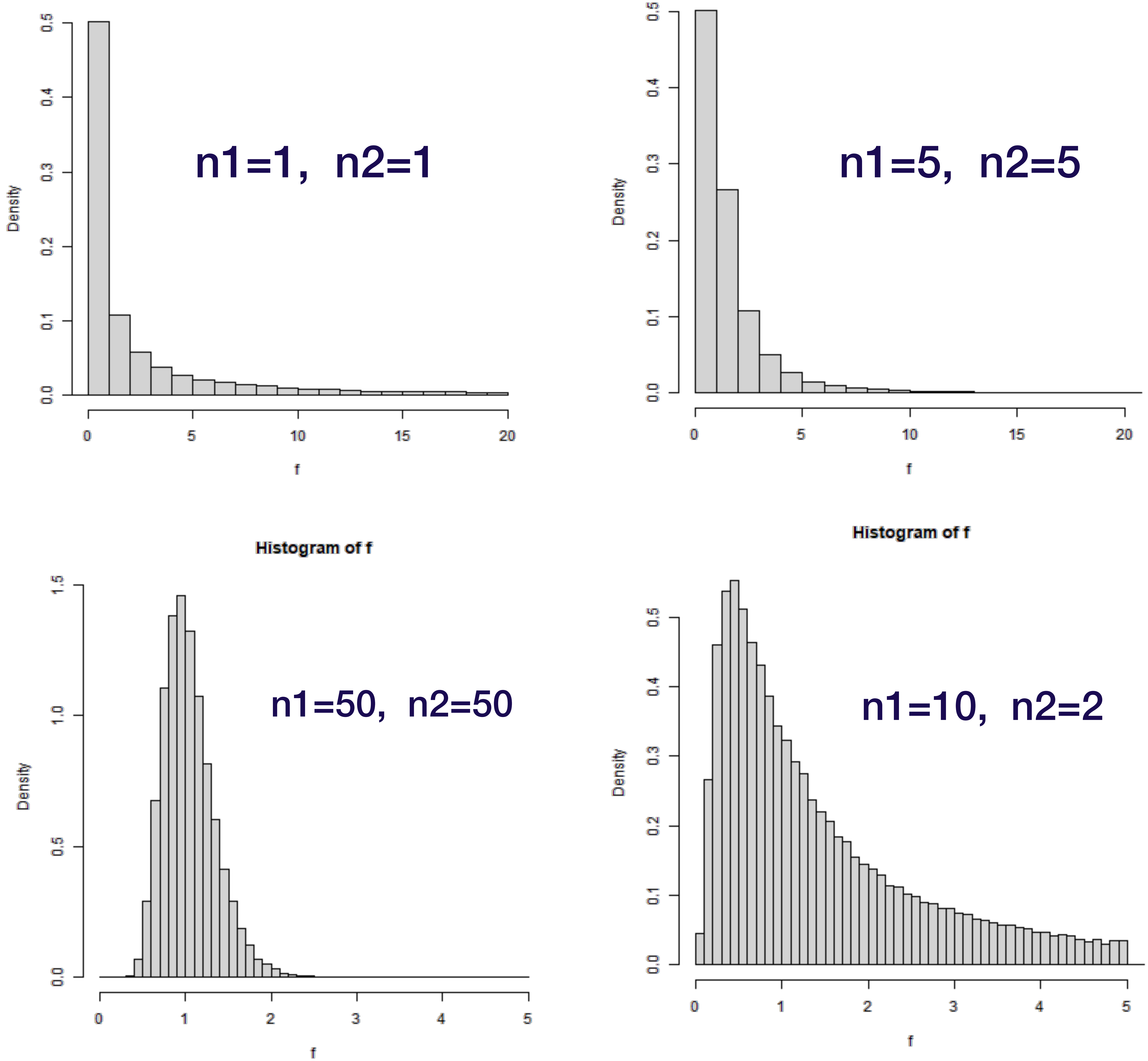
Suppose that X_1 and X_2 are independent random variables with

$$X_1 \sim \chi^2(n_1) \quad \text{and} \quad X_2 \sim \chi^2(n_2)$$

Define a new random variable

$$F = \frac{X_1/n_1}{X_2/n_2}$$



n1=1, n2=1
Density
0.0
0.1
0.2
0.3
0.4
0.5
0
5
10
15
20
f
n1=5, n2=5
Density
0.0
0.1
0.2
0.3
0.4
0.5
0
5
10
15
20
f
Histogram of f
n1=50, n2=50
Density
0.0
0.5
1.0
1.5
0
1
2
3
4
5
f
Histogram of f
n1=10, n2=2
Density
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
f

pdf:

$$f(x; n_1, n_2) =$$

$$\frac{1}{B(n_1/2, n_2/2)} \left(\frac{n_1}{n_2}\right)^{n_1/2} x^{n_1/2-1} \left(1 + \frac{n_1}{n_2} x\right)^{-(n_1+n_2)/2}$$

for x>0.

mean: $\frac{n_2}{n_2 - 2}$ **if** $n_2 > 2$

variance: $\frac{2n_2^2 (n_1 + n_2 - 2)}{n_1(n_2 - 2)^2(n_2 - 4)}$

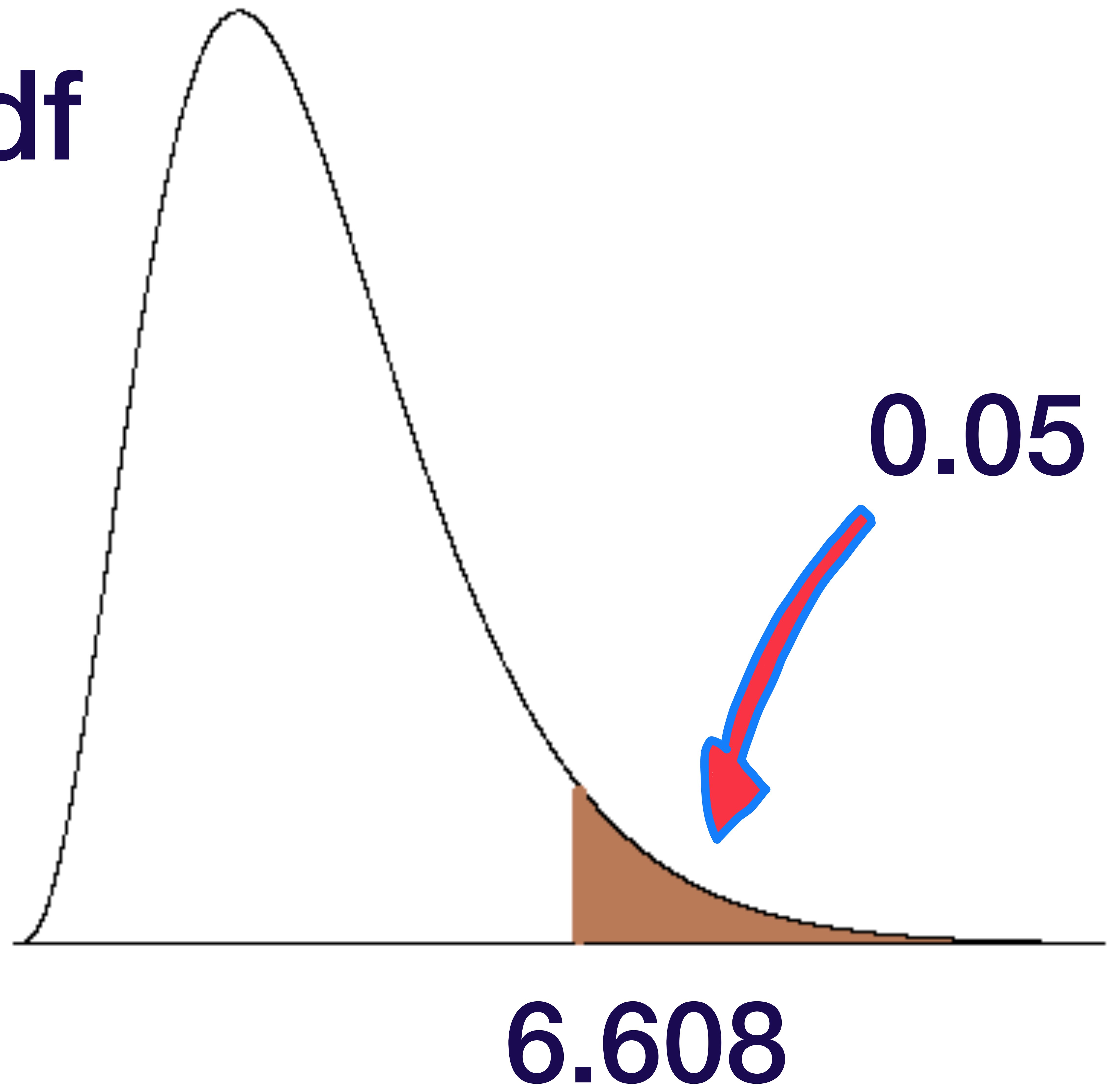
if $n_2 > 4$

In R:

qf(0.95,5,1) = 6.608

pf(6.608,5,1) = 0.9499824

F(5,1) pdf



The Mean:

$$E[F] = E\left[\frac{X_1/n_1}{X_2/n_2}\right] = \frac{n_2}{n_1} E\left[\frac{X_1}{X_2}\right]$$

$$\overset{\text{indep}}{=} \frac{n_2}{n_1} \underbrace{E[X_1]}_{n_1} \cdot E\left[\frac{1}{X_2}\right]$$

$$= n_2 E\left[\frac{1}{X_2}\right]$$

$$= n_2\, E\left[\frac{1}{X_2}\right] = n_2 \int_{-\infty}^{\infty} \frac{1}{x}\, f_{X_2}(x)\, dx$$

$$= n_2 \int_0^{\infty} \frac{1}{x} \frac{1}{\Gamma(n_2/2)} \left(\frac{1}{2}\right)^{n_2/2} x^{n_2/2-1}\, e^{-x/2}\, dx$$

$$= n_2 \int_0^{\infty} \frac{1}{\Gamma(n_2/2)} \left(\frac{1}{2}\right)^{n_2/2} \underbrace{x^{n_2/2-2}\, e^{-x/2}}_{\text{like a } \Gamma(n/2-1,1/2) \text{ pdf}}\, dx$$

$$= n_2 \frac{\Gamma(n_2/2-1)}{\Gamma(n_2/2)} \frac{1}{2} \cdot$$

$$\underbrace{\int_0^\infty \frac{1}{\Gamma(n_2/2-1)} \left(\frac{1}{2}\right)^{n_2/2-1} x^{n_2/2-2} e^{-x/2} dx}_{1}$$

$$= n_2 \frac{\Gamma(n_2/2-1)}{(n_2/2-1)\Gamma(n_2/2-1)} \frac{1}{2}$$

$$= \frac{n_2}{n_2-2}$$

And the point is... ?

- Suppose that $X_{11}, X_{12}, \ldots, X_{1,n_1}$ is a random sample of size n_1 from the $N(\mu_1, \sigma_1^2)$.
- Suppose that $X_{21}, X_{22}, \ldots, X_{2,n_2}$ is an independent random sample of size n_2 from the $N(\mu_2, \sigma_2^2)$.

Find a $100(1-\alpha)\%$ confidence interval for the ratio σ_1^2/σ_2^2.

Let S_1^2 and S_2^2 be the sample variances for the first and second samples, respectively.

We know that

$$\frac{(n_1 - 1)\, S_1^2}{\sigma_1^2} \sim \chi^2(n_1 - 1)$$

and

$$\frac{(n_2 - 1)\, S_2^2}{\sigma_2^2} \sim \chi^2(n_2 - 1)$$

are independent

So, define an statistic F as

$$F := \frac{\sigma_2^2}{\sigma_1^2} \cdot \frac{S_1^2}{S_2^2} = \frac{[(n_1 - 1)S_1^2/\sigma_1^2]/(n_1 - 1)}{[(n_2 - 1)S_2^2/\sigma_2^2]/(n_2 - 1)}$$

Then

$$F \sim F(n_1 - 1, n_2 - 1)$$

Example:

Fifth grade students from two neighboring counties took a placement exam.

Group 1, from County A, consisted of 18 students. The sample mean score for these students was 77.2.

Group 2, from County B, consisted of 15 students and had a sample mean score of 75.3.

Example:

From previous years of data, it is believed that the scores for both counties are normally distributed, and that the variances of scores from Counties A and B, respectively, are 15.3 and 19.7.

You wish to create a confidence interval for $\mu_1 - \mu_2$, the difference between the true population means.

Example:

You are thinking of using a pooled-variance two-sample t-test, however this requires that the true population variances, σ_1^2 and σ_2^2 are the same.

Find a 99% confidence interval for the ratio σ_1^2/σ_2^2. From your results, do you think it is plausible that $\sigma_1^2 = \sigma_2^2$?

Example:

$$n_1 = 18, \quad s_1^2 = 15.3$$

$$n_2 = 15, \quad s_1^2 = 19.7$$

$$F := \frac{\sigma_2^2}{\sigma_1^2} \cdot \frac{S_1^2}{S_2^2} = \frac{15.3}{19.7} \frac{\sigma_2^2}{\sigma_1^2}$$

Critical values:

$$F_{0.005,17,14} = 3.98267$$

$$F_{0.995,17,14} = 1.00217$$

Example:

$$1.00217 < \frac{15.3}{19.7}\frac{\sigma_2^2}{\sigma_1^2} < 3.98267$$

$$\Downarrow$$

A 99% confidence interval for σ_1^2/σ_2^2 is given by (0.19501,0.77497).

Since this interval doesn't include 1, it does not seem plausible that $\sigma_1^2 = \sigma_2^2$ at the 99% level.

It instead seems that $\sigma_1^2 < \sigma_2^2$.