Counting: Permutations and Combinations

Probability Theory: Foundation for Data Science

with **Anne Dougherty**

Data Science
UNIVERSITY OF COLORADO **BOULDER**

Learning Goals for Module 1

In this module, we'll learn about the difference between a population and a sample and why probability is the foundation for statistics and data science. At the end of this Module, students should be able to:

- Explain why probability theory is relevant to statistics and data science.
- Describe what it means to predict the outcome of an experiment and organize the outcomes into sample spaces.
- Calculate probabilities of events using the Axioms of Probability.
- **Understand permutations and combinations and be able to calculate probabilities when each simple event is equally likely.**

Counting

Recall that the goal of **probability** is to assign some number, $P(A)$, called the probability of event A, which will give a precise measure to the chance that A will occur. If a sample space, S, has N single events, and if each of these events is equally likely to occur, then we need only count the number of events to find the probability.

For example, if $S = \{E_1, E_2, \dots, E_N\}$ and if $P(E_k) = 1/N$ for $k = 1, 2, \dots, N$, and if A is an event in S, then

$$P(A) = \frac{\text{number of simple events in A}}{N}$$

Examples

Experiement: Roll a six-sided dice twice.
$S = \{(i,j) \mid i,j \in \{1,2,3,4,5,6\}\}$, $|S| = 36$ and each of the 36 outcomes of S is equally likely.

- Let A be the event of rolling a 1 on the first roll.
  $P(A) =$
- Let B be the event that the sum of the two rolls is 8.
  $P(B) =$
- Let C be the event that the value of the second roll is two more than the first roll.
  $P(C) =$

Permutations

Any **ordered** sequence of k objects taken from a set of n distinct objects is called a **permutation of size** k.
Notation: $P_{k,n}$.

Example: Suppose an organization has 60 members. One person is selected at random to be the president, another person is selected as the vice-president, and a third is selected as the treasurer. How many ways can this be done? (This would be the cardinality of the sample space.)

Definition: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ for any positive integer n. By definition, we take $0! = 1$.

Combinations

Given n distinct objects, any unordered subset of size k of the objects is called a **combination**. Notation: $C_{k,n}$

Example: Suppose we have 60 people and want to choose a 3 person team (order is not important). How many combinations are possible?

Example - continued

Notation: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, this represents the number of combinations of size k chosen from n distinct objects.

Example - continued

Example: Suppose we have the same 60 people, 35 are female and 25 are male. We need to select a committee of 11 people.

- How many ways can such a committee be formed?

- What is the probability that a randomly selected committee will contain at least 5 men and at least 5 women? (Assume each committee is equally likely.)

Example: A city has bought 20 buses. Shortly after being put into service, some of them develop cracks in the frame. The buses are inspected and 8 have visible cracks.

- How many ways can the city select a sample of 5 for thorough inspection? (Assume each bus is equally likely to be chosen.)

- If 5 buses are chosen at random, find the probability that exactly 4 have cracks.

- If 5 buses are chosen at random, find the probability that at least 4 have cracks.