APPM 4/5520

Solutions to Final Exam Review Problems

1. The pdf is

$$f(x;a,b) = \frac{1}{\mathcal{B}(a,b)} x^{a-1}(1-x)^{b-1}\, I_{(0,1)}(x).$$

The joint pdf is

$$\begin{aligned} f(\vec{x};a,b) &= \tfrac{1}{[\mathcal{B}(a,b)]^n}[\prod_{i=1}^n x_i]^{a-1}[\prod_{i=1}^n (1-x_i)]^{b-1}\, \prod_{i=1}^n I_{(0,1)}(x_i) \\ &= \underbrace{\frac{1}{[\mathcal{B}(a,b)]^n}}_{a(\theta)} \underbrace{\prod_{i=1}^n I_{(0,1)}(x_i)}_{b(\vec{x})} \exp[\underbrace{(a-1)}_{c_1(\theta)}\underbrace{\sum_{i=1}^n \ln x_i}_{d_1(\vec{x})} + \underbrace{(b-1)}_{c_2(\theta)}\underbrace{\sum_{i=1}^n \ln(1-x_i)}_{d_2(\vec{x})}] \end{aligned}$$

So, "by two-parameter exponential family", we have that $S = (\sum \ln X_i, \sum \ln(1-X_i))$ is complete and sufficient for this model.

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2. The pdf is

$$f(x;\beta) = \frac{1}{2}\beta^3 x^2 e^{-\beta x}\, I_{(0,\infty)}(x).$$

The joint pdf is

$$\begin{aligned} f(\vec{x};\beta) &= \tfrac{1}{2^n}\beta^{3n}\prod_{i=1}^n x_i^2 e^{-\beta\sum_{i=1}^n x_i}\, \prod_{i=1}^n I_{(0,\infty)}(x_i) \\ &= \tfrac{1}{2^n}\beta^{3n}\cdot \prod_{i=1}^n x_i^2 I_{(0,\infty)}(x_i)\cdot \exp[\beta\sum x_i]. \end{aligned}$$

By one-parameter exponential family, we have that

$$S = d(\vec{X}) = \sum_{i=1}^n X_i$$

is complete and sufficient for β.

To find the UMVUE for β, we need to find a function of S that is unbiased for β. We start by considering S itself.

$$\mathsf{E}[S] = \mathsf{E}[\sum X_i] = \sum \mathsf{E}[X_i] = \sum 3/\beta = 3n/\beta.$$

This is no good. We want to see β in the numerator. So, we will try $\mathsf{E}[1/S]$. In order to compute this, we need to realize that $S \sim \Gamma(3n,\beta)$.

$$\begin{aligned} \mathsf{E}\left[\tfrac{1}{S}\right] &= \int_0^\infty \tfrac{1}{s}\cdot\tfrac{1}{\Gamma(3n)}\beta^{3n}s^{3n-1}e^{-\beta s}\,ds \\ &= \int_0^\infty \tfrac{1}{\Gamma(3n)}\beta^{3n}s^{3n-2}e^{-\beta s}\,ds \\ &= \tfrac{\Gamma(3n-1)}{\Gamma(3n)}\beta\int_0^\infty \tfrac{1}{\Gamma(3n-1)}\beta^{3n-1}s^{3n-2}e^{-\beta s}\,ds \\ &= \tfrac{1}{3n-1}\beta. \end{aligned}$$

Therefore, by the Lehmann-Scheffé Theorem,

$$\hat{\beta} = \frac{3n-1}{S} = \frac{3n-1}{\sum X_i}$$

is the UMVUE for β.

The variance of this estimator is

$$\begin{aligned} Var[\hat{\beta}] &= \mathsf{E}[\hat{\beta}^2] - \left(\mathsf{E}[\hat{\beta}]\right)^2 \\ &= \mathsf{E}[\hat{\beta}^2] - \beta^2 \end{aligned}$$

since $\hat{\beta}$ is an unbiased estimator of β.

Now,

$$\begin{aligned} \mathsf{E}[\hat{\beta}^2] &= (3n-1)^2 \mathsf{E}\left[\tfrac{1}{S^2}\right] \\ &= (3n-1)^2 \int_0^\infty \tfrac{1}{s^2} \cdot \tfrac{1}{\Gamma(3n)} \beta^{3n} s^{3n-1} e^{-\beta s}\, ds \\ &= (3n-1)^2 \int_0^\infty \tfrac{1}{\Gamma(3n)} \beta^{3n} s^{3n-3} e^{-\beta s}\, ds \\ &= (3n-1)^2 \tfrac{\Gamma(3n-2)}{\Gamma(3n)} \beta^2 \int_0^\infty \tfrac{1}{\Gamma(3n-2)} \beta^{3n-2} s^{3n-3} e^{-\beta s}\, ds \\ &= (3n-1)^2 \tfrac{\Gamma(3n-2)}{\Gamma(3n)} \beta^2 = \tfrac{3n-1}{3n-2} \beta^2. \end{aligned}$$

So, the variance is

$$Var[\hat{\beta}] = \frac{3n-1}{3n-2}\, \beta^2 - \beta^2 = \frac{1}{3n-2} \beta^2$$

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3. The pdf is

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}\, I_{\{0,1,2,\ldots\}}(x)$$

The joint pdf is

$$\begin{aligned} f(\vec{x};\lambda) &= \tfrac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!} \cdot \prod I_{\{0,1,2,\ldots\}}(x_i) \\ &= e^{-n\lambda} \prod \tfrac{I_{\{0,1,2,\ldots\}}(x_i)}{x_i!} \cdot \exp[\ln\lambda \cdot \sum x_i] \end{aligned}$$

By one-parameter exponential family

$$S = d(\vec{X}) = \sum X_i$$

is complete and sufficient for λ.

To find the UMVUE for $\tau(\lambda) = \lambda^2$, we need to find a function of S that is unbiased for λ^2. We start by considering S itself.

$$\mathsf{E}[S] = \mathsf{E}[\sum X_i] = \sum \mathsf{E}[X_i] = n\lambda.$$

Since we really want to see λ^2, we'll now try

$$\mathsf{E}[S^2] = Var[S] + (\mathsf{E}[S])^2$$

Since $S \sim Poisson(n\lambda)$, this is

$$\mathsf{E}[S^2] = n\lambda + (n\lambda)^2 = \mathsf{E}[S] + n^2\lambda^2$$

and the UMVUE for λ^2 is

$$\widehat{\tau(\lambda)} = \frac{S^2 - S}{n^2} = \frac{(\sum X_i)^2 - \sum X_i}{n^2}.$$

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4. We want to find a function of $X_{(n)}$ that is unbiased for θ^p. Let's try

$$\begin{aligned} \mathsf{E}[X_{(n)}] &= \textstyle\int_0^\theta x \cdot \frac{n}{\theta^n} x^{n-1}\, dx \\ &= \textstyle\frac{n}{n+1}\theta \end{aligned}$$

From that integral, we can see that we will get θ^p if we compute

$$\mathsf{E}[X_{(n)}^p] = \int_0^\theta x^p \cdot \frac{n}{\theta^n} x^{n-1}\, dx = \frac{n}{n+p}\theta^p.$$

Therefore, the UMVUE for $\tau(\theta) = \theta^p$ is

$$\widehat{\tau(\theta)} = \frac{n+p}{n} X_{(n)}^p.$$

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5. (a) The likelihood ratio based on this sample of size 1 is

$$\lambda(x_1; 0, \theta_1) = \frac{f(x_1; 0)}{f(x_1; \theta_1)} = \frac{1}{1 - \theta_1^2(x_1 - 1/2)}$$

Setting this less than or equal to k and flipping we get

$$1 - \theta_1^2(x_1 - 1/2) \geq \frac{1}{k}$$

$$-\theta_1^2(x_1 - 1/2) \geq \frac{1}{k} - 1$$

$$x_1 - 1/2 \leq -\frac{1}{\theta_1^2}\left(\frac{1}{k} - 1\right)$$

Note that, if θ_1 is negative or positive, θ_1^2 is always positive and $-\theta_1^2$ is always negative. So, the inequality direction at this point is independent of the sign of θ_1.

$$x_1 \leq -\frac{1}{\theta_1^2}\left(\frac{1}{k} - 1\right) + \frac{1}{2}$$

So, the form of the test is to reject if $X_1 \leq k_1$.
Now to find k_1...

$$\begin{aligned} \alpha &= P(\lambda(X_1; 0, \theta_1) \leq k; H_0) \\ &= P(X_1 \leq k_1; H_0) = k_1 \end{aligned}$$

That last inequality comes from the fact that when H_0 is true, $X_1 \sim unif(0,1)$.
So, we take $k_1 = \alpha$ and the best (most powerful) test of the given simple versus simple hypotheses is to reject H_0 when $X_1 \geq \alpha$.

(b) Since the test from part (a) does not depend on the particular θ_1 that was chosen (and, in this problem we specifically did not flip an inequality based on θ_1 being greater or less than 0, it is also uniformly most powerful for

$$H_0 : \theta = 0 \qquad H_1 : \theta \neq 0.$$

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6. We first consider the simple versus simple hypotheses

$$H_0 : \sigma^2 = \sigma_0^2 \qquad H_1 : \sigma^2 = \sigma_1^2$$

for some fixed $\sigma_1^2 > \sigma_0^2$.
The joint pdf is

$$f(\vec{x}; \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum x_i^2}.$$

The likelihood ratio is

$$\begin{aligned} \lambda(\vec{x}; \sigma_0^2, \sigma_1^2) &= \frac{f(\vec{x};\sigma_0^2)}{f(\vec{x};\sigma_1^2)} \\ &= \frac{(2\pi\sigma_0^2)^{-n/2} e^{-\frac{1}{2\sigma_0^2}\sum x_i^2}}{(2\pi\sigma_1^2)^{-n/2} e^{-\frac{1}{2\sigma_1^2}\sum x_i^2}} \\ &= (\sigma_1^2/\sigma_0^2)^{n/2} \cdot e^{-\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum x_i^2} \end{aligned}$$

Setting this less than or equal to k and starting to move things, we get

$$e^{-\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum x_i^2} \leq (\sigma_0^2/\sigma_1^2)^{n/2} k$$

$$-\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum x_i^2 \leq \ln\left[(\sigma_0^2/\sigma_1^2)^{n/2} k\right]$$

$$\sum x_i^2 \geq \frac{\ln\left[(\sigma_0^2/\sigma_1^2)^{n/2} k\right]}{-\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)}$$

since $\sigma_1^2 > \sigma_0^2$.

So, the best test of

$$H_0 : \sigma^2 = \sigma_0^2 \qquad H_1 : \sigma^2 = \sigma_1^2$$

for some fixed $\sigma_1^2 > \sigma_0^2$ will be to reject H_0 if

$$\sum X_i^2 \geq k_1$$

where k_1 is chosen to give a size α test.

Now let's find k_1.

$$\alpha \;=\; P\left(\sum X_i^2 \geq k_1; H_0\right)$$

Since, under H_0, $X_i \sim N(0, \sigma_0^2)$ so $X_i/\sigma_0^2 \sim N(0, 1)$. Squaring a $N(0, 1)$ gives a χ^2 random variable. Adding independent χ^2-random variables gives another χ^2 with all the degrees of freedom added up.

So,

$$\frac{\sum_{i=1}^n X_i^2}{\sigma_0^2} = \sum_{i=1}^n \frac{X_i^2}{\sigma_0^2} = \sum_{i=1}^n \left(\frac{X_i}{\sigma_0}\right)^2 \sim \chi^2(n)$$

So,

$$\begin{aligned}
\alpha &= P\left(\sum X_i^2 \geq k_1; H_0\right) \\
&= P\left(\frac{\sum X_i^2}{\sigma_0^2} \geq k_1/\sigma_0^2; H_0\right) \\
&= P(W > k_1/\sigma_0^2)
\end{aligned}$$

where $W \sim \chi^2(n)$.

So, we have that k_1/σ_0^2 is the $\chi^2(n)$ critical value that cuts off area α to the right. Our notation for this is $\chi^2_\alpha(2n)$. So

$$k_1 = \sigma_0^2\, \chi^2_\alpha(2n).$$

So, the best test of size α of

$$H_0 : \sigma^2 = \sigma_0^2 \qquad H_1 : \sigma^2 = \sigma_1^2$$

for some fixed $\sigma_1^2 > \sigma_0^2$ will be to reject H_0 if

$$\sum X_i^2 \geq \sigma_0^2\, \chi^2_\alpha(2n).$$

This test does not depend on the specific chosen value of σ_1^2 (with the exception that the form of the test depends on the fact that $\sigma_1^2 > \sigma_0^2$). So, this is a UMP test of size α for

$$H_0 : \sigma^2 = \sigma_0^2 \text{ versus } H_1 : \sigma^2 > \sigma_0^2.$$

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7. The power function is

$$\begin{aligned}\gamma(\sigma^2) &= P(\text{Reject } H_0\,;\sigma^2)\\ &= P(\textstyle\sum X_i^2 \geq \sigma_0^2\,\chi^2_\alpha(2n)\,;\sigma^2)\end{aligned}$$

We are under the assumption that $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} N(0, \sigma^2)$. We don't know the distribution of these squared, but we would if they were $N(0,1)$. ($N(0,1)$ random variables squared are $\chi^2(1)$ random variables.) Note that

$$\frac{\sum X_i^2}{\sigma^2} = \sum\left(\frac{X_i}{\sigma}\right)^2 \sim \chi^2(n)$$

So, back to the power function...

$$\begin{aligned}\gamma(\sigma^2) &= P(\textstyle\sum X_i^2 \geq \sigma_0^2\,\chi^2_\alpha(2n)\,;\sigma^2)\\ &= P\left(\frac{\sum X_i^2}{\sigma^2} \geq \frac{\sigma_0^2\,\chi^2_\alpha(2n)}{\sigma^2}\,;\sigma^2\right)\\ &= P\left(W \geq \frac{\sigma_0^2\,\chi^2_\alpha(2n)}{\sigma^2}\right) = 1 - F_W\left(\frac{\sigma_0^2\,\chi^2_\alpha(2n)}{\sigma^2}\right)\end{aligned}$$

where $W \sim \chi^2(n)$.

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8. (a) We should reject H_0 if the minimum is small. So a test based on $X_{(1)}$ should look like

"Reject H_0 if $X_{(1)} < c$"

Now find c.

$$\begin{aligned}\alpha &= P(\text{ Reject } H_0 \text{ when true })\\ &= P(X_{(1)} < c; \theta_0)\\ &= 1 - e^{-n\theta_0 c}\end{aligned}$$

since the minimum of exponentials with rate θ_0 is exponential with rate $n\theta_0$.
So

$$c = -\frac{1}{n\theta_0}\ln(1-\alpha)$$

So a test of size α of the given hypotheses and based on $X_{(1)}$ is to reject H_0 if

$$X_{(1)} < -\frac{1}{n\theta_0}\ln(1-\alpha).$$

(b) The UMP test is to reject H_0 if

$$\sum X_i \leq \chi^2_{1-\alpha}(2n)/(2\theta_0).$$

(I'm rushing to get these solutions out. Regarding the details here, this is similar to a problem we did in class!)

(c) The power functions...

For the test from part (a):

$$\begin{aligned}\gamma_{(a)}(\theta) &= P(\text{ reject } H_0 \text{ when the parameter is } \theta\,) \\ &= P\left(X_{(1)} < -\tfrac{1}{n\theta_0}\ln(1-\alpha);\theta\right)\end{aligned}$$

When the parameter is θ, $X_{(1)}$ is exponential with rate $n\theta$. So

$$\begin{aligned}\gamma_{(a)}(\theta) &= 1 - e^{n\theta\left(-\frac{1}{n\theta_0}\ln(1-\alpha)\right)} \\ &= 1-(1-\alpha)^{(-\theta/\theta_0)}\end{aligned}$$

For the test from part (b):

$$\begin{aligned}\gamma_{(b)}(\theta) &= P(\text{ reject } H_0 \text{ when the parameter is } \theta\,) \\ &= P\left(\sum X_i \le \chi^2_{1-\alpha}(2n)/(2\theta_0);\theta\right)\end{aligned}$$

When the parameter is θ, $\sum X_i \sim \Gamma(n,\theta)$. So $2\theta\sum X_i \sim \Gamma(n,1/2) = \chi^2(2n)$. Therefore

$$\begin{aligned}\gamma_{(b)}(\theta) &= P\left(2\theta\sum X_i \le 2\theta\chi^2_{1-\alpha}(2n)/2\theta_0;\theta\right) \\ &= P\left(W \le \theta\chi^2_{1-\alpha}(2n)/\theta_0\right)\end{aligned}$$

This function is the cdf of a $\chi^2(2n)$ random variable evaluated at $\theta\chi^2_{1-\alpha}(2n)/\theta_0$ for a fixed n, a fixed θ_0 and regarded as a function of θ. There is no nice closed form expression for comparison to the other power function. For fixed n and θ_0, you could numerically plot the cdf– when plotted along with $\gamma_{(a)}(\theta)$ you should see that $\gamma_{(b)}(\theta)$ is above $\gamma_{(a)}(\theta)$ for all values of θ.

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9. To begin, we need to find any unbiased estimator. Note that, for this Poisson distribution, $P(X=0)=e^{-\lambda}$. So, an unbiased estimator is $I_{\{X_1=0\}}$.

   By one-parameter exponential family, it is easy to see that $S=\sum X_i$ is complete and sufficient for this distribution. By the Rao-Blackwell Theorem, we know that $\mathsf{E}[I_{\{X_1=0\}}|S]$ is also an unbiased estimator for $\tau(\lambda)$ and furthermore that it is a function of S. Since S is complete and sufficient, we will then have found the UMVUE.

   For ease of computation, we will begin by putting in a value for S:

$$\begin{aligned}\mathsf{E}[I_{\{X_1=0\}}|S=s] &= P(X_1=0|S=s) \\ &= \tfrac{P(X_1=0,S=s)}{P(S=s)} \\ &= \tfrac{P\left(X_1=0,\sum_{i=1}^n X_i=s\right)}{P(S=s)} \\ &= \tfrac{P\left(X_1=0,\sum_{i=2}^n X_i=s\right)}{P(S=s)} \\ &\overset{indep}{=} \tfrac{P(X_1=0)\cdot P\left(\sum_{i=2}^n X_i=s\right)}{P(S=s)}\end{aligned}$$

Since $\sum_{i=1}^{n} X_i \sim Poisson(n\lambda)$ and $\sum_{i=2}^{n} X_i \sim Poisson((n-1)\lambda)$, this is equal to

$$\frac{e^{-\lambda} \cdot \frac{e^{-(n-1)\lambda}[(n-1)\lambda]^s}{s!}}{\frac{e^{-n\lambda}[n\lambda]^s}{s!}} = \left(\frac{n-1}{n}\right)^s.$$

Removing the specific value of S, we have that

$$\widehat{\tau(\lambda)} = \left(\frac{n-1}{n}\right)^S = \left(\frac{n-1}{n}\right)^{\sum_{i=1}^{n} X_i}.$$

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10. Consider first the simple versus simple hypotheses:

$$H_0 : \theta = \theta_0 \qquad H_1 : \theta = \theta_1$$

for some $\theta_1 < \theta_0$. The ratio for the Neyman-Pearson test is

$$\lambda(\vec{x}; \theta_0, \theta_1) = \frac{\frac{1}{\theta_0^n} I_{(0,\theta_0)}(x_{(n)}) \cdot I_{(0,x_{(n)})}(x_{(1)})}{\frac{1}{\theta_1^n} I_{(0,\theta_1)}(x_{(n)}) \cdot I_{(0,x_{(n)})}(x_{(1)})} = \left(\frac{\theta_1}{\theta_0}\right)^n \frac{I_{(0,\theta_0)}(x_{(n)})}{I_{(0,\theta_1)}(x_{(n)})} \stackrel{set}{\leq} k$$

The k should be something non-negative since λ is a ratio of pdfs and therefore is always non-negative. Note that if the indicator in the numerator is zero if $x_{(n)} > \theta_0$. In this case, we absolutely know that H_0 is not true since it states that all values in the sample will be between 0 and θ_0. This is reflected in the fact that $x_{(n)} > \theta_0 \Rightarrow \lambda = 0$ which is less than or equal to any valid k, so we will always reject.

On the other hand, if the indicator in the denominator is zero, this means that $x_{(n)} > \theta_1$. The N-P ratio λ becomes infinite (in a sense) which makes it NOT less than or equal to any cut-off k, so we would never reject H_0. This makes sense because $x_{(n)} > \theta_1$ implies that H_1 could not possibly be true since it says that all values in the sample are between 0 and θ_1.

All of these comments aside, this test is garbage if $x(n)$ is greater than both θ_0 and θ_1 since, in hypothesis testing, the assumption is that one of the two hypotheses is true. Since $\theta_1 < \theta_0$, and the sample came from either the $unif(0, \theta_0)$ or $unif(0, \theta_1)$ distribution, we must have that $x_{(n)} < \theta_0$, and so the indicator in the numerator is one. Thus, we have

$$\left(\frac{\theta_1}{\theta_0}\right)^n \frac{1}{I_{(0,\theta_1)}(x_{(n)})} \leq k$$

$$\Rightarrow \qquad \frac{1}{I_{(0,\theta_1)}(x_{(n)})} \leq \left(\frac{\theta_0}{\theta_1}\right)^n k$$

$$\Rightarrow I_{(0,\theta_1)}(x_{(n)}) \geq k_1$$

Now the indictor will be "large" (ie: 1) if $x_{(n)}$ is small, so this is equivalent to

$$X_{(n)} \leq k_2$$

for some k_2 such that

$$P(X_{(n)} \leq k_2; \theta_0) = \alpha$$

ie:

$$\left(\frac{k_2}{\theta_0}\right)^n = \alpha$$

$$\Rightarrow \qquad k_2 = \theta_0 \alpha^{1/n}$$

So, the UMP test of

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta = \theta_1$$

is to reject H_0 if $X_{(n)} \leq \theta_0 \alpha^{1/n}$. Since this test does not involve θ_1 (only that $\theta_1 < \theta_0$), it is UMP for

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta < \theta_0$$

Finally, the composite null hypothesis will only chage the way the level of significance is defined

$$\begin{aligned} \alpha &= \max_{\theta \geq \theta_0} P(X_{(n)} \leq k_2; \theta) \\ &= \max_{\theta \geq \theta_0} \left(\tfrac{k_2}{\theta}\right)^n = \left(\tfrac{k_2}{\theta_0}\right)^n \end{aligned}$$

$$\Rightarrow \qquad k_2 = \theta_0 \alpha^{1/n}$$

So, a UMP test of size α of

$$H_0 : \theta \leq \theta_0 \qquad \text{versus} \qquad H_1 : \theta < \theta_0$$

is to reject H_0 if $X_{(n)} \leq \theta_0 \alpha^{1/n}$.

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11. (a) First note that, when the parameter is in the indicator like this, the exponential family factorization for find a complete and sufficient statistic will never work. That factorization is about complete separation of the x's and θ ($a(\theta)$, $b(\vec{x})$, $c(\theta)$, $d(\vec{x})$) but they are stuck together in the indicator.

First, we need to find a sufficient statistic. We'll use the Factorization Criterion:

$$f(\vec{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \cdots = e^{n\theta - \sum x_i} I_{(\theta,\infty)}(x_{(1)}) = \underbrace{e^{-\sum x_i}}_{h(\vec{x})} \underbrace{e^{n\theta} I_{(\theta,\infty)}(x_{(1)})}_{g(s(\vec{x});\theta)}$$

Thus, we see that $S = X_{(1)}$ is sufficient for θ.

To show that S is complete, we need to find the pdf for the minimum. I am running out of time and need to get these solutions posted, so I am omitting the details, but the pdf for the minimum is

$$f_{X_{(1)}}(x) = n e^{n(\theta - x)} I_{(\theta,\infty)}(x)$$

To show completeness, assume that g is any function such that $\mathsf{E}[g(X_{(1)})] = 0$ for all θ. Then

$$0 = \mathsf{E}[g(X_{(1)})] = \textstyle\int_{\theta}^{\infty} g(x)\, n\, e^{n(\theta - x)}\, dx = n e^{n\theta} \int_{\theta}^{\infty} g(x)\, e^{-nx}\, dx$$

for all θ. This implies that

$$\int_{\theta}^{\infty} g(x)\, e^{-nx}\, dx = 0$$

or, equivalently,

$$-\int_{\infty}^{\theta} g(x)\, e^{-nx}\, dx = 0$$

and thus

$$\int_{\infty}^{\theta} g(x)\, e^{-nx}\, dx = 0$$

for all θ.

Taking the derivative of both sides with respect to θ gives

$$g(\theta)e^{-n\theta} = 0$$

for all θ. Since $e^{-n\theta} \neq 0$, we get that $g(\theta)$ must be zero for all θ. Thus, $g(X_{(1)}) = 0$ and we have that $S = X_{(1)}$ is complete for θ.

(b) We need to find a function of $X_{(1)}$ that is unbiased for θ. We consider $X_{(1)}$ itself.

$$\begin{aligned}
\mathsf{E}[X_{(1)}] &= \textstyle\int_{-\infty}^{\infty} x f_{X_{(1)}}(x)\, dx \\
&= \textstyle\int_{\theta}^{\infty} x n e^{n(\theta - x)}\, dx \\
&= n e^{n\theta} \textstyle\int_{\theta}^{\infty} x e^{-nx}\, dx \\
&= e^{n\theta}[\theta e^{-n\theta} + \tfrac{1}{n} e^{-n\theta}] \\
&= \theta + \tfrac{1}{n}
\end{aligned}$$

So, $\widehat{\theta} = X_{(1)} - 1/n$.

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12. The joint pdf is

$$f(\vec{x};\theta) = \theta^n \left[\prod_{i=1}^{n}(1 - x_i)\right]^{\theta - 1} \prod_{i=1}^{n} I_{(0,1)}(x_i).$$

A likelihood is

$$L(\theta) = \theta^n \left[\prod_{i=1}^{n}(1 - x_i)\right]^{\theta - 1}.$$

The MLE (work not shown) is

$$\widehat{\theta} = \frac{-n}{\sum \ln(1 - X_i)}.$$

The restricted MLE is $\widehat{\theta}_0 = 1$.

The GLR is

$$\lambda(\vec{X}) = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}.$$

Note that $L(\widehat{\theta}_0) = L(1) = 1$. Thus, the GLR is

$$\lambda(\vec{X}) = \left(\frac{\sum \ln(1 - X_i)}{-n}\right)^n \left[\prod_{i=1}^{n}(1 - x_i)\right]^{\frac{n}{\sum \ln(1 - X_i)} + 1}.$$

The form of the GLRT is to reject H_0 in favor of H_1 if

$$\left(\frac{\sum \ln(1-X_i)}{-n}\right)^n \left[\prod_{i=1}^{n}(1-x_i)\right]^{\frac{n}{\sum \ln(1-X_i)}+1} \leq k$$

where k is to be determined so that $P(\lambda(\vec{X}) \leq k; 1) = \alpha$.

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13. The restricted MLE is $\hat{\mu}_0 = \mu_0$ (Here, μ_0 is notation for the constant that is given in the setup of the hypotheses and $\hat{\mu}_0$ is notation for the MLE estimator for μ restricted to when H_0 is true.)

    The unrestricted MLE is $\overline{X}$.

    Therefore, the GLR is

$$\lambda(\vec{X}) = \frac{L(\hat{\mu}_0)}{L(\hat{\mu})} = \frac{(2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(X_i-\mu_0)^2}}{(2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(X_i-\overline{X})^2}} = e^{-\frac{1}{2\sigma^2}\sum\left[(X_i-\mu_0)^2-(X_i-\overline{X})^2\right]}$$

$$= e^{-\frac{1}{2\sigma^2}\sum\left[(X_i-\mu_0)^2-(X_i-\overline{X})^2\right]}$$

    (Note that the σ^2's in the front of the e's could cancel because, in this problem, σ^2 is fixed and known.)

    After a bit of simplification, this can be expressed as

$$\lambda(\vec{x}) = \exp[-n(\overline{x}-\mu_0)^2/2\sigma^2]$$

    We reject H_0 if $\lambda(\vec{x}) \leq k$ which is equivalent to

$$\begin{aligned} & \frac{-n(\overline{x}-\mu_0)^2}{2\sigma^2} \leq k_1 \\ \Rightarrow \quad & \frac{n(\overline{x}-\mu_0)^2}{\sigma^2} \geq k_2 \\ \Rightarrow \quad & \left(\frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}}\right)^2 \geq k_2 \qquad (1) \end{aligned}$$

    We now could choose to take the square root of both sides which would give us

$$\frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} \geq k_3 \qquad \text{or} \qquad \frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} \leq -k_3, \qquad (2)$$

    (where $k_3 = \sqrt{k_2}$) or we could leave things in the form of (1). Either answer would be correct.

    Case 1: Leave things in the form of (1).

    Here, we choose k_2 such that

$$P\left(\left(\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\right)^2 \geq k_2; \mu_0\right) = \alpha$$

When $\mu = \mu_0$, $\overline{X} \sim N(\mu_0, \sigma^2/n) \Rightarrow (\overline{X - \mu_0})/(\sigma/\sqrt{n}) \sim N(0,1)$ *Rightarrow* $[(\overline{X} - \mu_0)/(\sigma/\sqrt{n})]^2 \sim \chi^2(1) \Rightarrow k_2 = \chi^2_\alpha(1)$.

So, the GLRT of size α is to reject H_0 if

$$\left(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right)^2 \geq \chi^2_\alpha(1).$$

Alternatively, we have....

<u>Case 2</u>: Leave things in the form of (2).

Here we find k_3 such that

$$P\left(\frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} \geq k_3 \text{ or } \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} \leq -k_3; \mu_0\right) = \alpha$$

Since $\mu = \mu_0$, this is equivalent to

$$P(Z \geq k_3 \text{ or } Z \leq -k_3) = \alpha$$

$$\Rightarrow \qquad k_3 = z_{\alpha/2}.$$

So, the GLRT of size α is to reject H_0 if

$$\frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} \geq z_{\alpha/2} \text{ or } \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} \leq -z_{\alpha/2}$$

If you used "Case 2", the GLRT is exactly the "common sense" two-tailed test from an earlier part of the course. Using "Case 1", we get the chi-squared test exactly without having to resort to asymptotics.

---

14. The joint pdf for X and Y is

$$f_{X,Y}(x,y) = \binom{n_1}{x} p_1^x (1-p_1)^{n_1-x} \cdot \binom{n_2}{y} p_2^y (1-p_2)^{n_2-y}$$

(a) The resctricted MLE:
We assume that $p_1 = p_2$ and denote the common value denoted simply by p. Then

$$f_{X,Y}(x,y) = \binom{n_1}{x}\binom{n_2}{y} p^{x+y}(1-p)^{n_1+n_2-(x+y)}$$

$$\Rightarrow \qquad L(p) = p^{x+y}(1-p)^{n_1+n_2-(x+y)}$$

$$\ln L(p) = (x+y)\ln p + (n_1 + n_2 - (x+y))\ln(1-p)$$

$$\frac{\partial}{\partial p} \ln L(p) = \frac{x+y}{p} - \frac{n_1 + n_2 - (x+y)}{1-p} \stackrel{set}{=} 0$$

$$\Rightarrow \qquad \hat{p}_0 = \frac{x+y}{n_1+n_2}$$

where $\hat{p}_0$ denotes the restricted MLE for p.
The unrestricted MLE's for p_1 and p_2:
Recall that the joint pdf for X and Y is

$$f_{X,Y}(x,y) = \binom{n_1}{x} p_1^x (1-p_1)^{n_1-x} \cdot \binom{n_2}{y} p_2^y (1-p_2)^{n_2-y}$$

So, a likelihood function is

$$L(p_1,p_2) = p_1^x (1-p_1)^{n_1-x} \cdot p_2^y (1-p_2)^{n_2-y}$$

and the log is

$$\ln L(p_1,p_2) = x \ln p_1 + (n_1-x)\ln(1-p_1) + y \cdot \ln p_2 + (n_2-y)\ln(1-p_2)$$

$$\begin{aligned} \tfrac{\partial}{\partial p_1} \ln L(p_1,p_2) &= \tfrac{x}{p_1} - \tfrac{n_1-x}{1-p_1} \overset{set}{=} 0 \\ \tfrac{\partial}{\partial p_2} \ln L(p_1,p_2) &= \tfrac{y}{p_2} - \tfrac{n_2-y}{1-p_2} \overset{set}{=} 0 \end{aligned}$$

$$\Rightarrow \qquad \hat{p}_1 = \frac{x}{n_1}, \ \ \hat{p}_2 = \frac{y}{n_2}$$

So, the GLR is

$$\lambda(\vec{x}) = \frac{\left(\frac{x+y}{n_1+n_2}\right)^{x+y} \left(1 - \frac{x+y}{n_1+n_2}\right)^{n_1+n_2-(x+y)}}{\left(\frac{x}{n_1}\right)^x (1-\left(\frac{x}{n_1}\right))^{n_1-x} \cdot \left(\frac{y}{n_2}\right)^y (1-\left(\frac{y}{n_2}\right))^{n_2-y}}$$

(b) The approximate large sample GLRT of size α is to reject H_0 if

$$-2\ln\lambda(\vec{X}) \geq \chi^2_\alpha(2)$$

(2 is the number of parameters restricted in the null hypothesis.)

---

15. We know that the unrestricted MLE is $\hat{\theta} = X_{(n)}$. The restricted MLE for this one point null hypothesis is simply $\hat{\theta}_0 = \theta_0$.

Since the likelihood is

$$L(\theta) = \frac{1}{\theta^n} I_{[0,\theta]}(x_{(n)}),$$

the GLR is

$$\begin{aligned}
\lambda(\vec{x}) &= \tfrac{L(\hat{\theta}_0)}{L(\hat{\theta})} \\
&= \tfrac{L(\theta_0)}{L(x_{(n)})} \\
&= \tfrac{\theta_0^{-n}\, I_{(0,\theta_0]}(x_{(n)})}{x_{(n)}^{-n}\, I_{(0,x_{(n)}]}(x_{(n)})} \\
&= \left(\tfrac{x_{(n)}}{\theta_0}\right)^n I_{(0,\theta_0]}(x_{(n)})
\end{aligned}$$

Under H_0, that indicator is 1 and so

$$-2\ln\lambda(\vec{X}) = -2\ln\left[\left(\frac{X_{(n)}}{\theta_0}\right)^n\right] = -2n\ln(X_{(n)}/\theta_0)$$

We know that, under H_0, $X_{(n)}$ has pdf

$$f_{X_{(n)}}(x) = \frac{n}{\theta_0^n} I_{(0,1)}(x).$$

So, we need the distribution of $Y = -2n\ln(X_{(n)}/\theta_0)$.

Letting $y = g(x) = -2m\ln(x/\theta_0)$, we have $x = g^{-1}(y) = \theta_0 e^{-y/(2n)}$ and so

$$\begin{aligned}
f_Y(y) &= f_{X_{(n)}}(g^{-1}(y)) \cdot \left|\tfrac{d}{dy} g^{-1}(y)\right| \\
&= \cdots \\
&= \tfrac{1}{2} e^{-w/2}\, I_{(0,\infty)}(w)
\end{aligned}$$

$$\Rightarrow Y = -2\ln\lambda(\vec{X}) \sim exp(rate = 1/2)$$

We want to reject H_0 when $\lambda(\vec{X})$ is below some k. This translates to rejecting if $-2\ln\lambda(\vec{X}) \geq k_1$ for some constant k_1. Since $\ln\lambda(\vec{X}) \sim exp(rate = 1/2)$, we can, using the cdf of the exponential find that k_1 must be $-2\ln\alpha$.

Alternatively since,

$$Y = -2\ln\lambda(\vec{X}) \sim exp(rate = 1/2) = \Gamma(1, 1/2) = \Gamma(2/2, 1/2) = \chi^2(2).$$

we can take $k_2 = \chi^2_\alpha(2)$. (This, necessarily, must be the same number as $-2\ln\alpha$.)

---

16. We think of the parameter λ as a random variable. I'll call it Λ. The joint pdf of the X's given that $\Lambda = \lambda$ is

$$f(\vec{x}|\lambda) = \frac{e^{-\lambda n}\lambda^{\sum x_i}}{\prod(x_i!)} \prod_{i=1}^{n} I_{\{0,1,2,\ldots\}}(x_i)$$

The prior pdf for Λ is

$$f_\Lambda(\lambda) = \frac{1}{\Gamma(\alpha)}\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda} I_{(0,\infty)}(\lambda).$$

Since the posterior pdf for Λ is proportional to the likelihood times the prior, we have the posterior pdf as

$$\begin{aligned}
f_{\Lambda|\vec{X}}(\lambda|\vec{x}) &\propto \tfrac{e^{\lambda n}\lambda^{\sum x_i}}{\prod(x_i!)} \textstyle\prod_{i=1}^{n} I_{\{0,1,2,\ldots\}}(x_i) \cdot \tfrac{1}{\Gamma(\alpha)}\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda} I_{(0,\infty)}(\lambda) \\
&\propto e^{-\lambda n}\lambda^{\sum x_i}\lambda^{\alpha-1}e^{-\beta\lambda} I_{(0,\infty)}(\lambda) \\
&= \lambda^{\sum x_i+\alpha-1}e^{-(\beta+n)\lambda}
\end{aligned}$$

We recognize this posterior pdf (with missing constants) as that of the $\Gamma(\sum x_i + \alpha, \beta + n)$ distribution. That is,

$$\Lambda|\vec{X} \sim \Gamma(\textstyle\sum X_i + \alpha, \beta + n)$$

and the posterior Bayes estimator of λ is

$$\widehat{\lambda}_{PBE} = \mathsf{E}[\Lambda|\vec{X}] = \frac{\sum X_i + \alpha}{\beta + n}.$$

---

17. We think of the parameter μ as a random variable. I'll call it M. The joint pdf of the X's given that $M = \mu$ is

$$f(\vec{x}|\mu) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}.$$

The prior pdf for μ is

$$f_M(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2}.$$

Since the posterior pdf for M is proportional to the likelihood times the prior, we have the posterior pdf as

$$\begin{aligned}
f_{M|\vec{X}}(\mu|\vec{x}) &\propto e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2} \cdot e^{-\frac{1}{2}\mu^2} \\
&\propto e^{-\frac{1}{2\sigma^2}(-2\mu\sum x_i + n\mu^2)} \cdot e^{-\frac{1}{2}\mu^2} \\
&= e^{-a\mu^2 + b\mu}
\end{aligned}$$

where $a = \frac{n}{2\sigma^2} + \frac{1}{2} = \frac{n+\sigma^2}{2\sigma^2}$ and $b = \frac{\sum x_i}{\sigma^2}$.

Completing the square on that exponent gives

$$
\begin{aligned}
f_{M|\vec{X}}(\mu|\vec{x}) &\propto \exp\left[-a(\mu - \tfrac{b}{2a})^2\right] \cdot \exp\left[\tfrac{b^2}{4a^2}\right] \\
&= \exp\left[-a(\mu - \tfrac{b}{2a})^2\right] \\
&= \exp\left[-\tfrac{n+\sigma^2}{2\sigma^2}(\mu - \tfrac{\sum x_i}{n+\sigma^2})^2\right]
\end{aligned}
$$

We recognize this posterior pdf (with missing constants) as another normal distribution. Specifically

$$
M|\vec{X} \sim N\left(\frac{\sum X_i}{n+\sigma^2}, \frac{\sigma^2}{n+\sigma^2}\right).
$$

This, the posterior Bayes estimator of μ is

$$
\widehat{\mu}_{PBE} = \frac{\sum X_i}{n+\sigma^2}.
$$