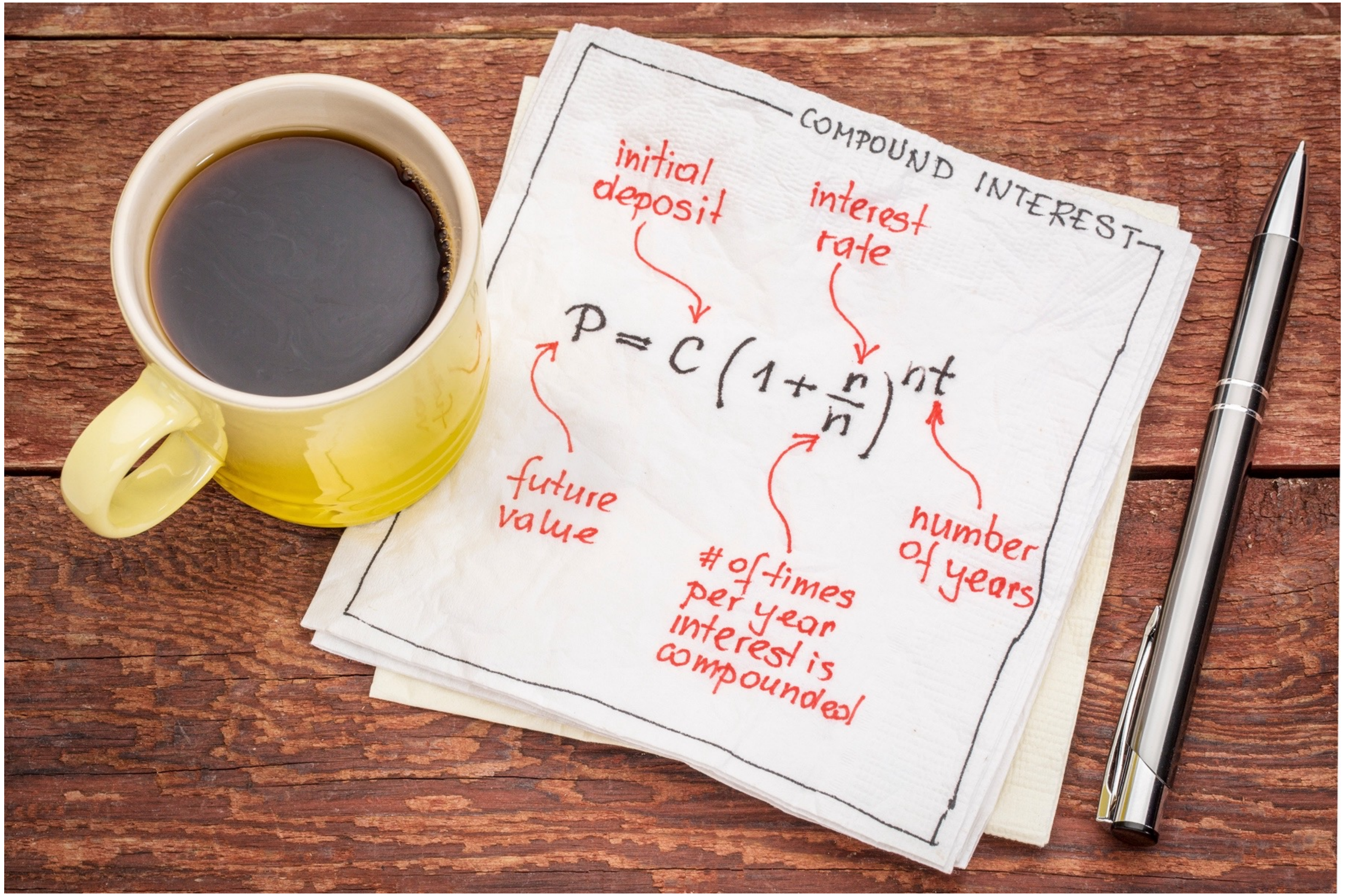
Compounding Interest Rates



Compounding Interest Rates

Interest:

- *what you receive when you loan your money (i.e., to your bank)*
- *what you pay when you borrow money (i.e., your car loan)*

Interest Rate:

- *the amount of interest expressed as a % of the principal.*

Thus far we have assumed that interest compounds yearly.

What happens when interest compounds quarterly? monthly?

Interest Rates

Nominal Interest Rate: the interest rate expressed on an annual basis ("compounding annually"). Often called the APR: Annual Percentage Rate.

Ex. You deposit \$100 in an investment earning 12% per year, compounded annually. What is the value at the end of the year?

PV = \$100 $i = 12\%$ $N = 1$

$FV = PV\,(1+i)^N$ FV = \$100 (1.12)

FV = \$112

A nominal interest rate of 12%, compounded annually earns you \$12 on your original \$100 investment when compounded annually.

Interest Rates

What happens when interest compounds quarterly?

Ex. You deposit $100 in an investment earning a nominal rate of 12% per year, but is now compounded *quarterly*. What is the value at the end of the year?

If the annual rate is 12%, then the *quarterly* rate is 3% (12% / 4).

At the end of the 1st quarter: FV_{Q1} = \$100.00 + (\$100.00 x 0.03) = \$103.00

At the end of the 2nd quarter: FV_{Q2} = \$103.00 + (\$103.00 x 0.03) = \$106.09

At the end of the 3rd quarter: FV_{Q3} = \$106.09 + (\$106.09 x 0.03) = \$109.27

At the end of the 4th quarter: FV_{Q4} = \$109.27 + (\$109.27 x 0.03) = \$112.55

FV = $112.55

What is the actual interest rate? 12.55%

Compounding quarterly increases your return slightly.
You now earn $12.55 on your original $100 investment.

Interest Rates

What happens when interest compounds monthly?

Ex. You deposit \$100 in an investment earning a nominal rate of 12% per year, but is now compounded *monthly*. What is the value at the end of the year?

If the annual rate is 12%, then the *monthly* rate is 1.0% (12% / 12).

At the end of Jan: FV_{Jan} = \$100.00 + (\$100.00 x 0.01) = \$101.00

At the end of Feb: FV_{Feb} = \$101.00 + (\$101.00 x 0.01) = \$102.01

At the end of Mar: FV_{Mar} = \$102.01 + (\$102.01 x 0.01) = \$103.03

…

At the end of Dec: FV_{Dec} = \$111.57 + (\$111.57 x 0.01) = \$112.68

FV = \$112.68

What is the actual interest rate? 12.68%

Compounding monthly increases your return even more.
You now earn \$12.68 on your original \$100 investment.

Interest Rates

Effective Annual Interest Rate (EAR): the "actual" annual interest rate that accounts for multiple compounding events.

Nominal Rate = 12.00%	(the actual annual rate when compounding annually)
Effective Rate = 12.55%	(the actual annual rate when compounding quarterly)
Effective Rate = 12.68%	(the actual annual rate when compounding monthly)

What is the relationship between "nominal" and "effective" interest rates?

Nominal vs. Effective Interest Rates

Nominal Interest Rate: the interest rate expressed on an annual basis (the APR)

Effective Interest Rate: the *actual* interest rate expressed on an annual basis (the EAR)

$$i_{EAR} = \left(1 + \frac{i}{M}\right)^{M} - 1$$

Where:

i_{EAR} = the effective annual interest rate (sometimes referred to as i_e)

i = the nominal interest rate (the APR)

M = the number of compounding periods per year

Nominal vs. Effective Interest Rates

The relationship between nominal and effective interest rate is:

$$i_{EAR} = \left(1 + \frac{i}{M}\right)^{M} - 1$$

For an 18% nominal rate, compounded quarterly, the effective annual interest rate is:

$$i_{EAR} = \left(1 + \frac{0.18}{4}\right)^{4} - 1 = 0.1925 = 19.25\% \text{ per year}$$

For a 7% nominal rate, compounded monthly, the effective annual interest rate is:

$$i_{EAR} = \left(1 + \frac{0.07}{12}\right)^{12} - 1 = 0.0723 = 7.23\% \text{ per year}$$

Nominal vs. Effective Interest Rates

The relationship between nominal and effective interest rate is:

$$i_{EAR} = \left(1 + \frac{i}{M}\right)^{M} - 1$$

Nominal Interest Rate, i	Effective Annual Interest Rate when compounding is:		
	Yearly	Quarterly	Monthly
2%	2.0%	2.015%	2.018%
5%	5.0%	5.095%	5.116%
10%	10.0%	10.381%	10.471%
15%	15.0%	15.865%	16.075%
25%	25.0%	27.443%	28.073%

Multiple Compounding Events & TVM Formulae

How does multiple compounding impact our TVM equations:

$i_{EAR} = (1 + i/M)^M - 1 \quad \Rightarrow \quad (1 + i_{EAR}) = (1 + i/M)^M$

Example:

The interest rate per period

$FV = PV\,(1+i_{EAR})^N \quad \Rightarrow \quad FV = PV(1+i/M)^{MN}$

Multiple Compounding Events & TVM Formulae

How does multiple compounding impact our TVM equations:

$$i_{EAR} = (1 + i/M)^M - 1 \quad \Rightarrow \quad (1 + i_{EAR}) = (1 + i/M)^M$$

The total # of periods

Example:

$$FV = PV\,(1+i_{EAR})^N \quad \Rightarrow \quad FV = PV(1+i/M)^{MN}$$

Ex. You invest $1000 into an account earning an APR = 10%, compounded quarterly. What is the future value of your investment after two years?

$$FV = \$1000\,(1 + 0.10/4)^{4*2}$$

$$FV = \$1000\,(1 + 0.025)^{8} \qquad FV = \$1000\,(1.2184)$$

FV = $1,218.40

Multiple Compounding Events & TVM Formulae

It is a lot easier than it looks…

To obtain the values for any of our TVM equations, simply:

1. Determine the interest rate per compounding period (i/M)
   - *Nominal Rate / # of Compounding periods per year*

2. Determine the total number of compounding periods (M*N)
   - *The # of compounding periods per year * the # of years*

3. Substitute these values into the equations…

Multiple Compounding Events & TVM Formulae

It is a lot easier than it looks…

Ex. You invest \$10,000 into an account earning an APR = 12%, compounded monthly. What is the future value of your investment after 5 years?

What are the # of compounding periods per year? M = 12

What is the interest rate per compounding period? i/M = 12% / 12 = 1% / month

What is the total # of periods? MN = (12 periods per year) x (5 years) = 60 periods

$FV = PV\,(1+i)^N$

$FV = \$10{,}000\,(1 + 0.12/12)^{5*12}$

$FV = \$10{,}000\,(1 + 0.01)^{60}$ $FV = \$10{,}000\,(1.8167)$

FV = \$18,167

Multiple Compounding Events & TVM Formulae

Ex. You have a credit card balance of $5000 at an APR = 15%, compounded monthly. If you want to pay this off in 3 years (w/o charging any more on it), what is your monthly payment?

What is the monthly interest rate?

$$A = PV\left[\frac{i\,(1+i)^N}{(1+i)^N - 1}\right]$$

How many payment periods are there?

In this case, i = 0.15/12 = 0.0125 , and N = 12*3 = 36 months

Therefore:

$$A = \$5000\left[\frac{0.0125\,(1+0.0125)^{36}}{(1+0.0125)^{36} - 1}\right]$$

A = $5000 (0.0347)

A = $173.40 per month

Multiple Compounding Events & TVM Formulae

Ex. You have a credit card balance of $5000 at an APR = 15%, compounded monthly. If you pay want to pay this off in 3 years (w/o charging any more), what is your monthly payment?

How would you set this up in your "PMT" Excel Worksheet?

	A	B	C	D	E
1	**TVM with Excel**				
2	**Annuity (Payment) Calculations**				
3					
4	PV =	$5,000	*this is our P, at time = 0*		
5	FV =	$0	*this is our F, at time = N*		
6	RATE =	1.25%	*this is our i*		
7	NPER =	36	*this is our N*		
8					
9	PMT =	-$173.33	*PMT(rate, nper, pv, fv, type)*		
10			*=PMT(B6,B7,B4,B5)*		
11					

Main Takeaways…

Nominal Interest Rate: the interest rate expressed on an annual basis (the APR).

Effective Annual Interest Rate (the EAR): the actual annual interest rate, accounting for multiple compounding periods per year.

In our TVM formulae, the interest rate is the interest rate per compounding period (i/M), and the number of periods is the total number of compounding periods (MN).

In Excel, just remember that "RATE" is the interest rate per compounding period, and "NPER" is the total number of compounding periods.

The more compounding periods per year, the higher the actual interest rate.

Next Time…

Some Practical Examples

Credits & References

Slide 1: Compound interest equation by MarekPhotoDesign.com, Adobe Stock (109532881.jpeg).

Slide 17: Cash flow illustration with laptop money and graph chart by teguhjatipras, Adobe Stock (106672218.jpeg).