ECE 603
Probability and Random Processes

Lesson 17
Chapter 7
Limit Theorems and Convergence of Random Variables

Objectives

- Examine the use of Law of Large Numbers (LLN).
- Examine the use of Central Limit Theorem (CLT).

Rationale

This lesson will focus on limit theorems and convergence modes for random variables. Limit theorems are among the most fundamental results in probability theory.

You will explore how these theorems are applied in practice.

Prior Learning

- Basic Concepts
- Counting Methods
- Random Variables
- Access to the online textbook: https://www.probabilitycourse.com/

Summary of Probability Bounds

Markov's Inequality

If X is any nonnegative random variable, then

$$P(X \geq a) \leq \frac{EX}{a}, \quad \text{for any } a > 0.$$

Summary of Probability Bounds

Chebyshev's Inequality

For any random variable X, with $EX = \mu$ and $\mathrm{Var}(X) = \sigma^2$, we have

$$P\big(|X - \mu| \geq \epsilon\big) \leq \frac{Var(X)}{\epsilon^2}.$$

$$||$$

$$P(\underbrace{\mu - \epsilon}_{a} < X < \underbrace{\mu + \epsilon}_{b})$$

Law of Large Numbers

Definition. For i.i.d. random variables $X_1, X_2, \ldots, X_n$ with $EX_i = \mu_i$ and $\mathrm{Var}(X_i) = \sigma_i^2$, the sample mean, denoted by $\overline{X}$, is defined as

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

Law of Large Numbers

The sample mean, $\overline{X}$, is also a random variable, then we have

$$
\begin{aligned}
E[\overline{X}] &= \frac{EX_1 + EX_2 + \dots + EX_n}{n} && \text{(by linearity of expectation)} \\
&= \frac{nEX}{n} && (\text{since } EX_i = EX) \\
&= EX.
\end{aligned}
$$

Law of Large Numbers

The variance of $\overline{X}$ is given by

$$\begin{aligned}
\mathrm{Var}(\overline{X}) &= \frac{\mathrm{Var}(X_1 + X_2 + \ldots + X_n)}{n^2} && (\mathrm{Var}(aX) = a^2\mathrm{Var}(X)) \\
&= \frac{\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)}{n^2} && (X_i\text{'s are independent}) \\
&= \frac{n\mathrm{Var}(X)}{n^2} && (\mathrm{Var}(X_i) = \mathrm{Var}(X)) \\
&= \frac{\mathrm{Var}(X)}{n} = \frac{\sigma^2}{n}.
\end{aligned}$$

Law of Large Numbers

The weak law of large numbers (WLLN)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a finite expected value $EX_i = \mu < \infty$. Then, for any $\epsilon > 0$,

$$\lim_{n \to \infty} P(|\overline{X} - \mu| \geq \epsilon) = 0.$$

Law of Large Numbers

Proof:

We assume $\mathrm{Var}(X) = \sigma^2$ is finite. In this case we can use Chebyshev's inequality to write

$$P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\mathrm{Var}(\overline{X})}{\epsilon^2}$$

$$= \frac{\mathrm{Var}(X)}{n\epsilon^2},$$

which goes to zero as $n \to \infty$.

Central Limit Theorem

Note:

if $EX = \mu,\ \mathrm{Var}(X) = \sigma^2$ **and the normalized random variable is defined:**

$$Z = \frac{X - \mu}{\sigma},$$

then,

$$EZ = 0,\ \mathrm{Var}(Z) = 1.$$

Central Limit Theorem

Proof:

$$EZ = \frac{EX - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0,$$

$$\mathrm{Var}(Z) = \frac{\mathrm{Var}(X)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1.$$

Central Limit Theorem

The Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with expected value $EX_i = \mu < \infty$ and variance $0 < \mathrm{Var}(X_i) = \sigma^2 < \infty$. Then,

$$Z_n = \frac{\overline{X} - E\overline{X}}{\sqrt{\mathrm{Var}(\overline{X})}} = \frac{\sum_{i=1}^{n} X_i/n - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma},$$

Central Limit Theorem

Converges in distribution to the standard normal random variable as n goes to infinity, that is

$$\lim_{n\to\infty} P(Z_n \leq x) = \Phi(x), \qquad \text{for all } x \in \mathbb{R},$$

Where $\Phi(x)$ is the standard normal CDF.

Note: This true regardless of the distribution of $\boldsymbol{X}$.

Central Limit Theorem

Example. Let $X_i \sim Uniform(a, b)$ and $\boldsymbol{Y_n = X_1, X_2, \cdots, X_n}$.

Central Limit Theorem

In practice n finite, but we still can approximate $Y = X_1, X_2, \cdots, X_n$ by a Normal random variable.

Central Limit Theorem

Two steps to solve problems using CLT:

$$Y = X_1, X_2, \cdots, X_n, \quad X_i \text{ i.i.d.}$$

a) Find $EY = \mu_Y = \sum_{i=1}^{n} EX_i$, and $\mathrm{Var}(Y) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$.

b) Use $Y_n \sim N\left(\mu_{Y_n}, \mathrm{Var}(Y_n)\right)$, so we can use Φ function.

Orchestrated Conversation: Central Limit Theorem

Example.

In a digital communication system $n = 1000$ bits are transmitted over a wireless channel. Each bit will be received in error with probability $P_e = 0.1$ (error probability) independently from other bits. Let W be the number of error.

a) Find $P\{W > 125\}$.

b) Find $P\{75 < W < 125\}$.

Central Limit Theorem

$$Y = X_1, X_2, \cdots, X_n, \quad X_i \text{ i.i.d.}$$

$$EX_i = \mu, \ \mathrm{Var}(X_i) = \sigma^2$$

$$EY = n\mu, \ \ \mathrm{Var}(Y) = n\sigma^2 \ \xrightarrow{CLT} \ Y \sim N(n\mu, n\sigma^2)$$

Orchestrated Conversation: Central Limit Theorem

Example.

A multiple choice test has 100 questions, each with four alternative. At least 80 correct answers are required for a passing grand. On each question, you know the correct answer with probability 0.75, otherwise you guess at random. There is no penalty for wrong answers.

Post-work for Lesson

- Complete homework assignment for Lesson 17:

HW#9

Go to the online classroom for details.

To Prepare for the Next Lesson

- Read Chapter 10 in your online textbook:

  https://www.probabilitycourse.com/chapter10/10_1_0_basic_concepts.php

- Complete the Pre-work for Lessons 18-20.

Visit the online classroom for details.