

# ECE 603

# Probability and Random

# Processes

Lesson 4

Chapter 2

Counting Methods



# Objectives

- **Examine counting as a result of the multiplication principle.**
- **Apply the basic introductions of the material to probability.**

# Rationale

- **Counting is necessary for solving some probability problems. This lesson will focus on methods for counting elements in an efficient manner.**
- **Almost everything you need to know about counting comes from the multiplication principle.**
- **This lesson will take what you previously reviewed about the Cartesian viewpoint and explore a different perspective.**

# Prior Learning

- Basic Concepts
- Access to the online textbook: <https://www.probabilitycourse.com/>

# Counting Methods

For a finite sample space  $S$  with equally likely outcomes, the probability of an event  $A$  is given by

$$P(A) = \frac{|A|}{|S|} = \frac{M}{N}$$

# Counting Methods

## Multiplication Principle:

If we are to perform  $r$  experiments in order such that there are  $n_1$  possible outcomes of the first experiment,  $n_2$  possible outcomes of the second experiment,  $\dots$ ,  $n_r$  possible outcomes of the  $r^{th}$  experiment, then there is a total of  $n_1 \times n_2 \times n_3 \times \dots \times n_r$  outcomes of the sequence of the  $r$  experiments.

# Counting Methods

## Example:

**A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 Seniors. A subcommittee of one person from each class is to be chosen. How many different subcommittees are possible?**

# Counting Methods

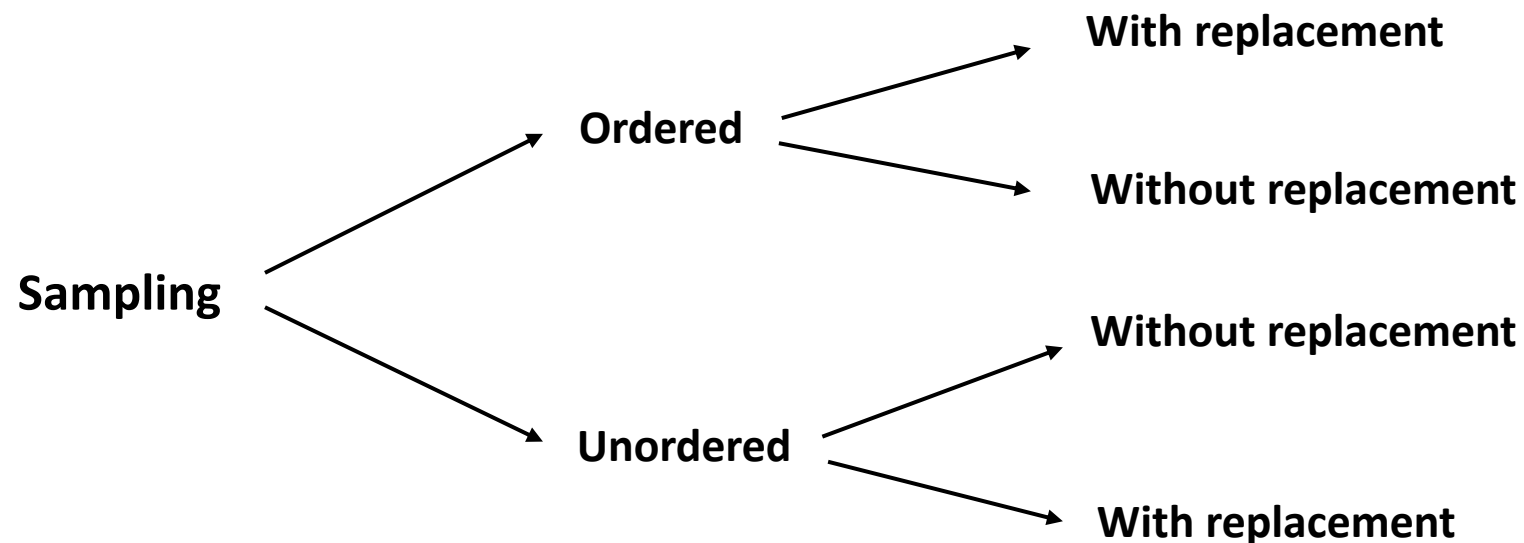
- Drawing (choosing) objects from a set  $A = \{a_1, a_2, \dots, a_n\}$  is referred to as sampling.
- We will often draw multiple samples from a set. If we put the object back after each draw, this is called **sampling with replacement**; if not it is called **sampling without replacement**.
- The result of drawing multiple samples can be **ordered** (order of draws matters;  $1, 2, 3 \neq 2, 3, 1$ ) or **unordered** ( $1, 2, 3 = 2, 3, 1$ ).



# Counting Methods

## General scenario:

We have a set of  $n$  elements, e.g.,  $A = \{1, 2, \dots, n\}$  and we draw  $k$  samples from the set:



# Counting Methods

Remember:  $n! = n \times (n - 1) \times \cdots \times 1$

$$\text{e.g., } 3! = 3 \times 2 \times 1 = 6$$

# Counting Methods

For  $A = \{1, 2, 3\}$ ,  $k = 2$

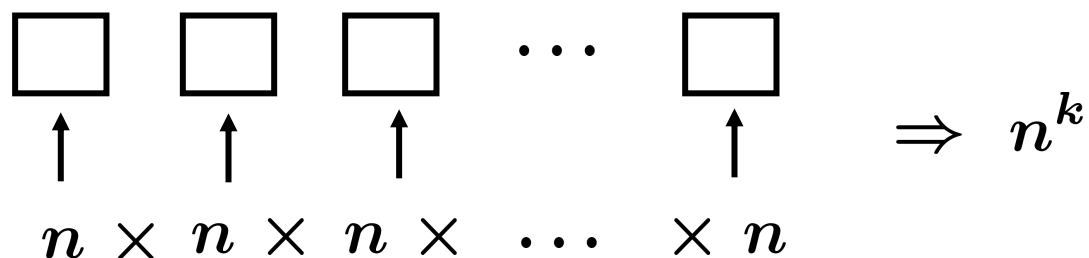
## 1) Ordered Sampling with Replacement (repetition allowed)

$(1, 1)$   $(2, 1)$   $(3, 1)$

$(1, 2)$   $(2, 2)$   $(3, 2)$   $\longrightarrow$  9 Possibilities

$(1, 3)$   $(2, 3)$   $(3, 3)$

In general:  $A = \{1, 2, \dots, n\}$



# Orchestrated Conversation: Counting Methods

## Example:

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

# Counting Methods

## 2) Ordered Sampling without Replacement (repetition not allowed)

$$A = \{1, 2, 3\}, k = 2$$

$$\begin{array}{l} (1, 2) \quad (2, 1) \quad (3, 1) \\ (1, 3) \quad (2, 3) \quad (3, 2) \end{array} \longrightarrow 6 \text{ Possibilities}$$

In general:  $A = \{1, 2, \dots, n\}$

$$\begin{array}{ccccccc} \square & \square & \square & \dots & \square \\ \uparrow & \uparrow & \uparrow & & \uparrow \\ n \times & (n-1) \times & (n-2) \times & \dots & \times (n-k+1) \end{array}$$

# Counting Methods

Number of  $k$ -permutations of  $n$ -objects:

$$P_k^n = n \times (n - 1) \times \dots \times (n - k + 1) = \frac{n!}{(n - k)!}.$$

The number of  $k$ -permutations of  $n$  distinguishable objects is given by

$$P_k^n = \frac{n!}{(n - k)!}, \text{ for } 0 \leq k \leq n.$$

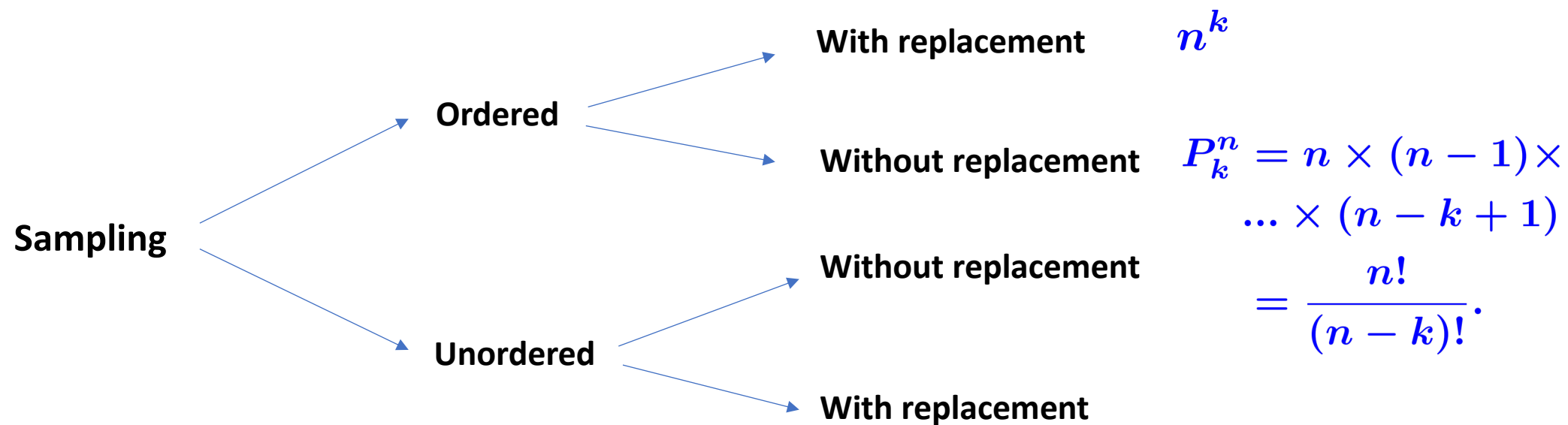
# Orchestrated Conversation: Counting Methods

## Example:

(Birthday Paradox) In a group of  $k$  people, what is the probability that at least two have the same birthday?

# Counting Methods

Sample of size  $k$  from  $A = \{1, 2, \dots, n\}$





# Counting Methods

## Unordered Sampling without Replacement (Combinations):

There are  $n$  distinguishable objects; we want to choose  $k$  objects, but ordering does not matter:  $1, 2, 3 = 2, 3, 1 = 3, 2, 1$ .

Let  $A = \{1, 2, 3\}$  and  $k = 2$ , then

$\{1, 2\} \quad \{1, 3\} \quad \{2, 3\} \longrightarrow 3$  possibilities

# Counting Methods

In general:

$\binom{n}{k}$  : # of ways to choose  $k$  elements from  $n$  elements (Unordered):  $k$ -Combinations

If ordered:  $P_k^n = \frac{n!}{(n-k)!} = k! \binom{n}{k}$ .

If unordered:  $\binom{n}{k} = \frac{P_k^n}{k!} = \frac{n!}{k!(n-k)!}$ .

# Counting Methods

Thus the number of  $k$  -combinations of  $n$  objects is:

$$\binom{n}{k} = \frac{(n)k}{k!} = \frac{n!}{k!(n-k)!}.$$

The number of ways to choose  $k$  objects out of  $n$  distinguishable objects is equal to  $\binom{n}{k}$ .

# Counting Methods

**Example.**

The number of five-card poker hands is  $\binom{52}{5}$ .

The number of  $k$ -combinations of an  $n$ -element set is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } 0 \leq k \leq n.$$

# Orchestrated Conversation: Counting Methods

## Example.

A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

# Orchestrated Conversation: Counting Methods

Another interpretation of  $\binom{n}{k}$  :

- The number of possible divisions of  $n$  distinct objects to two groups of sets of sizes  $k$  and  $n - k$  is also equal to  $\binom{n}{k}$ .

**Example:** We toss a coin 5 times and observe the sequence of heads and tails. How many different outcomes are possible if we know two tails and three heads have been observed?

# Counting Methods

- The number of observation sequences for  $n$  sub-experiments with the sample space  $S = \{0, 1\}$  (or  $\{T, H\}$ ) with 0 appearing  $n_0$  times and 1 appearing  $n_1 = n - n_0$  times is  $\binom{n}{n_0}$ .

**Example.** How many distinct sequences can we make using 3 As and 5 Bs?  
(AAABBBBB, AABABBBB, ....)

# Orchestrated Conversation: Counting Methods

**Example.** We toss a coin  $n$  times and observe the sequence of heads and tails. How many different outcomes are possible if we know  $n_0$  tails and  $n_1 = n - n_0$  heads have been observed?



# Counting Methods

**Multinomial Coefficients:** More generally if  $n = n_1 + n_2 + \dots + n_r$ , we define

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}.$$

$\binom{n}{n_1, n_2, \dots, n_r}$  is the number of possible divisions of  $n$  distinct objects into  $r$  distinct groups of respective sizes  $n_1, n_2, \dots, n_r$ .

# Counting Methods

**Theorem.** For  $n$  repetitions of sub-experiment with sample space

$$S = \{s_0, s_1, \dots, s_{m-1}\},$$

the number of length  $n = n_0 + n_1 + \dots + n_{m-1}$  observation sequences with  $s_i$  appearing  $n_i$  times is

$$\binom{n}{n_0, n_1, \dots, n_{m-1}}.$$

# Counting Methods

## Bernoulli Trials:

**Example.** We toss an unfair coin ( $P(H) = p$ )  $n$  times. What is the probability of observing  $k$  heads?

# Counting Methods

## Binomial Formula:

For  $n$  independent Bernoulli trials where each trial has success probability  $p$ , the probability of  $k$  successes is given by

$$P(k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

# Counting Methods

Generally, assume the sub-experiment has sample space  $S = \{s_0, s_1, \dots, s_{m-1}\}$ , with  $P(\{s_i\}) = p_i$ . For  $n = n_0 + n_1 + \dots + n_{m-1}$  **independent** trials, the probability that  $s_i$  appears  $n_i$  times for all  $i \in \{0, 1, \dots, m-1\}$  is

$$P(s_0, s_1, \dots, s_{m-1}) = \binom{n}{n_0, n_1, \dots, n_{m-1}} p_0^{n_0} p_1^{n_1} \dots p_{m-1}^{n_{m-1}}.$$

# Counting Methods

## Unordered Sampling with Replacement (repetition allowed):

**Example:**  $A = \{1, 2, 3\}$ ,  $n = 3$ ,  $k = 2$

(1, 1) (2, 2)

(1, 2) (2, 3)  $\longrightarrow$  6 Cases.

(1, 3) (3, 3)

# Counting Methods

## Lemma.

The total number of distinct  $k$  samples from an  $n$ -element set such that repetition is allowed and ordering does not matter is the same as the number of distinct solutions to the equation

$$x_1 + x_2 + \dots + x_n = k, \text{ where } x_i \in \{0, 1, 2, 3, \dots\}.$$

$$\binom{n - k + 1}{k} = \binom{n - k + 1}{n - 1}.$$

# Review

Let's summarize the formulas for the four categories of sampling.

ordered sampling with replacement	$n^k$
ordered sampling without replacement	$P_k^n = \frac{n!}{(n-k)!}$
unordered sampling without replacement	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
unordered sampling with replacement	$\binom{n+k-1}{k}$



# Summary of this Lesson

- You examined the necessity of counting for solving some probability problems. You also focused on methods for counting elements in an efficient manner.

# Post-work for this Lesson

- Complete the homework assignment for Lesson 4: HW#2

**Go to the online classroom for details.**

# To Prepare for the Next Lesson

- Read Chapter 3 in your online textbook:

[https://www.probabilitycourse.com/chapter3/3\\_1\\_1\\_random\\_variables.php](https://www.probabilitycourse.com/chapter3/3_1_1_random_variables.php)

- Complete the Pre-work for Lessons 5-6.

**Visit the online classroom for details.**