$\therefore\quad Z_{in} = \mathbf{24.01\angle 3.22^\circ\ \Omega = 23.97 + j\ 1.35\ \Omega}$$

Applications

1. Plotting an Impedance

Any complex impedance can be represented by a single point on the Smith chart. This point is nothing but the intersection of constant r_i circle i.e. $r_i = \frac{R_R}{R_0}$ circle and x_i circle i.e. $x_i = j\frac{X_R}{R_0}$ circle.

Consider above example

$$Z_R = R_R + j\,X_R = (60+j40)\ \Omega\ ,\ R_0 = 50\ \Omega$$

$$\therefore \text{ The normalized impedance } Z_R = \frac{Z_R}{R_0} = \frac{60+j40}{50} = 1.2 + j\ 0.8$$

Locate a point P on the the Smith chart as shown in the Fig. Q.11.1, where r_i = 1.2 circle and x_i = 0.8 circle meet together. The intersection of the two circles is represented by the dotted lines and the point P indicates the normalized impedance on the chart.

2. Measurement of VSWR

After plotting the normalized impedance, we can determine the value of VSWR by drawing constant S circle with center of the chart [i.e. point (1,0) on the u-axis] and radius equal to distance between centre O and point indicating the normalized impedance. Then the point of intersection of S-circle with the real axis at the right side of the centre indicates a VSWR for given line.

Consider above example. Select a centre of the circle as point O(1,0). Take a distance from O to the point P indicating normalized impedance and then draw a circle. The circle cuts the horizontal real axis at

point Q(2.1, 0). This indicates value of the VSWR for the line considered as shown in the Fig. Q.29.1.

From the Fig. Q.29.1 the value of VSWR is 2.1. approximately.

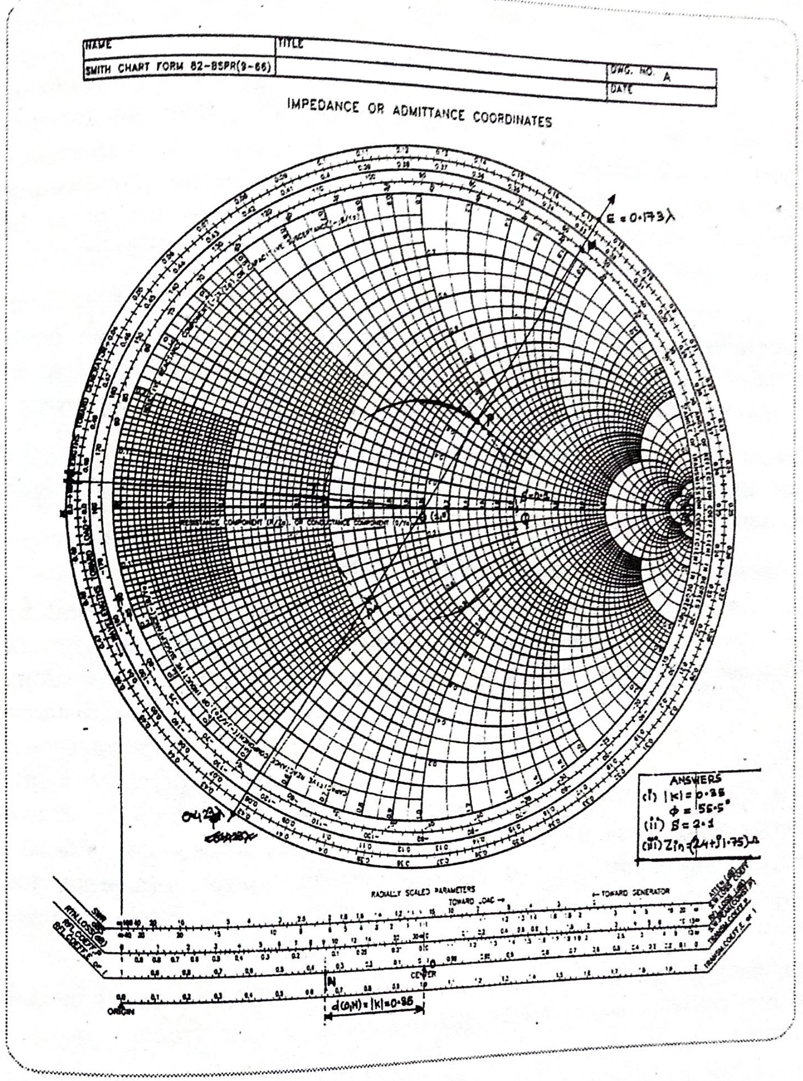


Fig. Q.29.1

3. Measurement of reflection coefficient K [magnitude and phase]

The angle of the reflection coefficient K can be obtained by extending a line from center to the outer rim of the chart through the point which indicates the normalized impedance. The point at which the extended line cuts the outer rim gives directly the value of angle of the reflection coefficient K.

In the commercial Smith chart, the K - scale is provided at the bottom of the chart. Then by selecting point center on this Scale draw an arc just intersecting the straight line of voltage reflection coefficient, with radius equal to distance between the centre of the chart and a point indicating normalized impedance. Then the distance from center to the point of intersection on the horizontal K - scale gives directly the magnitude of K.

For the transmission line considered in above example, draw a straight line starting form center and extending upto the outer rim through point P. This line intersects outer rim at point M, which indicates angle of k, as shown in the Fig. Q.29.1. At point M, angle of $k = \phi = 55.5$ (approx.)

Similar on the K-scale, at the bottom of the chart as shown in the Fig. Q.26.1, the arc draw with radius equal to OP, intersects with horizontal line at N. At N, $|K| = 0.35$.

4. Measurement of Input Impedance of the Line

To find the input impedance of the line, first locate the load point by plotting normalized load impedance and extending upto the outer rim of the chart. As we have to find the impedance at the generator, move along the outer rim in clockwise direction, towards generator, with a distance equal to the length of the transmission line. This is the point which indicates the generator side. Mark the point on outer rim and draw a line from point O to the generator point. This line cuts the circle drawn corresponding to the SWR of the line. This point of intersection indicates the generator point. This point gives the normalized input impedance and the actual input impedance can be obtained by multiplying this normalized impedance by R_0.

For the transmission line in above example, the total length is 30 m. Let us first calculate length, interms of λ or degrees.

$$\lambda = \frac{v}{f} = \frac{0.6c}{f} = \frac{0.6 \times 3 \times 10^8}{2 \times 10^6} = 90 \text{ m}$$

$$\text{But } S = 30 \text{ m} = \frac{30}{90}\lambda = \frac{\lambda}{3} \text{ m or } \frac{720}{3} = 240° \quad \dots (\lambda = 720°)$$

Now in the the chart, the line OP extended upto outermost scale cuts at point E say. The distance corresponding to point E is 0.173λ. Then move from point E, a distance equal to 0.3333 λ in clockwise direction to reach generator point F as shown in the Fig. Q.26.1. Now the total distance to be travelled is 0.3333λ. From point E to the extreme right point corresponding to 0.25 λ is given by 0.25 λ – 0.173 λ = 0.077 λ. Then from extreme right point on the chart to the extreme left point the distance of travel equals 0.25 λ. Then to reach point F from extreme left point corresponding to 0 or 0.3 λ is given by 0.333 λ – (0.077 + 0.25 λ) = 0.0063 λ. The point of intersection of line 0T and constant S circle is represented by point T, which is the intersection of $r_i = 0.48$ circle and $x_i = 0.035$ circle. As point T is above horizontal axis, the reactance is positive. Hence the normalized input impedance represented by point T is given by

$$z_{in} = 0.48 + j\,0.035$$

Hence the actual value of the input impedance is given by,

$$Z_{in} = R_0\,(z_{in}) = 50\,(0.48 + j\,0.035) = 24 + j\,1.75\ \Omega$$

Q.30 A transmission line has a characteristic impedance of 300 Ω and terminated in a load $Z_L = 150 + j\,150\,\Omega$. Find the following using Smith chart.

i) VSWR ii) Reflection coefficient iii) Input impedance at distance 0.1 λ from the load iv) Input admittance from 0.1 λ from load v) position of first voltage minimum and maximum from the load.

☞[SPPU : May-16, 17, 18, Marks 8]

Ans. : Given $Z_0 = R_0 = 300\ \Omega$

$$Z_L = 150 + j\,150\ \Omega$$

1. The normalized load impedance is given by,

$$z_L = \frac{Z_L}{Z_0} = \frac{Z_L}{R_0} = \frac{150 + j\,150}{300} = 0.5 + j\,0.5$$

Hence locate point A at the intersection of r = 0.5 circle and x = + 0.5 circle as shown in the Fig. Q.30.1. Point A represents the load point.

2. Draw a circle with point O(0,1) as center and radius equal to distance OA. This is constant S - circle. This circle cuts the real axis at 2.6. Hence the value of VSWR is given by ,

$$S = 2.6$$

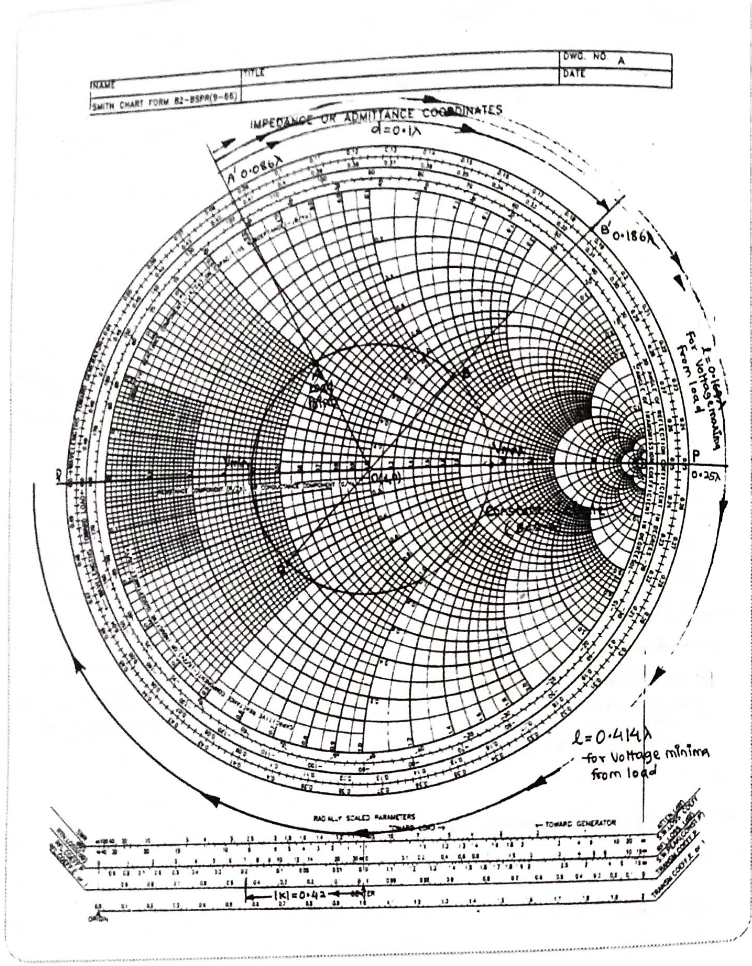


Fig. Q.30.1

3. Draw a line OA and extend it upto point A' on the outer rim. This line intersects the scale for the angle of reflection coefficient at 118°. This is the angle of reflection coefficient.

To get magnitude of $\bar{K}$, select the linear scale shown at the bottom of the chart - E. Marks a point from centre at a distance equal to distance OA. This gives value of $|K|$ as 0.42. Hence the reflection coefficient is given by,

$$\mathbf{K = 0.42 \angle 118^\circ}$$

4. To find input impedance at 0.1 λ from load, move in clockwise direction from point A' a distance equal to reach point B'. A' is at 0.086 λ. Hence B' will be located at (0.086 + 0.1) λ = 0.186 λ as shown in the chart E.

5. Draw a line OB' which cuts S - circle (S = 2.6) at point B. The point B is intersection of r = 1.38 circle and x = 1.12 circle. Hence input impedance 0.1 λ away from the load is given by,

$$Z_d = \frac{Z_d}{R_0} = 1.38 + j\,1.12$$

Hence actual impedance at 0.1 λ away from load is given by,

$$Z_d = R_0\,(1.38+j\,1.12) = 300\,(1.38+j\,1.12) = (414+j\,336)\ \Omega$$

6. To obtain input admittance 0.1 λ away from load, draw a diameter through O and B. This will cut constant - S circle at point C. As this point is the intersection of g = 0.44 circle and b = – 0.38 circle. Hence input admittance is

$$y_d = 0.44-j\,0.38 = \frac{Y_d}{G_0}$$

Hence actual admittance is given by

$$\therefore\ Y_d = G_0(0.44-j\,0.38) = \frac{1}{R_0}(0.44-j\,0.38) = \frac{0.44-j\,0.38}{300}$$

$$\therefore\qquad Y_d = \mathbf{(1.4667-j\,1.2667)\times 10^{-3}\ \mho}$$

7. The voltage minima occurs to the left of centre on real axis at 0.39 while the voltage maxima occurs to the right of centre on the real axis at S = 2.6.

The find location of the votlage maxima from load, move in clockwise direction from point A' (load point) to point P (voltage maxima point). Total travel is (0.25 – 0.086) λ = 0.164 λ. Hence **the first voltage maxima is located at 0.164 λ from the load.** Now to find location of voltage minima from load move in clockwise to find location of voltage minima from load move in clockwise direction from point A' to Q (voltage minima point). The total travel is 0.25 λ+0.164 λ=0.0414 λ. Thus **the first voltage minima exists at 0.414 λ from the load.**

Q.31 Consider a 30 m long lossless transmission line with the characteristic impedance of 50 Ω operating at 2 MHz. If the line is terminated in impedance $(60+j40)\ \Omega$ calculate the reflection coefficient, the standing wave ratio, the input impedance, if the velocity on the line is $v = 0.6\,c\ \left(c = 3\times10^8 m/s\right)$.

☞[SPPU : Dec.-99, 02, 04, 14, 17, 19, May-06, 09, Marks 10]

Ans. : The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{Z_R - R_0}{Z_R + R_0} = \frac{(60+j\,40)-50}{60+j\,40+50}$$

$$= \frac{10+j40}{110+j40} = \frac{41.231\angle 75.96^\circ}{117.0469\angle 9.98^\circ}$$

$$\therefore \quad K = \mathbf{0.3523\angle 56^\circ}$$

The standing wave ratio is given by,

$$S = \frac{1+|K|}{1-|K|} = \frac{1+0.3523}{1-0.3523} = \frac{1.3523}{0.6477} = \mathbf{2.088}$$

The velocity on the line given by

$$v = \frac{\omega}{\beta} \qquad \text{i.e.} \qquad \beta = \frac{\omega}{v}$$

But the electrical length of the line is βs where s = 30 m.

$$\therefore \quad \beta s = \frac{\omega}{v} s = \frac{2\times\pi\times f}{v}.s = \frac{2\times\pi\times 2\times10^6\times30}{0.6\times3\times10^8} = (0.6666)\pi = \mathbf{120^\circ}$$

Then the input impedance is given by,

$$Z_{in} = R_0\left[\frac{Z_R + jR_0\tan\beta s}{R_0 - jZ_R\tan\beta s}\right]$$

$$\therefore \quad Z_{in} = 50\left[\frac{(60+j40)+j\,50\tan120^\circ}{50-j(60+j\,240)\cdot\tan120^\circ}\right]$$

$$\therefore \quad Z_{in} = \mathbf{24.01\angle 3.22^\circ\ \Omega = 23.97 + j\ 1.35\ \Omega}$$

Q.32 The lossless 100 Ω transmission line is terminated in an impedance $50+j60\ \Omega$. Calculate VSWR, reflection coefficient, impedance of 0.35 λ from the load using Smith chart.

☞[SPPU : Dec.-16, 18, Marks 10]

Ans. :

$$Z_R = 50+j60\ \Omega$$

$$Z_0 = R_0 = 100\ \Omega$$

1) The normalized impedance is given by,

$$Z_R = \frac{50+j60}{100} = 0.5+j0.6$$

Plot this impedance on figure as point A which is intersection of r = 0.5 circle and x = j0.6 arc. This is load point.

2) With 'O' as centre and radius equal to OA draw a circle. Extend line OA to point A'.

3) The circle drawn in step 2 is constant S circle which cuts horizontal axis at point P on the right hand side. Through point P, the circle r = S passes. Hence

$$S = \text{VSWR} = 2.85$$

4) With radius OA draw an arc DE on the reflection coefficient line locked at lower side of the chart. This arc cuts line at F.

At point F, $|K| = 0.48$

Now angle corresponding to A′ is 108°

$$\therefore \quad \bar{K} = |K|\angle\phi = 0.48\angle 108^\circ$$

5) Point A′ is locked at 0.4 λ. To find impedance at 0.35 λ from load, move in clockwise direction thro 0.35 λ or 252° and locate point B′.

6) Join 'O' and B′. This line OB′ cuts constant S circle at point 'B'. Point B is intersection of r = 0.38 circle and x = − j0.28 arc

$$\therefore \quad Z = 0.38 - j0.28 = \frac{Z}{R_0}$$

$$\therefore \quad Z_{\text{at } 0.35\,\lambda \text{ from load}} = R_0\ (0.38 - j0.28) = \mathbf{38 - j28\ \ W}$$

Refer Fig. Q.32.1 on next page.

Q.33 A 50 Ω line is terminated by a load impedance of (75 − j 69) Ω. The line is 3.5 meter long and is excited by 50 MHz source. Propagation velocity is 3×10^8 m/sec. Find the input impedance, reflection coefficient, VSWR, position of minimum voltage.

☞ [SPPU : May-17, Marks 8]

Ans. : $Z_0 = 50\,\Omega \quad l = 3.5\text{ m} \quad v = 3\times10^8$ m/sec.

$$Z_L = 75 - j\,69\,\Omega \quad f = 50\text{ MHz}$$

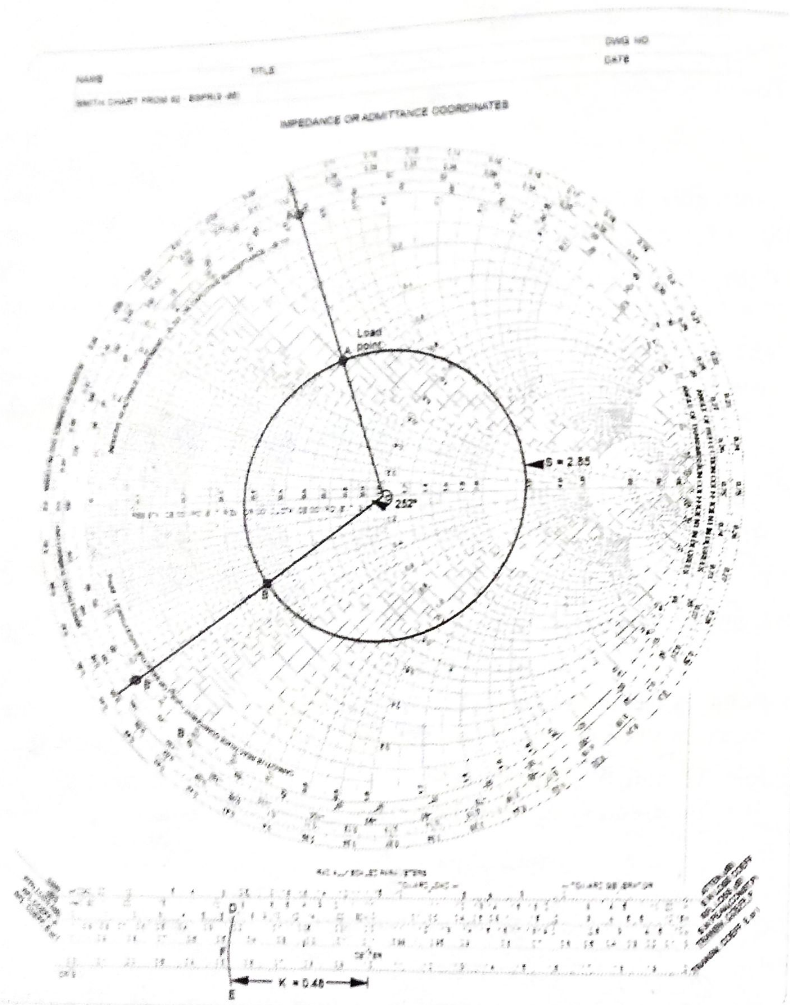


Fig. Q.32.1

a) The normalized load impedance Z_L is given by,

$$Z_{L_n} = \frac{Z_L}{Z_0} = \frac{75 - j\,69}{50} = 1.5 - j\,1.38$$

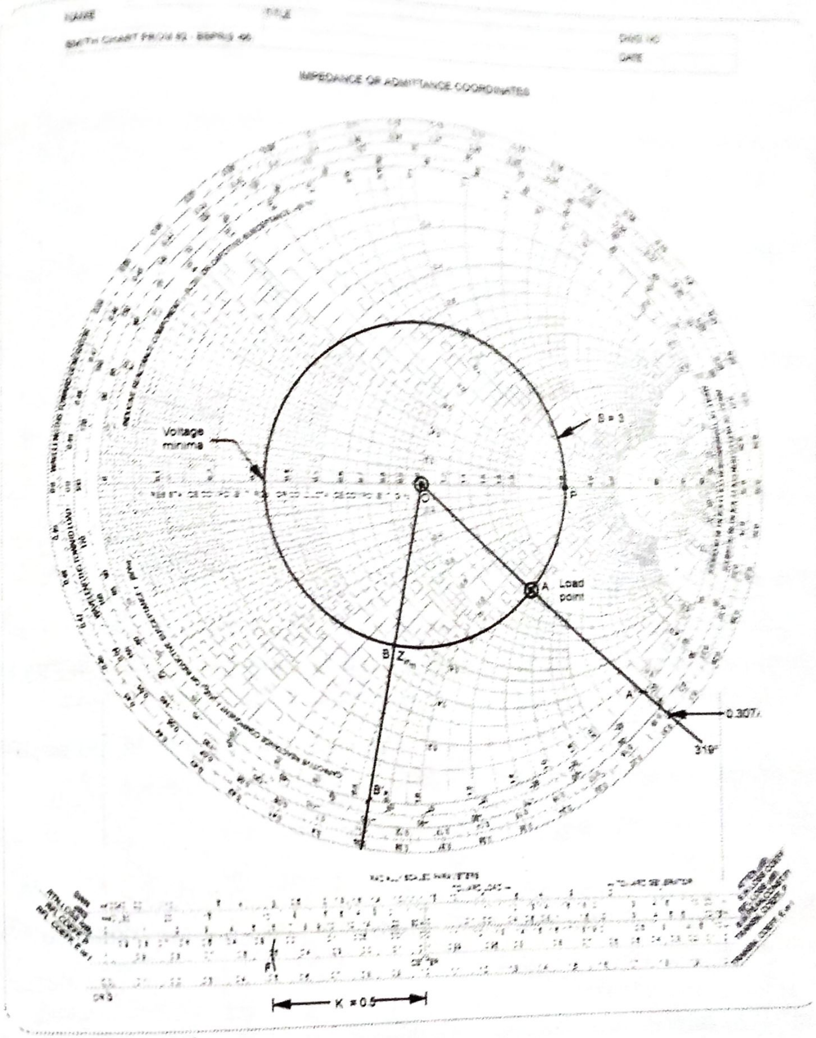


Fig. Q.33.1

Plot this impedance on Smith chart as point 'A' which is the intersection of r = 1.5 circle and x = j 1.38 arc. This is load point.

b) With 'O' as centre and radius equal to OA draw a circle. Extend line OA to point A'.

c) The circle drawn in step (b) is constant S circle which cuts horizontal axis at point P on the right hand side. Through point P, the circle r = S passes.

Hence $S = VSWR = 3$

d) With radius OA draw an arc DE on the reflection coefficient line located at lower side of the chart. The arc cuts line at F.

At point F, $|K| = 0.5$

Angle corresponding to A′ is

$$\therefore \quad \bar{K} = |K|\angle\phi = 0.5\angle 319^\circ$$

e) $$\text{Wavelength } \lambda = \frac{300}{f_{MHz}} = \frac{300}{50} = 6 \text{ m}$$

$$\text{Since } l = 3.5 \text{ m}$$

$$l = \left(\frac{3.5}{6}\right)\lambda = 0.5833\,\lambda$$

Since $\frac{\lambda}{2}$ corresponds to 360°, $l = 0.5833 \times 2 \times \frac{\lambda}{2} = 0.5833 \times 720$

$$= 419.98 \simeq 420^\circ$$

f) To find input impedance more in clockwise direction through $0.5833\,\lambda$ or 420° and locate point B′

g) Join 'O' and 'B′'. This line OB′ cuts constant S circle at point 'B'. Point B is intersection of r = 0.52 circle and x = – 0.7 arc

$$\therefore \quad Z_{in_n} = 0.52 - j\,0.7 = \frac{Z_{in}}{Z_0}$$

$$\therefore \quad \text{Input impedance} = Z_0(0.52 - j\,0.7) = 26 - j\,0.35\,\Omega$$

h) Position of minima from load will be at a distance of $\frac{\lambda}{2} - 0.307\lambda = 0.193\,\lambda$ from load.

Q.34 A lossless transmission line with Z_0 = 75 Ω is 30 m long and operates at 2 MHz. The line terminated with a load Z_L = 90 + j 60 Ω . If u = 0.6c on the line, using Smith chart find :

i) Reflection coefficient ii) Standing wave ratio

iii) Input impedance iv) Load admittance

[SPPU : May-19, Marks 10]

Ans. : Given data :

$Z_0 = 75\ \Omega \quad Z_L = 90 + j60\ \Omega \quad f = 2\text{ MHz}$

$l = 30$ m $\quad u = 0.6$ c

$Z_{L_n} = 1.2 + j\,0.8$

Reflection coefficient $K = 0.36\angle 56^\circ$

$$S = 2.1$$

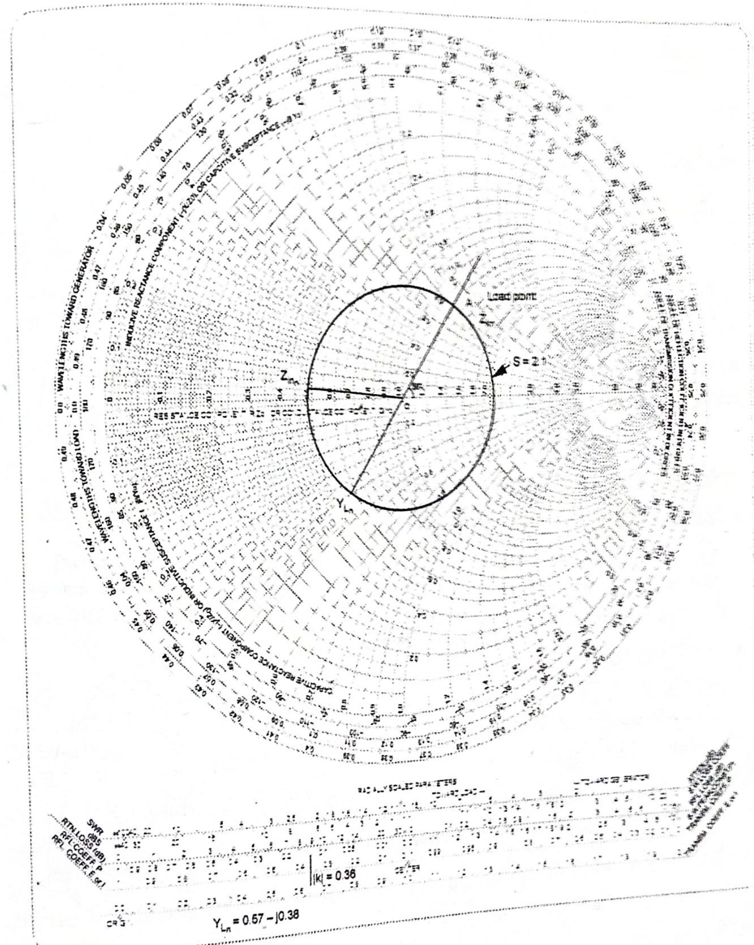


Fig. Q.34.1 Complete Smith chart

$$\lambda = \frac{u}{f} = \frac{0.6\times 3\times 10^{8}}{2\times 10^{6}} = 90 \text{ m}$$

$$l = 30 \text{ m} = \frac{\lambda}{3} \rightarrow 240^{\circ}$$

$$Z_{in_n} = 0.48 + j\,0.03$$

$$Z_{in} = 75(0.48 + j\,0.03) = 36 + j\,2.25\ \Omega$$

$$Y_{L_n} = 0.57 - j0.38$$

$$\therefore \quad Y_L = \frac{1}{75}(0.57 - j0.38) = 7.6 - j\,5.06 \text{ mS}$$

Check :

$$Y_L = \frac{1}{Z_L} = \frac{1}{90 + j60} = \frac{1}{108.1665\angle 33.69^{\circ}}$$

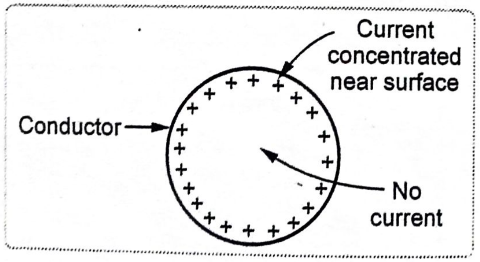
$$Y_L = 9.245\times 10^{-3}\angle -33.69^{\circ} = 7.69\times 10^{-3} - j5.12\times 10^{-3}$$

6.22 : Skin Effect

Q.35 Write a note on skin effect.

Ans. : Skin effect : When a conductor carries a d.c. current, it is uniformly distributed over the whole cross-section of the conductor.

- But for a.c. current, this distribution is nonuniform.
- For a.c., the current density is higher at the surface than at its centre.
- The current is concentrated near the surface of the conductor as shown in the Fig. Q.35.1.
- This behaviour of alternating current to concentrate near the surface of the conductor is called **skin effect.**



Q.35.1 Skin effect

- The effective resistance of the conductor is more for a.c. than d.c. which causes large power loss.
- The skin effect is significant for large and solid conductors even at 50 Hz.
- Use of stranded conductor reduces the skin effect.

- The factors affecting the skin effect are,

i. Nature of material ii. Diameter of wire

iii. Frequency of supply iv. Shape of wire

6.23 : Application Case Study

Q.36 What is coaxial cable and twisted pair cable ? State the types of twisted pair cables.

Ans. : Co-axial Cable

Co-axial cables are used to transmit large frequency signals. If consists of two co-axial cables conductors. The central conductor is solid formed by copper clad steel which carries main signal. The second conductor is in the form of mesh of copper forming shield which protects cable from electromagnetic interference. The two conductor are separated by polythelene insulator. The outermost layer is plastic jacket which provides mechanical protection.

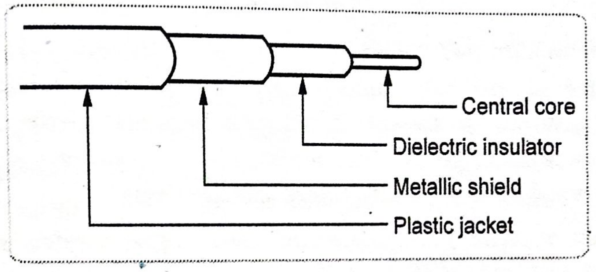


Fig. Q.36.1

The advantages to co-axial cable are easy to install, good resistance to EMI and transmission speed high as 10 MBPS. The electromagnetic field carrying the signal exists only in the space between the inner and outer conductors which gives it added advantage of being installed next to metal objects without power loss.

The main disadvantage of co-axial cable is that single cable failure can affect entire network.

Applications : Cable with 75 Ω impedance are used for video signals while 50 Ω cables are suitable data and wireless communication.

Twisted Pair Cable

Twisted pair cable is a type of cabling used for telephone communications and ethernet networks. A pair is wires forms a circuit that can transmit data. The two insulated copper wires are twisted in one pair to reduce crosstalk or electromagnetic induction between adjacent pairs of wires. When current is possed through a wire, it creates a small circular magnetic field around it. Because of twisting the opposite magnetic field of two wires cancel each other.

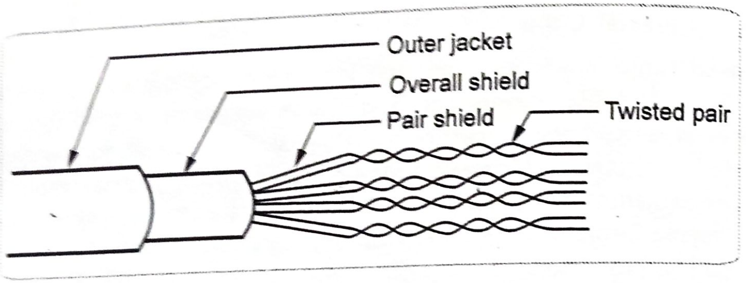


Fig. Q.36.2

Types of twisted pair cables

Unshielded twisted pair cables : Rely on cancellation effect due to twisting to reduce electromagnetic and radio frequency interferences. The number of twists in pair of wire reduces crosstalk. Available with 100 Ω impedance and 10 to 1000 Mbps speed and throughput.

Advantage : More prone to electrical noise. Signal boosters are closer than in case of co-axial cable.

Application : Ethernets to current computers onto a local area network LAN.

- **Shielded twisted pair cable :** Each pair of wire is wrapped in metallic foil and all pairs of wires are shielded combinedly. This reduces electrical noise both between pairs as well as reduce interferences. It is available with 150 Ω impedance. Speed and throughout are 10 to 100 mpbs.

Disadvantage : Metallic shielding must be properly grounded otherwise it act as anknna to pick up unwanted signal.

Application : In ethernet network installations

Q.37 Write a note on waveguide.

Ans. : A hollow conducting tube which guides electromagnetic waves is considered as waveguide. They are used to transmit energy at microwave frequencies due to low losses. The electric and magnetic field existing within waveguide is the solution of Maxwell's field equations. The tangential component of electric field is zero at the walls of the wave guide if the walls are perfect conductor.

Waveguides operate in following modes.

Transverse electric or TE mode or H mode : Electric field is everywhere transverse to the axis of guide. No component exists in the direction of axis for electric field but magnetic field component in the direction of axis is present.

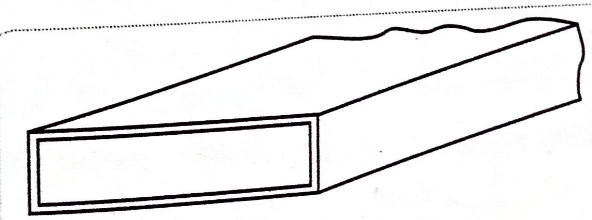
In other words TE mode $\bar{E}$ field is transverse to the direction of wave propagation.

Transverse magnetic or TM moodes or E modes :

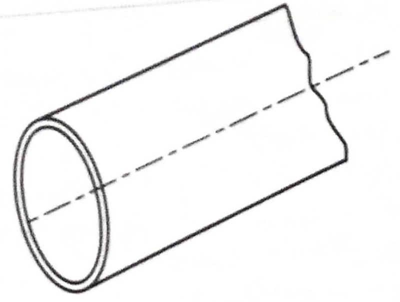
The magnetic field has its components transverse (normal) to the direction of wave propagation.

Transverse electromagnetic mode or TEM mode :

Both electric and magnetic fields have components transvarse to the direction of wave propagation.



Rectangular waveguide



Cylindrical waveguide

Fig. Q.37.1

Formulae at a Glance

- **Characteristic Impedance and Propagation Constant of a line**

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

- **Wavelength and Velocity of a Line**

Wavelength $\lambda = \frac{2\pi}{\beta}$

Phase velocity $v = \frac{2\pi f}{\beta} = \frac{\omega}{\beta}$

Group velocity $v_g = \frac{d\omega}{d\beta}$

- **Input and Transfer Impedance**

1. Input impedance

$$Z_{in} = Z_S = \frac{Z_0\left[Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)\right]}{\left[Z_0 \cosh(\gamma l) + Z_R \sinh(\gamma l)\right]}$$

2. Transfer impedance

$$Z_T = Z_R\left[\frac{e^{\gamma l}+e^{-\gamma l}}{2}\right] + Z_0\left[\frac{e^{\gamma l}-e^{-\gamma l}}{2}\right]$$

- **Reflection Coefficient**

$$K = \frac{\text{Reflected voltage at load}}{\text{Incident voltage at load}} = \frac{Z_R - Z_0}{Z_R + Z_0}$$

- **Reflection Loss, Reflection Factor and Return Loss**

1. Reflection Factor

$$K = \left|\frac{2\sqrt{Z_R Z_0}}{Z_R + Z_0}\right|$$

2. Reflection Loss

$$\text{Reflection loss} = 20 \log \frac{|Z_R + Z_0|}{|2\sqrt{Z_R Z_0}|} \text{ dB} = 20 \log \frac{1}{|K|}$$

3. Return Loss

$$\text{Return loss} = 10 \log \frac{P_1}{P_3} \text{ dB} = 20 \log \left|\frac{Z_R + Z_0}{Z_R - Z_0}\right| \text{ dB}$$

- **Line Constants for Dissipationless line**

$$Z_0 = R_0 = \sqrt{\frac{L}{C}}$$

$$\gamma = \alpha + j\beta = 0 + j\,\omega\sqrt{LC}$$

$$\alpha = 0 \text{ and } \beta = \omega\sqrt{LC} \text{ radian/m}$$

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \text{ m/sec}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} \text{ m}$$

- **Expressions for Voltages and Currents on Dissipationless Line**

$$E = E_R \cos\frac{2\pi s}{\lambda} + j\, I_R \cdot R_0 \sin\frac{2\pi s}{\lambda} \text{ and}$$

$$I = I_R \cos\frac{2\pi s}{\lambda} + j\frac{E_R}{R_0}\sin\frac{2\pi s}{\lambda}$$

1. When line is open circuited at the receiving end, $I_R = 0$

$$E_{OC} = E_R \cos\frac{2\pi s}{\lambda} \quad \text{And} \quad I_{OC} = j\frac{E_R}{R_0}\sin\frac{2\pi s}{\lambda}$$

2. When line is short circuited at the receiving end, $E_R = 0$

$$E_{SC} = j\, I_R \cdot R_0 \cdot \sin \frac{2\pi s}{\lambda} \quad \text{And} \quad I_{SC} = I_R \cos \frac{2\pi s}{\lambda}$$

3. When a line is terminated in an impedance $Z_R = R_0$

$$E = E_R \cdot e^{j\beta s} \quad \text{and} \quad I = I_R \cdot e^{j\beta s}$$

- **SWR in terms of Reflection Coefficient**

$$S = \frac{E_{max}}{E_{min}} = \frac{1+|K|}{1-|K|}$$

- **Input Impedance of the Dissipationless Line**

$$Z_S = Z_{in} = R_0 \left[\frac{Z_R + j\, R_0 \tan \beta s}{R_0 + j\, Z_R \tan \beta s} \right] \quad \text{..... General Expression}$$

1. Input Impedance of Short Circuited Line

$$Z_{SC} = j\, X_S = j\, R_0 \tan \left(\frac{2\pi s}{\lambda} \right)$$

2. Input Impedance of Open Circuited Line

$$Z_{OC} = Z_S = -j\, R_0 \cot \left(\frac{2\pi s}{\lambda} \right)$$

END...

b) *State and explain Biot-Savart law with mathematical expression.* **(Refer Q.3 of Chapter - 2)** **[7]**

OR

Q.4 a) *Define magnetic flux density and state its unit. State the relation between B and H.* **(Refer Q.2 of Chapter - 2)** **[3]**

b) *Obtain* $\overline{\mathbf{H}}$ *due to infinitely long straight conductor carrying I, using Ampere's circuital law.* **(Refer Q.12 of Chapter - 2)** **[6]**

c) *In cylindrical co-ordinates magnetic field is given by*

$$\overline{H} = (2\rho - \rho^2)\, \overline{a}_\phi \ A/m \ \text{for}\ 0 \le \rho \le 1\ m$$

i) Determine the current density as a function of ρ within the cylinder.

ii) Determine total current passing through surface z = 0, $z \le \rho \le 1$ *in* $\overline{a}_z$ *direction.* **(Refer Q.17 of Chapter - 2)** **[6]**

SOLVED MODEL QUESTION PAPER (End Sem)
Electromagnetic Field Theory
T.E. (E&Tc) Semester - V [As Per 2019 Pattern]

Time : $2\frac{1}{2}$ Hours] **[Maximum Marks : 70**

N.B. : ***i) Attempt Q.1 or Q.2, Q.3 or Q.4, Q.5 or Q.6, Q.7 or Q.8.***

ii) Neat diagrams must be drawn wherever necessary.

iii) Figures to the right side indicate full marks.

iv) Assume suitable data, if necessary.

Q.1 a) *Derive the boundary conditions between two perfect dielectrics.* **(Refer Q.6 of Chapter - 3)** **[8]**

b) *The region x < 0 is medium 1 with* $\mu_{r1} = 4.5$ *and* $\overline{H}_1 = 4\hat{a}_x + 3\hat{a}_y - 6\hat{a}_z$ *A/m. The region x > 0 is medium 2 with* $\mu_{r2} = 6$. *Find* $\overline{\mathbf{H}}_2$ *in medium 2 and angle made by* $\overline{\mathbf{H}}_2$ *with normal to interface.* **(Refer Q.21 of Chapter - 3)** **[10]**

OR

Q.2 a) *Derive the expression for capacitance of a parallel plate capacitor with single dielectric.* **(Refer Q.9 of Chapter - 3)** [6]

b) *Obtain an expression for an energy stored and energy density in a parallel plate capacitor.* **(Refer Q.15 of Chapter - 3)** [6]

c) *Derive the expression for a force between two current carrying wires assuming the currents are in the same direction.*
(Refer Q.24 of Chapter - 3) [6]

Q.3 a) *State and explain in brief, scalar and vector magnetic potentials.* **(Refer Q.1 of Chapter - 4)** [6]

b) *A circular loop lies in* $z = 0$ *plane has radius of 0.2 m and resistance of 10 ohm. Find the current flowing through the conductor due to field* $\overline{B} = 0.2 \sin 10^3 t \, \hat{a}_z$. **(Refer Q.6 of Chapter - 4)** [6]

c) *Derive the expression for average power density.*
(Refer Q.23 of Chapter - 4) [5]

OR

Q.4 a) *Derive an expression for continuity equation for time varying fields.* **(Refer Q.15 of Chapter - 4)** [8]

b) *Explain Maxwell's equations for time varying electromagnetic fields.* **(Refer Q.16 of Chapter - 4)** [9]

Q.5 a) *Explain the term uniform plane wave and its transverse nature.*
(Refer Q.2 of Chapter - 5) [4]

b) *A 10 GHz plane wave travelling in free space as an amplitude* $E_x = 10$ *V/m. Find v,* λ, β η *and amplitude and direction of* $\overline{H}$.
(Refer Q.8 of Chapter - 5) [6]

c) *Find the depth of penetration for Cu at 1 MHz,*
$\sigma = 6 \times 10^6 \; S/m \, \mu_r = 1$. **(Refer Q.17 of Chapter - 5)** [8]

OR

Q.6 a) *Derive expression of electromagnetic wave equation in phasor form. Also derive expression of* α *and* β *from it.*
(Refer Q.11 of Chapter - 5) [9]

b) *Briefly explain about the wave incident normally on a perfect conductor.* **(Refer Q.20 of Chapter - 5)** [9]

Q.7 a) *A cable has an attenuation of 3.5 dB/km and a phase constant of 0.28 rad/km. If 3 V is applied to the sending end then what will be the voltage at point 10 km down the line when line is terminated with Z_0.* **(Refer Q.5 of Chapter - 6)** [8]

b) *Write the equations for voltage and current at any point along the length of transmission line and hence explain physical significance of general solution of transmission line.* **(Refer Q.13 of Chapter - 6)** [9]

OR

Q.8 a) *Derive the expression for characteristics impedance (Z_0) and propagation constant (γ) in terms of primary constants of transmission line.* **(Refer Q.7 of Chapter - 6)** [8]

b) *A loss less transmission line with characteristic impedance 50 ohm is 30 m long and operates at 2 MHz. The line is terminated with a load of (60 + j40). If phase velocity is 0.6 C.*
where c is speed of light then find using Smith chart.
1) Reflection coefficient
2) The standing wave ratio
3) The input impedance. **(Refer Q.29 of Chapter - 6)** [9]

END...