Time Response Analysis

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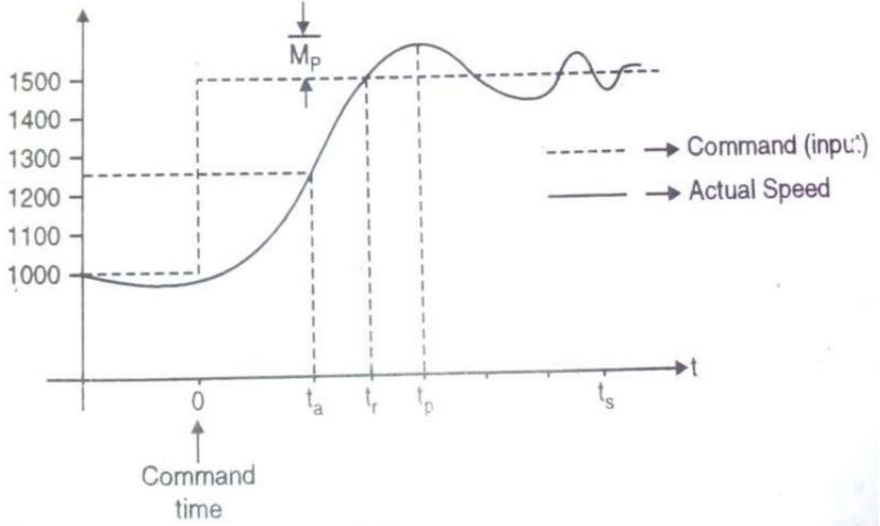
Time domain analysis

The response of the system to an applied excitation is called **Time Response** and it is function of c(t).

Example:The response of motor’s speed when a command is given to increase the speed.



As seen from figure, the motor’s speed gradually picks up from 1000 rpm and moves towards 1500 rpm.It overshoots and again corrects itself and finally settles down at the last value.

Generally speaking, the response of any system thus has two parts

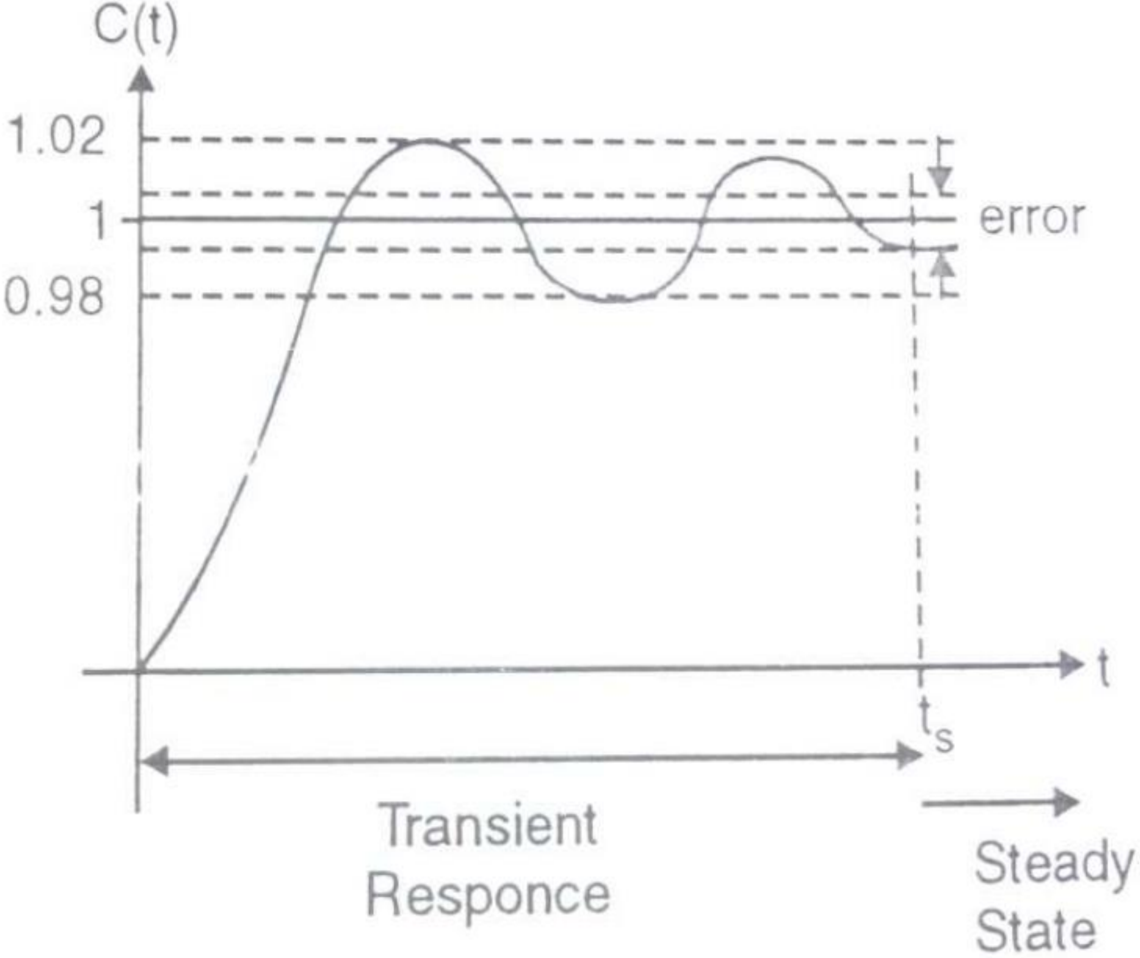
1. Transient Response
2. Steady state Response

Transient Response

- The part of the response that goes to zero as time becomes very large is called as Transient Response.
- As the name suggests that transient response remains only for some time from initial state to final state.

Steady State Response

- The part of the response that remains after the transients have died out is called Steady State Response


C(t)
1.02
1
0.98
error
t
t_s
Transient
Responce
Steady
State

Need of standard test inputs

- The most of the control systems do not know what their inputs are going to be.
- Thus system design cannot be done from input point of view as we are unable to know in advance the type of input.

The input may be

1. A sudden change
2. A momentary shock
3. A constant velocity
4. A constant acceleration

Hence these signals form standard test signals.The response to these signal is analyzed. The above inputs are called as,

1. Step input-Signifies a sudden change
2. Impulse input-Signifies momentary shock
3. Ramp input-Signifies a constant velocity
4. Parabolic input-Signifies constant acceleration

Thus, the output of systems is tested by using this standard test signals.

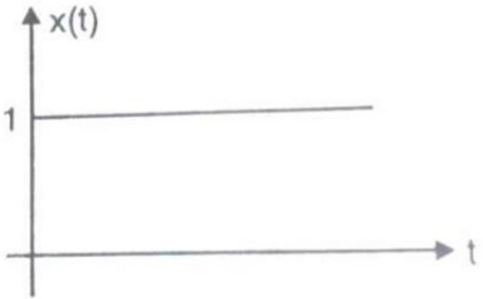
Unit Step Input

Mathematical Representation

x(t)=u(t)   for t>0

=0   for t>0

Graphical Representation



Laplace Representations-

$$L\{u(t)\} = \frac{1}{s}$$

Note-This signal signifies a s e reference input x(t) at t=0 and it is a DC signal

Ramp Input

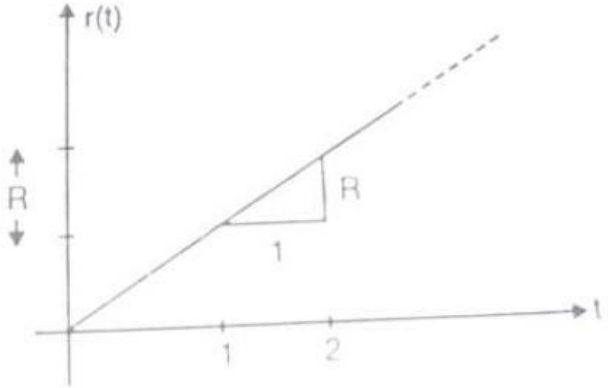
Mathematical Representation

Representation

x(t)=R.t for t>0

=0 for t<0

Graphical



Laplace Representation- $L\{\mathrm{Rt}\} = \dfrac{R}{s^2}$

Note-Signal have constant velocity i.e. constant change in it's value w.r.t. time

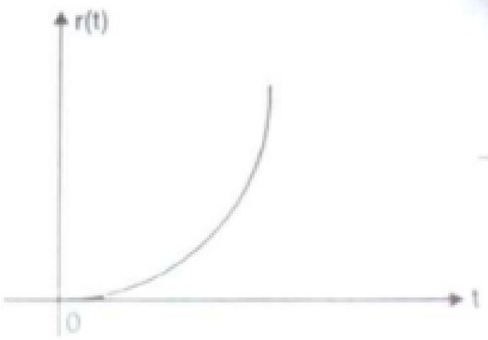
Parabolic Input

Mathematical Representation

x(t)= $\frac{Rt^2}{2}$ for t>0

= 0 for t<0

Graphical Representation



Laplace Representation- $L\{Rt\} = \frac{R}{s^3}$

Note-This type of signal exponentially changes w.r.t. time.

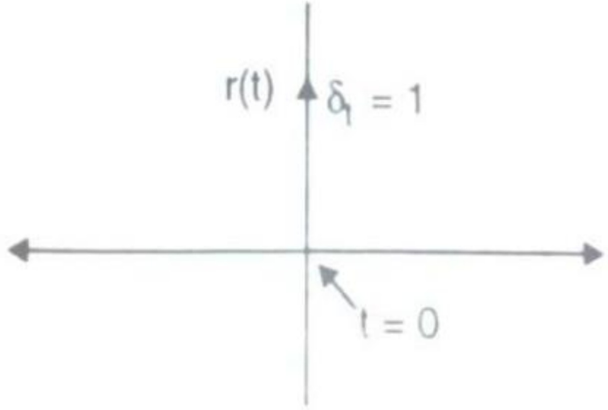
Impulse Input

Mathematical Representation

$x(t) = \delta(t) = 1$ for $t=0$

$= 0$ for $t<0$ & $t>0$

Graphical Representation



Laplace Representation- $L\{\delta(t)\} = 1$

Note-The signals has unit value only at t=zero.

Poles & Zeros of Transfer Function

The transfer function is given by,

$$G(s) = \frac{C(s)}{R(s)}$$

Both C(s) and R(s) are polynomials in s

$$\therefore G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots\ldots\ldots\ldots + b_o}{s^n + a_{n-1} s^{n-1} + \ldots\ldots\ldots\ldots\ldots\ldots + a_n}$$

$$= \frac{K(s-b_1)(s-b_2)(s-b_3)\ldots\ldots\ldots\ldots(s-b_m)}{(s-a_1)(s-a_2)(s-a_3)\ldots\ldots\ldots\ldots(s-a_n)}$$

Where, K= system gain

n= Type of system

Poles

The value of **s** for which transfer function magnitude |G(S)| becomes infinite after substitution in the denominator of system are called as **poles** of transfer function.

Example 1

Determine the poles of given transfer function.

$$G(s) = \frac{s(s+2)(s+4)}{s(s+3)(s+4)}$$

Solution: The poles can be obtained by equating denominator with zero

$$s(s+3)(s+4) = 0$$

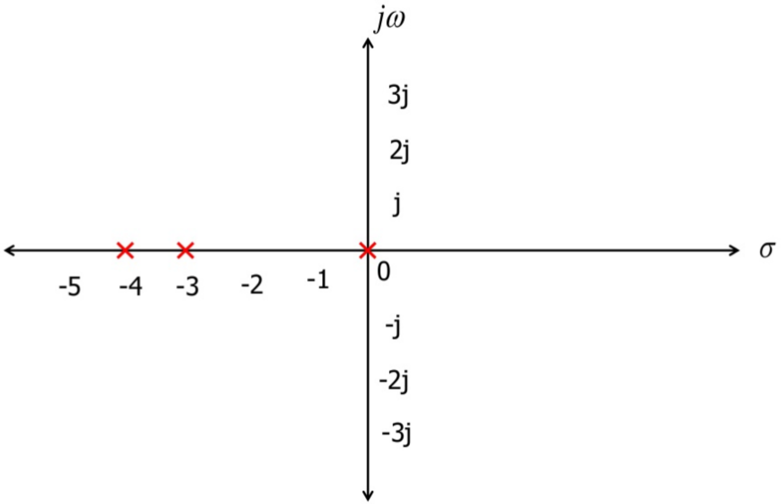
$$\therefore s = 0$$

$$\therefore s+3 = 0 \qquad \therefore s = -3$$

$$\therefore s+4 = 0 \qquad \therefore s = -4$$

The poles are s=0, -3, -4

S-plane Representation of Poles



zeros

The value of **s** for which the transfer function magnitude |G(S)| becomes zero after substitution in the numerator of the system are called as **Zeros** of transfer function.

Example 2

Determine the zeros of given transfer function.

$$G(s) = \frac{s(s+2)(s+4)}{s(s+3)(s+4)}$$

Solution: The zeros can be obtained by equating numerator with zero

$$s(s+2)(s+4) = 0$$

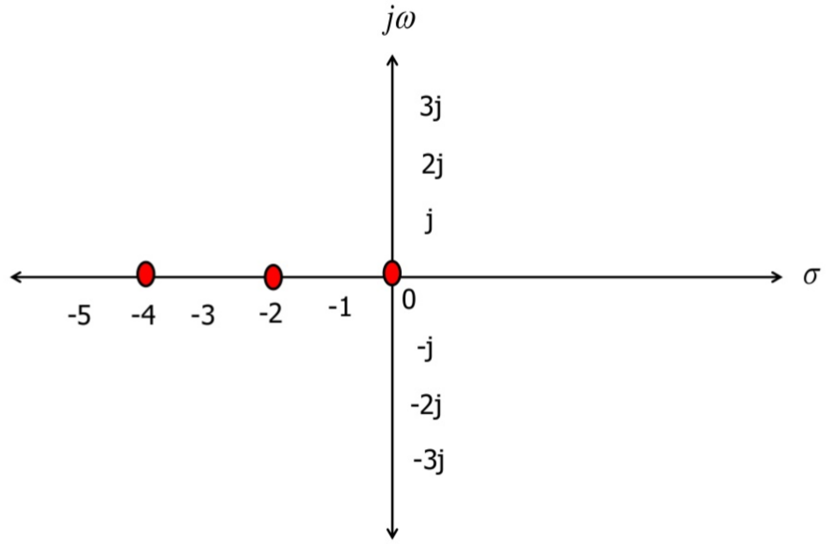
$$\therefore s = 0$$

$$\therefore s+2 = 0 \qquad \therefore s = -2$$

$$\therefore s+4 = 0 \qquad \therefore s = -4$$

The poles are s=0, -2, -4

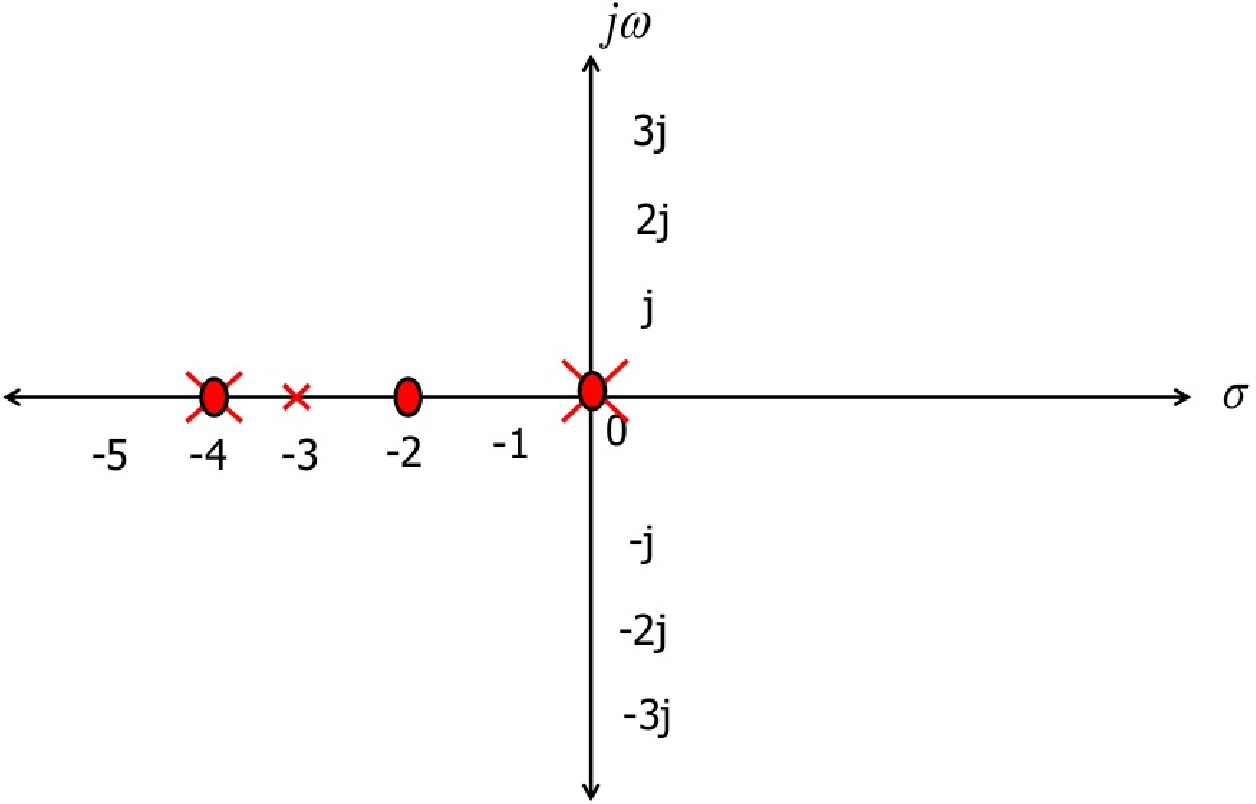
S-plane Representation of Zeros



Pole- Zero Plot

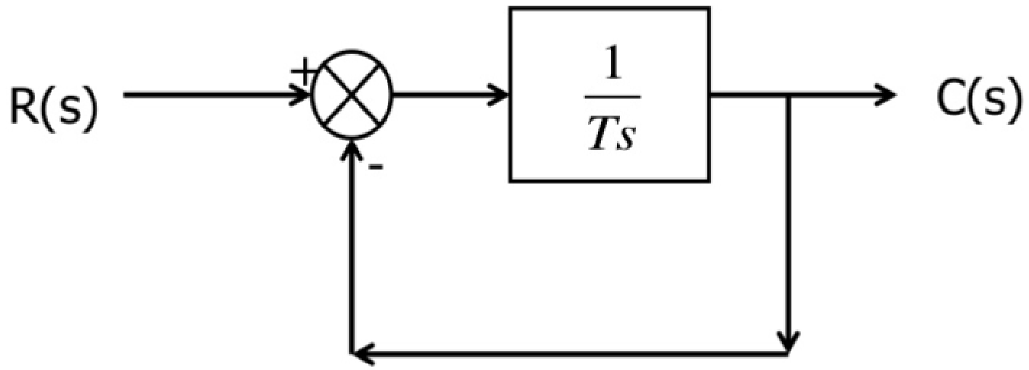
- The diagram obtained by locating all poles and zeros of the transfer function in the s-plane is called as "Pole-zero plot".
- The s-plane has two axis real and imaginary. Since $s = \sigma + j\omega$, the X-axis stands for real axis and shows a value of σ .
- Similarly, Y-axis stands for $j\omega$ and represents the imaginary axis.

Pole- Zero Plot for Example 1 and 2



Analysis of first order system for Step input

Consider a first order system as shown;



Here $G(s) = \dfrac{1}{Ts}$ and $H(s) = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{1}{Ts}}{1+\frac{1}{Ts}} = \frac{1}{1+Ts}$$

Analysis of first order system for Step input

For step input;

$$r(t) = u(t) \qquad t>0$$
$$= 0 \qquad t<0$$

Taking Laplace transform;

$$R(s) = L\{\mathrm{Ru(t)}\} = \frac{1}{s}$$

but

$$\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$$

$$\therefore C(s) = \frac{1}{1+Ts} \times R(s)$$

Analysis of first order system for Step input

$$\therefore C(\mathrm{s}) = \frac{1}{1+Ts} \times \frac{1}{s}$$

Using partial fraction;

$$\therefore C(\mathrm{s}) = \frac{A}{s} + \frac{B}{s + \frac{1}{T}}$$

Solving;

$$\therefore A = s.C(\mathrm{s})\,|_{s=0} = 1$$

$$\therefore B = (s + \frac{1}{T})C(\mathrm{s})\,|_{s=-\frac{1}{T}} = -1$$

Analysis of first order system for Step input

$$\therefore C(\mathrm{s}) = \frac{1}{s} - \frac{1}{s + \frac{1}{T}}$$

Taking Inverse Laplace transform;

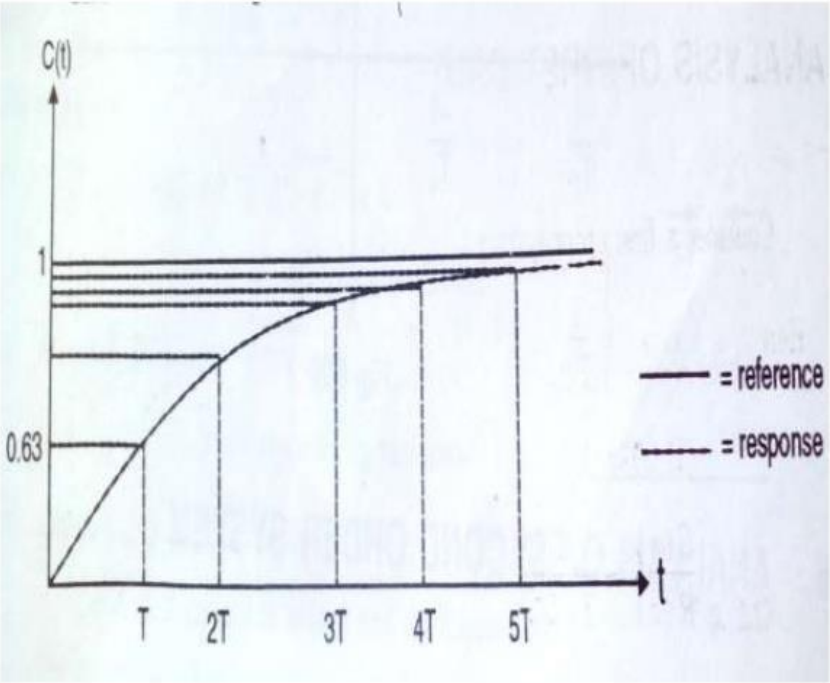
$$\therefore c(\mathrm{t}) = L^{-1}\{C(\mathrm{s})\} = L^{-1}\{\frac{1}{s}\} - L^{-1}\{\frac{1}{s + \frac{1}{T}}\}$$

$$\boxed{\therefore c(\mathrm{t}) = 1 - e^{-\frac{1}{T}t}}$$

Analysis of first order system for Step input

Plot c(t) vs t;

Sr. No.	t	C(t)
1	T	0.632
2	2T	0.86
3	3T	0.95
4	4T	0.982
5	5T	0.993
6	∞	1



Time Constant (T)

- ✓ The value of $c(t)=1$ only at $t=\infty$.
- ✓ Practically the value of $c(t)$ is within 5% of final value at $t=3T$ and within 2% at $t=4T$.
- ✓ In practice $t=3T$ or $4T$ may be taken as steady state.
- ✓ How quickly the value reaches steady state is a function of the time constant of the system.
- ✓ Hence smaller T indicates quicker response.

Damping and Damping Ratio

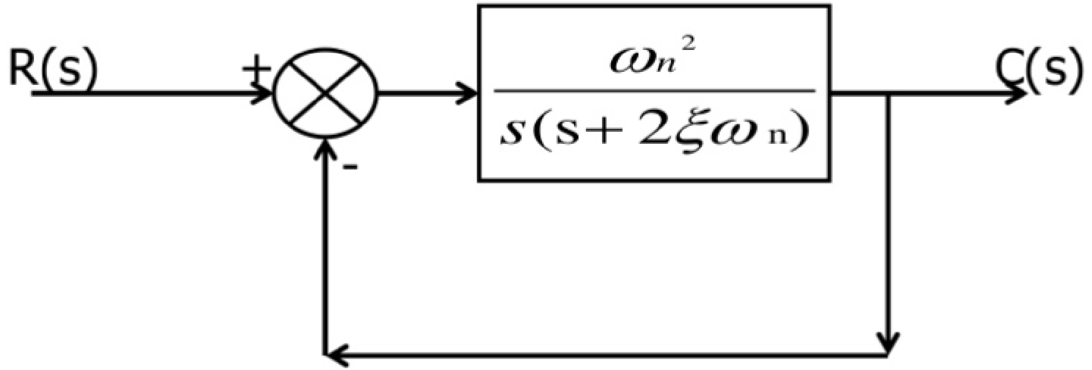
Every system has a tendency to oppose the oscillatory behaviour of the system which is known as **Damping.**

The damping in a system is measured by a factor or ratio which is known as **damping ratio**

It is denoted by ξ (Zeta)

Analysis of second order system for Step input

Consider a second order system as shown;



Here $G(s)=\frac{\omega_n^2}{s(s+2\xi\omega_n)}$ and $H(s)=1$

$$\therefore \frac{C(s)}{R(s)}=\frac{G}{1+GH} = \frac{\frac{\omega_n^2}{s(s+2\xi\omega_n)}}{1+\frac{\omega_n^2}{s(s+2\xi\omega_n)}} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Analysis of second order system for Step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

This is the standard form of the closed loop transfer function

These poles of transfer function are given by;

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\therefore s = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4(\omega_n)^2}}{2}$$

$$= -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Analysis of second order system for Step input

The poles are;

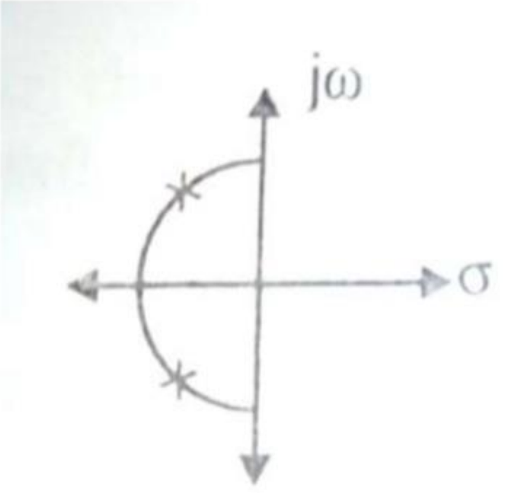
(i) Real and Unequal if $\sqrt{\xi^2 - 1} > 0$
i.e. $\xi > 1$ They lie on real axis and distinct

(ii) Real and equal if $\sqrt{\xi^2 - 1} = 0$
i.e. $\xi = 1$ They are repeated on real axis

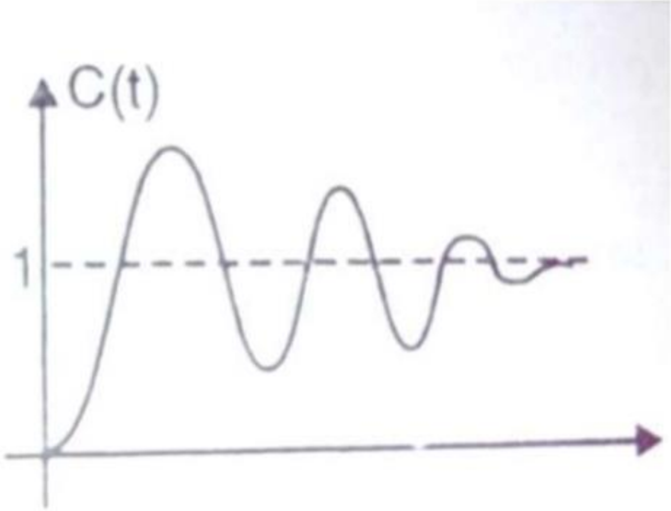
(iii) Complex if $\sqrt{\xi^2 - 1} < 0$
i.e. $\xi < 1$ Poles are in second and third quadrant

Relation between ξ and pole locations

(i) $\mathbf{O} < \xi < \mathbf{1}$ Under damped



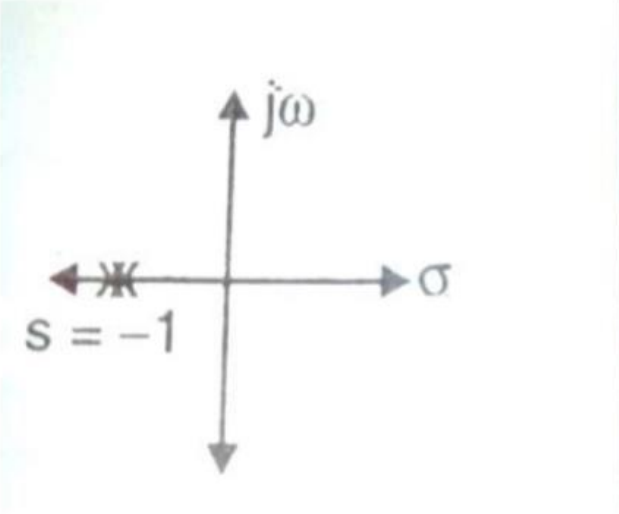
Pole Location

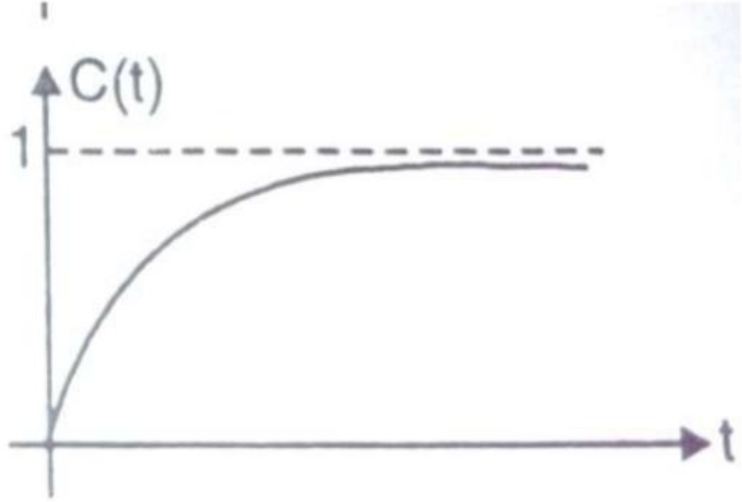


Step Response c(t)

Relation between ξ and pole locations

(ii) $\xi = 1$ Critically damped



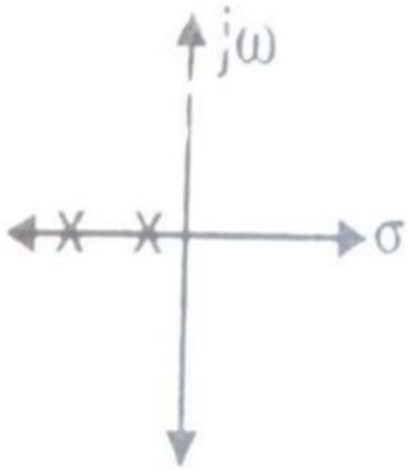


Pole Location

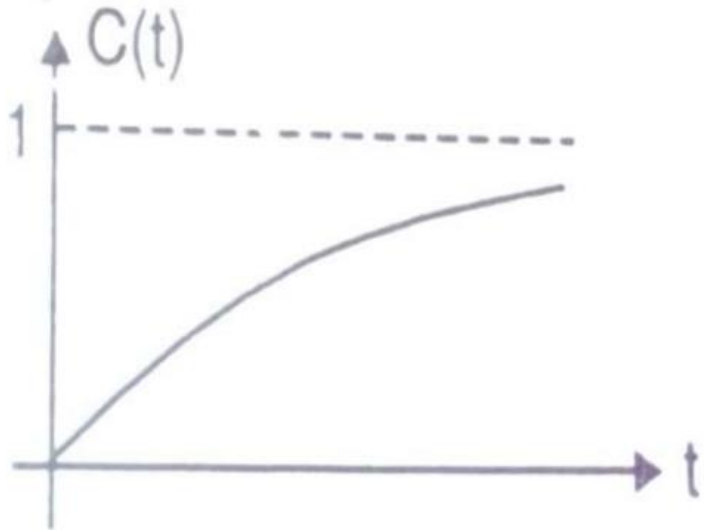
Step Response c(t)

Relation between ξ and pole locations

(iii) $\xi > 1$ over damped

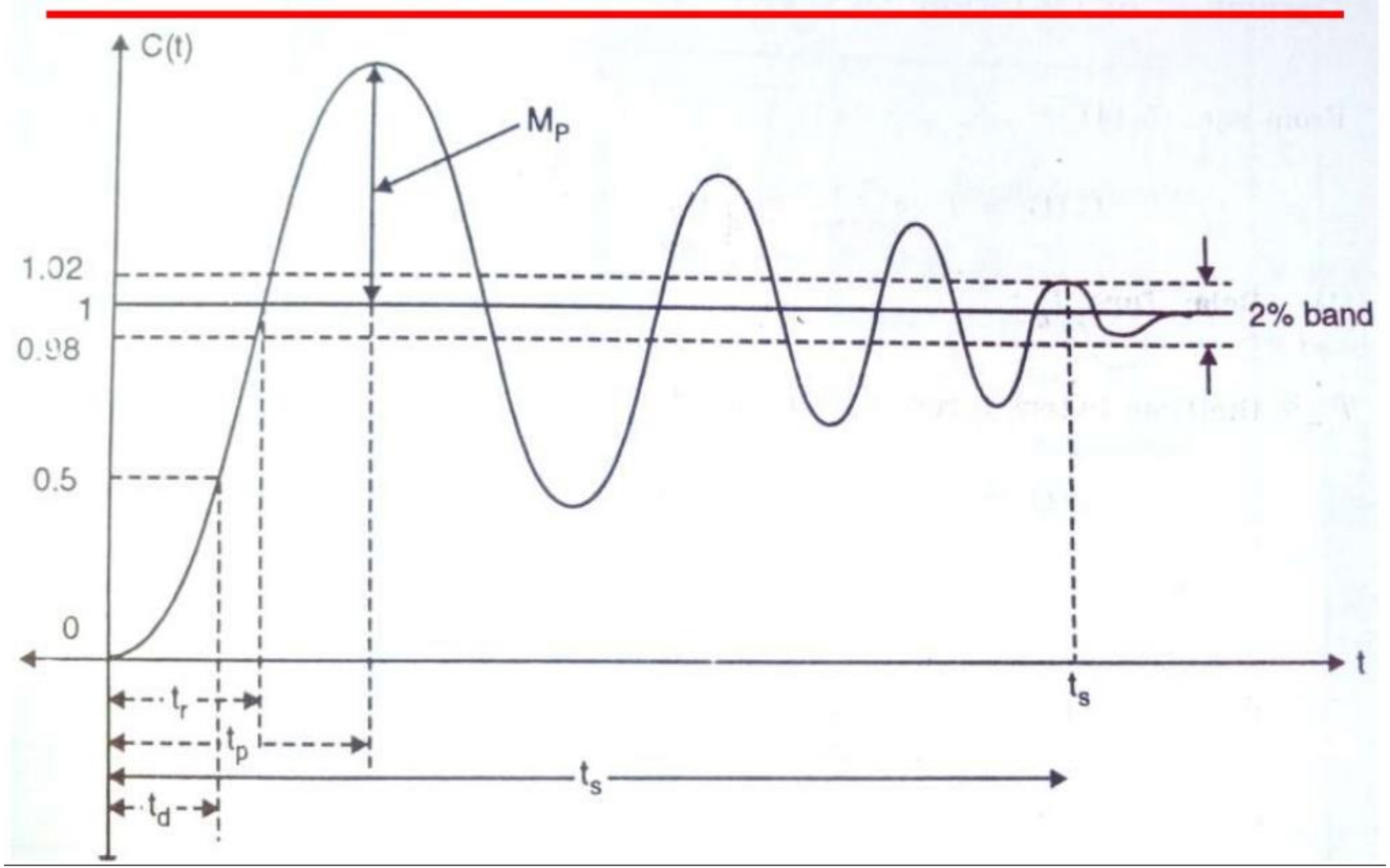


Pole Location



Step Response c(t)

Time Response Specifications



Delay time (Td)

It is time required for the response to reach 50% of the final value in the first attempt.

$$t_d = \frac{1+0.7\xi}{\omega_n}$$

Rise time (tr)

It is the time required for the response to rise from 10% to 90% of final value for overdamped systems.

(It is 0 to 100% for underdamped systems)

$$t_r = \frac{\pi - \beta}{\omega_d}$$ Where, $$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$ and $$\omega_d = \omega_n\sqrt{1-\xi^2}$$

Peak overshoot (Mp)

The maximum overshoot is the maximum peak value of the response curve measured from unity. It is therefore largest error between input and output during transient period.

$$\% M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

Peak Time (tp)

It is the time required for the response to reach the first peak.

$$T_P = \frac{\pi}{\omega_d}$$

Settling time (ts)

It is the time required for the response curve to reach and stay within a specified percentage (usually 2% or 5%) of the final value.

$$T_s = 4T = \frac{4}{\xi\omega_n}$$

Problems on time response specifications-Example 1

A unity feedback system has

$$G(s)=\frac{16}{s(s+5)}$$

If a step input is given calculate

1. Damping Ratio
2. Overshoot
3. Settling Time

Solution: $G(s)=\frac{16}{s(s+5)}$ $H(s)=1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)}=\frac{G}{1+GH}=\frac{\frac{16}{s(s+5)}}{1+\frac{16}{s(s+5)}}=\frac{16}{s^2+5s+16}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2 + \underline{2\xi\omega_n s} + \underline{\omega_n^2}} = \frac{16}{s^2 + \underline{5s} + \underline{16}}$$

Compare denominators of both

Natural Frequency;

$$\omega_n^2 = 16 \qquad \therefore \omega_n = 4 \ \ rad/\sec$$

Damping Ratio;

$$2\xi\omega_n s = 5s \qquad \therefore \xi = \frac{5}{2\times\omega_n} = \frac{5}{2\times 4} = 0.625$$

Settling Time;

$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{(0.625)\times(4)} = 1.6 \sec$$

Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.625)\pi}{\sqrt{1-(0.625)^2}}\right\}} \times 100$$

$$\%M_p = 8.08\%$$

Example 2

A unity feedback system has

$$G(s) = \frac{100}{s(s+5)}$$

If it is subjected to unit step input determine;

1. Damped frequency of oscillations
2. Time for first overshoot
3. Settling Time
4. Maximum Peak Overshoot

Solution: $G(s) = \frac{100}{s(s+5)}$ $\quad H(s) = 1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{100}{s(s+5)}}{1+\frac{100}{s(s+5)}} = \frac{100}{s^2+5s+100}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n{}^2}{s^2+2\xi\omega_n s+\omega_n{}^2} = \frac{100}{s^2+5s+100}$$

Compare denominators of both

Natural Frequency;

$$\omega_n{}^2 = 100 \qquad \therefore \omega_n = 10 \quad rad/\sec$$

Damping Ratio;

$$2\xi\omega_n s = 5s \qquad \therefore \xi = \frac{5}{2\times\omega_n} = \frac{5}{2\times 10} = 0.25$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 10\sqrt{1-(0.25)^2} = 9.68\ rad/\sec$$

Time for first overshoot (Peak Time);

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{9.68} = 0.324\ \sec$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.25)\times(10)} = 1.6\ \sec$$

Maximum Peak Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.25)\pi}{\sqrt{1-(0.25)^2}}\right\}} \times 100$$

$$\%M_p = 44.48\%$$

Example 3

For the unity feedback control system having open loop transfer function

$$G(s) = \frac{10}{s(s+4)}$$

Determine;

1. Delay Time
2. Rise Time
3. Peak Time
4. Settling Time
5. Maximum Peak Overshoot

Solution: $G(s) = \dfrac{10}{s(s+4)}$ $H(s) = 1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\dfrac{10}{s(s+4)}}{1+\dfrac{10}{s(s+4)}} = \frac{10}{s^2+4s+10}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n{}^2}{s^2+2\xi\omega_n s+\omega_n{}^2} = \frac{10}{s^2+4s+10}$$

Compare denominators of both

Natural Frequency;

$$\omega_n{}^2 = 10 \qquad \therefore \omega_n = 3.16 \;\; rad/\sec$$

Damping Ratio;

$$2\xi\omega_n s = 4s \qquad \therefore \xi = \frac{4}{2\times\omega_n} = \frac{4}{2\times 3.16} = 0.633$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 3.16\sqrt{1-(0.633)^2} = 2.44\ rad/\sec$$

Delay Time;

$$T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.633)}{3.16} = 0.457\ \sec$$

Rise Time;

$$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-(0.633)^2}}{(0.633)} = 0.885 \text{ rad}$$

$$T_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 0.885}{(0.244)} = 0.92 \text{ sec}$$

Peak Time;

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.44} = 1.273 \text{ sec}$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.633)\times(3.16)} = 1.997 \text{ sec}$$

Maximum Peak Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.633)\pi}{\sqrt{1-(0.633)^2}}\right\}} \times 100$$

$$\%M_p = 7.66\%$$

Steady state analysis-Type of system

The open loop transfer function of unity feedback system can be written in two standard forms: the time constant form and the pole-zero form.

$$G(s) = \frac{K(s+z1)(s+z2)\ldots\ldots\ldots\ldots\ldots\ldots}{s^n(s+p1)(s+p2)\ldots\ldots\ldots\ldots\ldots} \quad \text{(Pole-zero form)}$$

$$G(s) = \frac{K(1+Tz1s)(1+Tz2s)\ldots\ldots\ldots\ldots\ldots\ldots}{s^n(1+Tp1s)(1+Tp2s)\ldots\ldots\ldots\ldots\ldots} \quad \text{(Time constant form)}$$

Type 0 system

Definition: A control system with no integration in the open loop transfer function and no pole of transfer function G(s) at the origin of s-plane is designated as "**Type-0** " system.

$$G(s) = \frac{K(1+T_{z1}s)(1+T_{z2}s)\ldots\ldots\ldots\ldots\ldots\ldots}{(1+T_{p1}s)(1+T_{p2}s)\ldots\ldots\ldots\ldots\ldots}$$

(Standard form)

An amplifier type control system is a practical example of Type-0 system

Type 1 system

Definition: A control system with one integration in the open loop transfer function and one pole of transfer function G(s) at the origin of s-plane is designated as "**Type-1** " system.

$$G(s) = \frac{K(1+T\,\mathrm{z}1s)(1+T\,\mathrm{z}2s)\ldots\ldots\ldots\ldots\ldots\ldots}{\mathrm{s}(1+T\,\mathrm{p}1s)(1+T\,\mathrm{p}2s)\ldots\ldots\ldots\ldots\ldots}$$ (Standard form)

An pneumatic type control system is a practical example of Type-1 system

Type 2 system

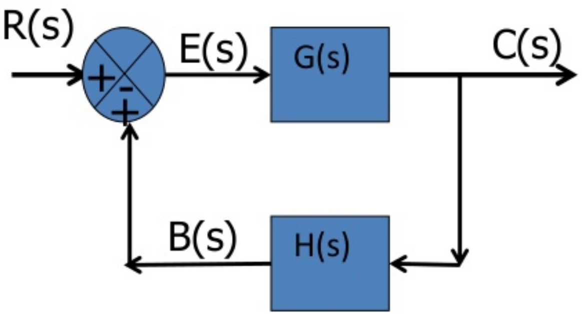
Definition: A control system with two integration in the open loop transfer function and two pole of transfer function G(s) at the origin of s-plane is designated as "**Type-2** " system.

$$G(s) = \frac{K(1+Tz1s)(1+Tz2s)\ldots\ldots\ldots\ldots\ldots\ldots}{s^2(1+Tp1s)(1+Tp2s)\ldots\ldots\ldots\ldots\ldots} \quad \text{(Standard form)}$$

A mechanical displacement system is a practical example of Type-2 system

Derivation for steady state error

The steady state response is important to judge the accuracy of the output. The difference between the steady state response and desired reference gives the steady state error.



For given figure,

$$E(s) = R(s) - B(s)$$

But

$$B(s) = C(s).H(s)$$

$$E(s) = R(s) - C(s).H(s)$$

But

$$C(s) = G(s).E(s)$$

$$E(s) = R(s) - G(s).E(s).H(s)$$

$$R(s) = E(s) + G(s).E(s).H(s)$$

$$R(s) = E(s)\{1 + G(s).H(s)\}$$

$$E(s) = \frac{R(s)}{1 \pm G(s).H(s)}$$

In time domain,

$$e(\mathrm{t}) = \mathrm{L}^{-1}\, E(\mathrm{s})$$

and is the expression of error valid for all time. Steady state error is defined as,

$$e_{ss}(\mathrm{t}) = \lim_{t \to \infty} e(t)$$

From the final value theorem in Laplace transform,

$$e_{ss}(\mathrm{t}) = \lim_{s \to 0} sE(\mathrm{s})$$

Steady state error,

$$e_{ss}(\mathrm{t}) = \lim_{s \to 0} sE(\mathrm{s})$$

$$\boxed{e_{ss}(\mathrm{t}) = \lim_{s \to 0} \frac{sR(\mathrm{s})}{1 + G(\mathrm{s})\,\mathrm{H}(\mathrm{s})}}$$

Steady state error and type of system

Sr. No.	Type of System	Step Input		Ramp Input		Parabolic Input	
		K_P	e_{ss}	K_V	e_{ss}	K_a	e_{ss}
1	Zero	K	$\frac{A}{1+K}$	0	∞	0	∞
2	One	∞	0	K	$\frac{A}{K}$	0	∞
3	Two	∞	0	∞	0	K	$\frac{A}{K}$

THANK YOU