Introduction to the Central Limit Theorem

Probability Theory: Foundation for Data Science

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Central Limit Theorem

At the end of this module, students should be able to

- Understand the definition of a random sample.
- Understand the Law of Large Numbers.
- Understand and use the Central Limit Theorem (CLT).
- Explain the implications of the CLT to the calculation and estimation of the mean.

For a random variable X, we need either the probability mass function $p(k) = P(X = k)$ or density function $f(x)$ to compute a probability or to find

- $\mu_X = E(X) = \sum_k kP(X = k)$ or $\mu_X = \int_{-\infty}^{\infty} xf(x)\, dx$
- $\sigma_X^2 = V(X) = E[(X - \mu_X)^2] = \sum_k (k - \mu_X)^2 P(X = k)$ or $\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)\, dx$

Question: What if we don't know how a random variable is distributed? What if we don't know the mean or the variance?

Statistical Inference: In future courses, we will be focusing on making "statistical inferences" about the true mean and true variance of a population by using sample datasets. Before we do, we need to finish laying the groundwork.

Definition: $X_1, X_2, \ldots, X_n$ are a **random sample** of size n if

- $X_1, X_2, \ldots, X_n$ are independent
- each random variable has the same distribution

We say that these X_i's are *iid*, independent and identically distributed.

We use **estimators** to summarize our iid sample. For example, suppose we want to understand the distribution of adult female heights in a certain area. We plan to select n women at random and measure their height. Suppose the height of the i^{th} woman is denoted by X_i. $X_1, X_2, \ldots, X_n$ are iid with mean μ.

An **estimator** of μ is denoted $\bar{X}$ and $\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k$

$E(\bar{X}) =$

The Law of Large Numbers is fairly technical. However, it says that under most conditions, if $X_1, X_2, \ldots, X_n$ is a random sample with $E(X_k) = \mu$, then $\bar{X} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} X_k$, converges to μ in the limit as n goes to infinity.

Example: Let $X_1, X_2, \ldots, X_n$ each have a uniform distribution on $[0, 1]$.

What about the variance? Given a random sample $X_1, X_2, \ldots, X_n$ with $V(X_i) = \sigma^2$,

$$V(\bar{X}) =$$

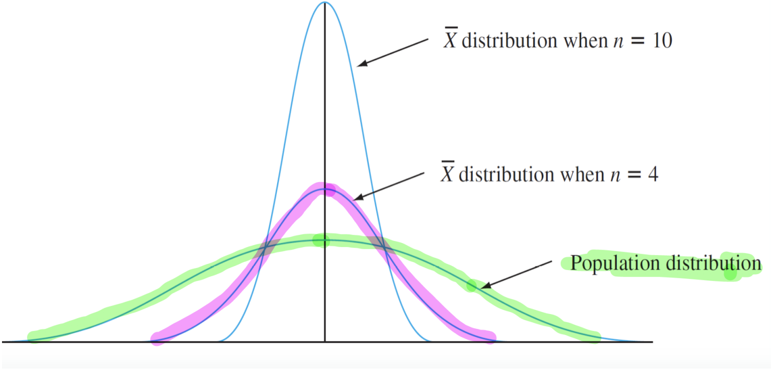
We use estimators to summarize our iid sample. Any estimator, including the sample mean, $\bar{X}$, is a random variable (since it is based on a random sample).

This means that $\bar{X}$ has a distribution of it's own, which is referred to as the **sampling distribution of the sample mean**. This sampling distribution depends on:

- the sample size n
- the population distribution of the X_i
- the method of sampling

Great, but what is the **distribution** of the sample mean?

Proposition: If $X_1, X_2, \ldots, X_n$ is iid with $X_i \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N(\mu, \sigma^2/n)$.



We know everything there is to know about the distribution of the sample mean when the population distribution is normal.

What if the population distribution is not normal?

- When the population distribution is non-normal, averaging produces a distribution that is more bell-shaped than the one being sampled.
- A reasonable conjecture is that if n is large, a suitable normal curve will approximate the actual distribution of the sample mean.
- The formal statement of this result is one of the most important theorems in probability and statistics: **Central Limit Theorem**

Central Limit Theorem Let $X_1, X_2, \ldots, X_n$ be a random sample with $E(X_i) = \mu$ and $V(X_i) = \sigma^2$. If n is sufficiently large, $\bar{X}$ has approximately a normal distribution with mean $\mu_{\bar{X}} = \mu$ and variance $\sigma^2_{\bar{X}} = \sigma^2/n$.

We write $\bar{X} \approx N(\mu, \frac{\sigma^2}{n})$

The larger the value of n, the better the approximation. Typical rule of thumb: $n \geq 30$.