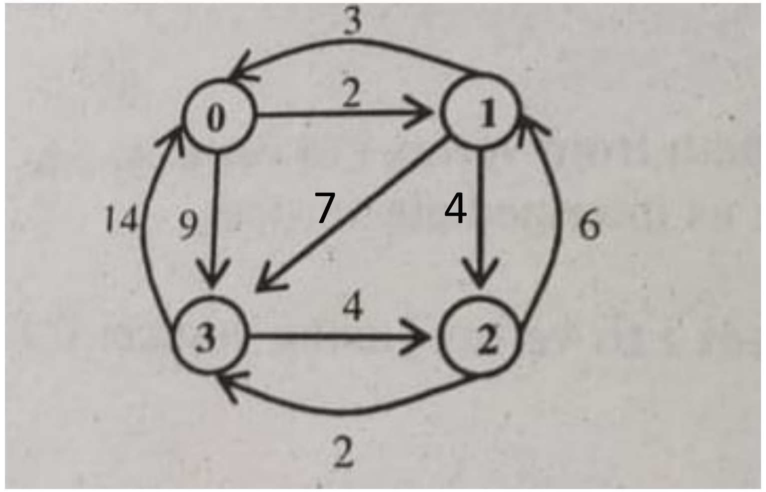
Floyd - Warshall Algorithm

Shortest Path

- to find the shortest distances between every pair of vertices in a given edge-weighted directed Graph.



$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 5 & 2 & 6 & 8 \\
1 & 3 & 5 & 4 & 6 \\
2 & 9 & 6 & \underline{6} & 2 \\
3 & 13 & 10 & 4 & 6
\end{array}
$$

All pair Shortest Path Problem

It is an algorithm for finding the shortest path between **all the pairs of vertices** in a **weighted graph**.

This algorithm follos the dynamic programming approach to find the shortest path.

Floyd - Warshall algorithm

- *Initialize the solution matrix same as the input graph matrix as a first step.*
- *Then update the solution matrix by considering all vertices as an intermediate vertex.*
- *The idea is to one by one pick all vertices and updates all shortest paths which include the picked vertex as an intermediate vertex in the shortest path.*
- *When we pick vertex number k as an intermediate vertex, we already have considered vertices {0, 1, 2, .. k-1} as intermediate vertices.*
- *For every pair (i, j) of the source and destination vertices respectively, there are two possible cases.*
    - *k is not an intermediate vertex in shortest path from i to j. We keep the value of dist[i][j] as it is.*
    - *k is an intermediate vertex in shortest path from i to j. We update the value of dist[i][j] as dist[i][k] + dist[k][j] if dist[i][j] > dist[i][k] + dist[k][j]*

Weighted adjacency matrix for this graph is-

$$W = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 9 \\ 3 & 0 & 4 & 7 \\ 0 & 6 & 0 & 2 \\ 14 & 0 & 4 & 0 \end{bmatrix} \end{array}$$

We can easily write matrices D_{-1} and $Pred_{-1}$ from matrix W.

$$D_{-1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \infty & 2 & \infty & 9 \\ 3 & \infty & 4 & 7 \\ \infty & 6 & \infty & 2 \\ 14 & \infty & 4 & \infty \end{bmatrix} \end{array} \qquad Pred_{-1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 1 \\ -1 & 2 & -1 & 2 \\ 3 & -1 & 3 & -1 \end{bmatrix} \end{array}$$

$$
D_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
\infty & 2 & \infty & 9 \\
3 & \underline{5} & 4 & 7 \\
\infty & 6 & \infty & 2 \\
14 & 16 & 4 & 23
\end{bmatrix}
\end{array}
$$

$$
\text{Pred}_0 \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
-1 & 0 & -1 & 0 \\
1 & \underline{0} & 1 & 1 \\
-1 & 2 & -1 & 2 \\
3 & 0 & 3 & 0
\end{bmatrix}
\end{array}
$$

Similarly we can find matrices D_2, $Pred_2$, D_3 and $Pred_3$.

$$D_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5 & 2 & 6 & \underline{8} \\ 3 & 5 & 4 & \underline{6} \\ 9 & 6 & 10 & 2 \\ 13 & 10 & 4 & 6 \end{bmatrix} \end{array} \qquad Pred_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & \underline{2} \\ 1 & 0 & 1 & \underline{2} \\ 1 & 2 & 1 & 2 \\ \underline{1} & \underline{2} & 3 & \underline{2} \end{bmatrix} \end{array}$$

$$D = D_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5 & 2 & 6 & 8 \\ 3 & 5 & 4 & 6 \\ 9 & 6 & \underline{6} & 2 \\ 13 & 10 & 4 & 6 \end{bmatrix} \end{array} \qquad Pred = Pred_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & \underline{3} & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix} \end{array}$$

All the best!!!