Stability Analysis of Control System

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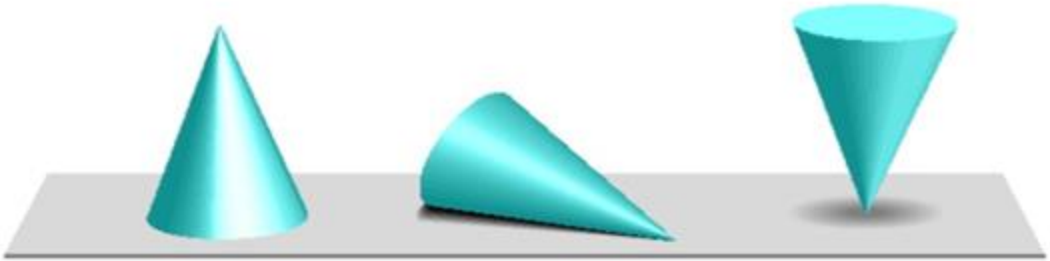
Concept of stability

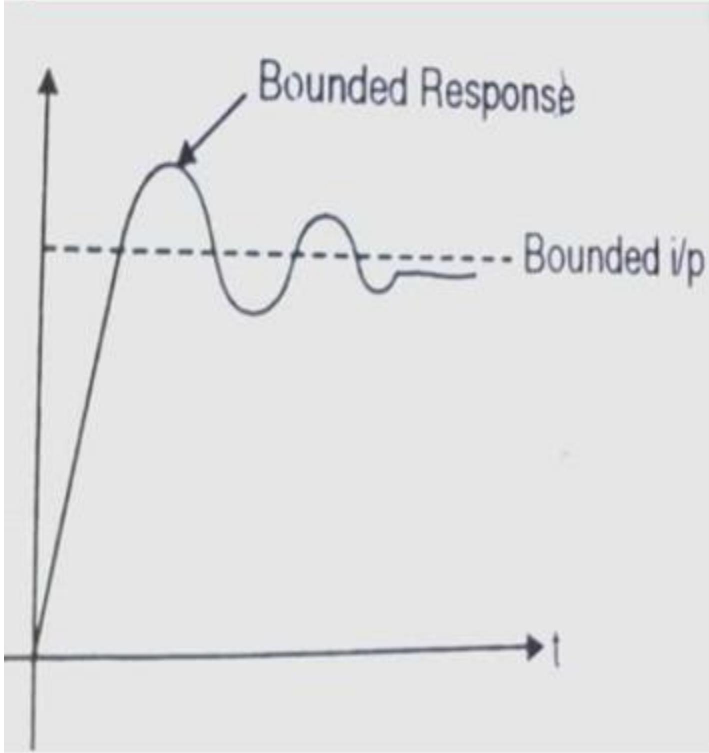
The concept of stability can be illustrated by a cone placed on a plane horizontal surface.



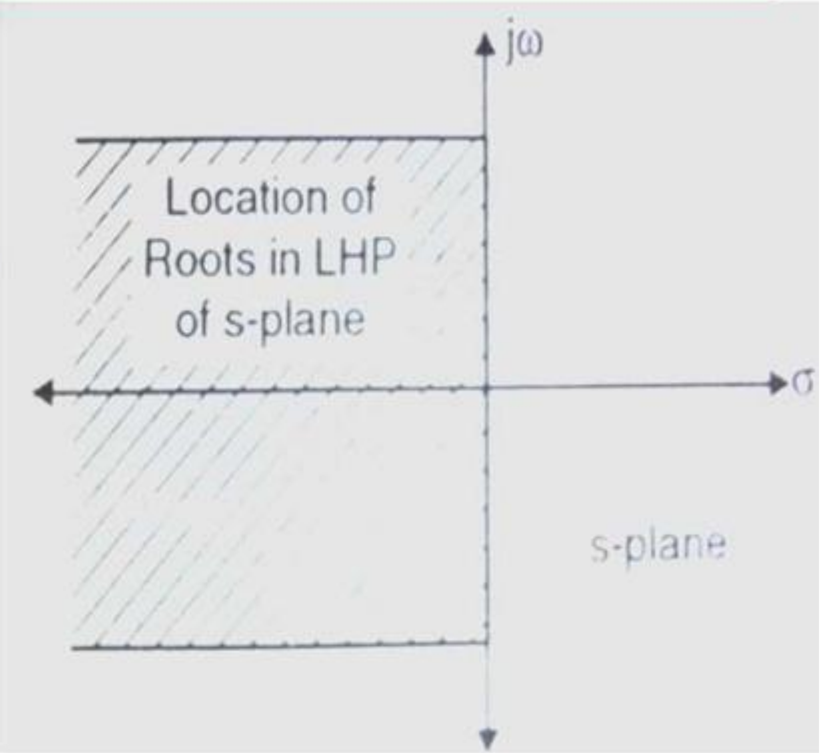
(a) Stable (b) Neutral (c) Unstable

Stable system

- A linear time invariant system is stable if following conditions are satisfied.
- A bounded input is given to the system, the response is bounded and controllable.
- In absence of the inputs, the output should tend to zero as time increases.



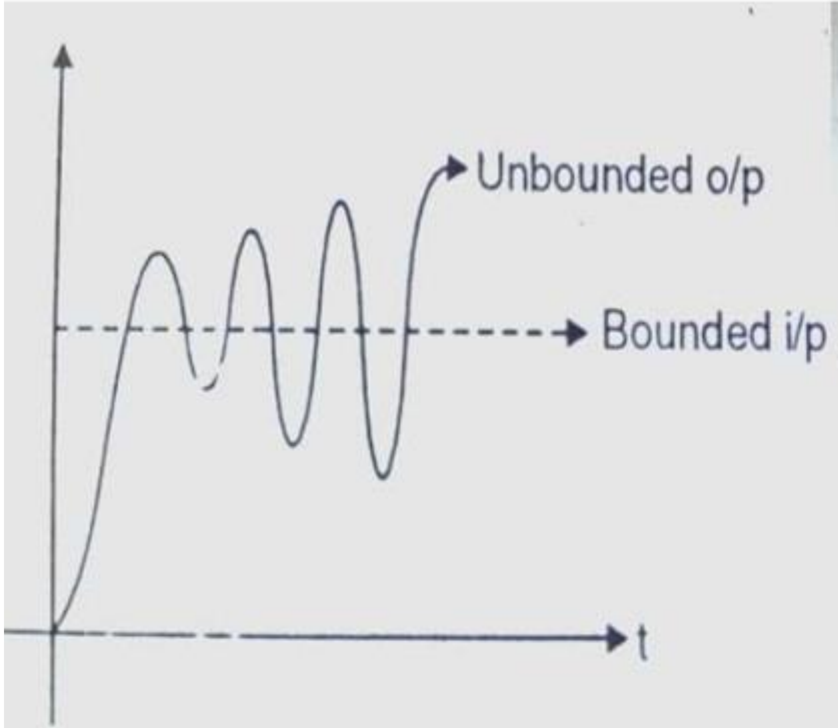
Bounded i/p bounded o/p
for stable system



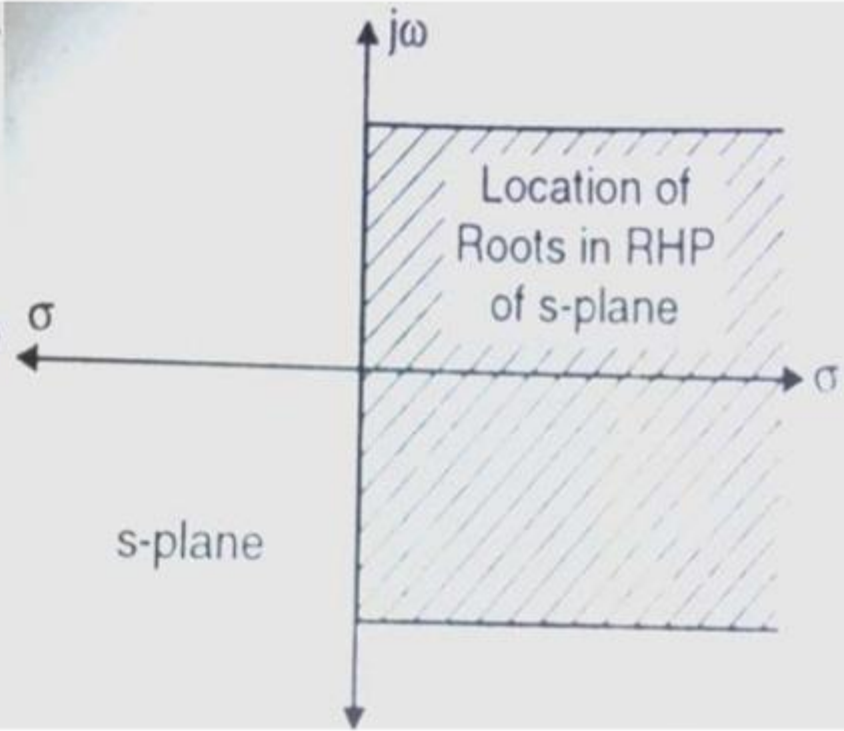
Location of roots
for stable system

Unstable system

- A linear time invariant system is said to be unstable if system is excited by a bounded input, response is unbounded.
- i.e. output goes on increasing and does not have any control on it.



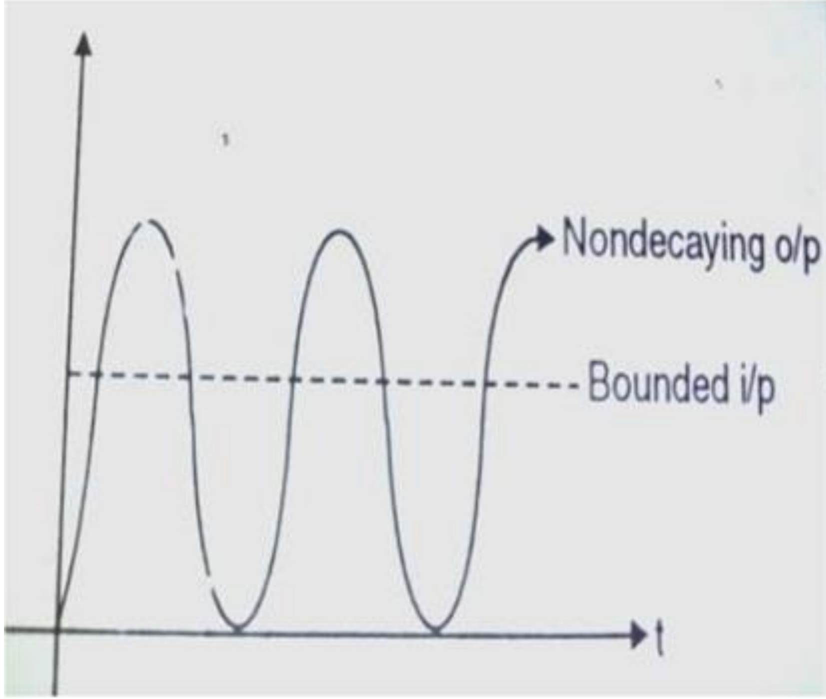
Bounded i/p Unbounded o/p
for unstable system



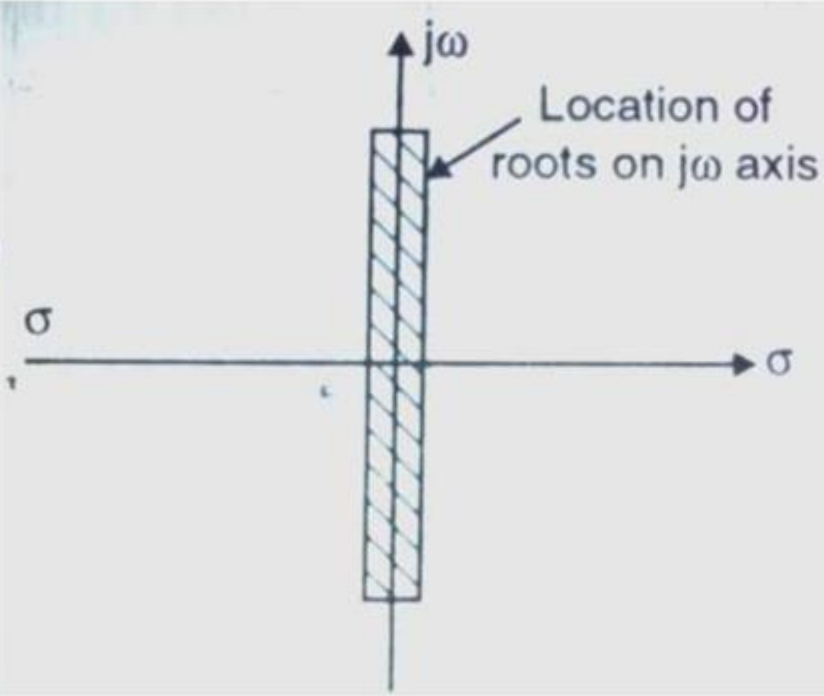
Location of roots
for unstable system

Critically stable systems

- When the input is given to a linear time invariant system, for critically stable systems the output does not go on increasing infinitely nor does it go to zero as time increases.
- The output usually oscillates in finite range or remains steady at some value.
- This systems are neither stable nor unstable.



Bounded i/p & o/p response
for critically stable system



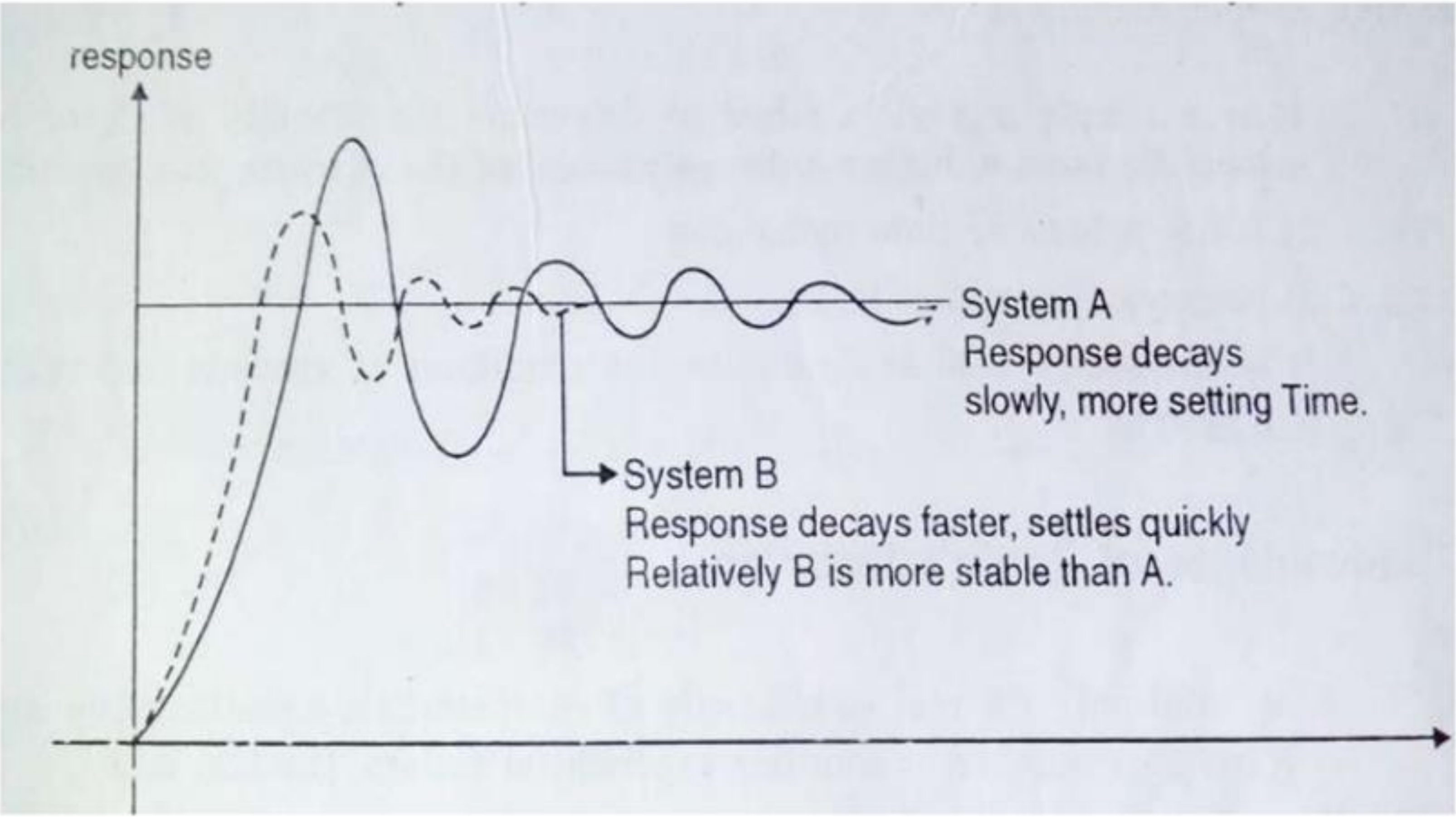
Location of roots
for critically stable system

Conditionally stable systems

This type of systems are stable if a particular condition is satisfied otherwise it is unstable.

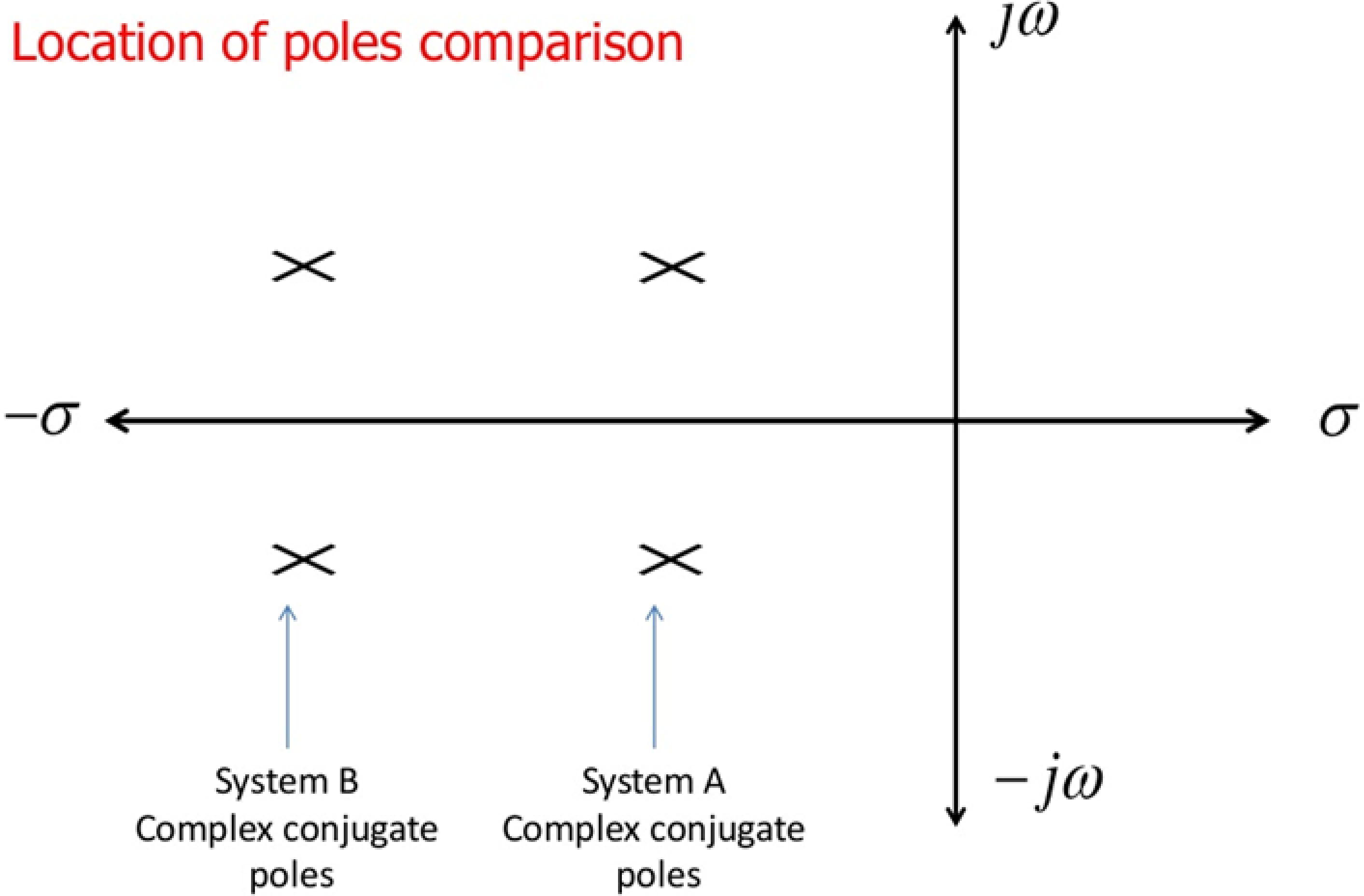
Relative stable systems

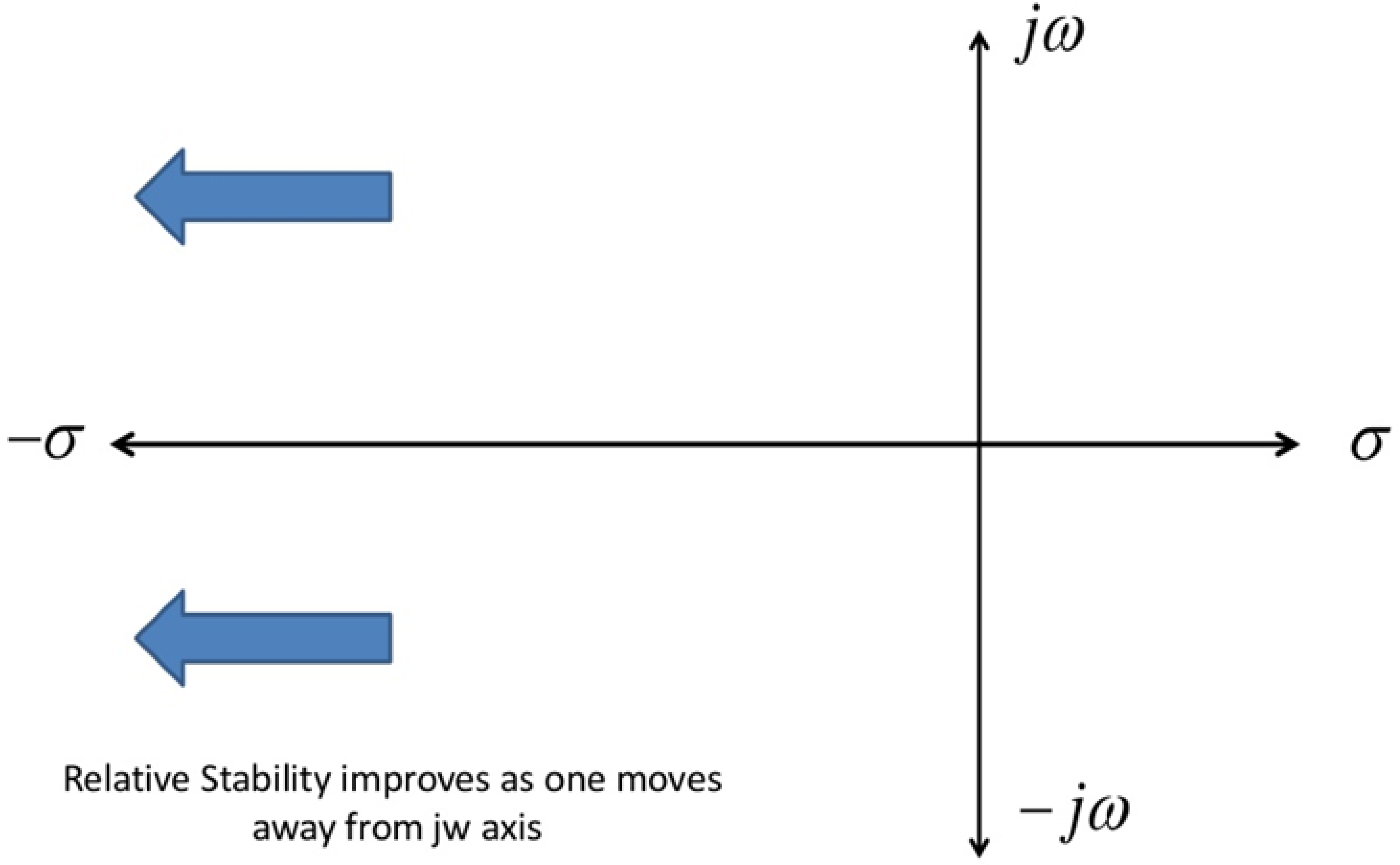
- A system may be absolutely stable i.e. it may have passes the Routh's stability test.
- As a result response decays to zero under zero input conditions.
- The ratio at which these decay to zero is important to check the concept of relative stability.
- When the poles are located far away from jw axis in LHP of s-plane, the response decays to zero much faster, as compared to the poles close to jw axis.
- The more the poles are located far away from jw-axis the more is the system relatively stable.



Response comparison

Location of poles comparison





Relative Stability improves as one moves away from jw axis

Routh’s stability criterion

For the transfer function;

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots\dots\dots\dots\dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots\dots\dots\dots\dots + a_n}$$

In this criterion, the coefficients of denominator are arranged in an Array called “Routh’s Array”;

$$a_0 s^n + a_1 s^{n-1} + \dots\dots\dots\dots\dots + a_n = 0$$

$$a_0 s^n + a_1 s^{n-1} + \ldots\ldots\ldots\ldots\ldots + a_n = 0$$

The Routh's array as below;

$$\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \\
s^{n-2} & b_1 & & & \\
s^{n-3} & & & & \\
 & & & & \\
s^0 & & & &
\end{array}$$

For next row i.e. s^{n-2} ;

$$b_1 = \frac{a_1 . a_2 - a_3 a_0}{a_1}$$

The coefficients of s^n and s^{n-1} row are directly written from the given equation.

$$a_0 s^n + a_1 s^{n-1} + \ldots\ldots\ldots\ldots\ldots + a_n = 0$$

The Routh's array as below;

$$\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \\
s^{n-2} & b_1 & b_2 & & \\
s^{n-3} & & & & \\
\vdots & & & & \\
s^0 & & & &
\end{array}$$

For next row i.e. s^{n-2} ;

$$b_1 = \frac{a_1.a_2 - a_3 a_0}{a_1}$$

$$b_2 = \frac{a_1.a_4 - a_0.a_5}{a_1}$$

The coefficients of s^n and s^{n-1} row are directly written from the given equation.

$$a_0 s^n + a_1 s^{n-1} + \ldots\ldots\ldots\ldots\ldots + a_n = 0$$

The Routh's array as below;

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	
s^{n-3}				
s^0				

For next row i.e. s^{n-2} ;

$$b_1 = \frac{a_1.a_2 - a_3 a_0}{a_1}$$

$$b_2 = \frac{a_1.a_4 - a_0.a_5}{a_1}$$

$$b_3 = \frac{a_1.a_6 - a_0.a_7}{a_1}$$

Now the same technique is used, for the next row i.e. s^{n-3} row, but only previous two rows are used i.e. s^{n-1} and s^{n-2}

The Routh's array as below;

$$
\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \\
s^{n-2} & b_1 & b_2 & b_3 & \\
s^{n-3} & c_1 & & & \\
 & & & & \\
s^0 & & & &
\end{array}
$$

For next row i.e. s^{n-2} ;

$$b_1 = \frac{a_1.a_2 - a_3a_0}{a_1}$$

$$b_2 = \frac{a_1.a_4 - a_0.a_5}{a_1}$$

$$b_3 = \frac{a_1.a_6 - a_0.a_7}{a_1}$$

For next row i.e. s^{n-3} ;

$$c_1 = \frac{b_1.a_3 - b_2a_1}{b_1}$$

Now the same technique is used, for the next row i.e. s^{n-3} row, but only previous two rows are used i.e. s^{n-1} and s^{n-2}

The Routh's array as below;

$$
\begin{array}{c|cccc}
s^{n} & a_0 & a_2 & a_4 & a_6 \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \\
s^{n-2} & b_1 & b_2 & b_3 & \\
s^{n-3} & c_1 & c_2 & & \\
\vdots & & & & \\
s^{0} & & & &
\end{array}
$$

For next row i.e. s^{n-2} ;

$$b_1 = \frac{a_1.a_2 - a_3a_0}{a_1}$$

$$b_2 = \frac{a_1.a_4 - a_0.a_5}{a_1}$$

$$b_3 = \frac{a_1.a_6 - a_0.a_7}{a_1}$$

For next row i.e. s^{n-3} ;

$$c_1 = \frac{b_1.a_3 - b_2a_1}{b_1}$$

$$c_2 = \frac{b_1.a_5 - b_3a_1}{b_1}$$

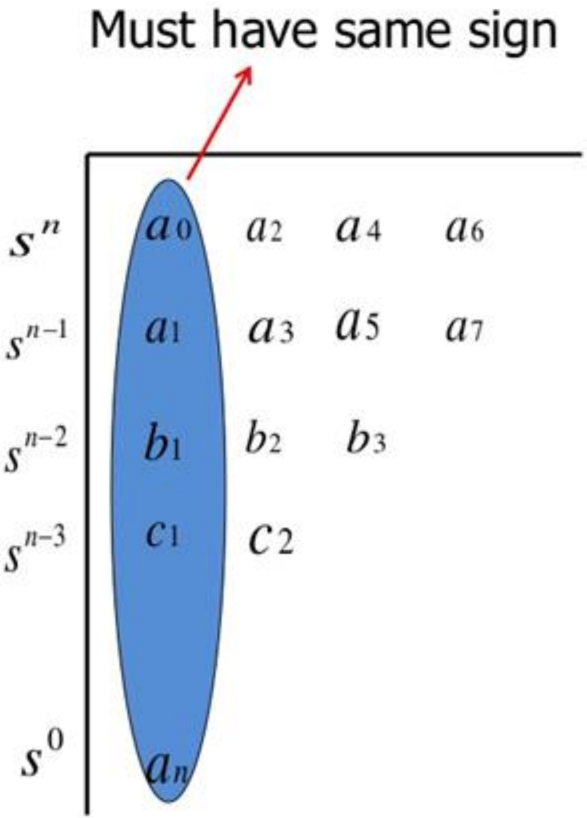
- Each column will reduce by one as we move down the array.

- This process is obtained till last row is obtained.

$$
\begin{array}{c|cccc}
s^{n} & a_0 & a_2 & a_4 & a_6 \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \\
s^{n-2} & b_1 & b_2 & b_3 & \\
s^{n-3} & c_1 & c_2 & & \\
 & & & & \\
s^{0} & a_n & & &
\end{array}
$$

Routh's Criterion

- The necessary & sufficient conditions for a system to be stable is all terms in the first column at Routh's Array should have same sign.
- There should not be any sign change in first column.



Routh's Criterion

- When there are sign changes in the first column of Routh's array then the system is unstable.
- There are roots in RHP.
- The number of sign changes equal the number of roots in RHP.

$$\begin{array}{c|cccc} s^{n} & +a_0 & a_2 & a_4 & a_6 \\ s^{n-1} & +a_1 & a_3 & a_5 & a_7 \\ s^{n-2} & -b_1 & b_2 & b_3 & \\ s^{n-3} & +c_1 & c_2 & & \\ & & & & \\ s^{0} & a_n & & & \end{array}$$

Example 1

$$\underset{a_0}{s^3} + \underset{a_1}{6}s^2 + \underset{a_2}{12}s + \underset{a_3}{8} = 0$$

$$\begin{array}{c|cc}
s^3 & 1 & 12 \\
s^2 & 6 & 8 \\
s^1 & b_1 & b_2 \\
s^0 & a_n &
\end{array}$$

s^3	1	12
s^2	6	8
s^1	b_1	b_2
s^0	a_n	

$$b_1 = \frac{a_1.a_2 - a_3.a_0}{a_1}$$

s^3	1	12
s^2	6	8
s^1	b_1	b_2
s^0	a_n	

$$b_1 = \frac{(6 \times 12) - (8 \times 1)}{6}$$

s^3	1	12	0
s^2	6	8	0
s^1	10.67	b_2	
s^0	a_n		

$$b_2 = \frac{a_1.a_4 - a_0.a_5}{a_1}$$

s^3	1	12	0
s^2	6	8	0
s^1	10.67	b_2	
s^0	a_n		

$$b_2 = \frac{1\times 0 - 6\times 0}{6}$$

s^3	1	12	0
s^2	6	8	0
s^1	10.67	0	
s^0	a_n		

$$b_2 = \frac{10.67 \times 8 - 6 \times 0}{10.67}$$

$$s^3 + 6s^2 + 12s + 8 = 0$$

s^3	1	12	0
s^2	6	8	0
s^1	10.67	0	
s^0	8		

Elements in first column are of same sign

s^3	1	12	0
s^2	6	8	0
s^1	10.67	0	
s^0	8		

As all elements in first column are of same sign. Hence system is stable

Example 2

Comment on stability. $$2s^3 + 4s^2 + 4s + 12 = 0$$

s^3	2	4	0
s^2	4	12	
s^1	b_1	b_2	
s^0	a_n		

$$b_1 = \frac{a_1.a_2 - a_3.a_0}{a_1}$$

$$b_1 = \frac{(4\times 4)-(2\times 12)}{4}$$

$$b_1 = -2$$

s^3	2	4	0
s^2	4	12	
s^1	-2	b_2	
s^0	a_n		

$$b_2 = \frac{(4\times 0)-(2\times 0)}{4}$$

$$b_2 = 0$$

s^3	2	4	0
s^2	4	12	
s^1	-2	0	
s^0	a_n		

$$a_n = \frac{(-2\times 12)-(4\times 0)}{-2}$$

$$a_n = 12$$

s^3	2	4	0
s^2	4	12	
s^1	-2	0	
s^0	12		

There are two sign changes +4 to -2 and -2 to +12. Hence two roots are in RHP S-plane and system is unstable

Example 3

Comment on stability. $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

s^5	1	2	3
s^4	1	2	5
s^3	b_1	b_2	
s^2	c_1	c_2	
s^1	d_1		
s^0	a_n		

$$b_1 = \frac{(1\times 2)-(2\times 1)}{1}$$

$$b_1 = 0$$

$$b_2 = \frac{(1\times 3)-(5\times 1)}{1}$$

$$b_2 = -2$$

Comment on stability. $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	
s^2	c_1	c_2	
s^1	d_1		
s^0	a_n		

Case I – "When the first of any row is zero and the rest of the row is non-zero."

Case I – "When the first of any row is zero and the rest of the row is non-zero." Here the next row cannot be formed as division by 0 will take place.

Method to Overcome: A method to overcome above problem is to replace s by $\frac{1}{z}$ and complete the Routh's test for z.

Replace s by (1/z)

$$(\frac{1}{z})^5 + (\frac{1}{z})^4 + 2(\frac{1}{z})^3 + 2(\frac{1}{z})^2 + 3(\frac{1}{z}) + 5 = 0$$

Take L.C.M

$$\frac{1 + z + 2z^2 + 2z^3 + 3z^4 + 5z^5}{z^5} = 0$$

$$1 + z + 2z^2 + 2z^3 + 3z^4 + 5z^5 = 0$$

$$5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0$$

Use above characteristics equation and complete Routh's Test

$$
\begin{array}{c|ccc}
z^5 & 5 & 2 & 1 \\
z^4 & 3 & 2 & 1 \\
z^3 & -\frac{4}{3} & -\frac{2}{3} & \\
z^2 & \frac{1}{2} & 1 & \\
z^1 & 2 & & \\
z^0 & 1 & &
\end{array}
$$

There are two sign changes in first column.
Hence two roots are in RHP S-plane and system is unstable

Example 4

Comment on stability. $s^4 + 6s^3 + 11s^2 + 6s + 10 = 0$

s^4	1	11	10
s^3	6	6	0
s^2	b_1	b_2	
s^1	c_1		
s^0	a_n		

$$b_1 = \frac{(6\times 11) - (1\times 6)}{6}$$

$$b_1 = 10$$

$$b_2 = \frac{(6\times 10) - (1\times 0)}{6}$$

$$b_1 = 10$$

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	c_1		
s^0	a_n		

$$c_1 = \frac{(10\times 6)-(10\times 6)}{10}$$

$$c_1 = 0$$

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	0		
s^0	a_n		

$$c_1 = \frac{(10\times 6)-(10\times 6)}{10}$$

$$c_1 = 0$$

Case II – "When all elements in any one row is zero."

Case II – “When all elements in any one row is zero.”

Method to Overcome:

- ✓ Here form an auxillary equation with the help of the coefficients of the coefficients of the row just above the row of zeros.
- ✓ Take the derivative of this equation and replace it’s coefficients in the present row of zeros.
- ✓ Then proceed for Routh’s test.

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	0		
s^0	a_n		

Here s^1 row breaks down.
Hence write auxiliary equation for s^2 .

$$A(s) = 10s^2 + 10$$

(Note each term of next column differs by degree of 2)

Take derivative of auxiliary equatior

$$\frac{d}{ds}A(s) = 20s$$

Use these for s row coefficients.

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	20		
s^0	a_n		

$$a_n = \frac{(20\times 10)-(10\times 0)}{20}$$

$$a_n = 10$$

s^4	1	11	10
s^3	6	6	0
s^2	10	10	
s^1	20		
s^0	10		

As no sign change in first column; system is stable

Example 5

Comment on stability. $s^6+3s^5+5s^4+9s^3+8s^2+6s+4=0$

s^6	1	5	8	4
s^5	3	9	6	
s^4	b_1	b_2	b_3	
s^3	c_1	c_2		
s^2	d_1			
s^1	e_1			
s^0	a_n			

$$b_1=\frac{(3\times5)-(9\times1)}{3}$$

$$b_1=2$$

$$b_2=\frac{(3\times8)-(6\times1)}{3}$$

$$b_2=6$$

$$b_3=\frac{(3\times4)-(0\times1)}{3}$$

$$b_3=4$$

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	c_1	c_2		
s^2	d_1			
s^1	e_1			
s^0	a_n			

$$c_1 = \frac{(2\times 9)-(3\times 6)}{2}$$

$$c_1 = 0$$

$$c_2 = \frac{(2\times 6)-(4\times 3)}{2}$$

$$c_2 = 0$$

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	0	0		
s^2	d_1			
s^1	e_1			
s^0	a_n			

Here s^3 row breaks down.
Hence write auxiliary equation for s^4.

$$A(s) = 2s^4 + 6s^2 + 4$$

(Note each term of next column differs by degree of 2)

Take derivative of auxiliary equation

$$\frac{d}{ds}A(s) = 8s^3 + 12s$$

Use these for s^3 row coefficients.

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	8	12		
s^2	d_1	d_2		
s^1	e_1			
s^0	a_n			

$$d_1 = \frac{(8\times 6)-(12\times 2)}{8}$$

$$d_1 = 3$$

$$d_2 = \frac{(8\times 4)-(0\times 2)}{8}$$

$$d_2 = 4$$

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	8	12		
s^2	3	4		
s^1	e_1			
s^0	a_n			

$$e_1 = \frac{(3\times 12)-(8\times 4)}{3}$$

$$e_1 = 4$$

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	8	12		
s^2	3	4		
s^1	4			
s^0	a_n			

$$a_n = \frac{(4\times 4)-(3\times 0)}{4}$$

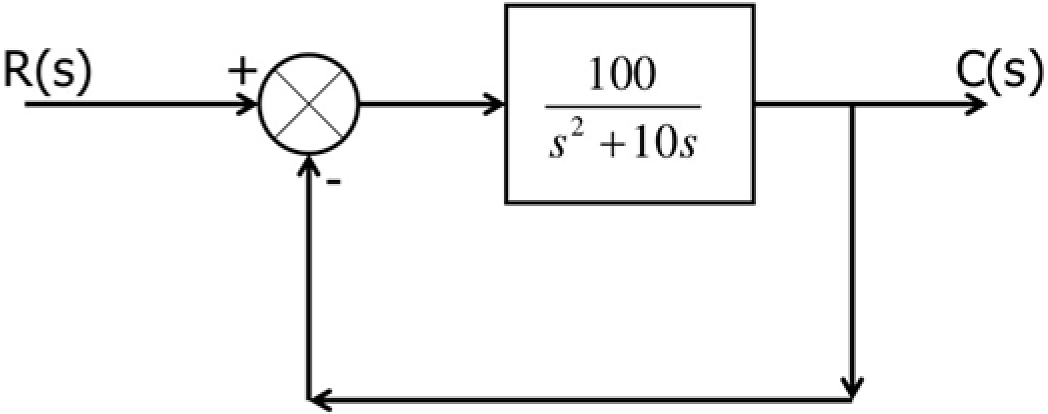
$$a_n = 4$$

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	8	12		
s^2	3	4		
s^1	4			
s^0	4			

As no sign change in first column; system is stable

Example 6

Problem: Using routh's criteria find the stability for given figure.



$$G(s) = \frac{100}{s^2 + 10s} \qquad \mathrm{H}(s) = 1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{100}{s^2 + 10s}}{1 + \dfrac{100}{s^2 + 10s}} = \frac{\dfrac{100}{s^2 + 10s}}{\dfrac{s^2 + 10s + 100}{s^2 + 10s}} = \frac{100}{s^2 + 10s + 100}$$

Characteristics equation is the denominator of the CLTF

Characteristics equation $\qquad s^2 + 10s + 100 = 0$

$$s^2 + 10s + 100 = 0$$

s^2	1	100
s^1	10	
s^0	a_n	

$$a_n = \frac{(10 \times 100) - (1 \times 0)}{10}$$

$$a_n = 100$$

$$s^2 + 10s + 100 = 0$$

$$\begin{array}{c|cc} s^2 & 1 & 100 \\ s^1 & 10 & \\ s^0 & 100 & \end{array}$$

As no sign change in first column; system is stable

Applications of Routh's criterion

- ✓ The gain is kept in terms of k and Routh's array is solved to find k for stable operation.

Example 1

Determine the range of k for stable system. $s^4+5s^3+5s^2+4s+k=0$

s^4	1	5	k
s^3	5	4	0
s^2	4.2	k	
s^1	$\dfrac{16.8-5k}{4.2}$		
s^0	k		

For stability all elements of first column 1 should be positive

i.e. $k > 0$ For s^0 row ------------------------(1)

and $\frac{16.8 - 5k}{4.2} > 0$ For s^1 row

i.e. $16.8 > 5k$ or $k < \frac{16.8}{5}$

$k < 3.36$ ------------------------(2)

Thus combining equations (1) and (2), $0 < k < 3.36$

This is the range of k stable operation.

Example 2

Determine the range of k for stable system. $s^4+4s^3+4s^2+3s+k=0$

s^4	1	4	k
s^3	4	3	0
s^2	$\frac{13}{4}$	k	
s^1	$\frac{39-16k}{13}$		
s^0	k		

For stability,

i.e. $k > 0$ and $\frac{39-16k}{13} > 0$

i.e. $39-16k > 0$ or $k < \frac{39}{16}$

$$k < 2.43$$

Thus $0 < k < 2.43$

This is the range of k stable operation.

Advantages of Routh's criterion

- It is a simple algebraic method to determine the stability of closed loop without solving for roots of higher order polynomial of the characteristics equation.
- It is not tedious or time consuming.
- It progress systematically.
- It is frequently used to determine the conditions of absolute & relative stability of a system.
- It can determine range of k for stable operation.

Disadvantages of Routh's criterion

- It is valid only for real coefficients of characteristics equation. Any coefficient that is a complex number or contains exponential factors, the test fails.
- It is applicable only to the linear systems.
- Exact location of poles is not known.
- Only idea is obtained about stability. A method to stabilize the system is not suggested.

Thank You