Derivation of the Uniform Series Annuity Equations

Consider the cash flow diagram in Figure 1. By definition, a series of uniform cash flows coming at the end of each time period are known as Annuities (or more specifically, Ordinary Annuities).

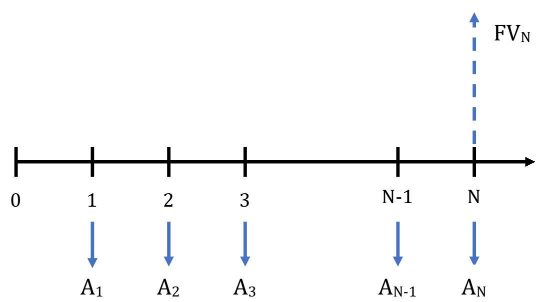

Figure 1. A uniform series of cash flows.

Our goal is to derive an analytical expression that relates the annuity cash flows, A, to the future value of the cash flows "N" time periods in the future and at an interest rate, i.

The way to do this is to treat each "A" as an individual cash flow and using our expression for single payment cash flows, $F = P(1+i)^N$, and then sum these all together.

For example, consider the cash flow in Figure 1. If we break it up into single cash flows, then treat each "A" as the "P" in our single payment expression, then we see how things add up.

By definition, ordinary annuities have cash flows that start at the end of the first year, and then continue at the end of each subsequent year, including the final year, N.

We know that for an investment that starts at time = 0 (e.g., the beginning of the first year), we have:

$$FV = PV(1+i)^N$$

But as annuities start at the end of the first year, (or equally, at the beginning of Year 2), therefore we would anticipate that the contribution of this first payment to the Future Value, FV, is just:

$$FV_{A1} = A_1 (1+i)^{N-1}$$

The second annuity payment comes at the end of year 2 (or equally, the beginning of Year 3), therefore, the contribution from this payment to the future value is:

$$FV_{A2} = A_2 (1+i)^{N-2}$$

If you carry this through, then the final payment at the end of year N would contribute to the future value by:

$$FV_{AN} = A_N (1+i)^{N-N}, \text{ or just } A_N$$

The total future value is just the sum of the contributions from the individual annuity payments:

$$FV_N = FV_{A1} + FV_{A2} + FV_{A3} + \ldots + FV_{AN}$$

Which after substituting our expression for each year, becomes:

$$FV_N = [A_1\,(1+i)^{N-1}] + [A_2\,(1+i)^{N-2}] + [A_3\,(1+i)^{N-3}] + \ldots + A_N$$

Now by definition of a uniform series of payments, all the A's are exactly the same:

$$A_1 = A_2 = A_3 = \ldots = A_N.$$

Therefore:

$$FV_N = A(1+i)^{N-1} + A(1+i)^{N-2} + A(1+i)^{N-3} + \ldots + A(1+i)^3 + A(1+i)^2 + A(1+i) + A$$

If we multiply both sides by (1+i):

$$(1+i)\,FV_N = A(1+i)^N + A(1+i)^{N-1} + A(1+i)^{N-2} + \ldots + A(1+i)^3 + A(1+i)^2 + A(1+i)$$

then subtract the former equation from the latter one, we end up with:

$$(1+i)FV_N - FV_N = A(1+i)^N - A$$

which, upon simplifying becomes:

$$FV_N + i\,FV_N - FV_N = A\,[(1+i)^N - 1]$$

Eliminating terms and simplifying once again yields our final expression:

$$\boxed{FV_N = A\left[\frac{(1+i)^N - 1}{i}\right]}$$

It is now easy to flip the equation around to get the required Annuity, A, to achieve a future value, FV, given an interest rate, i, and period of time N:

$$\boxed{A = FV_N\left[\frac{i}{(1+i)^N - 1}\right]}$$