ECE 603
Probability and Random Processes

Lesson 4
Chapter 2
Counting Methods

Objectives

- Examine counting as a result of the multiplication principle.
- Apply the basic introductions of the material to probability.

Rationale

- Counting is necessary for solving some probability problems. This lesson will focus on methods for counting elements in an efficient manner.
- Almost everything you need to know about counting comes from the multiplication principle.
- This lesson will take what you previously reviewed about the Cartesian viewpoint and explore a different perspective.

Prior Learning

- Basic Concepts
- Access to the online textbook: https://www.probabilitycourse.com/

Counting Methods

For a finite sample space S with equally likely outcomes, the probability of an event A is given by

$$P(A) = \frac{|A|}{|S|} = \frac{M}{N}$$

Counting Methods

Multiplication Principle:

If we are to perform r experiments in order such that there are n_1 possible outcomes of the first experiment, n_2 possible outcomes of the second experiment, … , n_r possible outcomes of the r^{th} experiment, then there is a total of $n_1 \times n_2 \times n_3 \times \cdots \times n_r$ outcomes of the sequence of the r experiments.

Counting Methods

Example:

A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 Seniors. A subcommittee of one person from each class is to be chosen. How many different subcommittees are possible?

Counting Methods

- Drawing (choosing) objects from a set $A = \{a_1, a_2, \cdots, a_n\}$ is referred to as sampling.
- We will often draw multiple samples from a set. If we put the object back after each draw, this is called sampling with replacement; if not it is called sampling without replacement.
- The result of drawing multiple samples can be ordered (order of draws matters; $1, 2, 3 \neq 2, 3, 1$) or unordered ($1, 2, 3 = 2, 3, 1$).

Counting Methods

General scenario:

We have a set of n elements, e.g. , $A = \{1, 2, \cdots, n\}$ and we draw k samples from the set:

- Sampling
  - Ordered
    - With replacement
    - Without replacement
  - Unordered
    - Without replacement
    - With replacement

Counting Methods

Remember: $n! = n \times (n-1) \times \cdots \times 1$

$$\textbf{e.g., } \mathbf{3! = 3 \times 2 \times 1 = 6}$$

Counting Methods

For $A = \{1, 2, 3\},\ k = 2$

1) Ordered Sampling with Replacement (repetition allowed)

$$\begin{matrix} (1,1) & (2,1) & (3,1) \\ (1,2) & (2,2) & (3,2) \\ (1,3) & (2,3) & (3,3) \end{matrix} \longrightarrow 9 \text{ Possibilities}$$

In general: $A = \{1, 2, \cdots, n\}$

$$\begin{matrix} \square & \square & \square & \cdots & \square \\ \uparrow & \uparrow & \uparrow & & \uparrow \\ n \times & n \times & n \times & \cdots & \times\, n \end{matrix} \Rightarrow n^k$$

Orchestrated Conversation: Counting Methods

Example:

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Counting Methods

2) Ordered Sampling without Replacement (repetition not allowed)

$$A = \{1, 2, 3\},\ k = 2$$

$$\begin{matrix} (1,2) & (2,1) & (3,1) \\ (1,3) & (2,3) & (3,2) \end{matrix} \longrightarrow 6 \quad \textbf{Possibilities}$$

In general: $A = \{1, 2, \cdots, n\}$

$$\begin{matrix} \square & \square & \square & \cdots & \square \\ \uparrow & \uparrow & \uparrow & & \uparrow \\ n\ \times & (n-1)\ \times & (n-2)\ \times & \cdots & \times\ (n-k+1) \end{matrix}$$

Counting Methods

Number of k-permutations of n-objects:

$$P_k^n = n \times (n-1) \times \ldots \times (n-k+1) = \frac{n!}{(n-k)!}.$$

The number of k-permutations of n distinguishable objects is given by

$$P_k^n = \frac{n!}{(n-k)!}, \text{ for } 0 \leq k \leq n.$$

Orchestrated Conversation: Counting Methods

Example:

(Birthday Paradox) In a group of k people, what is the probability that at least two have the same birthday?

Counting Methods

Sample of size k from $A = \{1, 2, \cdots, n\}$

Sampling → Ordered → With replacement: n^k

Sampling → Ordered → Without replacement: $P_k^n = n \times (n-1) \times \ldots \times (n-k+1) = \frac{n!}{(n-k)!}.$

Sampling → Unordered → Without replacement

Sampling → Unordered → With replacement

Counting Methods

Unordered Sampling without Replacement (Combinations):

There are n distinguishable objects; we want to choose k objects, but ordering does not matter: $1, 2, 3 = 2, 3, 1 = 3, 2, 1.$

Let $A = \{1, 2, 3\}$ and $k = 2$, then

$\{1, 2\}$ $\{1, 3\}$ $\{2, 3\}$ $\longrightarrow 3$ possibilities

Counting Methods

In general:

$\binom{n}{k}$: # of ways to choose k elements from n elements (Unordered): k-Combinations

If ordered: $P_k^n = \dfrac{n!}{(n-k)!} = k!\binom{n}{k}.$

If unordered: $\binom{n}{k} = \dfrac{P_k^n}{k!} = \dfrac{n!}{k!(n-k)!}.$

Counting Methods

Thus the number of k -combinations of n objects is:

$$\binom{n}{k} = \frac{(n)k}{k!} = \frac{n!}{k!(n-k)!}.$$

The number of ways to choose k objects out of n distinguishable objects is equal to $\binom{n}{k}$.

Counting Methods

Example.

The number of five-card poker hands is $\binom{52}{5}$.

The number of k-combinations of an n-element set is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } 0 \leq k \leq n.$$

Orchestrated Conversation: Counting Methods

Example.

A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Orchestrated Conversation: Counting Methods

Another interpretation of $\binom{n}{k}$:

- The number of possible divisions of n distinct objects to two groups of sets of sizes k and $n - k$ is also equal to $\binom{n}{k}$.

Example: We toss a coin 5 times and observe the sequence of heads and tails. How many different outcomes are possible if we know two tails and three heads have been observed?

Counting Methods

- The number of observation sequences for n sub-experiments with the sample space $S = \{0,1\}(\text{or } \{T,H\})$ with 0 appearing n_0 times and 1 appearing $n_1 = n - n_0$ times is $\binom{n}{n_0}$.

Example. How many distinct sequences can we make using 3 As and 5 Bs?

(AAABBBBB, AABABBBB, ….)

Orchestrated Conversation: Counting Methods

Example. We toss a coin n times and observe the sequence of heads and tails. How many different outcomes are possible if we know n_0 tails and $n_1 = n - n_0$ heads have been observed?

Counting Methods

Multinomial Coefficients: More generally if $n = n_1 + n_2 + \ldots + n_r$, we define

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}.$$

$\binom{n}{n_1, n_2, \ldots, n_r}$ is the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Counting Methods

Theorem. For n repetitions of sub-experiment with sample space

$$S = \{s_0, s_1, \dots, s_{m-1}\},$$

the number of length $n = n_0 + n_1 + \dots + n_{m-1}$ observation sequences with s_i appearing n_i times is

$$\binom{n}{n_0, n_1, \dots, n_{m-1}}.$$

Counting Methods

Bernoulli Trials:

Example. **We toss an unfair coin ($P(H) = p$) n times. What is the probability of observing k heads?**

Counting Methods

Binomial Formula:

For n independent Bernoulli trials where each trial has success probability p, the probability of k successes is given by

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Counting Methods

Generally, assume the sub-experiment has sample space $S = \{s_0, s_1, ..., s_{m-1}\}$, with $P(\{s_i\}) = p_i$. For $n = n_0 + n_1 + ... + n_{m-1}$ independent trials, the probability that s_i appears n_i times for all $i \in \{0, 1, \cdots, m-1\}$ is

$$P(s_0, s_1, \cdots, s_{m-1}) = \binom{n}{n_0, n_1, ..., n_{m-1}} p_0^{n_0} p_1^{n_1} ... p_{m-1}^{n_{m-1}}.$$

Counting Methods

Unordered Sampling with Replacement (repetition allowed):

Example: $A = \{1, 2, 3\}, \quad n = 3, \quad k = 2$

$$\begin{array}{ll} (1,1) & (2,2) \\ (1,2) & (2,3) \\ (1,3) & (3,3) \end{array} \quad \longrightarrow \quad 6 \text{ Cases.}$$

Counting Methods

Lemma.

The total number of distinct k samples from an n-element set such that repetition is allowed and ordering does not matter is the same as the number of distinct solutions to the equation

$$x_1 + x_2 + \ldots + x_n = k, \text{ where } x_i \in \{0, 1, 2, 3, \ldots\}.$$

$$\binom{n-k+1}{k} = \binom{n-k+1}{n-1}.$$

Review

Let's summarize the formulas for the four categories of sampling.

ordered sampling with replacement	n^k
ordered sampling without replacement	$P_k^n = \dfrac{n!}{(n-k)!}$
unordered sampling without replacement	$\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$
unordered sampling with replacement	$\dbinom{n+k-1}{k}$

Summary of this Lesson

- You examined the necessity of counting for solving some probability problems. You also focused on methods for counting elements in an efficient manner.

Post-work for this Lesson

- Complete the homework assignment for Lesson 4: HW#2

Go to the online classroom for details.

To Prepare for the Next Lesson

- Read Chapter 3 in your online textbook:

    https://www.probabilitycourse.com/chapter3/3_1_1_random_variables.php

- Complete the Pre-work for Lessons 5-6.

Visit the online classroom for details.