

an introduction to artificial intelligence: can computers think?

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Richard Bellman

An Introduction to Artificial Intelligence: Can Computers Think?

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Manufactured in the United States of America.

Library of Congress Cataloging in Publication Data:

Bellman, Richard Ernest, 1920—

 An introduction to artificial intelligence.

Bibliography: p.

Includes index.

1. Artificial intelligence. 2. Computers.

I. Title.

Q355.B43 001.53'5 78-9474

ISBN 0-87835-066-7

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preface

Computers are here to stay. Like many other products of technology, fire, the wheel, the printing press, electricity, television, and nuclear energy, they have influenced everyone's life and the influence will increase.

Consequently, it is essential that everyone have some knowledge of what a computer can accomplish, what it cannot, and why. In the following, we try to discuss these questions.

In the first chapter we show how the verbal question, Can computers think? gives rise to many mathematical questions. Furthermore, we show how the mathematician plays an important part in considering these questions. In Chapter 2, we consider the properties of the commercial digital computer which we will use. In Chapter 3, we turn to decision making. In Chapter 4, we show that this methodology can be used to treat many common puzzles. In Chapter 5, we consider decision making under uncertainty. In doing so, we discuss what is meant in many cases by uncertainty. In Chapter 6, we discuss simulation. In Chapter 7, we turn to learning. In Chapter 8, we consider consciousness. In Chapter 9, we consider humor as an aspect of consciousness. The consideration of humor gives us an opportunity to say some words about the paradoxes that occur in logic. These paradoxes have an important bearing on the possibilities that exist for the future use of digital computers. In the next chapter, Chapter 10, we consider local logics. In Chapter 11, we turn to a more mathematical discussion. Here, we are interested in mathematical models of the mind. In Chapter 12, we consider the important problem of communication. Finally, in the concluding chapter we ask the question, Can computers really think?

Throughout, we have used one method. We have shown that many examples of thinking by computer can be regarded as tracing a path

through a network. Obviously, other methods of human thinking exist. The method of tracing a path through a network has not worked well in pattern recognition, language translation, or theorem proving, to name a few. It is possible that a more adroit use of these techniques will be successful. It is more probable that new ideas are required, and it is even more probable that a digital computer may never be able to accomplish these tasks well.

Another way of putting it is that we have studied thinking as rational behavior. We have converted the question, Can computers think? to Can computers exhibit rational behavior? What we have shown is that in many cases we can have a digital computer exhibit rational behavior. But, the point we stress is that the digital computer can only exhibit rational behavior if we understand the process.

In any case, the field of artificial intelligence, as it is often called, is a very fascinating field for the young mathematician. It contains many challenges and many interesting problems.

—R. B.

acknowledgments

This manuscript has gained from discussion with many people.

We particularly want to thank Victor Aladyev, Paul Brock, Kenneth Colby, and Art Lew.

**an introduction
to artificial intelligence**

CHAPTER ONE

can computers think?

There is nothing good or bad, but thinking makes it so.

—Hamlet (Shakespeare)

1. INTRODUCTION

In this chapter we shall make a few remarks about what we mean by the question “Can computers think?” In addition, we shall discuss the roles of the mathematician in dealing with these questions. It will turn out that this apparently simple verbal question conceals many interesting mathematical questions, indeed an endless number. What we shall show is that there is work for many generations of mathematicians in answering these questions and indeed there will never be a final answer. This is very good news to a mathematician, since a good problem is worth much more than a good solution.

2. PRECISE QUESTIONS

A question that invariably creates a debate is “Can computers think?” Yet the interesting point is that in its original form the question is meaningless. We have not made precise what we mean by “think,” what we mean by “computer,” nor even what we mean by “can.”

One of the functions of the mathematician is to formulate precise questions. We have spoken above about the value of mathematical problems. Indeed, his aim is to see how many mathematical problems are contained in this one apparently simple query.

Let us begin then by saying what we shall mean by these terms. By “computer” we shall mean a commercially available digital computer. By “think” we shall mean a performance of activities that we associate with human thinking, activities such as decision making, problem solving, learning, creating, game playing, and so on. Notice that we avoid the morass of trying to define what we mean by human thinking.

The question arises as to how these activities are carried out by the

computer. The answer is that we shall use mathematical theories, theories such as probability theory and dynamic programming, but we shall avoid analysis as far as possible. In the discussion of probability theory, we shall use both the classical form and new forms such as the theory of fuzzy systems. Why we need new forms will be discussed further on. We are more interested now in describing the activity. What is the connection with human thinking? Probably very little. We say "probably" since we have essentially little idea of how the mind operates. Our aim then is certainly not to duplicate human cognitive abilities, nor even to use similar methods.

Finally, let us discuss what we mean by "can." We wish to use standard programming methods that do not require a high level of mathematical expertise. Furthermore, we wish to accomplish these tasks in a reasonable time, one minute, one hour, one day, but not ten years or one million years.

What is interesting is that ingenuity frequently makes a method feasible. Thus, the computer generates a number of novel mathematical problems, as will be described below.

Naturally, with more expertise and more ingenuity, more complicated tasks can be done. However, they are not different in quality from the operations we describe here. In other words, significantly different things cannot be done at present.

These questions are part of an exciting new field, "artificial intelligence." As computers are getting more powerful and cheaper these questions become more and more important. It is obvious that if we have good methods for accomplishing these tasks there are many applications we can think of. Certainly, in the future we will use computers more and more. But it should be remembered, we run computers or computers will run us.

It is essential that every educated person understand how computers operate.

3. THE ROLES OF THE MATHEMATICIAN

At social gatherings, a frequent question is, "What does a mathematician do?" The average person has a good idea of what a doctor or a lawyer does, or how a chemist or engineer earns his living. But he does not have a good idea of what a mathematician does. Indeed this is a hard question to answer to a non-mathematician.

One of the things that a mathematician does is to make problems precise. By this we mean that the mathematician constructs mathe-

mathematical models of various phenomena. With the aid of these mathematical models, various experiments can be interpreted. Galileo said that mathematics was the language of science. The mathematician has done that in physics and many other parts of science. The process is still going on as experiments are done.

In the field of artificial intelligence, one of the main roles of the mathematician is to make questions exact. It is hard to realize that four hundred years ago terms such as "velocity" or "acceleration" were too vague. Today they are quite precise. Similarly, today such terms as "intelligence" or "thinking" are quite imprecise. In what follows we want to show that many well-defined questions about human intelligence can be formulated.

4. DISCUSSION

The purpose of this short chapter is to make meaningful what we mean by the question "Can computers think?" In what follows, we shall describe what we mean by a digital computer, and then we shall discuss decision making, learning, and a few other attributes of human intelligence.

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CHAPTER TWO

the digital computer

God created the integers; man created everything else.

—Kronecker

1. INTRODUCTION

In this chapter we want to describe the properties of the digital computer that we shall employ. How these operations are done is of no interest to us here. Commercially, the operations are done by electronics. Special-purpose computers employ mechanics or pneumatics, and other techniques as well. We shall say a few words about this below.

Back in the old days, about forty years ago, when computers were first developed, people were quite anthropomorphic. Unfortunately, people spoke of giant brains and so on, and some of the terms have remained. We shall point out some of these old-fashioned terms as we proceed, but try to use modern terminology.

2. DESCRIPTION OF A DIGITAL COMPUTER

We think of a machine that can do the elementary operations of arithmetic, by which we mean addition, subtraction, multiplication, and division.

In addition, it can store, retrieve, and print out the results of these arithmetic operations. We shall talk below of how long it takes to perform these operations, how many it can store, and how it prints out this data.

Finally, it can follow instructions to accomplish these tasks. This set of instructions is called a program. Writing a program is not a routine operation and is more of an art than a science. Why this is so will be described in subsequent chapters.

Important parts of any program are how to start the calculation and how to end it, a “stop rule.”

This ability to do arithmetic is what makes the digital computer so important in science. It has been responsible for two scientific revolutions. First of all, it allows us to carry out the calculations required for existing equations. Thus, we can now do easily a great deal of eighteenth- or nineteenth-century mathematics.

Secondly, new theories and new techniques have been developed under the impetus of the digital computer. A new technology produces new theories. In addition, many new questions have arisen.

The digital computer makes theory feasible. We must always keep in mind the dictum of Boltzmann, "There is nothing as practical as a good theory." In particular, the digital computer makes automation possible.

3. COMPARISON OF TWO NUMBERS

A machine that can do arithmetic can compare two numbers. By this we mean it can determine which of two numbers is the larger, and which is the smaller. It follows that a digital computer can determine the maximum or the minimum of a finite number of quantities. We shall employ this property extensively in the following chapters.

One way of doing this is recursively. It compares the first two and then the larger or the smaller with the third number and so on. We shall make extensive use of this property below.

4. BINARY OPERATIONS

It turns out to be quite convenient to operate in a scale of two rather than a scale of ten. This means that every number, which we ordinarily write in the scale of ten, must be converted into the scale of two before the digital computer can perform its tasks, and then the answer must be converted back to the scale of ten. The reason for this is that a zero can be interpreted as a current not going through a component, while a one can be interpreted as a current going through a component. In other words, a zero represents an off and a one represents an on.

When electronics gets better, we can return to the scale of ten, or any other scale that is convenient.

5. COMMUNICATION WITH THE COMPUTER

The computer displays its results usually by printout on a paper. Especially in recent times some of these results may be displayed in the

form of a curve. This is called computer graphics. In general, we don't have good ways of communicating with a computer, either input or output.

Sometimes, particularly in simulation, it is convenient to display some results on a television screen. This television screen is usually called a CRT, standing for cathode ray tube.

6. NEW GAMES

What is interesting is that the existence of the digital computer creates many new games, or problems as we prefer to say, for the mathematician. One of these is the following. In order to obtain a number, it is necessary to make all the operations arithmetic. Thus, for example, if we want to use the digital computer to calculate the square root of two, we must somehow transform this nonarithmetic problem into an arithmetic problem. There are many ways of doing this. The square root of two is a very real geometric entity. It represents the hypotenuse of an isosceles right triangle.

The fact that the square root of two is not an arithmetic quantity caused a great deal of consternation about twenty-five hundred years ago when this was first discovered.

Thus the digital computer requires more mathematical ingenuity, not less. We cannot stress in strong enough terms that the use of a digital computer is seldom routine.

What also should be stressed is that there are many languages available. The fact that the square root of two is a real geometric quantity but not an arithmetic quantity tells us that one language may be more convenient than another for a description of a certain quantity. Thus, the mathematician has a choice of the language in which to work.

In addition to designing algorithms that are specifically designed for the digital computer, there is also the problem of solving an equation under the time and storage constraints.

7. ROUND-OFF ERROR

What is interesting about the digital computer, a symbol for accuracy, is actually that the computer may make a mistake on each calculation. To see this, consider the multiplication of two numbers. If we want to store a number in the computer we must agree to use ten places, or twenty places, or more. But, whatever we agree upon, we must generally use a finite number of places.

When we multiply two numbers, we increase these places. Thus, for example, if we multiply 63 by 14, each a two-digit number, we obtain a three-digit number. If we had two ten-digit numbers, the product could be a nineteen-digit number.

In order to keep the numbers of digits the same, first, we calculate the exact product, then we use a rule to round off the answer. Clearly, this procedure makes little difference if a single multiplication is involved. Unfortunately, in many problems millions of multiplications are involved. If a small error is committed at each stage, the final answer may be meaningless.

It follows that a calculation must be carefully designed so that round-off error does not swamp the final answer. This design of calculations is part of the field called "numerical analysis" and requires, in general, a great deal of ingenuity and mathematical ability. What is so dangerous about the computer is that it may produce numbers that are completely meaningless, and one may not be aware of it.

Obviously we want a precise answer if we are getting a gas, electric, or tax bill. However, for many purposes in applied mathematics, such as weather prediction or geological exploration, we don't need a very precise answer. Unfortunately, the state of mathematics today is such that either we get a very accurate answer or a very inaccurate one. What is needed is a set of theories that yield fairly accurate results quickly.

Because computers can hold only a finite number of digits, even rational numbers like $1/3$ cannot be stored exactly.

8. STORAGE

We have said that a digital computer is a machine that can store and retrieve numbers. Let us now say a few words about how many it can store and retrieve, and how long these operations take.

Obviously, the amount of storage is practically unlimited. It is the time required to find a number that counts. Consequently, we talk about fast storage and slow storage. As mentioned above, in the old days people were quite anthropomorphic and used the terms "fast memory" and "slow memory." Unfortunately, the term "memory" has persisted, despite the fact that the so-called memory of a machine has little in common with human memory. As a matter of fact, we don't even know how many human memories there are, or what mechanism they use. We do know that humans possess associative memories.

Despite the best efforts of many people, it has so far proved impossible to create an associative memory for a digital computer. Let us think then of storage as an operation like storing wood. It is no more sophisticated.

Fast storage and retrieval takes about the time of a multiplication, about 10^{-9} seconds. Slow storage depends upon the mechanism used.

The amount of fast storage is limited. At the present, we can think of about 10^6 ten-digit numbers in fast storage. At first, this seems more than adequate. As we shall show below, it is not nearly enough for the kind of problems we want to tackle in a study of human thinking.

Consequently, there are two limits on the use of a digital computer, time and fast storage.

The "associative memories" mentioned in computer literature (as used for "caches") are small special-purpose storage devices that permit parallel rather than serial access to one of only a handful of numbers.

9. TIME

One important feature of a digital computer is the time that is required to do the elementary operations. At the present, with micro-miniaturization a multiplication takes about 10^{-9} seconds. In other words, we can perform 10^9 multiplications of ten-digit numbers per second. Addition takes about one-tenth of that time. Addition and subtraction take the same time, while division takes about the same time as multiplication. Because of round-off error, we try to avoid division and subtraction as much as possible. It is clear that subtraction of two numbers that are almost equal can destroy the significance of the answer. Thus, if we possessed a number that was accurate to ten significant figures and subtracted a number that was also accurate to ten significant figures, the result could have no significance at all.

10. SPECIAL-PURPOSE COMPUTERS

Many devices have been employed to solve particular problems.

The time requirements and the storage available that we gave are for commercially available computers. These are commonly called "general purpose" computers. In other words, they can be used for many purposes. If we design a digital computer for one purpose, which is to say, one algorithm, we can considerably improve some estimates given above. For example, if we want a device that will solve

only differential equations or dynamic programming processes, we can be much more efficient.

We want to stress that a general-purpose computer is very inefficient. We pay for flexibility with this inefficiency.

11. ANALOG COMPUTERS

One useful idea is the following. We try to find an equation that describes a particular process. Then we try to find a device that is ruled by this equation. Observing this device, we can solve the original problem. This is the basic idea of analog computers, which we shall not discuss.

The advantage of using an analog computer is that it is very easy to consider changes in the equations. Consequently, where an analog computer can be used, it is very effective.

The disadvantages are that the results obtained are seldom very accurate. Furthermore, only certain types of equations can be treated by analog computers. This is due to the limitations of hardware. It is very difficult to get parts that are very accurate themselves. These parts are usually inductances, capacitors, and resistors.

As technology improves, accuracy will improve and we will be able to handle many different types of equations.

The use of a device that yields the same equation as that under consideration is the basis of the Monte Carlo method. Again, a great deal of ingenuity goes into thinking up processes that yield the desired equation.

12. DIGITAL-ANALOG DEVICES

Often, the results of a digital computer are used directly by the human being. In many important cases in industry and in medicine, the results of a digital computer go directly to another machine. How this other machine is hooked into the computer obviously requires a great deal of engineering expertise.

A device of this type is an ideal tool for monitoring, and is used in that way in many hospitals. A patient is connected to a digital computer which in turn is connected to a CRT or a bell or a buzzer. This is not a routine operation. Apart from the engineering involved, we need a mathematical theory of what to look for.

In the near future, computers will play a vital role in surgery. What we want is a computer hooked up to a hologram. Thus, the surgeon can see a three-dimensional picture of what he is operating on.

13. SYMBOL MANIPULATION

A machine that can store and retrieve numbers can store and retrieve symbols. All that is necessary is to have a code that converts a symbol into a number and conversely. This code can be carried out by a simple program. Thus, for example, we can call each letter of the alphabet by its numerical value, A = 1, B = 2, etc. It follows that we can store and retrieve words.

This means that the computer can play a significant role in philology and history, as well as many other fields.

Galileo asserted that the language of science was mathematics, a dictum well substantiated over the intervening years. The outstanding success of this methodology in the physical sciences has held out the hope for many years that a number of aspects of human affairs could similarly be studied by using these procedures. Many people, however, maintained that mathematics had no role in these areas because of the presence of so many qualitative rather than quantitative factors. They believed that mathematical reasoning can be fruitfully employed only in domains where numbers, formulas, and clockwork regularities abound. This belief is fortunately not correct, as we wish to explain briefly in what follows.

Some of this pessimism stems from a basic misunderstanding of the nature of mathematics, some from the usual prejudice that impedes relations between cultures, and some from a fear instilled by Sunday Supplement scientific propaganda frequently centering around computers. In this connection, let us note that it is sad that as a general rule mathematicians and scientists are far more familiar with the humanities than conversely.

Let us describe mathematics as the study of conceptual structures, their transformations over time, and their interactions. If we replace the carefully chosen vague term "structure" by the equally vague term "system" and speak of the study of human systems over time, we obtain a reasonably good quick definition of the field of history.

Accepting the fact that any human activity must possess structure, indeed many different types of structures, it is plausible that any field can profit by the use of mathematical thinking. This was a popular idea during the Renaissance with great influence upon art, architecture and music. The interaction of mathematics with music goes back to the Greeks.

The classic use of mathematics is quite stylized. The structure (sys-

tem, process, etc.) under consideration is first endowed with properties or qualities such as "position," "velocity" and so on which can be described in numerical terms. Some, but not all, of these can actually be measured. This modeling, as the activity is called, of course requires intimate knowledge of the field and a great deal of trial and error.

No mathematical theories are intrinsic; all are superimposed. In order to appreciate the effort involved, it is essential to note that when we view existing approaches, we see only the successes. It is very difficult to estimate the ratio of successful to total attempts.

Next, certain rules are introduced to tell how these numerical quantities change over time. For example, there is the famous law of Newton, $F = ma$, force equals mass times acceleration. Acceleration is change of velocity; velocity is change of position. Using calculus, these statements translate into simple equations (usually differential equations) which can be applied by a scientist, or mathematician, to predict the future behavior of the system, i.e., to predict the future given the present, and sometimes to discover the past. This usually requires an enormous amount of arithmetic, whence the great significance of a digital or analog computer. This device has been responsible for two scientific revolutions, as discussed above. But this is a story in itself.

In many applications the transformation of a symbol depends upon the presence of other symbols. Thus, the rules (algorithms) may read: *A* transforms into *C* if *B* is present, otherwise into *D*, and so on. We can combine this more complex algorithm with stochastic behavior to obtain more realistic descriptions of processes.

It is essentially impossible for the human mind to perform an enumeration of cases systematically for a large number of stages in a process of this nature. The computer can, however, with the aid of simple programs, carry out a thorough examination of cases and display the desired data in various ways.

The methodology we have been describing is known as simulation, described more fully in Chapter 6. The reader versed in the classical uses of mathematics in science will realize that there is nothing conceptually new so far despite the absence of numbers.

We do begin to encounter conceptually novel processes when we do not allow the luxury of specific symbolic description, the existence of explicit rules of transformation, or the presence of a criterion for behavior. Nonetheless, we insist upon decision making. This is typical of much political, economic, business and military decision making.

Although this kind of process can fruitfully be studied by simulation techniques, some new ideas are required which will be discussed briefly in Chapter 6.

Let us give a simple example. Suppose that we wish to study the process involved in a South American or African country changing from one form of government to another.

We might begin by listing several qualities:

- A: strength of Catholic church
- B: economic level
- C: strength of military
- D: climate
- E: strength of middle class
- F: strength of revolutionary movement

In place of a single symbol, we might now use A, AA, or AAA. Thus, A denotes weak influence of Church, AA moderate influence, AAA strong influence; D denotes poor climate, DD moderate, DDD excellent; etc. Thus, we might write some rules:

$F \rightarrow FF$ if B, DDD, and A or AA, and so on.

Each country will possess its own descriptions and its own transformation rules. We can then ask for some long-term trends and predictions.

Similarly, we can study the evolution of certain legal concepts from the Magna Carta to Holmes, the decline of feudalism, the decline and fall of the Roman empire, and in general the workings over time of any identifiable historical, economic, and political forces on a particular system.

It is essential to note that the computer here is a logic machine, exploring the consequences of the data and hypotheses that the experts furnish. Different experts, different predictions.

This also means that we can use the computer for creativity. We shall not discuss this very fascinating topic here since it deserves a work of its own, except in Chapter 9, where we discuss humor.

In the references, we shall give some works that the reader may wish to read.

14. ROLES OF THE MATHEMATICIAN

The mathematician has many tasks in using a digital computer. First, he must formulate the problem. This is often the most difficult stage. Then he must convert the problem into a series of arithmetic opera-

tions. We call this series of operations an algorithm. Then, he must write a program for the computer. In other words, he must tell the computer how to perform the algorithm. Finally, he must interpret the results.

What we want to stress again is that the use of a digital computer is seldom routine. It requires mathematical ability of the highest level for effective use.

15. HYBRID COMPUTER

A computer that combines digital and analog features is called a hybrid computer. Obviously, we can do many more things with a hybrid computer than with a digital computer or an analog computer.

Interesting as hybrid computers are and interesting as the tasks are that they can accomplish, we shall not discuss them here.

16. PARALLEL COMPUTERS

In the above sections, and in the following pages, we shall talk about operations being performed one at a time. That determines the time and storage considerations. Obviously, this is very inefficient. In general, it is very difficult for a mathematician to work with a device designed by engineers. If mathematicians had designed the digital computer from the beginning it would have many more desirable features. Many operations can be performed simultaneously.

Ideally, what we would desire is a digital computer that rearranges its components according to the problems it is solving. Such computers will be available in the future, and they will enable us to handle certain types of problems more quickly.

In general, the time for a method is very short. Consequently, it is desirable to use several algorithms simultaneously and to compare the results. Often, we do not know ahead of time whether a method will work or not. If we have a man-machine interaction, we can stop a method that is not working. If we do not want to watch the results of the computer we can easily write a computer program that will do this. There is a great deal of flexibility with a digital computer.

Many interesting scheduling problems arise in the use of parallel computers. These problems are quite difficult.

17. MAN-COMPUTER INTERACTION

The results of the computer will be used by human beings. Originally the idea was that the results would be untouched by human hands.

Clearly, that is inefficient in many cases. What is much more preferable is that the human can observe the results of the computer and make decisions as to what to do next. Thus, a man-computer interaction is very desirable. How this can be accomplished will not be considered here. Obviously, all the methods described here can be considerably enhanced by this interaction.

Some examples of man-computer interaction will be discussed in the chapter on simulation.

18. COMPUTERS AND SOCIETY

Any change in technology has a corresponding effect on society. Thus, it is to be expected that the computer will have a great effect on the structure of society.

Many sociological changes can be observed. For example, the computer makes automation feasible. Secondly, the computer changes the interactions between the sexes. No muscles are required to operate a computer.

A separate volume would be required to discuss the effects of the computer on society.

19. DISCUSSION

We could go on about time and storage, but it is best to illustrate these ideas by means of specific examples. Consequently, in the next chapter, we shall discuss a particular question that will illustrate many of these points.

As stated in the preface, we shall employ one method systematically. Other methods exist. However, we feel that this method will illustrate the capabilities of the digital computer and the many difficulties we face in employing it.

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Section 3. Very interesting problems arise in finding the maximum or minimum if we impose some structure on the set of numbers. See the book

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We use the scale of 10 because we have 10 fingers. Some cultures use the scale of 20 because of 10 fingers and 10 toes. There are remnants of this use of the scale of 20 in French.

The Mesopotamians used the scale of 12 to avoid fractions. We have remnants of this in 60 seconds in a minute, 60 minutes in an hour, 12 inches in a foot, and so forth.

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Section 7. See the book by Booth and Chien cited above, for an introduction to numerical analysis and further references.

In many cases, we can avoid these difficulties by means of a description of these quantities in arithmetic terms.

Thus, we can avoid the square root of 2 by means of the equation $x^2 - 2 = 0$. We can avoid, as was pointed out by Cauchy, the use of imaginaries by means of the equation $x^2 + 1 = 0$.

There is a great deal of flexibility in the use of a digital computer. Thus, it is very hard to make absolute statements. What we want to do in the text is to give the reader an idea of what the computer can and cannot do.

Section 8. See

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Section 9. The design of a calculation to avoid subtraction and division often requires a great deal of ingenuity.

For example, the exact solution of a linear system of equations is seldom useful for this reason. We have to use many other methods for large systems.

Section 10. Some work has been done on the design of special-purpose computers for special purposes. Unfortunately, because of commercial limitations, these devices have not been put into practice.

Section 12. This monitoring method is particularly valuable in cardiology.

The use of holograms requires the solution of many interesting mathematical problems. For example, we must be able to solve the partial differential equations connected with a three-dimensional region of irregular shape. In addition, many problems concerning pattern recognition arise.

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CHAPTER THREE

decision making

My object all sublime, I shall achieve in time.

—*The Mikado* (Gilbert and Sullivan)

1. INTRODUCTION

In this chapter we shall discuss decision making by computer. We could talk in general terms. However, we feel that it is better to illustrate the methods that are used and the difficulties that occur by means of a particular example. This example seems quite special. Actually, it turns out to be a prototype of many decision processes, as we shall see.

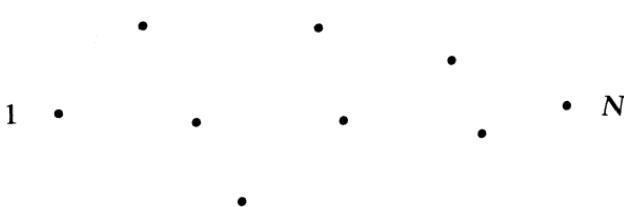
2. AN OPERATIONAL POINT OF VIEW

We shall take an operational point of view, "the proof is in the program." Let us then discuss some particular problems and see if we can get the computer to exhibit intelligence, according to our definition of what we mean by this term.

3. DESCRIPTION OF A PARTICULAR PROBLEM

We begin with what appears to be a puzzle. Actually, it represents a very general process, as we shall see.

Consider a set of cities.

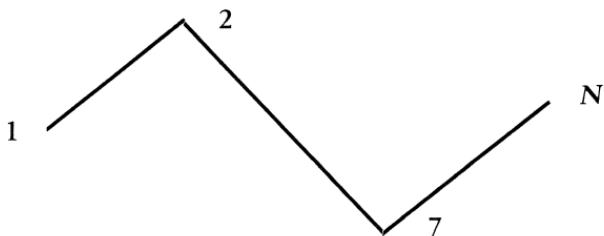


The objective is to start at 1, the initial point, and trace a path to the final point that requires the shortest time. We are given the time array

$[t_{ij}]$ which tells us the time required to go from city i to city j in a direct link. One path, for example, is $1N$, which is to say we go directly from the initial city to the terminal city.

1 ————— N

Another path is $127N$, and so forth.



The procedure that we follow is not the best in all cases. This problem has been studied by many people, and many ingenious techniques have been discovered, because this problem occurs in many contexts. However, the method that we shall follow has many advantages. In the first place, it is easy to understand and to implement, either by hand or by computer. In the second place, it generalizes to many cases, particularly where chance events are concerned, as we shall see in Chapter 5.

4. IMBEDDING

We use a very powerful technique to study this problem. We imbed the particular problem within a family of problems. In addition to studying the original problem, we determine the time to go from any city to the terminal city.

At first sight, it seems very inefficient to solve these additional problems. It turns out that it is easy to get an equation for all the members of the family at the same time.

Obviously this imbedding may be done in many ways. Any particular problem can be studied by various means. We want to stress that anything that can be done one way usually can be done another way.

5. A FUNDAMENTAL EQUATION

Let us then define a function, the minimum time to go from any city i to the terminal city. The time required to go from any city to the terminal city obviously depends upon the city at which we start. The

mathematical translation of "depends upon" is the concept of function. Let us agree to call this function f_i .

The next step is to get an equation for this function, and in order to get an equation we ask for some intrinsic property of the process and convert that property into an equation.

Let us then consider the problem of going from any city i to city N in minimum time. The first question is, "How do we determine the function f_i ?" Now we begin to feel much happier because we have changed the original combinatorial problem over to the problem of determining a certain function. We are now on familiar ground. The way we determine a function is we obtain an equation for this function, the standard way. Thus, the next question is, "Can we find an equation that will determine the function f_i ?"

Of course, we have two problems. We want not only to determine the function but we want also to find the method for determining the optimal path. These two problems are not the same. It might turn out that we could find some method that would give us the time for the minimal path without telling us how to find the minimal path. This is, after all, a very common situation in mathematics. We may not get all the information we need from one equation.

The first problem then is, can we find an equation for f_i ? In order to find an equation, we say, "Is there some intrinsic property of the process that we can use to get an equation?" The answer fortunately is "Yes." Looking at the problem, if we start at city i , we have to go someplace. The first observation to make is that we have to go from i to some city j . We do not know what j is, but we have to go to some j . Now the second observation is that if f_i is the minimum time from i to N , then no matter what city j we go to, we must go from j to N in the shortest time. This is obvious, and a simple proof is by contradiction.

If the original path from i to N were a minimum path, then the part of the path from j to N must be a path of minimum time too, because if it were not we could convert it into the path of minimum time and make the total time shorter, which it can not be, because f_i is the minimum time by definition of the function.

This is an essential intrinsic property of the process: the tail of the process, the end of the process, is always a path of shortest length. What does that mean as far as the equation is concerned? We say it means this: if f_i is going to be the time required to go from i to N along the shortest path, it will consist of t_{ij} , the time required to go from i to j (we do not know what j is yet), plus f_j , the shortest time from j to N for some j .

Now the question is, "What j do we choose?" We choose the j that minimizes the sum $t_{ij} + f_j$. We get then the equation

$$f_i = \min_{j \neq i} (t_{ij} + f_j), \quad f_N = 0, \quad i = 1, 2, \dots, N - 1. \quad (3.1)$$

This is a fundamental equation. It is the equation that extends to all kinds of applications, as we shall see in subsequent chapters.

6. GEOMETRIC IDEAS

What we have done is convert a geometric problem into an analytic problem. Frequently, in applications it is important to go in the other direction to convert an analytic problem into a geometric problem. In general, one has to use both techniques, analytic and geometric. This is where mathematical ingenuity and training are important. The more one knows, the better one is at attacking problems.

We have stressed the analytic approach here because the digital computer does arithmetic. If we had a device that could do geometry as well as a digital computer does arithmetic, we would use a geometric approach.

7. CONVERSION OF A DECISION PROCESS INTO AN EQUATION

We have used a fundamental device. We have converted a decision process into an equation. Obviously, there are many equations that can be used. In other words, there are many ways of doing this. How we do this depends upon ingenuity. There is no getting around the fact that mathematics requires ingenuity. We try to use general methods, and this is very desirable.

When we teach, we very seldom describe how a method was really discovered. Partly we do this because it is easier to give a logical method, partly we do it because it would usually be embarrassing to describe how we really found something.

8. THE CONCEPT OF A SOLUTION

This, of course, does not complete the problem. It is, however, the beginning of the second phase. What we have to show is that this is a useful equation. It may turn out that it is a tautology, an intrinsic part of the process that does not tell us anything that can be used to determine the function f_i .

At least, we have made ourselves happy because we have changed

the original combinatorial problem into a problem involving a non-linear equation for the unknown function. There are four questions that we can think of immediately:

1. Does a solution exist?
2. Is the solution unique?
3. How can it be determined?
4. What is the connection between the solution and the original problem?

It might very well be that there is no connection at all between the solution of the equation and the original combinatorial problem. This is a standard problem that one always has in science. When we change a scientific problem into a mathematical problem, we may have lost a great deal in the process. Let us, therefore, discuss this last question first. It is the most important at the moment. The first two are just technique: the third is a different kind of problem.

What does equation 3.1 determine? It determines two things: first of all it determines the minimum time; secondly, it determines the minimal path, because the question that we ask ourselves at i is what city we go to next. There are two questions then:

1. What is the minimal time?
2. What is the minimal path?

This last question can be answered in two ways. One is to say: We start at i then we go to some city i_1 , then i_2, i_k and then N .

This is your minimal path. Another way of answering the question is to say: That isn't the way we want to give instructions as to how to proceed. All we want to do at every city i is to furnish a rule as to where to go next, that is to say: if we are at city i , we go to j according to the function $j(i)$. Thus, there are two forms of the solution: one is the actual set of indices, the actual set of numbers $[i, i_1, i_2, \dots, i_k, N]$; the other is a rule, a *policy*, which tells us in terms of where we are now, what to do next. We see the connection with decision making. This, of course, can be interpreted as a decision. What we are saying is that we are looking at this as a multi-stage decision process.

What is interesting is that the set of indices $[i, i_1, \dots, i_k, N]$ is the conventional mathematical approach, the conventional approach of mathematical analysis. The function $j(i)$ is the approach of dynamic programming, a new mathematical theory, but it is also the intuitive approach. If we were not educated mathematically, this is the way we would think of it.

9. DETERMINATION OF THE SOLUTION

Let us now see if we can determine the answers to the two questions, the determination of the function f_i , and the optimal path, at the same time. Equation 3.1 really determines two functions. It is obviously an equation for f_i , but it is also an equation for $j(i)$. If we have the function f_i , then the j that minimizes tells us what city to go to next. Thus, we have one equation that determines two functions. This is strange at first sight, but it's what we want. We want an equation that determines the actual time, but we also want an equation that determines the policy—the optimal policy.

Now, which is more fundamental? Is the function more fundamental or is the policy more fundamental? We would say that the policy is more fundamental because the concept of minimum time does not carry over to situations in which there is no ordering. If we do not have complete ordering, if we do not have a concept of the best or the worst, we cannot talk about a variational problem. But if we have a situation where we have to make decisions, where we have a multistage decision process, we can still think in terms of policies. Thus, the more fundamental concept which carries over to many different fields of decision making is the concept of policy.

10. DETERMINATION OF A NUMBER

In using digital computers for making decisions, what we expect is a policy. We have converted this problem of thinking by computer to the problem of determining a policy. The determination of a policy will involve a number and it is important to realize that the problem of decision making has been converted into a problem of generating a number. This is why digital computers can be used for certain types of decision making. If we cannot reduce a problem to numbers we have great difficulty in using computers. The new theory of fuzzy systems due to Lotfi Zadeh can be used in certain cases, but little has been done in this direction using computers. Consequently, we will say very little about this theory here.

11. DECISION MAKING BY COMPUTERS

Let us now review how we are going to tackle decision making by computer. What we are going to try to do is take a particular problem and show that it can be considered to be a multistage decision process, and furthermore a multistage decision process of essentially the type

we have been talking about. If we can show that a particular problem is a multistage decision process of this type, then we can get an equation for the function f_i . If we can get an equation for the function f_i , we can, theoretically at least, determine the optimal policy. If we can determine the optimal policy, then we have a way of solving this problem by means of computers.

Now, in order to solve the problem by means of a digital computer, we have to show that there is an arithmetic algorithm that can be used. If we look at equation 3.1, we see that it is nonlinear but that the operations are arithmetic. We have a sum and we have a minimum over a finite set. Both of these operations can be readily carried out using a computer. The difficulty, however, is that we have the unknown function on both sides. What we have to do is convert the equation into a recursive algorithm; we have to introduce iteration, an operation that the digital computer performs very well, in some fashion.

In this case, iteration means that we have to employ successive approximations in some way. There are many ways in which successive approximations can be used. Any application of successive approximations requires that we examine the storage question. What we want to do is to hold one approximation in storage while we calculate the next. Thus, the method will be analytically sound in all cases, but if the storage requirements are too great we cannot employ the computer. There are many ways that we can reduce the storage requirements. However, we shall not discuss them here as these require a great deal of mathematics.

Successive approximations can be used in many ways. Not only do we have the storage problem but we also have to establish convergence, and convergence to the solution we want.

12. ENUMERATION

There are several ways of solving the particular routing problem using the computer. The simplest method, and the most sensible, is enumeration. If we have a digital computer, why not just explore all possible paths?

Let us discuss enumeration. How many different paths are there? It is easy to convince ourselves that there are at least $(N - 2)!$. We can go from the first city to any of the other $N - 1$ cities, then to any of the other $N - 2$, and so on. Hence, we have at least $(N - 2)!$ paths. These are the paths that go through every possible city. We also have

to consider those that omit one city, two cities, etc., but let us ignore these. This number $(N - 2)!$ is an interesting number. It gets large rather quickly. Ten factorial is a convenient unit of how large factorials are. That is only 3,628,800 possibilities. Consider, however, 20!. We see that it is obviously larger than $10^{10} \times 10!$. If we had a good, efficient way of enumerating and we wanted a time scale, we would say: whatever time it took to search through 10! possibilities, 20! possibilities would take us at least 10^{10} times as long.

Another very good unit is one year, equal to approximately 3×10^7 seconds. If we want to have some idea of how long it takes us to perform a certain number of operations, and each operation takes one second, then 3×10^7 seconds is one year. If we change the seconds to microseconds, and then multiply by 10^{10} and so on, we get a very large number of years. Think now of how large 100! is. It is a number beyond belief, yet that is still a consequence of only a very small network. Consider where the question involves 1,000! . . . 10,000!. These are numbers beyond imagination.

Enumeration remains the best way to solve problems, if we can do it. Of course, mathematicians do not like to solve problems by enumeration (technological unemployment!), but if we have a fast digital computer, why not solve problems by enumeration? The answer is that most problems cannot be tackled by enumeration because you rather easily start getting into numbers like 1,000! or 10,000!. These are small numbers compared to some encountered in combinatorial problems. We can easily get numbers like $2^{100}!$ or $2^{1000}!$ in pattern recognition.

Thus, we cannot say, "We can always solve problems by enumeration." It is just not true.

Even when enumeration is possible, we have to worry about error build-up. Thus, a solution by enumeration may be theoretically possible, but practically impossible.

13. SUCCESSIVE APPROXIMATIONS

Enumeration is not usually feasible, as we have discussed. Although it is theoretically possible to use enumeration, this means that we must use some mathematics. When people say that a problem can be solved by looking at all the cases, we have to say, "What do you really mean by 'can'?" It may be a finite process, but it is outside of any present time scale.

Hence, we must talk about other methods, say successive approximations. There are two general ways to approximate. The classical approach to solving a nonlinear functional equation is to approximate in function space. We select some initial function for f_i and then we iterate the equation,

$$f_i^{(k+1)} = \min_{j \neq i} [t_{ij} + f_j^{(k)}]. \quad (3.2)$$

This is now a recursive approach. We can show convergence for any choice of a nonnegative initial function. This is then a feasible analytic approach. The only constraints are storage and the possible slowness of convergence.

One of the nice things about these multistage decision processes is that we have a second approach—approximation in policy space. It turns out that it is much more important to think in terms of policy spaces because in general the function is an artificial construct. There is nothing intrinsic about the minimum time, but there is something intrinsic about the optimal policy. This is where intuition and experience really help. We always know a great deal more about the process than we can ever state explicitly in mathematical terms. This helps in guessing approximate policies.

14. APPROXIMATE POLICY

What would be an approximate policy for this particular routing problem? One type of policy that we could have is to say: we go directly. This gives an approximation $f_i^{(0)} = t_{iN}$.

Another approximation is to say, we stop at most at one city. That gives us another approximation. A further approximation says we stop at most at two cities, and so on. One attractive feature about an approximation in policy space is that we always get an approximation from above. In this case, if we are trying to find the minimum time, any policy gives an approximation to the actual time from above. The time obtained from any approximate policy must be at least as large as the minimum time.

One of the difficulties, in general, with approximation in function space is that the function that we guess initially need not actually correspond to a policy. If we approximate in policy space, however, every policy produces a function, despite the fact that not every function comes from a policy.

15. WRITING A PROGRAM

The point we want to indicate, without going into details, is that by these methods of successive approximation we end up solving an equation like 3.2, and that is an arithmetical operation. Thus, it is an operation that can easily be carried out on a digital computer. We think we can see that the computer program required to solve an equation of this type is a very, very simple one. It can be written by a beginner; a person who has just learned something about Fortran programming in a couple of hours could write a program for that equation.

With a little bit more experience, we can write a little bit better program, but there is no need for any sophistication or expertise. We might also say that for $N \leq 100$ or so, it is a hand calculation. We have had people test this for about a hundred different cities. This would take about an hour, using a hand calculator. If one can do the calculation by hand in one hour, a digital computer could do it in one second or something like that.

16. TIME

Generally, the execution time of a program is provided as part of the accounting printout, but only to a precision of, say, one-hundredth of a second. For programs running less than this time (0.01 sec.), we cannot tell how long they really take. A minimum charge for computer usage, the equivalent of several seconds of computer time, is assessed against such small programs. Computer charges are commonly also made for memory and input/output usage.

17. STORAGE

There is one interesting question of feasibility—the question of storage. We are talking about ordinary, commercial digital computers. How do we store the distance array? There are several aspects to this question. If it is an actual geographical problem, where the times t_{ij} represent ordinary distances, then we do not have to store them at all. We just store the set of cities and an algorithm for computing the distance given each two cities:

$$t_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2}$$

In other words, we generate rather than store. In any particular problem, it is a very interesting question whether it is more efficient

to generate rather than store. Here is where experience and mathematical knowledge count. Often, a problem is feasible only if certain functions are generated rather than stored.

It is interesting to note that to solve problems involving, say, certain trigonometric functions on paper, we commonly look up values in tables stored in handbooks. To determine the same value using a digital computer, a stored library program may be called upon to generate the answers (usually by means of a power series formula.) If we wish to know a sequence of adjacent answers, instead of using a power series (polynomial), we may instead solve a differential equation.

18. DIMENSIONALITY

The second point is that, in general, $t_{ij} = \infty$. That is to say, if we have a network, it is not possible to go from one city to every other city in one stage. We can only go from one city to other nearby cities. Thus, most of the time t_{ij} is going to be ∞ . Therefore, when we are storing the matrix t_{ij} we only have to store the non-infinite elements with the convention that, if we cannot find a value, it must have the value infinity. These are some of the local tricks and devices that we use when we are trying to solve high-dimensional problems.

If we were to ask, what is the principal difficulty of applying these analytic techniques to problems of decision making (problems of problem solving), the answer is dimensionality. As we shall show, there is no difficulty in applying this formalism to all kinds of problems in the field of artificial intelligence and decision making.

The difficulty is that, in general, it does not yield a feasible approach. Just as enumeration was not feasible for the routing problem, so this type of approach will not be feasible for problems of very high dimension, because we get into storage and time difficulties. These are new and formidable mathematical problems.

19. DETERMINATION OF SECOND BEST PATHS

The same methods, slightly modified, can be used to determine the second best, third best, and so on, paths.

Often, there are other considerations which we have not included in the original equation. Consequently, it is important to know the best path, the second best path, etc.

20. STRUCTURE

What saves the day in many cases is that the network has structure.

How this structure is used is a matter of mathematical ingenuity.

In some cases, we do not want to examine the structure. Thus, for example, we have said above that it is not possible in general to go from every city to every other city. Thus, the equation should read

$$f_i = \min_{j \in S(i)} (t_{ij} + f_j) \quad (3.3)$$

The set $S(i)$ tells us what cities to examine. This information will considerably reduce the search. However, it requires a preliminary examination of the network. If we do not want to do this, it is practical to use a very large number for forbidden paths. Either we can use the largest number available in the computer, or we can determine a number from an approximate policy. This avoids any geometric consideration of the network. As we said above, sometimes we want to do it and sometimes we do not. It is nice to have a choice.

21. DETERMINISTIC SCHEDULING

Another interesting problem is that of scheduling. We could study these problems profitably in the framework of a hospital. We could ask: Can we use computers to do decision making, or thinking, in a hospital situation? This is a very precise problem.

Suppose we have a patient who has to go to various places for various purposes. He may have to have an X-ray taken, he may have to have a blood test, and so on. A very interesting question is: If we have a patient who requires a certain number of examinations, certain types of tests, etc., how do we schedule him through the hospital in such a way as to minimize the total time required, or to make most efficient use of all the resources, or to have as many patients as possible go through the facilities? This is a very real problem in connection with the national health care of patients. In general, at the present time, this is done by a person. A patient enters and there is a nurse or doctor who says: I want you to go first here, then here, then here and so on.

It would be very nice if one could have a computer do the thinking, have a computer make the decisions, and have a computer do the scheduling.

We can think of this as a problem of the following type: we have to trace a path through a network. We can make the problem more interesting by saying: some places have to be visited before others, i.e., if we are in one place we can go only to some nearby place. One thing we want to point out is that in general, when we face an optimization

problem, constraints in the usual formulation cause a great deal of difficulty. To the dynamic programming approach, however, this normally causes no difficulties, and indeed actually speeds up the solution, because it simplifies the search for an optimal policy.

If we think of that for a moment it should be true that the more constraints there are the easier it should be to determine the optimal solution because of the smaller number of possibilities that have to be examined. If this is not true for the ordinary approach, that means that something is wrong with the ordinary approach. There should be an approach in which the more realistic constraints there are, the quicker it is to scan the allowable possibilities. If constraints are present we have the following equation:

$$f_i = \min_{j \neq i} (t_{ii} + f_j) \quad j \in S(i). \quad (3.4)$$

Instead of minimizing over all possible j 's at each state, we are minimizing only over a small set, $S(i)$, of j 's. This reduces the search time for the computer algorithm considerably. Thus, the more constraints there are in the scheduling problem, the more realistic the formulation of the problem, the better off we are using dynamic programming. This is an essential point.

Let us return to what we were talking about. We showed that in a deterministic problem of this type we can have a computer do the scheduling. All we have to do is to input a set of places that the patient has to visit, the time required at each, the constraints, and the computer will determine the optimal path using any of a number of available mathematical techniques. Thus the computer can start "thinking" and determine the efficient path.

We can, of course, say that the computer is not doing it in the way the human being does it. In the first place, we do not know what the human being does; in the second place, we do not really care, at the moment. We have not pledged ourselves to do exactly what the human being does to solve this problem. We are trying to get an efficient solution; we are trying to obtain an efficient thought process.

22. HEURISTICS

If the storage capacity of the digital computer is exceeded, we have to use approximate policies. These approximate policies, particularly in artificial intelligence, are called "heuristics." The determination of

these approximate policies is not easy. Often we are guided by experience with the actual process.

In general, many interesting problems arise in the determination of these approximate policies. The determination of these approximate policies depends upon the structure of the process and how we wish to describe it.

23. DISCUSSION

In the previous pages, we have shown how a digital computer can be used for decision making in certain cases. One of the tasks of a mathematician is to recognize that a given problem is of this form. What would be desirable would be a computer method that would recognize a problem. At present, we seem far from this.

Consequently, at the moment we need the mathematician to formulate the problem for the computer. Even if we solve this problem, and there are signs that we shall, we need the mathematician to determine which problems are worth solving. There is no way of getting rid of this pernicious breed. The computer requires more mathematicians, not less.

In the next chapter we shall show how the same procedure can solve some familiar puzzles.

We have spent a considerable amount of time on this problem for several reasons. In the first place it illustrates many important points about the use of digital computers. In the second place, the same method can be used for quite general systems. Naturally, the method cannot be used automatically. It requires that a great deal be known about the system.

Let us summarize the essence of the method. We must describe the system by means of numbers, state variables, and the possible constraints on decisions; then we must describe the effect of a decision, action, upon these state variables; finally we must state the objective, the criterion function, explicitly.

BIBLIOGRAPHY AND COMMENTS

Section 1. We are employing the theory of dynamic programming in what follows. However, there is no reason to bother the reader with various mathematical details. The reader who wants to learn more about the theory may wish to consult the following books:

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The routing problem can be used to obtain approximations to general control processes. See

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A basic question in structural theory in all fields of culture concerns the reconstruction of evolutionary or cladistic trees on pathways by inferences from the characteristics of organisms, systems, or data surviving at the present time. Let us cite the fields of biology and anthropology, the use of fossils in the case of archaeology, and the domain of philology.

In recent years, methods have been developed for deducing trees that satisfy the condition of requiring a minimal number of evolutionary steps or changes in characters to explain the evolutionary history of the set of existing structures. See

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The principle of minimum evolution or "parsimony" is generally assumed in these papers as a suitable hypothesis in the absence of empirical laws of evolution. General algorithms for these "most parsimonious" trees have not been completely studied, although algorithms for close approximations called "Wagner trees" do exist. See

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from protein amino acid sequence data under some restrictions about all distance measures is given by

W. A. Beyer, M. L. Stein, T. F. Smith, and S. M. Ulam, "A Molecular Sequence Metric and Evolutionary Trees," *Mathematical Biosciences*, vol. 17, 1973, pp. 444-461.

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R. Bellman and S. Dreyfus, *Adaptive Dynamic Programming*. Princeton University Press, Princeton, N.J., 1961.

Section 4. We have used the concept of imbedding in many contexts. In mathematical physics it can be used widely. See

R. Bellman and G. M. Wing, *An Introduction to Invariant Imbedding*. John Wiley & Sons, New York, 1975.

Section 5. What we have done here is convert a combinatorial problem into the problem of solving an equation. In many cases, we want to go in the other direction and take advantage of the geometry in order to do the analysis.

It should be stressed that a well-trained mathematician is familiar with algebra and geometry as well as analysis. In solving a problem, he often has to use all of these.

Section 7. It is very interesting to observe a given behavior and see whether this behavior can be interpreted in terms of some optimal policy. This leads to many interesting problems which are often called "inverse problems." See for example the paper

R. Bellman, "Dynamic Programming and Inverse Optimal Problems in Mathematical Economics," *Journal of Mathematical Analysis and Applications*, vol. 29, no. 2, 1970, pp. 424-428.

Many further references are given in this paper.

Often, a particular equation can be considered as a minimization equation. In this way, we get upper and lower bounds which are very valuable. See, for example,

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The conventional approach described in the text in the continuous version is that of the calculus of variations. For a discussion of these matters, see

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The approach of dynamic programming is a dual approach to that of the calculus of variations. The calculus of variations regards a curve as a locus of points. Dynamic programming regards a curve as an envelope of tangents.

We can use the computer by means of a very simple program, to convert one solution into the other.

Section 9. In this case, the equivalence between a policy and the function f_i is the fundamental duality of Euclidean space. The locus of points is the same as the envelope of tangents. In the stochastic case, this equivalence does not hold. We shall discuss these matters in Chapter 4.

Section 10. See

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See also

R. Bellman and L. Zadeh, "Decision-Making in a Fuzzy Environment," *Management Science*, vol. 17, no. 4, 1970, pp. 141-164.

Section 13. Rigorous details may be found in the R. Bellman/K. L. Cooke/J. Lockett book mentioned above.

Section 17. These questions are discussed in detail in

R. Bellman and S. Dreyfus, *Applied Dynamic Programming*. Princeton University Press, Princeton, N.J., 1962.

R. Bellman, *Methods of Nonlinear Analysis*, vol. 1. Academic Press, New York, 1970.

Section 19. For many reasons we may want to know alternate paths. As mentioned in the text, similar procedures can be used to determine alternate paths, second best, third best, etc. See the paper

R. Bellman and R. Kalaba, "On k th Best Policies," *J. SIAM*, vol. 8, 1960, pp. 582-588.

Section 21. Scheduling problems are very difficult and very interesting. They give rise to many new mathematical problems. See the papers

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L. D. Stone, *Theory of Optimal Search*. Academic Press, New York, 1975.

Section 22. Algebraic topology can often be used to determine the structure of an approximate policy.

Section 23. The methods used here can also be used to study many location problems. This problem in mathematical literature is called "the problem of Steiner." See the paper cited above.

The method can also be used to locate "missing links." It has application in many fields. In philology, see

A. Kaufman and R. Faurre, *Introduction to Operations Research*. Academic Press, New York, 1968.

The general problem of finding a logical path from one phenomenon to another leads to many interesting mathematical problems.

In addition to general decision making problems, numerous specialized problems in artificial intelligence (e.g., pattern recognition) can be solved using the same methods. See

K. S. Fu, *Sequential Methods in Pattern Recognition and Machine Learning*. Academic Press, New York, 1968.

The method can also be applied when the state variables are sets. See, for example,

R. Bellman and B. Gluss, "On Various Versions of the Defective Coin Problem," *Information and Control*, vol. 4, 1961, pp. 118-131.

R. Bellman, "An Application of Dynamic Programming to the Coloring of Maps," *ICC Bulletin*, vol. 4, 1965, pp. 3-6.

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CHAPTER FOUR

puzzles

Man never shows more ingenuity than in his games.

—Leibniz

1. INTRODUCTION

Let us now apply the method of the preceding chapter to some familiar puzzles. Puzzle solving is one of the attributes of human intelligence. Consequently, it is important to show how a digital computer may be used to solve these puzzles. We shall consider two classical puzzles, the wine pouring problem and cannibals and missionaries. In addition, we shall consider a very amusing game invented by Lewis Carroll, and the Chinese fifteen puzzle. Finally, we shall say a few words about chess and checkers.

2. MATHEMATICAL ABSTRACTIONS

We want then to use the power of mathematics; of mathematical abstraction.

We gave the routing problem in terms of tracing a path through a network when, in actuality, what we were talking about was N states. We called these N states the numbers 1 to N , just for convenience. We could also have called them p_1, p_2, \dots, p_N and so on, and we could have used Greek letters or German letters, etc. These are dummy symbols. It does not make any difference what symbols we use.

What we are really talking about is the problem of transforming a system in state space from some initial state to some terminal state. The problem then is, in any particular case, how to determine the state space and how to determine the transformations. This is where mathematical ability and experience come in. We have this mathematical abstraction at our disposal but any particular realization is a trick. In any particular case, we have to look at the problem and determine what the states are, what the space is, and what the transformations are.

3. INTELLIGENT MACHINES

Hence, when we are talking about intelligent machines and artificial intelligence, there really are many levels of problems. One problem is to say, we as mathematicians shall teach a computer, we shall tell a computer how to solve this problem. Another problem which, at the moment, is completely beyond our ability is how to read this problem in ordinary English speech with a program that would tell the computer how to convert this problem into a dynamic programming problem. We do not think we are that far away from this at the present time, but it is a different problem. It is important that we realize the levels of problems.

4. THE WINE POURING PROBLEM

Let us now turn our attention to a puzzle that has amused and bemused people for hundreds of years.

Two men have a jug filled with eight quarts of wine which they wish to divide equally between them. They have two empty jugs with capacities of five and three quarts respectively. These jugs are unmarked and no other measuring device is available. How can they accomplish the equal division?

We shall show how the techniques of the preceding chapter are applicable.

5. FORMULATION AS A MULTISTAGE DECISION PROCESS

In stating the pouring problem, several assumptions were made tacitly. Let us see if we can make them explicit. First, we assume that no liquid is spilled (a conservation requirement), that none is lost due to evaporation, and that none is drunk (a prohibition requirement).

The fact that no measuring device is available means that the only operation that is allowed at any time is that of pouring wine from one jug to another. The pouring is stopped when one jug is empty, or the other is full, whichever event happens first.

Thus, for example, the first operation might be to pour five quarts from the full jug into the previously empty five-quart jug, or it might be to pour three quarts from the full jug into the previously empty three-quart jug. No other initial operation is possible.

It remains to identify the process, by which we mean we have to

choose state variables. Let us agree that the amounts of wine in each jug will specify the process. Next, we must see how decisions, actions, affect these state variables. It is easy to see the effect of various pourings.

Let us now examine how a digital computer can be used. In the first place, we want to examine the storage requirements. We see that there are four possibilities for the first jug, six for the second, and nine for the third, a total of $4 \times 6 \times 9 = 216$. This is a modest requirement for a digital computer.

However, this figure may be considerably reduced. Let us observe that it is sufficient to specify the amount of wine in two jugs. Since the total amount stays the same, it is sufficient to specify any two jugs. Naturally, we will choose the smaller. Thus, we have a total of 24 possibilities plus a small calculation. This is a very important reduction in many cases.

Once we have determined the state variables we can have the computer print out the results of one pouring, two pourings, etc. We can have the computer stop when no new states are generated. This will automatically determine what states are reachable. If we do not want to proceed that way, we can introduce a distance function. In other words, we can convert the original problem to that of getting as close as possible. This is quite a common device by which in many cases we avoid the difficult problem of determining whether a certain operation is possible.

What we have sketched above is essentially a solution by enumeration. If we do not want to employ this method we can use the functional equation approach of the preceding chapter.

6. CANNIBALS AND MISSIONARIES

In this section, we wish to consider the following classical conundrum: "A group consisting of three cannibals and three edible missionaries seeks to cross a river. A boat is available which will hold at most two people, and which can be navigated by any combination of cannibals and missionaries involving one or two people. If the missionaries on either side of the river, or in the boat, are outnumbered at any time by cannibals, dire consequences, which may be guessed at, will result. What schedule of crossings can be devised to permit the entire group of cannibals and missionaries to cross the river safely?"

Our experience with the wine pouring puzzle suggests the desira-

bility of looking at this puzzle, too, as a succession of transitions from one state, or condition, to another, and indeed such an interpretation is possible.

7. FORMULATION AS A MULTISTAGE DECISION PROCESS

Let us follow the same procedure used in the wine-pouring problem.

We first introduce state variables. In this case, the numbers of cannibals and missionaries on each bank and in the boat can be used as state variables. This yields a small number of possibilities. However, once again, we observe that we have a conservation condition that can be used to reduce dimensionality. The number of cannibals and missionaries stays constant. Consequently, we can use either the numbers on each bank, or the number on one bank and the number in the boat. Again, this is an important reduction in dimensionality.

The effect of a decision upon these state variables is readily determined.

Again, we have the problem of determining whether the original puzzle can be solved. Either of the methods sketched above may be used.

8. CHINESE FIFTEEN PUZZLE

Let us now turn to the Chinese fifteen puzzle. This is a disease that becomes epidemic every once in a while. We have fifteen squares with a number on each and a blank space X into which we can put a nearby square. The problem is to manipulate the blank space in such a way that the 14 and 15 are reversed. Occasionally, in newspaper advertisements, you see somebody offering a prize of ten thousand dollars to anybody who can do it. They are very safe because we can prove mathematically that it is impossible. Simple theoretical considerations will show which configurations are possible and which ones are not. Suppose then we take this task as a puzzle and say that we want our intelligent computer to solve this type of problem. Let us take a very simple version of it, and then we will explain why we cannot solve the original puzzle at this time using dynamic programming in any direct fashion. Suppose then we take a three-by-three version. We have the numbers 1, 2, 3, 4, 5, 6, 7 and 8, originally given in the order shown in figure 4.1. The question is: can we slide the sections around so that we get the numbers 8 and 7 reversed? This would be a test of decision making.

1	2	3
4	5	6
7	8	

FIGURE 4.1

9. THE PUZZLE AGAIN

Now, let us look at the puzzle. We will use the symbols as in the figure below. One of them denotes the blank space.

1	2	3
4	5	6
7	8	9

FIGURE 4.2

Each of these corresponds to a state. What are the allowable transformations? Wherever the blank is, we can change one of the states into at most four adjoining states, depending upon where the blank is. Sometimes there are only two or three adjoining states. Thus every transformation takes us from the original state to one of four at most nearby states. What is the “time” required? The time is a single operation.

10. FEASIBILITY

We originally asked the question, “Can it be done?” which is actually a harder problem than saying, “What is the shortest way of doing it?” Hence we solve the problem of “Can it be done?” by saying “What is the minimum number of operations?” Of course, one gets into such difficulties as, “How does one know whether it can be done at all?”

Let us assume that we are in an initial state that can be converted into the desired state. Then we can safely ask, “What is the minimum

number of operations?" Given any initial position, using the algorithm we mentioned above, we have a way of determining how one shifts the blank around to get the desired final position. We assume that we do not cheat, that we give a position that can be transformed into the final position.

11. DOABLE POSITIONS

The second part of the problem is, how do we know which are the positions that can be transformed? There are several ways of doing this. One way is to look at the time. We can get an upper bound on the time required. We did not mention this, but when we have N points, it turns out that N is an upper bound for the number of iterations required when we use an appropriate method of successive approximations. Hence, if we count the number of possible positions of this type, and set the program going, if the calculation takes more than N time units, we know it is impossible to solve the original problem.

12. ASSOCIATED QUESTIONS

Another way of approaching this, avoiding the problem of whether it is feasible or not, is to ask a different question. Instead of asking for the minimum time, we could ask: How close can we come to a given position? One of the ways we can measure distance is to say, we look at the number of symbols that are not correct. We want to get as close as possible to the specified configuration. This is a very important consideration in pattern recognition. We want to get one pattern to represent another as closely as possible.

We can use this technique in many puzzles to avoid the question of whether a puzzle can be done in its original form.

13. THE ORIGINAL PUZZLE

Why can we not use the foregoing methods on the Chinese fifteen puzzle? We are stymied by storage considerations. A simple calculation shows that there are $16!$ possible states. There are $15!$ states having the blank in the lower right corner, but $16!$ states into which the blank can move. Hence, it is easy to write down the relevant functional equation. However, we cannot use existing computers to resolve it. This dimensionality difficulty is typical of many combinatorial problems. It is easy to write down equations, but we cannot use digital computers to treat these equations routinely. It is essential to reduce

the dimensionality by using the structure of the process. In other words, we must use mathematical training.

14. LEWIS CARROLL'S GAME OF DOUBLETS

Lewis Carroll invented the game of "Doublets." The idea of the game is to construct a chain of English words connecting two given words subject to the condition that each word differs from the preceding word by the change of exactly one letter.

There is no difficulty in writing down an appropriate functional equation using the method given above. To resolve a given problem we may need an unabridged dictionary. Obviously, there will be a different problem for each language.

15. CHESS AND CHECKERS

Let us say a few words about chess and checkers.

Unfortunately, the two are often lumped together. As we shall see, they should not be since there is a vast difference in dimensionality between checkers and chess.

Let us begin with checkers. If we regard each position as a state, we see that the effect of a move is to change one state into another. Consequently, there is no difficulty in writing down an equation depending upon whether it is white's move or black's move. The basic question in using a digital computer is that of storage.

This means that checkers is a solvable puzzle. If we use the structure of the process, we can lower the storage requirements considerably.

Let us turn to chess. As before, there is no difficulty in writing down the relevant functional equation. However, the number of states, positions, is so great that this is not a feasible procedure, either now or in the near future.

How then does a chess master play chess? How does he recognize a favorable position and determine a good move? We wish we knew. Some people are born with the magical ability to play chess well. We can train people to play chess much better than they do, but we cannot make a chess genius.

The question is that of gestalt. If we understood this, we would know how to perform pattern recognition and language translation. At present, and most probably forever, we cannot use a computer to recognize structure. However, there is no proof of this conjecture, and it may well be possible that tomorrow someone will find a way to use a

computer for this purpose. If we can make this structure explicit, there is no difficulty. This means that we have considerable difficulty in using a computer for interviewing or for reading tissue smears for cancer.

16. SOLVING PUZZLES BY COMPUTER

What we want to indicate here is that we have very systematic ways for solving puzzles. We can, of course, object on many grounds. We can say, we give a method of solving puzzles by mathematics. Is this the way a human being would solve puzzles? We reply that we do not care. This is not the problem that interests us. In the first place, we do not care, and, in the second place, we do not know how humans solve problems. One good reason we do not care is because we do not know.

17. OPERATIONAL PHILOSOPHY

In general, one should not care about problems that one cannot answer. This is a very useful bit of philosophy. We think that most people make the mistake of not having a useful, operational philosophy. It is a question after all of who is master. One should pick a philosophy that helps one get around difficult problems in life. One of the ways of getting around difficult problems is to say, "I don't care," because immediately that makes them of no particular interest. Sometimes you can do this; sometimes there is no easy way of doing it. But, in general, this is a useful philosophy.

18. DISCUSSION

In the preceding pages we have discussed some puzzles using the methodology we have developed. We want to stress again that there are many other ways of attacking these puzzles. Again, we do not know how the human mind solves these puzzles. What we have given here is a mathematical method that can be used on a computer in some cases.

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Section 1. For details, see the book

R. Bellman, K. L. Cooke, and J. Lockett, *Algorithms, Graphs, and Computers*. Academic Press, New York, 1970.

Many popular puzzles are discussed there. The same method can be used for other puzzles. See

- R. Bellman, "An Application of Dynamic Programming to the Coloring of Maps," *ICC Bulletin*, vol. 4, 1965, pp. 3-6.
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See also the book

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Section 7. See the two articles cited for Section 1 above.

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- E. A. Feigenbaum and J. Feldman, *Computers and Thought*. McGraw-Hill, New York, 1963.

Present-day programs are much better, and can defeat "average" human players. Dimensionality problems are reduced by incorporating "learning" mechanisms (see Chapter 7).

CHAPTER FIVE

uncertainty

There are many masks to the face of uncertainty.

1. INTRODUCTION

In this chapter we want to consider some examples of how the digital computer may be used for decision making under uncertainty. We regard decision making under uncertainty as one of the attributes of human intelligence. Consequently, if we can get a computer to make decisions under uncertainty we can feel that it is imitating one of the aspects of human intelligence.

As usual, although we consider quite general processes, we try to illustrate the methods and difficulties by particular problems. We shall use the same method that we employed in the previous chapters.

2. CAVEAT

The first problem, of course, is what does one mean by uncertainty? Here again, the mathematician has to make mathematical problems out of verbal problems. One of the difficulties we would like to emphasize is that we have all been educated very badly; we have all been brainwashed so that we automatically say, uncertainty is probability. This is not true at all. In the universe of uncertainty there are only little parts that can be handled by classical probability theory. In general, classical probability theory is of little value in the study of uncertainty. In practice, we use it because we do not know anything better to do. One should understand this clearly.

Probability theory is very useful for insurance companies, and even so, insurance companies, as we know, overcharge considerably. Look at our auto insurance or our life insurance rates. Insurance companies have very brilliant actuaries who spend a great deal of time computing what the rates should be. Then the companies charge ten times as much. Unfortunately, this is not an exaggeration. This is a safety

factor. Since, justifiably, they do not have that much confidence in the mathematical theory of classical probability, they just increase the rates.

The question remains, how to handle uncertainty? One of the points to remember is that classical probability theory is an axiomatic theory, as much an axiomatic theory as geometry. We now know that geometry is an axiomatic theory. It is well understood, but not as widely appreciated, that classical probability theory is also an axiomatic mathematical theory.

The whole problem is that of application. As we know, there is a considerable amount of difficulty in applying the axiomatic geometric theory to the real world. The earth, for example, is not a flat plane, nor a perfect sphere. We have to take into account that there are hills and valleys. The same goes for classical probability theory. The world does not operate according to the axioms. Therefore when we want to handle uncertainty, we face a tremendous amount of difficulty. When we talk about decision making under uncertainty, we have on one hand the ability of the human mind to operate under real uncertainty, an ability not understood at all.

On the other hand we have the mathematical theory of decision making under uncertainty. What the mathematical theory is worth, it is hard to say. It does have the advantage, though, of providing definite rules. It probably has the advantage of saving the decision maker an embarrassing situation. If we were to ask ourselves, what is the most important value of mathematics in the world, the answer is that it is a way of avoiding responsibility. If we have to make a serious decision, we say "We are doing this on the basis of statistics," or "We are doing this on the basis of mathematical theory." Then if things go wrong and somebody comes and complains, "Why did you make this bad decision?" we say, "We did it according to theory."

This is the principal function of mathematical consultants in industry, government, the military, and so on. The principal decision makers want a way of avoiding responsibility, so that if anything goes wrong they can say: "My mathematical consultants told me to do this." If a plane does not fly properly, they say, "According to aerodynamical theory we constructed the plane in the following way," and so on. We may smile about this, but this is a very important point. If we are a major decision maker, we must have some buffer, some ways of avoiding taking direct responsibility. The people who do not understand it get into very serious problems. Mathematical theory also gives us a

way of doing something—a way of making a decision in situations where often it makes little difference what decision is made. One of the reasons why the world operates as well as it does is the fact that most decisions are probably of little importance. What is important is that a decision be made, and that the sequence of decisions be reasonably consistent. Hence, it might be just as well in many situations to do what the ancient Romans used to do—take a bird, spread the entrails on the ground, and see whether the signs were favorable or unfavorable. In most situations it would be just as useful to toss a coin if there are two possibilities, and people know just as much.

We may think this is very cynical, but if we look at economic policy, if we look at the question of should we increase taxes or decrease taxes, should we increase the interest rate or decrease the interest rate, it would be just as useful if people tossed a coin doing these. We have little theory really to guide us. The government, of course, has to pretend that there is some theory behind its decisions.

3. PROBABILISTIC PROCESSES

This whole question of decision making under uncertainty and how it should be used is a very deep problem which we do not want to go into now. It is not at all an easy problem to determine how major decisions should be made, who should make them, when they should be made, and so on. We are talking about the much easier problem, decision making within the mathematical theory of probability. This is an axiomatic theory in which we know what we mean by probability, we know what we mean by this, we know what we mean by that and so on. We know what we mean because we made up the rules. This is the only reason why we know what we mean. This is the whole point of it. It is not very satisfactory, but there is nothing much we can do about it. Life is complicated. It was said by someone, Huxley we believe, that the universe is stranger than we think. It may be stranger than we can think. Thus, when we talk about uncertainty, although real uncertainty is seldom probability, we shall begin by talking about a probabilistic process.

4. ROUTING UNDER UNCERTAINTY

There are many problems that we can think of. One problem, for example, is that of going from city 1 to city N . We are at i and we want to go to j . However, we find out after we start to go from i to j that the road is washed out. The question now is what do we do, and the

more important question is how do we start out, knowing in advance that this could happen.

Second problem: We are pilots flying planes and we set a course. There is a strong wind and we may actually be blown off course. When we find ourselves off course what do we do? Do we go back to the original course as fast as possible or do we take a different course? This is the type of problem and the kind of uncertainty we can think of.

5. UNCERTAINTY

Now the question is: What do we know about the uncertainty? We can assume that when we start out from i we know the probability that the road will be washed out, or we can say: We start out to j and with a certain probability, instead of getting to j , we find ourselves at k . Or we have a strong wind and when we start out flying we get pushed off course. The question would be, how do we make decisions in a situation like this? Can we formalize the problem in such a way that we can have this type of decision making done by computers?

6. INVESTMENT PROCESS

One practical application of this would be an investment process. There are many other applications to agriculture, pest control, and so on. There are important applications in medicine, in drug administration. If a doctor is treating a patient, the patient might be in condition i . For example, the doctor uses a certain treatment and he wants the outcome j . But he may get the outcome k or l .

If he knows these probabilities how does he start the treatment all the way back at condition i ? This is an example of the type of problem we encounter in radiotherapy. If we had planned a simple and effective sequence of treatment, let us say seven weeks in a row, what happens to the patient if he misses his turn? Do we just continue as if we started from a new point or do we increase the amount of radiation we give the next week, and so on.

7. THE AVERAGE OUTCOME

These problems of decision making under uncertainty occur in all fields. If we are talking about the use of computers in decision making then we require explicit algorithms that tell us what to do.

We agree that classical probability theory is an axiomatic theory. The first thing to examine is the concept of the classical probability p .

Intuitively, we can think of this in terms of a frequency approach. We say a particular event has probability p if in a very large number of events a certain fraction p of the events are of one type. Note that we have not made precise what we mean by "very large." This is why we require an axiomatic approach.

But remember we said it was an axiomatic theory. There is no way of giving an intuitive and rigorous definition of classical probability. At some point we have to say "these are the axioms."

Another major difficulty is the criterion. In the routing problem we were talking about the minimum time. In going through a network, however, where one path may be out, there are many different possibilities. What then do we mean by the minimum time? This is now a meaningless concept. We must have some way of replacing what occurs so naturally in the deterministic case for the stochastic case. What we do is we agree to deal with an average outcome where we measure the average in some way. Although there are many different ways of taking an "average," we will look at one particular average outcome. This ambiguity in the notion of average is one of the major problems that we have in the application of mathematical techniques to important decision making under uncertainty.

A conceptual advantage of using average outcome is that the average outcome possesses the same property that we used before. No matter where we are, if we want to minimize the total average outcome we continue in the best possible way.

Analytically, the equations that we derive for the stochastic case have the same form as for the deterministic case. This means that we can use the same technique.

8. DIFFICULTIES

Let us give two examples of the difficulty: The first is conceptual, when there is only one event, or let us say one trial. Something is going to be done once and only once. What meaning then is there to the average outcome? A decision must be made as to how to do something, and there is going to be just one trial. How do you measure the effectiveness of decision making?

One mathematical approach is to say, "Pretend that we are really carrying on a very large number of trials over a long period of time, or pretend that many people like ourselves are each carrying out one trial." This provides a way out of the difficulty, since it furnishes a precise meaning to the term average outcome.

A well-known difficulty as far as the applications of classical probability to practical affairs are concerned is that the obstacles become worse as the problems become more and more important. The more important the problem is, the greater the chances (whatever that means) that this series of events is only going to happen once in the history of the world. We are never going to be able to go back and say, "Let's try some different approaches; let's average over different approaches." This is a part of high-level decision making that few people understand.

Another point is a practical, operational, and moral point. We encounter it in medical treatment when planning the treatment of clinic patients. It turns out that the patient and the doctor have different criteria. To the patient, he is unique, which means that the patient is not interested in average outcome. He is interested quite sensibly in the outcome to himself. The doctor, however, operationally has to think in terms of averages. If he has a hundred patients a day, or if he has several thousand over the year, he has to act according to some average. This is an operational constraint. He has a certain amount of time, a certain amount of resources, a certain number of hospital beds, and so on. He certainly tries his best to make each patient unique, but he has to operate the clinic over time according to some average outcome. This is a very difficult problem which society has not solved.

9. POLICY FOR STOCHASTIC PROCESSES

One question to ask is, "How do we apply this?" Another question is, "How do we do it mathematically?" Mathematically, we can talk about an average outcome. Let us take then the following simple specific problem: We have a network. We want as before to trace some path ending up at N . Incidentally, for future reference, " N " may actually be a set of cities. We may not want to get to a specific point. It may suffice to get to a point somewhere in its vicinity.

For example, if we are driving home we may have a garage, in which case we want to go to our garage, or we may have no garage, in which case we really want to park somewhere along the street. Hence, if we were trying to determine the minimum path from our office to our home, one path would be determined when we have a fixed space to park, another path would arise if we have to park someplace along the street. This is a problem that students have in getting to school. This is a stochastic process because, when we are driving to school, if we have a fixed parking place, then we can follow one technique for driving. If

we have to worry about getting a parking place somewhere, then the question is, "If we have to get to class on time, should we gamble on driving very close to the lecture hall and then walking a short distance, or should we stop far away where we are guaranteed getting a parking place and then walk a long distance?" The question is, "What is the optimal policy to pursue?" It depends, of course, upon the probabilities of finding parking places and the speed with which we can walk, as well as traffic conditions. This is an example of the fact that we may not want to get to a particular place, but rather we may want to get to a set. This is an important problem as far as pattern recognition is concerned. What we know is that the human mind can operate very well according to clues. For example, a caricature may be sufficient to identify a person. At the present time computers are not very useful for pattern recognition.

Let us look at the simple problem we sketched. If we want to go from i to j , there is a certain probability we really land at j , and there is a certain probability we land at j_1 and a certain probability we land at j_2 . In other words, this is a typical stochastic decision process, where we cannot guarantee the outcome of a decision. The question then is, "What decisions do we make in order to get to N ?"

One thing we can do is minimize the expected time. We introduce this criterion very much as before. The average does not have to be a simple average. We could minimize the probability that the time does not exceed a certain fixed amount. This is the problem of getting to class on time. We want a path that minimizes the probability of being late. Essentially what we have now associated with every decision is a probability distribution. If we say we want to go to j , what we really do is we pick a probability distribution around j . This asserts that there is a certain probability we will use up the time t_{ij} . The same argument as before asserts that wherever we actually end up as a consequence of the first decision we minimize the average time from then on. Thus, we get the same type of equation as before:

$$f_i = \min_{j \neq i} (t_{ij} + \sum_j f_j dG(i,j)) \quad (5.1)$$

The arithmetic is a little bit more complicated, but not that much. The way we carry it out arithmetically is we make the probability distribution discrete instead of continuous. The mathematics tells us what the minimum expected time is; again, the solution is a policy $j(i)$. The policy says, when we are in state i , we make decision j . This

is much more like a decision than before, because we cannot predict what state we are going to end up in. But what we can predict exactly is what decision we are going to make in terms of where we are. It is interesting that the policy is deterministic even though the process is stochastic. This can be proved mathematically.

10. POLICIES

A policy is ideally designed for stochastic processes. In a stochastic process you cannot follow through the other method of saying, this is the set of all states we are going to go through. We cannot tell what those are. What we can say is the solution consists of a policy—what do we do in terms of where we are. We cannot tell where we are going to be at a particular time, but we can tell what we do if we are in a certain state.

11. EFFECTIVENESS

Thus, once again, we can use computers to do decision making under uncertainty, provided we introduce the proper axioms and mathematical formulation. We have to define what we mean by uncertainty. Uncertainty in this case is probability. What do we mean by effectiveness? That is answered by a criterion that defines average minimum expected time, maximum expected gain, minimum probability of failure, and so on. We have to deal with averages. Although the applications of the results can now be very much more complicated than before (for reasons we do not go into), we possess a firm conceptual basis.

What is important to stress is that we have become accustomed to the fact that we have to use probability in certain situations. Consequently, we must make it clear that we are using one kind of probability. In many cases we want to use a different kind of probability. But, whatever kind we use, mathematics can never compensate completely for uncertainty. We must pay a certain cost for not knowing the future.

12. THE DIMENSIONALITY PROBLEM

Finally, let us make some comments on the dimensionality problem. Theoretically, the technique we have sketched can be applied to all multistage decision processes of deterministic or stochastic type. Unfortunately, in many cases, the number of possible states becomes very large. Hence, we have to talk about feasibility, about computer

storage capacity, about the ability to put a number into fast storage and to take it out of fast storage: in other words, we must talk about fast storage and time.

If we want to take a number out of fast storage practically, which means if we want to do it say in a hundred-thousandth of a second or a microsecond, we can handle an N of about 10^6 . We can expect in ten years that this will go up to 10^9 or even up to 10^{12} . This is not important in many situations when we are talking about numbers like 10^{100} or $100!$.

The point we want to get across is that our computer capabilities are quite limited. There is a large set of important problems that we can tackle, but an infinitely larger set of important problems that we cannot tackle by this method. What we have to do is use approximate methods, and this means that we have to use policies.

13. HEURISTICS

In artificial intelligence people frequently use the word heuristics for policies. One of the interesting questions is, How do we determine heuristics—how do we determine policies? Types of approximate policies are furnished by stochastic approximation and branch-and-bound techniques; but the best types of approximate policies are furnished by experience.

We would like to emphasize that there are many problems in which questions of optimality, questions of minimizing or maximizing, are not at all important. It is important to realize that they are just mathematical devices that can help us to handle “can.” One of the ways of studying “can” is, “Can we do it in the shortest time?” There are many problems where other interpretations of “can” are much more important. Stochastic approximation and branch-and-bound techniques in many cases furnish techniques for finding heuristic policies, approximate policies, that are feasible.

Here again is another very interesting problem of a higher level—a problem for intelligent machines but far more difficult than those we have been considering. Namely, given a problem of this type, which is of such high dimensionality that we can’t use a standard algorithm, how can one devise heuristics that are very much better than enumeration? Let us point out that any policy, any rule that tells us what transformation to make in terms of a given state, is an approximate policy. In general, it won’t be an optimal policy with respect to some criterion but it is approximate policy—it is a heuristic. The question

remains, in particular situations how do we determine feasible, approximate policies.

14. FEASIBILITY

In many cases even the determination of feasible policies is difficult. Enumeration is one approximate policy. It just doesn't work very effectively.

Our personal opinion is that each large problem will have to be treated on its own, that we have to give up the idea of a general theory of decision making for high-dimensional processes. We have to look at each process and determine approximate policies that depend upon the structure of the process. We can do this in individual cases. This is essentially the history of what has been going on. We have not been doing it long enough, however, to have some idea about the structures of some of these problems. One could hope that in ten years or twenty years or some period like that, there will have been enough special cases so people will then synthesize once again and say, "Ah, but there is a general structure." In this case we will go up to a higher level of dimensionality and be back at the same general problem. This is what is very interesting about this field—that whatever power of 10 we think we can handle at the present time, we can keep developing very elementary combinatorial decision problems that require even higher factorials, higher powers of 10, and so forth.

It is thus a fascinating field, and a highly open field. At the present time, it is a matter of using experience and ingenuity.

15. STOCHASTIC SCHEDULING

There are several stochastic scheduling problems we can think of. One is the following: A patient has to go to a number of different agencies, and one agency is completely overcrowded. We have the feeling that if somebody would just reschedule things, most of these people could get through very quickly without wasting their time. The question is: where should the patient go if the office " j " is too busy, or if some emergency develops, so that the doctors who are supposed to take care of X-rays or blood tests and so on have been called to some emergency operation? This is a stochastic problem and we have pointed out that the dynamic programming approach works equally well. It says: this is the way we schedule the patient if we want to minimize the expected time, or to maximize the probability of getting through in one day and so on.

16. SCREENING

We want to find out if a stochastic decision process can also be handled by means of the computer up to a certain point. We mentioned already that a modern approach is the man-machine system. When we have to do decision making, we do not necessarily want the computer to make all the decisions. We may want to do some screening. Essentially we want to cut down the number of decisions that the human being has to make, so that we can use the power of the human mind to treat the difficult problems that escape the low-level thinking of the computer. There is no difficulty in doing this.

17. DIAGNOSIS

There is another very interesting stochastic process, the diagnosis of ailments. We can think of medical diagnosis as a stochastic multistage decision process. There are two approaches to medical diagnosis. There is the old-fashioned approach which has the patient fill out many, many pages of forms. First, we get all the possible answers to all possible questions, and then we decide what to do. This is bad for several reasons: it takes an enormous amount of time on the part of the patient and, secondly, the doctor who was looking at all these sheets has to spend a great deal of time reading them over. He may not be able readily to understand what is going on.

The second approach is to consider the process as a multistage decision process. We ask some questions. For example: "Do you have a fever?" The patient answers, "yes" or "no." Now, if he answers "yes" there are two questions: "high" or "low" fever. If he answers "no," we could ask: "Do you have any pain?" If the answer is "yes," we could ask: "For how many days?" and so on. Hence, we ask a number of questions and we get a number of responses. The essential feature now is that the question we ask next depends upon the responses of the patient.

At the very end, there is a certain preliminary diagnosis. This is used for screening. We want to make sure that the patient goes to the appropriate doctor, to one who recognizes whether that is a serious problem or not. Obviously, if the patient has a high fever it is a serious problem; the doctor may want to see the patient immediately. If the pain has been going on for a long time, maybe it is serious, maybe it is not, depending upon other data and so forth.

18. GENERATION OF STOCHASTIC EFFECTS

In the foregoing, we have several times said that there were various stochastic events. How does a computer, a deterministic device, generate a stochastic number? This is a very interesting question and at first sight rather paradoxical.

The answer is, of course, that it cannot generate a stochastic number. However, it can easily generate numbers that act like stochastic numbers. Any statistical test could not discern the difference.

There are several ways of doing this. One way is the following: We take a number, say 123456789, and square it. The result will be a number of 17 digits. We take the middle 9 numbers 157875019, and square this. We continue in this fashion, say 100 times. What we have finally appears to be a random sequence.

Another approach is to use a table of random numbers which is stored in the computer.

19. COMPETITION

Many interesting mathematical problems arise when there is conflict between two or more people. There are important applications in the field of economics.

If there are two people who are in direct conflict, which is to say, what one gains the other loses, there is a good mathematical theory. It is called the "theory of games," due to Borel and von Neumann. If there are more than two people there are serious difficulties and no satisfactory theory exists at present.

Any discussion of these very interesting questions would take us too far afield.

There is a mathematical theory due to Nash, which provides a concept of optimality. This is a good example of a problem that has no neat mathematical answer.

20. INFORMATION PATTERN

We have assumed in what has gone before that we know where we start. If we assume that we have only partial information about where we start, we obtain many interesting classes of problems. Different assumptions lead to different equations, which in many cases can be resolved by the computer. Thus, we can do decision making in many cases where we have partial information. For example, in many

situations we have a choice of acting on incomplete data or gathering more data.

21. DISCUSSION

In the foregoing pages, we have discussed various simple forms of uncertainty. Our purpose was to show how a digital computer can do certain types of decision making under uncertainty. We have stressed several times that uncertainty is not equivalent to probability. Consequently, what we have shown is that the computer can be used to treat certain types of mathematical probability. The identification of this with the uncertainty that occurs in many applications is up to the user. It is always a risky affair.

We have spent some time in discussing uncertainty for several reasons. In the first place, we wanted to point out that there are many types of uncertainties. In the second place, we wanted to show how the computer can frequently be used to do decision making under uncertainty. Finally, we want to state that the same methods can be used for many more complex processes.

In the foregoing we have considered uncertainty in the effect of decisions. We should also consider uncertainty in knowing where we are, i.e., the state of the system, and in the objective function. Many new and interesting mathematical problems arise in this way.

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Section 3. The problem of decision making can often be treated by simulation, a topic discussed in Chapter 6.

Section 4. The second problem was historically the one that aroused our interest in this area.

Section 5. What we want to stress again is that one process can lead to many mathematical problems.

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CHAPTER SIX

simulation

You can never step into the same stream twice.

—Anaximander

1. INTRODUCTION

In this chapter we shall show how a digital computer may be used for simulation, a basic technique of applied mathematics. This is a very powerful technique which is available when previous techniques fail for various reasons, as we shall discuss. This technique has many applications.

A great deal has been written about simulation. It is an art with an associated science. Rather than talk in general terms, we shall give an example. We shall show how simulation may be used to learn how to play blackjack. In general, the digital computer possess great capabilities for training. At the end of this chapter we shall say a few words about the Monte Carlo technique.

2. UNCERTAINTY DUE TO COMPLEXITY

As we have noted, we are primarily concerned with decision making, and more precisely, decision making under conditions of uncertainty. Let us begin, however, with simpler questions connected with the descriptive aspects of systems. As a point of reference let us consider a system S with a number of component systems S_1, S_2, \dots, S_N (Figure 6.1).

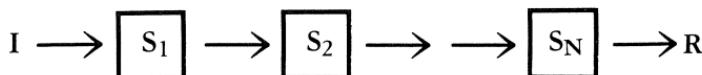


FIGURE 6.1

To begin with, let us suppose that the behavior of each individual system is known in the sense that we possess the required input-state-

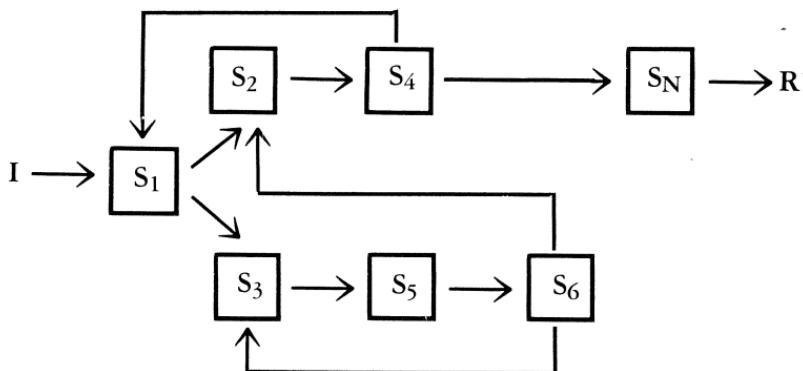


FIGURE 6.2

response table for each subsystem S_1, S_2, \dots, S_N . These may be of deterministic or stochastic nature.

We are interested in the case where there are a large number of these subsystems, although some of these subsystems may be the same system at different times. We do not, however, necessarily assume that their structure is as uncomplicated as that illustrated in Figure 6.1. For example, the actual structure may have the form shown in Figure 6.2, with associated time delays. Furthermore, the structure examined may be of far greater complexity.

Despite the fact that the behavior of the individual components is well documented, the complicated structure of the system may prevent us from readily predicting what R will ultimately result as a consequence of the initial influence I . In many investigations of this nature, we face the fact that the state of mathematical and scientific skill is not sufficient to use the available information concerning local interactions to determine global effects. We thus possess no useful rule which permits us to predict R , given I . This is a typical example of uncertainty due to complexity.

3. EXPERIMENTAL APPROACH

A standard approach for overcoming this barrier is the use of experiment. To find out how a system operates, we propose to observe it closely, identifying sets of variables and watching to see what the overall response is to various inputs. In addition, we postulate various internal mechanisms.

Occasionally, it is convenient to observe the performance of the

actual system. In general, however, we wish to determine the behavior of the system under a wide range of influences and operating conditions. Hence, if possible, small-scale replicas are devised to facilitate the experimentation; sometimes, however, large-scale models are necessary. To test these models, large or small, one uses such devices as towing tanks and wind tunnels, and many analogous structures.

The idea is readily carried over to man-machine systems. To determine how a telephone exchange might operate under peak load conditions, a typical telephone exchange complete with operators is constructed, tested, and subjected to characteristic demands for service. Many illustrations of this natural and fruitful idea can be given. To study reality, we observe reality or useful replicas of reality.

4. ADVANTAGES

There are many obvious advantages to the experimental approach. Many of the approximations inherent in any theoretical model are bypassed in this fashion. Furthermore, most of the difficulties associated with a mathematical model, consequences of our intellectual limitations, disappear in a direct examination of the actual system. The input-state response tables may be constructed on the basis of direct experimentation.

5. DISADVANTAGES

Unfortunately, it is also true that the experimental approach possesses some serious disadvantages.

To begin with, it may be impractical, or even impossible, to build a useful replica of the system of interest. Although we can study the actual behavior of a transportation system or a chemical factory by direct observation, we may not, however, be able to perform a number of important experiments without seriously interfering with normal operation. An automobile manufacturer can build a prototype of a new automobile, but it hardly qualifies as experimentation to build an ocean liner or a hospital complex. Despite the fact that it is desirable to understand the effects of a particular interviewing policy in an initial interview, there are reasons why we may prefer not to use an actual patient.

Secondly, there is the question of cost in time. It may not always be desirable to operate in real time. For example, it is essential to determine in some fashion the effect of a fiscal policy to be pursued for one

year in a national economy without spending one year in the process of study. It is desirable to study within a relatively short time patients with certain problems that might ordinarily be encountered only occasionally in a five-year or ten-year period; for example, patients with terminal illnesses.

Thus, there can be serious costs in resources (human and nonhuman), or time, or both, when the experimental method is employed.

6. DESTRUCTIVE TESTING

In some cases performing experiments may be dangerous to the experimenter, as witnessed by early experiments with Roentgen rays and nuclear reactors. In other cases, testing is destructive to the machine, or to the system, or to the individual, as, for example, experimentation with new drugs.

This is a particularly serious point in the practice of medicine. One takes a critical view of experimentation with people already suffering.

7. IDENTIFICATION

Let us briefly consider the question of observation. In studying a real system, there is the problem first of all of isolating cause and effect. Determining all of the influences actually operating on a particular system is never simple and, as a matter of fact, rarely completely feasible. *In vivo* a system is subject to many different kinds of influences, some beyond our control, and some possibly beyond our current comprehension. Hidden variables, or non-specifics as they may be termed, can result in serious difficulties in the interpretation of observed results and, ultimately, in the formulation of incorrect principles.

The history of science shows very clearly that the observer, the scientist, is strongly bound by his culture, his philosophy, and his training. There is much to observe in any real system and thus considerable choice in the selection of what to observe and the length of time that it is observed. It is well to note that no observation takes place without implicit or explicit theory guiding the selection of observables, the use of measuring devices, and the listing of data. There are no critical experiments, only critical experimenters.

It follows that before observing real systems we would do well to practice on simple idealized systems.

There is also the problem that experimentally observing a system may alter the behavior of the system.

8. IDENTIFICATION USING EXPERTS

One way to overcome the obstacles involved in this type of learning situation is to employ experts as guides and interpreters to teach us to identify and extract information from data. Let us recall that we classify information as that part of data useful for decision making and learning. For example, the resident learns to practice medicine under the supervision of an experienced doctor. There are, however, obvious drawbacks to this procedure, necessary as it is, since an observer can have a serious effect upon the behavior of both patient and resident.

9. MOTIVATION

The preceding considerations make it clear that it would be highly desirable to possess techniques that would enable us to observe some cause and effect relations and to test some policies without any use of the actual system. Furthermore, we want to introduce the novice to a complex system under carefully controlled and limited conditions. A suitable combination of mathematics and digital computers enables us to go quite far in this direction in a number of important situations. How far depends both upon the nature of the system and what we wish to accomplish.

10. MATHEMATICAL SIMULATION OF COMPLEXITY

Let us return to the phenomenon of uncertainty due to complexity discussed in Section 2. We have agreed that the behavior of the subsystems is known, along with their interactions. Nonetheless, we may possess no usable theoretical technique for determining R , given I .

We can, however, laboriously trace the effect of the specific influence I through the system. Thus, using the simpler schematic of Section 2, we have the schematic shown in Figure 6.3. By this schematic

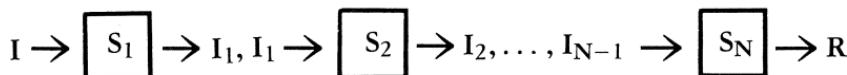


FIGURE 6.3. Tracing the effect of an influence through a system with loops is not so simple (cf. Fig. 6.2).

we mean, as usual, that I influences S_1 to produce an influence I_1 , which in turn becomes an influence to S_2 , which produces I_2 , and so on. Thus, by a direct enumeration of cases we can determine the IR

relation for the entire system, first for a particular I and then for the set of all admissible I 's.

The idea is conceptually sound, if not particularly elegant in thought or execution. We are merely imitating, or simulating, the behavior of the actual system. The computer, which can run through an incredibly large number of procedures of the foregoing type in a reasonable amount of time, makes the foregoing idea operational in many cases.

11. SAMPLING

In order to obtain a representative idea of the effect of a specific influence I_o upon a system with stochastic features, the process described above involving the systematic tracing of effects will necessarily have to be repeated many times. Although the probability is very small that a complex stochastic system will react the same way two times in a row, nonetheless, the probability is very high that there will be a regularity of response in some average sense if the influence stimulus is repeated a large number of times.

The technique of determining the properties of a stochastic system by repeatedly using certain influences is called sampling. Although the basic idea is simple, successful use requires considerable thought and experience.

All of the fundamental ideas of science are elementary. It is their operational use that requires a great deal of care, thought, and ingenuity.

12. FEASIBILITY OF MATHEMATICAL SIMULATION

It is clear that the mathematical simulation method described above can handle processes of some degree of complexity. Can it, however, handle processes of a high degree of complexity, and, if so, what factors limit its applicability to any type of process?

There are three considerations we must keep in mind when studying this question:

1. Can we reduce a particular process to a sequence of influence-response operations of the foregoing type?
2. Can we keep track of all of the instructions contained in the influence-response tables?
3. Will a simulation process of the type described above consume too much time?

Any answers we provide to these questions are clearly conditional ones, highly dependent on the properties of contemporary science and

digital computers, and on current mathematical theory. Note that these are two different kinds of questions. The first is conceptual, the second and third operational. The line of demarcation, however, is not clear-cut. To use a theory, we must be operational; to be operational requires a theory.

13. USE OF THEORY

Considerations of the foregoing type begin to make us appreciate having a theory to use. In this case, theory would provide us with a rule, an algorithm, for calculating S ; and R , given I and S . It is clearly very much more convenient to store the algorithm than to store the entire ISS, R table.

What do we do, however, when theory does not exist, or only a partial theory exists?

14. BLACKJACK

To illustrate some of the methods that can be used, let us consider a simplified version of the card game, Blackjack, often called Twenty-One.

Let us recall the way in which the game is commonly played.

Using a bridge deck, the dealer gives the player two cards face down and himself two cards, one face up. The cards have associated numerical values, Ace through Ten as indicated, Jack, Queen, and King each counting 10. To simplify some subsequent discussions and diagrams, we shall ignore the important point that in the actual game the Ace can also count as 11.

The player looks at his cards and then makes the first decision. He can decide to draw a card or not. If he does not draw, the decision passes to the dealer and the player has no further options. If the dealer has a total of 16 or over, he cannot draw. Play is ended and the totals of the two hands are compared. If the dealer's total exceeds the player's, the dealer wins; if the total is less, he loses; if equal, it is a draw.

If the dealer has a total of 15 points or less, he must draw a card. If his total with three cards is over 21, he automatically loses. If his total is between 16 and 21, inclusive, the game is decided as before by comparison of the two totals, the player's and the dealer's. If his total is still under 16, he must draw another card, and so on.

If, on the other hand, the player decides to draw another card, he loses automatically if his three-card total exceeds 21. He automatically wins if he achieves a total of 21. If he does not obtain either 21 or over,

he has the option of stopping, or drawing another card. Once he stops, the game proceeds as described above with the options now up to the dealer.

The game is an excellent example of a multistage stochastic decision process. We have simplified it, leaving out inessential details. This is of no great conceptual concern since many versions exist. The variations are, however, of significance in determining whether the game is favorable to dealer or player.

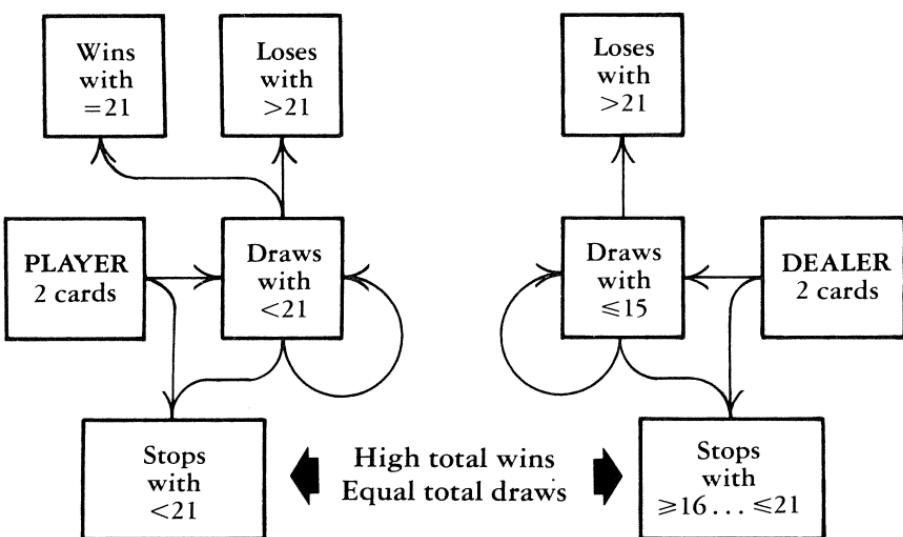


FIGURE 6.4

15. TESTING A SIMPLE POLICY

We will continue to illustrate these ideas with the game of Twenty-One. With the aid of the computer, thousands of experiments can be performed with only a small initial effort on our part.

Let us suppose we wanted to test the following simple policy for the player: "Never draw a card if there is any chance of ending up with a total exceeding 21, regardless of what the dealer has."^{*}

We would have the computer "deal" a hand to the dealer and a hand to the player. If the player has a total of 12 or more, his hand is frozen, in accordance with the foregoing policy; i.e., he stops. The dealer (i.e., the computer in the role of the dealer) then proceeds to draw or not

*In practice, this is not a bad policy. It loses slowly.

according to preassigned house rules: he draws a card if he has a total of 15 or under; he stops if he has 16 or over. If the player (i.e., the computer in the role of the player) has a total of 11 or less, he draws a card until 12 or more is attained.

The computer records the win, or tie, as well as playing both hands according to the assigned policies. It follows the simple instructions for play contained in the computer program. The process is now repeated until we attain some feeling of confidence in the statistical results. Using standard statistical techniques, we can determine a suitable number of plays required to evaluate this policy.

16. LEARNING TO PLAY

We may, however, be in the position of an individual who has never played this particular game before. In this case, he will certainly not know enough to possess a theory of play, i.e., a policy. Before fixing on a method of play, he will want the opportunity to play the game a number of times and to observe the consequences of various decisions. In other words, he would like the opportunity to learn the game.

He can readily do this, playing against the computer. The computer "deals" two cards to the player and two to itself. The player then makes various decisions which the computer records and to which it reacts, again, of course, according to assigned instructions. The process is repeated until the player feels that he has sufficiently mastered the play of the game.

We can use the foregoing procedure in the case where the rules are known but optimal play is not. Observe that we can also use it even in the case where the rules are *not* known initially. This is then a learning process; learning is discussed in Chapter 7.

An important point is that the computer can be instructed to test various policies and that this testing can be done in electronic time, millions of times faster than actual. Thus, we have a simple means of compressing time required for learning or, equivalently, accelerating the process in real time.

17. HOW DOES A COMPUTER DEAL CARDS?

Let the cards be numbered 1, 2, 3, . . . , 13, with Jack, Queen, and King assigned the numbers 11 through 13 respectively, the Ace assigned the number 1, and the others as indicated by their numbers. The computer picks two of these numbers at random for the player's hand and two for the dealer's hand. The player is then told, for example, "You

have been dealt two cards, a 4 and a Queen. The dealer has been dealt two cards, one unknown and a 10 showing." Note that the picture cards have the uniform value of 10 in the game although they are numbered 11, 12, 13 for identification purposes.

This information may be printed out on a typewriter, or displayed on the screen of a CRT. If we are using a CRT, we can even picture the hands, imitating what occurs at a casino.

The player now must make one of two decisions:

1. Stop; i.e., draw no cards.
2. Draw one card.

If the player decides not to draw, following, for example, the simple policy described above in Section 15, the computer has two possibilities:

1. If the hole card, the unknown card, is 6 or above, it stops.
2. If the hole card is 5 or below, it must draw a card, i.e., choose a third random number, and so on.

The decisions of the player are communicated to the computer by pushing buttons, using typewriter keys, or using a light pen. When the computer stops, comparison of the hands is made. In this case where the player has 14, 16 or more for the dealer is a win for the dealer.

If the computer draws a card, the new dealer total is computed. The decision to stop or draw is made as before, or an automatic loss is tabulated if the dealer total is over 21.

A schematic of the process is shown in Figure 6.5.

The computer dealer begins its play only after the player has terminated his play.

The essential point is that a decision by the computer is equivalent to the selection of a number according to a preassigned rule.

18. WHAT CONSTITUTES OPTIMAL PLAY?

We have previously agreed that in a process involving stochastic elements decision making is to be evaluated in terms of *average* behavior. For example, in the game of Twenty-One, we may wish to play to win as often as possible on the average. If, however, we are allowed to wager different amounts on different hands, we may wish instead to play so as to make the long-term average gain as large as possible.

The average, whichever one used, is evaluated in the expected fashion. The decision process is carried out a certain number of times with a tabulation of the results, say, wins and losses in the Twenty-One

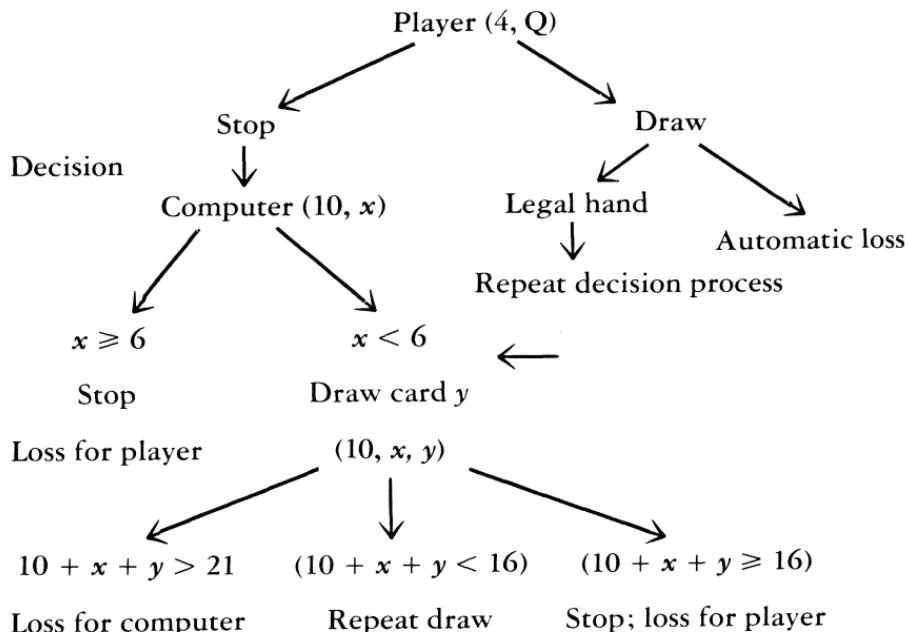


FIGURE 6.5

game. In this way, a frequency of wins is calculated. How many plays are required in order to provide a firm basis for judgment is usually a delicate matter. In general, if possible, it is best to err on the side of conservativeness and repeat the process until stabilization of the calculated frequency is clearly observed.

If we are accepting the simple arithmetic average as a criterion, an optimal policy is one that yields the greatest frequency of wins. It is clear, however, that use of an optimal policy cannot guarantee a win on a particular play, or even in any fixed sequence of plays. Every card player, every person who has engaged in a game with stochastic elements, can testify to this. The best of play can be vitiated by the vagaries of chance. Many examples exist of the occurrence of events of vanishingly small probability.

Risk is thus one price of uncertainty. We can diminish the price by repetition of the process a large number of times and by conservative action on any particular play, but we cannot eliminate risk entirely. Many perplexing questions remain as to the proper course of action in stochastic situations where we do not possess the luxury of repetition.

This is, of course, a particularly serious problem in the medical field where the cost of a wrong treatment is so high.

19. RULES FOR OPTIMAL PLAY

Let us suppose that we have agreed upon a specific criterion and seek to learn optimal play, which is now a well-defined concept. If rules for optimal play exist, which they do in a number of simple processes, it becomes a matter of recognition of the situation and implementation of the decision. Sometimes these are simple, often not. As a simple example, consider the game of Tic-Tac-Toe, a deterministic process.

The rule is for the first player to occupy the center square (Figure 6.6). If the second player makes an incorrect response, say an *O* where indicated in Figure 6.7, the first player should choose a corner, since he now has a forced win, by means of, say, the occupation of the lower right-hand square, no matter what move the second player makes (Figure 6.8). Observe that occupation of the lower left-hand square would be just as good. This illustrates the important point that optimal moves need not be unique. In many situations there are a number of alternate courses of equally effective action. In most complex situations there is a large set of feasible policies which for all practical purposes have outcomes that are indistinguishable. This blurring of fine detail is what makes the operation of large complex systems possible.

Very seldom can we evaluate a situation in such simple terms so as to make a correct move to guarantee a desired outcome. This means, as stated above, that risk-taking is an essential part of the important processes of life. In other words, stochastic decision processes are the rule rather than the exception.

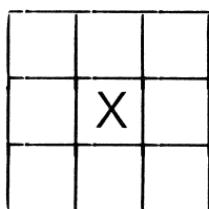


FIGURE 6.6
Optimal First Move

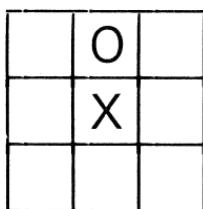


FIGURE 6.7
Incorrect Second Move

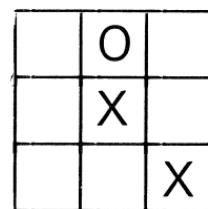


FIGURE 6.8
An Optimal Third Move

20. LACK OF CODIFIED RULES

In a number of situations of importance there are no codified rules for optimal decision making. In some cases this is due to the complexity of

the process (e.g., bridge and chess); in other cases it is due to inherent ambiguities (e.g., poker involving three or more players, or medical diagnosis). How then does one learn good, if not optimal, play?

The answer is that one learns good play by some mysterious combination of theory, teaching by experts, and personal experience. How the human mind learns to perform complex actions and to cope with genuine complexity is not well understood at the present time. We do observe, however, that this learning process occurs. For our purposes, then, we shall accept the fact that experience is a good teacher. Thus, we strive to provide some artificial experience that can be combined with theory. This is the basis for using simulation processes.

21. EXPERTS

We have questioned the meaningfulness of the concept of optimal policy. Intuitively, however, we accept the concept of some more and some less effective policies. But how are we to evaluate a particular policy or a computer simulation? The experts may not be able to describe completely what standards they are employing as they go along, but they can judge in a particular situation how well someone else has performed. They can point out appropriate and inappropriate actions, and explain to a great extent why these are of this nature; they can identify states and types of behavior.

As is true of experts in all fields, they can evaluate a situation when they see it, without necessarily being able to analyze the elements involved, or being able to anticipate what they expect to see.

An expert, or group of experts, is equally capable of examining a particular computer simulation and evaluating its various components and range of policies.

On the basis of the previous discussion, it is clear then that experts are required to construct meaningful simulation processes. Are they, however, needed after the simulation process has been constructed? The answer depends critically on the level of the simulation process. If, as in the game of Tic-Tac-Toe, it is easy to learn from computer experience, no supervision of the trainee is needed. In the game of Twenty-One, although no supervision is needed to learn optimal play, it is reasonable to believe that the learning experience can be substantially accelerated using the analysis of an expert player.

We have previously considered the problem of learning from experience. In a complex process, of course, there is always the danger that incorrect, as well as correct, behavior will be acquired since one

may be unaware of correct behavior. This is a well-known effect in tennis, golf, and bridge, and other games as well. Particularly in an intellectual process limited experience can readily lead to faulty conclusions. One can reach equilibrium on an undesirable plateau.

Experts are thus vitally needed to discuss and interpret the sequence of decisions. A simulation process is designed as a research tool and as a teaching aid, to supplement—not to replace—the experienced instructor.

22. CAVEAT SIMULATOR!

Having discussed the need for simulation at the beginning of the chapter, it is appropriate at the end of the chapter to indicate what is perhaps the principal danger of simulation. This consists of a tendency to confuse reality with the mathematical model constructed. The model is man-made, simpler, controllable, and therefore, ipso facto, psychologically more appealing. It becomes seductively easy to believe that optimal decision making in this model of reality, abstracted and simplified, is actually optimal decision making in the real world. The student must thus constantly be reminded of the many simplifying assumptions inherent in any simulation. This is one of the functions of the expert. Indeed, the search for these may be one of the most important pedagogical aspects of a simulation process.

23. MONTE CARLO

A very good example of simulation is the Monte Carlo technique.

This is based upon a very simple idea. If, for example, we want to know the probability of winning at dice, we can proceed mathematically and easily calculate the required chance. However, if we have a digital computer available, we do not need mathematical theory. We can have the computer roll dice, as explained previously, until the required regularity is observed. This will not take a significant amount of time.

The same procedure can be used to determine a nuclear shield. This technique is particularly valuable if the shield has an irregular structure. We can follow the neutron through the shield, allowing scattering in random directions. Although the basic idea is simple, it requires a great deal of ability to make it work. The difficulty is that the answers obtained are not very accurate if the procedure is done routinely. We shall give some references which explain this point.

Once the idea is observed, we can find an associated Monte Carlo

technique for many processes. It becomes quite an interesting mathematical game to figure out a Monte Carlo technique that will yield the required values. An advantage of this procedure is that we can calculate part of the answer without calculating the whole answer.

24. OPERATION OF THE SYSTEMS OF SOCIETY

We wish to gain some understanding of the systems of the society in which we live. It is crucial to understand these systems for survival.

To accomplish this, we combine some results of modern control theory, dynamic programming, fuzzy systems, and simulation, with sociology and psychology. The problems cut across many disciplines. This means that a team effort is required.

The advantages of simulation are that we can study important questions in electronic time without disturbing the systems or the people who are affected by the system.

The results can be used by three classes of people, decision makers, people who are training for decision making, and people doing research in the study of systems.

Simulation provides a vocabulary and a methodology. This vocabulary and methodology can be used for all systems. Thus, it will facilitate communication between people who are studying the same or different systems.

25. DISCUSSION

In the preceding pages we have discussed some aspects of simulation. Two points should be stressed. In the first place simulation involves a mathematical model. In the second place, considerable knowledge of the system is required for a successful simulation. The method is not routine.

As stated above, simulation is an art with an associated science.

Simulation can be used for three purposes. It is valuable for the decision maker who has to make decisions for a complex system. It is valuable in training people for this role, particularly in university courses. Finally, it can be a valuable research tool.

One of the advantages of the computer is that it forces us to be explicit. One cannot wave one's hands in front of a computer. In writing a program, we have to specify precisely the description, the effects of decisions and actions, and objectives. Often this precise examination of the system is the most valuable contribution of a simulation.

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Section 3. This is a useful idea. Unfortunately, in practice, often we cannot observe the inner workings of the system. In general, it is not possible to observe many important state variables.

Section 4. What must be remembered is that any experiment depends upon theory. The result of an experiment must always be interpreted.

Section 7. The procedure of experimentation is considerably complicated by time delays. Recognition of the fact that observation takes time and that there are time delays in the system, leads to very interesting analytic problems. See the book

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CHAPTER SEVEN

learning

A little learning is a dangerous thing.

—Pope

1. INTRODUCTION

One of the attributes of human intelligence is learning. In this chapter we want to show how a digital computer can be used to do certain types of learning.

In addition to the formulation, as usual, we will examine the feasibility of the procedure.

What we shall see is that there are levels of learning. What we mean by this will be discussed below.

2. LEARNING PROCESSES

We have agreed that we can do thinking of various types by computer, and we agree that we may say this is low-level thinking. What then about higher levels of thinking? One of the ways to think about thinking is to recognize the fact that we have levels. For example, we can talk about decision making. There are also decisions about decision making, there are decisions about decisions about decision making and so on. These problems arise in a very natural way when we talk about learning.

In the hospital context we would like to know whether we can learn something about the nature of an illness, about the nature of a chemical that can be used for treatment, and in general, whether we can imitate in some fashion the type of experimentation that is commonly done by humans. We want then to go up to a higher level and see if we can reproduce some of the procedures that human beings carry through. We will call this "learning."

3. NEW DRUGS

The problem we posed about determining the appropriate treatment

of patients using new drugs was the subject of research studies of W. R. Thompson in 1934. Thompson's name is not familiar to us. He was quite unfortunate, in the sense that he was too brilliant too soon. He formulated the idea of sequential analysis, and also independently thought of using Monte Carlo techniques. All of this was back in 1934, much too early.

The problem he was interested in is this: How does a doctor introduce a new drug? We consider the situation in which we have the population of people who are suffering from a certain illness, and where there is a conventional drug with a certain probability of success. There is an illness, and people die from not being given a proper drug. All of a sudden, however, a new drug comes on the market. The problem that the doctor faces is: How does he use this new drug? Does he just throw away the old drug and introduce the new drug? Does he wait until other people use the new drug for a certain period of time? Does he in some way mix these two techniques by introducing a certain amount of the new drug and a certain amount of the old drug? One thing we can do is change the entire treatment over to the new drug. This, of course, has obvious drawbacks. The new drug may turn out not to be that effective, which means we may be killing a certain number of people unnecessarily.

We are going to have to gamble with human life. The question is: how to do it in some reasonable fashion.

4. THOMPSON'S TECHNIQUE

What Thompson thought of was the following: if we have a sample of, say, a hundred people, we divide them initially into two groups, say ten and ninety. To the first group we give the new drug and to the second group we give the old. If this small sample does very well, next time we make it say twenty and eighty, and so on. The perplexing question is: how do we find out how to do this in some reasonable fashion? In other words, how do we learn what the properties of the new drug are? Do we use intuition and experience, or is there a mathematical theory that tells us how to do it?

5. MATHEMATICAL MODEL

Let us take a far simpler version of this problem in which we don't have emotional and moral aspects of dealing with human beings. This was the problem we started working on some years ago. Incidentally, Thompson's paper was found quite by accident while looking for

another paper (on number theory) which was right next to it in a mathematical journal. We were very curious to see what anybody was doing in decision making in 1934 and thus we looked through the paper. This stimulated us to make up the following problem, which now has a fancy name: two-armed bandit.*

The two-armed bandit problem is the following: suppose we have a slot machine with two arms and we are told that one arm has a fixed probability which is known. Every time we pull this arm we have a certain probability of winning one dollar. All we know about the other arm is that there is a fixed probability, but that is unknown.

This is a very simple model of learning and decision making under uncertainty. The question is: we have a certain amount of trials, which may be considered to be experiment, "What to do?" Do we take the conservative point of view and say: never waste any time learning or experimenting? Or, if we do some experiments, when do we decide that we know enough, and many similar questions.

This is a very simple mathematical model, very interesting and full of many intriguing features. One of the things we want to point out in connection with the problems of this type is how much we have simplified the original problem. The original verbal problem was: how does the doctor introduce a new drug. The problem that the mathematician faces is how to make precise mathematical problems out of this. One of the points that we want to emphasize is that none of these verbal problems ever has a precise one-to-one correspondence with the mathematical problem. It is always a one-to-many correspondence. We have one verbal problem, one real-life problem, and there are hundreds and thousands of mathematical problems that we can think of. Which one do we choose? We choose those we can solve most easily or those that have most applications and so on. There is always some mix of characteristics. Hence we have to mix some combination of the real application of the mathematical problem with the amount that we can do mathematically and computationally on the original problem.

6. ASSUMPTIONS

Some of the assumptions that are made are the following: Here is an unknown machine and we are told that one lever has a fixed probability of success. We do not know at all what happens when we pull

*After the well-known slot machines of Las Vegas.

the other lever. The first assumption that is made is that "unknown" means fixed probability. This is a very strong assumption. It may be that unknown doesn't necessarily mean that there is a probability of success. It may well be (as is the case in Las Vegas) that as we keep playing this game and doing well, there is somebody behind the machine who will change the probabilities. We must not automatically assume that unknown means a probability. We have discussed this point before in Chapter 5.

We face this difficulty in any process involving human beings. This is particularly true in the field of psychiatry. As people become familiar with the ideas their behavior changes. The practicing psychiatrist has to take this into account.

There is also no guarantee of reproducibility, by which we mean that there is no guarantee that the situation will repeat itself, since these are basic difficulties of experimentation. We cannot guarantee the reproducibility of a steady state process and so on. Hence, when we think about medical experiments on human beings we have to look at the results with a great deal of care.

At the moment, let us throw away all these practical problems which make life so complicated, and so interesting, and say: let us consider this mathematical problem which is hard enough.

We hope that we begin to realize how many levels of simplification we must go through in order to get a mathematical problem that we can treat. Often, we work very hard simplifying and simplifying reality and still emerge with a very difficult mathematical equation.

An important question is what to do for the mathematician who plays strictly according to the rules and says: I must work on this equation because this is an oath that I took, I swore to my mother on her dying bed, or to my professor when I left the university, that I shall always work on an equation until I solve it.

The mature mathematician says: I know from the beginning that these problems are only approximations to real problems. Hence, if I meet a mathematical problem that is too difficult why not go back and simplify some of the assumptions so as to get a mathematical problem that is a little bit easier. In other words, there should be some balance between scientific and mathematical complexity. This is a very important point. But eventually it becomes a question of the game we want to play.

What is interesting is that the simplifications often made by the scientist to simplify the task of the mathematician often complicate

life for the mathematician. Frequently, the more realistic problem is actually easier to handle. Consequently, the mathematician always does well if he knows the original scientific problem.

7. LEARNING

The problem we want to face is the following: we have N trials—what do we do? It is clear that we have to spend some time learning about the nature of the unknown lever. We have simplified the problem by assuming that it has fixed probability, but we still do not know what it is.

The question is: how are we going to find out? If N is very large, the problem is trivial. Because if N is very large, like a million, we do a thousand experiments on the unknown lever. If we have a fixed probability, then with a very high probability we will be able to know whether the known probability is greater than the unknown one or less. If it is less we keep using the known lever for the rest of the time.

Thus, if we have a very large number of trials available, the problem is simple. In other words, if the cost of experimentation is very little, then learning is simple. The important question is how do we learn when it is expensive to learn? It may be expensive in the sense of actual money, or in the sense of time, or both, or in human life and suffering.

8. FORMULATION AS A MULTISTAGE DECISION PROCESS

First we want to formulate the problem as a multistage decision process. We have made k trials and have observed m successes and n failures, let us say on the strange lever. There are N trials remaining. The question is: how do we use this knowledge? This is another way of saying how do we learn, because the use of acquired knowledge is learning. Do we try one more experiment or do we make a definite decision? What is interesting about a learning process is that the essential part of the process is not only making the decisions about what to do, but making decisions about what information to gain in order to make an ultimate decision. We thus have two types of decisions: we have decisions about performing an experiment and decisions about using the results of the experiment.

Let us define the function

$$f_{mn}(N) = \text{maximum gain expected in the foregoing situation}$$

The gain comes into this case by saying: if we pull the right-hand lever we have a probability p of winning one dollar; if we pull the left-hand lever we have an unknown probability of winning a dollar. We could, of course, make the problem a little more complicated by saying that there are different costs of experimentation and so on. It will be clear how one can consider this more general situation.

9. EXPECTED GAIN

The first problem that is faced is this: what do we mean by "expected"? We mean average, but how do we compute this average? Average with respect to what?

We know what we mean by the average gain when we pull the right-hand lever, because we have a known probability. The question is, what do we mean by an average when we have an unknown probability? What are we going to average over?

What we assume is that the unknown probability, which we will call q , has a distribution function, a density function $\phi(q)dq$, $0 \leq q \leq 1$.

The simplest assumption will be that this is a uniform probability distribution. But let us assume, as is always the case in practice, that one knows a little bit more about $\phi(q)$. That this function exists and is known is a very strong assumption indeed and many people can again complain: where did we get this function? There are many ways to answer. We can first of all say that we are talking about a fairly long-term process and thus it does not make too much difference what $\phi(q)$, the a priori estimate, is. The second thing we can do is to say that we are going to work the problem out as if we know the value of $\phi(q)$, and then we are going to use a theory of games approach. This means assuming that nature is trying to do its worst. After we have obtained the best policy for the fixed function $\phi(q)$, we are going to find out what function $\phi(q)$ minimizes our maximum expected gain.

10. VALUE OF INFORMATION

It turns out that this is rather interesting in many cases. What one can show from the mathematical analysis is that the policy that nature uses is to constantly reduce the value of information. In other words that nature, if it were trying to do its worst, would choose a probability distribution that would enable us to make the least use of the experimental results that we have obtained. This is a very reasonable intuitive policy.

We assume then that we are going to take averages as if q satisfied

some fixed probability distribution. This is now an a priori estimate, an estimate of q that we have when we begin the process.

One of the questions we face is how do we change this a priori estimate on the basis of an experiment. To accomplish this another assumption is made. We assume that we are going to modify the a priori estimate for q on the basis of Bayes' theorem. That means the following:

$$dG' = \frac{p^m(1-p)^n dG}{\int_0^1 p^m(1-p)^n dG} \quad (7.1)$$

These terms in the denominator, although complicated, are merely normalizing factors.

11. FURTHER ASSUMPTIONS

These are now a posteriori estimates. Another assumption is now made. The process is started using an a priori estimate and then all expected values and probabilities are calculated as if this a priori estimate were the actual probability distribution. After some additional data is obtained, a new a posteriori probability distribution is calculated and again we act as if this were the actual probability distribution—until we get some further data.

This we call one type of learning. We learn by changing our probability distribution on the basis of experience.

12. FUNCTIONAL EQUATION

How do we get the relevant functional equation? First of all, there are two possibilities: either we choose the left-hand lever or the right-hand lever. There are two possible decisions. One decision is to keep on experimenting, the other decision is to give up experimenting. If we renounce experimenting then the expected gain is going to be Np . That term is easy. Once we stop experimenting we know what the expected gain is for the remaining trials. A very simple mathematical analysis shows that when we stop experimenting once, we stop forever.

Suppose we experiment. What can happen if we experiment? There is going to be either a success or a failure. What is the probability of getting a success? An a priori probability distribution $\phi(q)$ is used for the unknown probability q and we assume as above that we are going to modify the a priori estimate of q on the basis of Bayes'

theorem. Hence if we have had m successes and n failures, the original $\phi(q)$ has been transformed to

$$\phi(q) = q^m(1-q)^n / \int_0^1 q^m(1-q)^n \phi(q) dq \quad (7.2)$$

The expected probability is computed as if this new a priori probability distribution were the actual probability distribution for q . The expected probability is thus

$$p_{mn} = \int_0^1 q^{m+1}(1-q)^n \phi(q) dq / \int_0^1 q^m(1-q)^n \phi(q) dq \quad (7.3)$$

Now, if we act as if this is the probability of success when we pull the left-hand lever, then with this probability we win one unit and we continue for $N-1$ trials remaining.

On the other hand, the probability that we are not going to gain one is $1-p_{mn}$ and we are in a situation where we have m successes, $n+1$ failures, and $N-1$ trials remaining. Thus, the expected gain that we get if we try one more learning operation by pulling the left-hand lever is

$$f_{mn}(N) = p_{mn}(1 + f_{m+1,n}) + (1-p_{mn})f_{m,n+1} \quad (7.4)$$

On the other hand, if we give up experimentation and learning and just pull the right-hand lever, the expected gain will be Np . To obtain the maximum expected gain we must maximize over these two possibilities, thus obtaining the relation

$$f_{mn}(N) = \max_{Np} p_{mn}(1 + f_{m+1,n})(N-1) + (1-p_{mn})f_{m,n+1}(N-1) \quad (7.5)$$

This is the functional equation that tells us at what point we stop trying to get additional information and either pull the left-hand lever or give up and pull the right-hand lever.

13. DISCUSSION OF ASSUMPTIONS

This is a very complicated formulation, one with a very large number of assumptions. It is interesting to see how much is involved in trying to construct a mathematical model of learning.

Let us again point out that this is something that we can give to a

computer. As we have noticed, $f_{mn}(N)$ is described in terms of the same function for $N-1$. This is an iterative relation that can be determined on a computer.

14. COMPUTATIONAL FEASIBILITY

Let us examine what is involved in using a digital computer to solve this functional equation. To begin with, we must compare two numbers. We have already remarked that this is one of the abilities of the digital computer.

The calculation of Np causes no difficulties. The function $f(m,n,N)$ is stored in the computer. Since we don't expect m and n to be more than a thousand each there will be no storage problem.

The calculation of the a priori probabilities requires some thought. We can make everything discrete, so that there is no difficulty in generating the required probability.

15. LEVELS OF LEARNING

What the mathematician does on a computer is like the magician's tricks. If we do not understand the mathematical theory, it all looks very mysterious and very sophisticated. If we understand the mathematical theory it all looks quite simple and we say, "Is that all there is to it?" and so on.

The mathematical trick here is to find an appropriate state space and then to define what we mean by learning. In our case, we explain learning in terms of a very simple idea: knowledge is a probability distribution. Let this then be our concept of knowledge.

What do we actually know? In general, we know something approximately with certain probabilities: learning we interpret as changing one probability distribution into another probability distribution. Our state space here has four components: the number of successes, the number of failures, the a priori probability distribution ($m,n,\phi(q)$), and the number of trials remaining.

The interesting thing mathematically is that the functional equation, although it looks more complicated, is actually, abstractly, the same functional equation that we looked at originally in Chapters 4 and 5. That is the nice feature about the abstract model. It looks very, very simple. All the complication, all the complexity, is hidden in the definition of state space, the definition of transformations, and the choice of a certain criterion function.

When we perform an action, we go from one point in state space to

another point in state space. If we enjoy a success we can go to $(m+1, n, \phi_1(q))$; if we experience a failure we go to $(m, n+1, \phi_2(q))$. The complication comes in determining what these transformations are. We now want to talk about levels of learning. We have assumed a fixed transformation, obtained by means of a theorem. We use Bayes' theorem because we don't have any more information available and because this is a very useful principle for changing the probability distribution.

It may turn out, however, that the process is either simpler or more complicated. It may be simpler in the sense that there is more structure. If there is more structure then we cannot use a simple Bayes transformation. We may have to use a more complicated transformation and then we may have to learn about the nature of this more complicated transformation. Hence, we can take the simple process above and superimpose a more complicated learning process.

16. UNKNOWN PARAMETER

For example, we can say that not only do we have an unknown probability distribution, but that the unknown probability has an unknown parameter. Let us give a very simple example of how this could occur. Suppose we have an unknown coin and we do not know what the probability of heads is. However, we do know that all the coins were made in a certain mint, and that they came in a certain barrel. We say that all the coins in that barrel have an unknown probability distribution of a certain form. Suppose we have another barrel and that coins in this barrel have the same type of probability distribution but with a different parameter. All we know at the moment is that the coin comes from one of these barrels and we have the probability distribution for the first barrel, second barrel, and so on. Hence, we can easily think of a higher level of uncertainty in which we have an unknown probability satisfying a probability distribution involving an unknown parameter. Which parameter we do not know, but we can assume if we wish that this parameter itself satisfies a probability distribution.

17. LEVELS OF LEARNING

Now it is clear that we can go up and up. We can say all these barrels are in a room and all the parameters in the room have a probability distribution $dG(a, b)$ where b determines the room. If we have different rooms there are different values of b . One way of thinking about this is that we have a sub-universe, a slightly bigger universe, a

still bigger universe, and so on. As we add more and more universes, enveloping universes, we get more and more parameters. What we want to emphasize is that there are levels of learning and levels of decision making. How far up we want to go depends upon what we want to do, which is to say what is of interest to us.

Let us now point out that although this model looks very complicated, it is abstractly the same kind of model we have started with, merely with different points in state space. Note that we did not tell how to define these points. In a particular process the points may have a quite complicated constitution—the components may actually be functions of probability distributions representing certain amounts of knowledge, and so on.

18. STATE SPACE

Nonetheless the abstract model is always the same. There are thus several levels to the model. The first level is finding out what the state space is. There are many, many different ways in which we can take the physical problem and consider it in different types of state space. We must be very careful not to assume that there is a one-to-one correspondence between a physical process and a mathematical formulation. Thus, there are many ways in which we can take this problem of computer learning and convert it into a problem of the foregoing type.

We also have to think of the transformation space. By the transformation space, we mean the space of decisions and actions. In any particular physical process you can allow all possible actions, or else you can simplify it. For example, we may take this slot machine problem and say: we can pull this lever or we can pull that lever.

Somebody might say, "I don't like this game at all. I want to take the top of the machine off and see what is inside." This is a perfectly reasonable idea.

19. SURGERY

If we return to the medical model, this is surgery. If we go into the hospital and we are not feeling very well, the doctor will try tests of different kinds and he will see what happens. If he gets enough information, fine. If he doesn't get enough information then he has to take the top off, he has to go in and perform an exploratory operation. Since this is very expensive, usually dangerous, and frequently painful, it is something that is only done as a last resort.

20. TRANSFORMATIONS

We want to point out that it is not obvious a priori what the set of allowable transformations is. There are many different decisions. We have to make up our mind what the transformation in decision space is and what the action space is. We have a state space and we have a transformation space. Next we have to decide how this affects the state space. We used Bayes' theorem. Some people might like a theory of games and some might want a min-max procedure.

Another point is why optimize? Why ask for the procedure that maximizes the expected gain, why not just talk about a feasible procedure. For example, going back into the hospital the doctor says, "I want to minimize the number of days that the patient stays in the hospital." This may be an unstable procedure, in the sense that he may be taking chances that are too serious when he does that. He might instead say, "I want a reasonable stay. I am going to cure the patient. If I can cure him in one week or two weeks that is good enough."

If we start talking about feasible procedures then again we are talking about approximate policies and we want to mention again the very powerful method of stochastic approximation. This is the most powerful modern method for handling complex problems.

We optimize because it is convenient to optimize. As we have mentioned before, we have a theory that enables us to handle multistage decision processes in which we want to optimize. But one of the questions that we should ask very carefully is: if we have all this uncertainty, and if we have all these assumptions we have made, the use of Bayes' theorem, the use of certain probability distributions to represent knowledge, and so on, what meaning is there in optimization? Is it sensible to formulate the problem as an optimal problem from the very beginning? Our personal feeling is: No.

In a realistic situation it makes no sense to talk about optimization. In realistic situations we want a reasonable approach, a feasible approach. The only trouble is that although we know how to define an optimal approach, we do not know how to define a "reasonable" approach. We can't write down the mathematical problem which says: we want this to be done using common sense.

21. FUZZY SETS

To some extent then we have the same difficulty with the mathematician that we have with the computer. Instead of turning to a computer

and saying: do something in a reasonable or feasible fashion, we turn to a computer and say: do this in the most efficient fashion. We want to minimize the expected time because we have formulas for that. But we can't say to the computer: give us a reasonable program. We would be able to do that if we could define what we mean by reasonable. There is a theory of that kind that is being developed—it is Zadeh's theory of fuzzy sets. The theory of fuzzy sets enables us to define what we mean by words like feasible and reasonable.

These are all sophisticated mathematical approaches. One of the things we would like to emphasize is that we wear two hats. One is that of a mathematician. As a mathematician we like this type of problem. But as a citizen, as a person, we would say this mathematical sophistication is probably a waste of time as far as real-life problems are concerned. We think that we ought to keep this in mind. If these problems can be handled by intelligence, we are not so sure that they can be handled by mathematics. It will, however, be harder to handle them by intelligence than it will be by mathematics. The advantage of mathematics is that it gives us a way of doing things, it provides a technique for decision making. We optimize, we minimize, we maximize, and so on.

22. INTELLIGENCE

If we want to use our intelligence and do things in a reasonable, feasible, sensible fashion, then we don't have a good theory of that type at the present time. Therefore we would say that if we are faced with a really important problem we will certainly not use the foregoing technique, we would use, most likely, stochastic approximation and simulation, which are again mathematical theories, but based on common sense. Stochastic approximation is the use of approximate policies and the mathematical part of stochastic approximation is that these approximate policies actually work, that you do converge to the point that you want to converge to in many cases.

23. LEARNING THE NATURE OF AN UNKNOWN SYSTEM

If no stochastic features are present, many interesting mathematical problems are still present. In particular, there is the mathematical theory of experimentation, a subject remarkably untreated.

24. DISCUSSION

In the foregoing pages we have discussed some aspects of learning. What we find is that it is very difficult to make this useful term precise. Any precise problems are quite difficult mathematically.

We have wanted to indicate how the digital computer can be used. We have emphasized throughout how many assumptions are required. Different assumptions will yield different results. Consequently, it is essential to make the assumptions explicit. The great value of the computer is that it forces us to make the assumptions explicit. Consequently, we know what we are talking about.

There is always a risk involved in uncertainty. No mathematical theory can eliminate this risk. What saves us frequently is the fact that we can learn. As we perform various actions we learn more and more about the system.

But it must be emphasized that mathematical theory cannot avoid the risk involved in uncertainty. We must pay something for ignorance.

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Heuristic game-playing programs (e.g., for checkers and chess) generally incorporate learning mechanisms.

CHAPTER EIGHT

consciousness

The universe was not made for consciousness.

—Aldous Huxley

1. INTRODUCTION

In this chapter we want to discuss some ideas about consciousness. Obviously consciousness is a very difficult concept and as far as we know nobody has made it precise. What we want to emphasize is that the general area of consciousness cannot be made precise.

What we want to do, as usual, is to show that many aspects of consciousness can be treated by mathematical means, which means that we can have a computer be conscious in certain ways.

At the end of the chapter we will say a few words about instinct, emotion, free will, and predestination.

2. PROCESSES

In what follows we shall be rather mathematical. That is the easiest way of making what we mean by consciousness precise.

We begin with the idea of a process. Consider a system described by a point p in a state space S . Let $T(p)$ be a transformation with the property that $p_1=T(p)$ belongs to S whenever p is in S . We call the pair $[p, T(p)]$ a process. More precisely, this is a particular description of a process. We want to emphasize again that there are many ways of describing a particular process.

When the transformation is repeated, yielding a sequence of states, p_1, p_2, \dots , where $p_1=T(p)$, $p_2=T(p_1), \dots$, we call it a multistage process. This is an abstract version of a dynamic process.

Assume next that $T(p)$ is replaced by a transformation of the form $T(p,q)$ having the property that for any p in S and any q in a decision space D , the point $p_1=T(p,q)$ is in S . A choice of vectors (decisions)

q_1, q_2, \dots , then yields a sequence of states, $p_1 = T(p, q_1)$, $p_2 = T(p_1, q_2)$, \dots . We call this a multistage decision process.

It is also an abstract version of a control process. From the mathematical point of view, control and decision processes are equivalent. Our approach to consciousness will be by way of a control process.

A determination of the q_i may be effected by maximizing a criterion function that depends on the history of the process $K = K(p, p_1, \dots; q_1, q_2, \dots)$. In many important cases this has a separable structure, as in Chapters 3, 4, 5, and 7, $K = k_1(p, q_1) + k_2(p_1, q_2) + \dots$, in other words an accumulation of single-stage effects. A criterion function is a measure of the effectiveness of a control process.

The maximizing q_i will be functions of the states p_1, p_2, \dots . In the most important cases the q_i that maximizes depends only upon the present and past history of the process $q_i = q_i(p, p_1, \dots, p_{i-1})$, and frequently only upon the current state, $q_i = q_i(p_{i-1})$.

Let us call any function of this type a policy, reserving the term optimal policy for a policy that maximizes K . The study of control and decision processes may then be considered to be the study and effective determination of optimal policies, once the criterion function is fixed.

3. ABSENCE OF A CRITERION FUNCTION

Unfortunately, in many of the control processes of greatest significance either no criterion function exists or there are too many of them. (The criterion function is a measure of the sequence of decisions.) This absence of a criterion function makes the application of mathematical theories such as the calculus of variations and dynamic programming quite difficult.

Nevertheless, the concept of policy remains meaningful. Furthermore, the powerful and flexible theory of simulation can be used to study the complex processes associated with animate and human systems.

4. FEASIBILITY

As usual, we have to examine the functional equation we obtained to see whether the procedure is feasible on a digital computer.

We have the usual problems. Time, storage, and the operations that have to be performed. We have already discussed some of these questions in connection with Chapters 3, 4, 5, and 6. Consequently, we shall not go into a discussion again. We do want to point out that many

new questions arise in the consideration of more complicated processes. We shall give some references at the end of the chapter for the reader who wants to see what these are.

5. INSTINCT

Let us now identify instinct as a policy, a policy that controls the behavior of an organism in a particular situation. We consider instinctive behavior a precise automatic response to a signal or stimulus acting on an organism. The evolutionary value of instinct is clear since in critical circumstances there may not be time for conscious behavior. The response must be preprogrammed to ensure survival.

The point we wish to emphasize, however, is that instinct is not solely a low-level intellectual activity. What seems to be the case is that there are levels of instinctive behavior, intermingled with conscious behavior. We shall return to this point below.

6. LEVELS OF PROCESSES

When the original system does not cope in a satisfactory style, survival of the species depends upon the development over time of control mechanisms to improve the performance. At the lowest level these are feedback control devices. These control systems, however, themselves require supervision and modification. Gradually, then, we see that the development of control systems requires a hierarchy of control systems.

We can identify this hierarchy of control systems with levels of consciousness. Indeed, if we wish to discuss consciousness in a meaningful fashion, we must utilize the concept of levels of consciousness. Similarly, to study thinking, we must consider levels and kinds of thinking. A confusion of levels leads to paradoxes and other difficulties, as will be discussed in the next chapter.

7. INSTRUCTIONS

Let us pursue this subject of hierarchies in another direction. Suppose that an organism contains certain types of mechanisms for conveying instructions. Some of these instructions may well be instructions for issuing instructions, and so on, again a hierarchical concept.

An error in one kind of instruction is thus seen to be far more critical than an error in another type of instruction. An error in the instructions for issuing instructions will lead to a far larger number of errors than an inaccurate instruction that leads to the development of

one faulty organ, for example. We see then that numerical probabilities of mutation are not inherently meaningful. We must take the structure of the organism into account and examine the responsiveness of the structure to a change in one component. These are questions of mathematical stability and sensitivity.

8. UNCERTAINTY

So far we have considered only deterministic processes, processes where cause and effect is assumed to hold. Let us now consider processes where uncertainty plays a major role, as in Chapter 5.

One way to construct mathematical models of uncertainty is to introduce random variables. This leads to the concept of a stochastic transformation $T(p, q, r)$. The point $p_1 = T(p, q, r)$ belongs to S whenever p is in S , q is in D and r is a random variable in R .

A choice of a sequence q_1, q_2, \dots , and a selection of random variables r_1, r_2, \dots leads as before to a sequence of states p_1, p_2, \dots . An optimal policy may be determined in a process of this nature by maximizing the expected value of a criterion function.

This, however, takes care of only first-level processes. There are far more complex levels of uncertainty. In the foregoing we assumed that the probability distributions for the random variables were known. Frequently, this is not the case, as was discussed in Chapter 7. Instead, we may possess some initial clues as to the nature of the uncertainty and then proceed to discover more of the structure of the process using observation of the events that occur. We call this an adaptive control process, and equate this operation with learning. One instinct associated with learning is curiosity. There are, however, levels of learning. We learn, we learn how to learn, and so on.

9. HIGHER-LEVEL INSTINCT

A mathematical technique for studying adaptive control processes is the theory of dynamic programming. In animals, however, it seems that adaptive control involves higher-level instincts. If so, this plausibly explains why it is so difficult to carry out mathematical studies of pattern recognition, language translation, and human communication. If these are instinctive, they are not based upon the mathematical principles of the last five thousand years, but instead upon techniques developed by evolutionary selection over hundreds of millions of years.

In other words, we possess in our brain very complex, genetically

determined, internal mechanisms specifically designed for particular tasks. Existing mathematical methods and computers cannot yet compete with these consequences of selective breeding.

10. LEVELS OF INSTINCT

Now, let us make a few final remarks regarding some phenomena that have bothered philosophers, biologists, psychologists, and scientists for a long time: instinct, intelligence, free will versus predestination, and so on.

How do we define instinct? We think of an organism as faced with multistage decision processes all the time. Life is a sequence of multi-stage decision processes. For example, take the insect. We say an insect has instinct. It is possible that some insects also have intelligence, but, at least, insects have instinct. What do we mean by that? We find an insect, and put some food nearby. What we observe is that the insect drags the food in. This is instinctive response. The interesting thing about this is that if we now place more food there, the insect reappears, drags it in, and will keep on doing this all day long. It will never stop; the instinct is preprogrammed.

What do we mean by intelligence? Certainly, instinct is a very important part of intelligence. One of the things we can say about intelligence is that intelligence possesses techniques for modifying instinct. Our instinct is to eat when we are hungry, but we don't keep on eating. We stop after a while, we don't keep on performing the same action; we are not completely preprogrammed. We have an override. We can learn, we can modify our probability distribution, we can modify our policy on the basis of what we have learned. Intelligence can be considered a way of modifying a policy.

A very interesting question is: where did we get this ability to learn? We say there are levels of instinct. Let us give an example. We have an unknown coin and we toss it 11 times. We find 5 heads and 6 tails. Then we ask ourselves to give an estimate for the probability of heads. We say $5/11$. We say, why? It is intuitively reasonable to us that this should be $5/11$. Now we maintain that this obvious response is what we call instinct. Hence, it may well be that we have levels of instinct. We have lower levels of instinct that give us strict policies. We then have policies for modifying policies, which are instinctive; we have policies for modifying policies for modifying policies, and so on. This may be instinct. This we may have gained through evolution. The organism had intuitively calculated the probability this way to

survive. Those organisms which said the probability of heads is one, died. There are organisms that can easily have their own probability calculations of this type. We might obtain probability calculations of this type in various tests.

11. FREE WILL VS. PREDESTINATION

This, of course, winds up with the fact that these are all philosophical questions of free will versus predestination. What is predestination? Predestination says that when we are at point p in state space our policy is going to be $q(p)$. Free will says that we have an override ability, that we do not have a fixed policy $q_1 \dots q_n$. But then, of course, we have seen that there are levels of free will and levels of predestination. We have purely instinctive preprogrammed techniques for modifying our behavior in the case of difficulty.

We don't insist upon all this. We just want to point out that the type of procedure that we have been talking about, multistage decision processes, gives us a way of looking at these questions of instinct and intelligence.

12. COMPUTER INTELLIGENCE

The important point when we talk about computer intelligence is to say: let us have levels of computer intelligence, level 1, level 2, level 3. Then we can ask a very precise question: where are we now, where can we expect to go? As we indicated earlier, we can say: level one is those deterministic and stochastic problems we have been talking about. Level two involves learning—where we have to estimate an unknown parameter, we have to estimate an unknown probability, an unknown probability distribution. It is clear that computers can do thinking on level one at the present time. There is no difficulty in using computers to carry out control processes of deterministic and stochastic type. The other type of process involving learning we call adaptive. We can ask ourselves what we mean by adaptation—we mean estimating parameters and the probability distribution. Adaptive behavior is learning how to estimate unknown parameters on the basis of experience. We may have preprogrammed policies, but then our experiences over time may not be very good, and we will have to learn from bad experiences that the original preprogrammed policy has estimated parameters that are not effective. We now learn to estimate parameters in a different way. Thus, we can go to the second level: estimating parameters. But what do we mean by the third level or

higher processes? Where do other aspects of thinking and problems of creativity come in? These questions are very, very difficult.

We want to point out again that we have to think of these problems in terms of levels. Levels of decision making, levels of intelligence, levels of creativity and so on. And one of the reasons why there is a tremendous amount of confusion is because people mix the levels. This will be discussed further in the next chapter. What we were trying to do, as we mentioned in the beginning, is not to give any answer, not to say that such and such is definitely the case, but merely to propose that we do have ways of thinking about these problems and ways of talking about them in a fashion that is reasonably clear. In the past what we had instead was just a great deal of argument between different schools of thought, with no understanding of what each school really meant and no way for them to communicate.

13. EMOTION

We think that there is no question that a great deal of our emotion is instinctive. Certainly, if we look at the dog and watch the way the dog behaves we realize how instinctive the emotions are and how close we are to animals. We certainly mean that most, if not all, of our emotion is a policy that is preprogrammed. The only thing that keeps us out of jail is the fact that we have an override. Most of the time we get angry at somebody and we would like to do some very serious things. Then we think about the consequences. Hence, we have a low-level policy of reaction, followed by a higher-level override which says, "Don't take that action."

We appear to operate on at least two levels but it may very well be that we operate not only on two levels, but on many levels. We have our emotions and instincts on level 1; there is a more intellectual level 2. Then there is also a part of our brain which is looking at these two at the same time and deciding to which of the two we shall pay attention. One of the interesting questions is how can an organism watch itself. What do we mean logically by an organism being self-aware? If we are self-aware we are aware that we are self-aware and again we have an infinite regression, an infinite hierarchy of levels. This has been pointed out time and time again, and it is a well-known conundrum. As a matter of fact, at the time of St. Augustine, one of the proofs of the existence of God was the fact that we could think about thinking. This was considered extraordinary, absolutely amazing. Now we know we can think about thinking about thinking. There

is nothing strange about that. It is a very interesting exercise to see how many of these levels are really meaningful, how many collapse. Maybe thinking about thinking about thinking is the highest level. Maybe there is nothing above that, but maybe there is. Until we look at these problems we can't tell how many there are.

One of our difficulties is that we don't accept reality. We have our own view of what logic and the mathematical world should be and this is what we try to superimpose upon the world that was not made for logic and mathematics. The difficulty is in our own psychology.

In our quest for some understanding of the complex and unexpected phenomenon of consciousness, we engage in thinking about thinking. Since there are many ways of doing this, we are forced to engage in thinking about thinking about thinking and so on, an infinite regression. How much then can we understand about our own thought processes? How well can the part understand the whole? Human behavior seems to be more of a task of partial control based upon partial understanding than complete control based upon complete understanding.

This brings to mind the jingle of de Morgan, "Big fleas have little fleas upon their backs to bite 'em. Little fleas have lesser fleas and so on ad infinitum."

Humor is a form of consciousness, as we shall discuss in the next chapter.

14. DISCUSSION

In the foregoing pages we have discussed some aspects of consciousness. Our approach was that consciousness could be viewed as a control process. Obviously, the reader will think of many other aspects of consciousness. What the mathematician can do is to take each phenomenon and try to construct a mathematical model.

Each model involves various assumptions. One of the great advantages of the computer is that these assumptions are made explicit.

It is well to remember the interchange between Margaret Fuller and Nietzsche. Margaret Fuller said, "Sir, I accept the universe." To which Nietzsche replied, "By God, madam, you'd better."

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CHAPTER NINE

humor

A paradox, a paradox, a most delightful paradox.

—Gilbert and Sullivan

1. INTRODUCTION

Humor, one of the most intriguing aspects of human consciousness, is an intellectual phenomenon that has been the subject of a great deal of philosophical, psychological, and psychiatric research. Diverse authorities such as H. Bergson, J. Gregory, M. Grotjahn, S. Leacock, and S. Freud, to cite only a few, have analyzed and attempted to explain the various mechanisms behind comedy, humor, and wit.

We wish to discuss here a breed of humor that appears at first sight to be rather esoteric, a type that is intimately connected with paradox, and, indeed, with some fundamental areas of logic. What is noteworthy is that this quite sophisticated brand of humor is becoming increasingly popular and is penetrating more and more deeply into our folk humor.

One of the results of our analysis is that we can use the computer to generate jokes. Although we have avoided the subject of creativity, we cannot avoid saying a few words about humor.

2. EXAMPLES OF HUMOR

We shall explore a number of examples taken from the current scene, particularly the comic strips, which illustrate the humor inherent in a certain kind of paradox; we shall present a tentative explanation of why we laugh at jokes of this genre; we shall discuss a classical logical paradox, the "Barber of Seville," and a more formidable cohort due to Russell which are abstract representations of all the jokes we cite.

There are many ramifications of the topics we discuss, some of which we shall briefly mention.

Let us begin with some simple examples. Bumper stickers containing pithy irreverence are now as common as graffiti. One might say

that they are graffiti on wheels, as opposed to graffiti on walls. Capitalizing on this fad are bumper stickers that simply read "BUMPER STICKER." Related to these are the signs in many offices that read "THIMK" or "PLAN AHEAD." It is not difficult to explain plausibly why the last two amuse us. The first, however, appears to have a deeper basis.

Let us give two examples from the cinema. In *Breakfast at Tiffany's* Holly Golightly has a cat named "Cat"; in *Bizarre, Bizarre* there is a fervent vegetarian who goes about butchering butchers to express his disapproval of their activities.

Finally, let us note the amusement we feel when we see a tow truck towing a tow truck, or read that a fire station has burned to the ground.

Basically we see that an interaction between the content of the message and the form in which the message is couched stimulates us in some fashion. There is a "feedback" effect which we will examine below.

Let me now cite some examples taken from syndicated comic strips:

Peanuts, February 10, 1968, *Los Angeles Times*

Lucy: You're a very boring person, Charlie Brown.

Lucy: (Yawn)

Lucy: Excuse me.

Lucy: I get bored just talking about how boring you are.

Andy Capp, February 26, 1969, *Los Angeles Times*

Wife: What's going to become of the younger generation, I often wonder?

Andy: Oh, they'll probably grow up and wonder what's goin' t' become of the younger generation.

Wife: We are sharp this mornin', aren't we?—mind you don't cut yourself.

Andy Capp, March 23, 1969, *Los Angeles Times*

Wife: Okay, Kid—Don't drink it dry, leave a drop for somebody else.

Andy: If I've told yer once, I've told yer a million times—don't exaggerate.

Wife: Heh! Heh! Heh!

Andy: I said something funny?

Peanuts, November 9, 1968, *Los Angeles Times*

Linus: You look kind of depressed, Charlie Brown.

Charlie: I worry about school a lot.

Charlie: I also worry about my worrying so much about school.

Charlie: My anxieties have anxieties.

Blondie, May 5, 1967, *Los Angeles Herald-Examiner*

Dagwood: I can't get to sleep.

Blondie: That means you must be terribly worried about something.

Dagwood: I am.

Blondie: What is it?

Dagwood: I'm worried about getting to sleep.

Grin and Bear It, July 4, 1969, *Los Angeles Times*

He: I'll tell you what you did on the old-fashioned 4th of July. You sat around asking whatever happened to the old-fashioned 4th of July.

The question that we wish to explore is that of isolating the humorous aspect of these panels.

That there is a systematic way of producing laughter using certain simple general techniques of the foregoing nature has been known for a long time. St. Augustine, when asked what the Creator was doing before he constructed the universe, responded, "Creating a hell for people who ask questions like that."

There is the ancient story about the philosopher who was asked why philosophers ask so many questions. "Why shouldn't philosophers ask so many questions?" he responded. This is also the standard stall of the teacher and psychiatrist, "Why do you ask that question?" The comedian who sees a joke falling flat comments about the failure of the joke, and so on.

The *New Statesman* competition #1863, due to Peter Rowlett, reads: Competitors are invited to compose a short portrait of a typical Weekend Competition entrant. Bored and weary English instructors familiar with freshman humor assign students to "write themes on anything—but how to write themes."

Let us cite also the story of the mother who proudly boasts that when her son goes to the psychiatrist all he talks about is his mother. Groucho Marx is reputed to have said that he would not join any club that would accept him as a member.

Finally there is the story that physicists and engineers love to tell which expresses their attitude toward mathematicians. It seems that they identify with the man who, walking down the street with a bag full of dirty clothes, was looking desperately for a laundry. After walking several blocks he finally spots a store with the sign "Laundry." He rushes in, dumps the bag down on the counter, and asks for the quickest service possible. The man behind the counter looks at him in amazement and says, "We don't clean clothes here." "But, what about

the sign on the front of the store?" "Oh," says the man behind the counter, "we only make the signs."

3. INCONGRUITY

Why then do we laugh at the foregoing?

Let us begin with the observation that incongruity is certainly one basic ingredient of humor. In place of "incongruity," let us use the term "illogic."

We postulate next that a need for order is instinctive in human beings. (For those who don't like the term "instinct," we can exacerbate the situation by making the further statement that all basic reasoning can be postulated to be instinctive, as has been discussed in the last chapter.) We can make the existence of this instinct plausible on the grounds that the animal that can discern regularity, and hence irregularity, can predict, anticipate, and thus survive. Logical thinking can be considered then as one expression of this desire for order. The ability to reason provides some control over the uncertainties of a menacing universe. We have already discussed how we can use computers to reason. It follows that we can use the discussion of this chapter to use the computer to generate jokes. We shall not explore this, just as we shall not explore the use of computers to generate art, music, or literature. We have already said that we shall not explore creativity.

A device that can generate logic can generate illogic.

4. LAUGHTER

Our feeling is the following: when an event assaults our logic and contradicts our preconception of the natural order of things, a basic instinct is affected. This threat to our well-being produces tension: a mild threat and a mild tension may be relieved by laughter. Laughter disengages us, and has the further value of reassuring us. It helps us feel that the threat is ephemeral and need not be taken seriously. (This is related to the technique of whistling in the dark to dissipate fear.)

5. VIOLATION OF LOGIC

With or without acceptance of the postulated mechanism, we know that violation of logic often produces humor. This is a familiar fact. Numerous jokes based on this principle can be cited, as well as many based on too rigid adherence to logic, the Gracie Allen type of humor.

Let us note in passing that this rigidity is the principal difficulty in dealing with computers, since neither level nor relationship is perceived by a computer. As completely logical devices, they possess no "common sense," no sense of proportion or balance, nor of their outgrowth, humor. As a matter of fact, it was in attempting to learn how to communicate with computers that interest in these questions was aroused.

6. THREATS TO LOGIC

Assuming the postulated mechanism, the next step is to determine what threat to logic is posed by jokes of the foregoing type. Do we scent some danger to the precarious order we have superimposed upon a chaotic world? It turns out that our instincts are very sound. The paradox that underlies all of the material quoted above is deep-rooted and troublesome. Indeed, it is hopeless but not serious.

It is clear upon reading any book on humor that discussions of humor are frequently not humorous. On the contrary, even the most serious study of pornography is frequently found to be pornographic. The study of logic need not be logical, a fact that seems very surprising only to nonmathematicians. By this statement we mean not only that the discovery of new facts and theories does not proceed in a logical way, but that logic itself contains numerous paradoxes or, more precisely, antinomies. What is amusing is that these antinomies are very similar in form to the foregoing sentences.

7. PARADOXES

The most famous of these paradoxes is that pertaining to the Barber of Seville, a paradox easily stated. At one time, it seems, the city possessed only one barber who was required by law to shave those people, and only those people, who could not shave themselves. The question inevitably arises: Who shaved the barber?

There are a number of ways of extricating ourselves from the cul-de-sac. Perhaps the easiest is to state categorically that it is a contrived paradox. Since we do not accept as reasonable the fundamental premise, the apparent contradiction does not unduly disturb us. We don't feel responsible for the consequences of an artificial situation. We experience the same disdain for the contrived joke, one perhaps that begins: A crocodile and an elephant were discussing their mothers-in-law one day . . .

8. CLASS OF ALL CLASSES

Fortunately for the gaiety of nations, or should we say notions, the example is bad—but the difficulty is real.

Far more disturbing, and actually profoundly upsetting is the observation that classical mathematical reasoning can lead to paradoxes. One of the finest examples of this is due to Russell, centering on a “class” of all classes.

For the purposes of discussion let us state that by a “class” we mean a set of objects sharing some specified property, e.g., the class of all psychoanalysts, the class of all patients, the class of all non-obscene four-letter words, the class of all concepts.

In the first of these examples, the class itself is not an object of the same kind as its members. It is neither a psychoanalyst, a patient, nor a four-letter word. But a class is a concept. Hence, the class of all concepts is an example of a class that belongs to itself.

We next divide all classes into two disjoint categories, those that belong to themselves, and those that don’t. Question: Does this latter class belong to itself or not?

9. THE SCHOOLBOY PARADOX

It is easy to see that either decision leads to a contradiction. Note, incidentally, that the classical schoolboy “paradox” is an example of this type of reasoning:

1. All generalizations are false.
2. This is a generalization.
3. Therefore, according to the original statement, this last statement must be false.
4. Therefore, there is one generalization at least that must be true.

We see the “feedback effect” working in full force.

10. AVOIDANCE OF THE PARADOX

Conventional mathematical, logical processes have led to a paradox, a self-contradiction. Our intuition then was not misled—tigers dwell within. What do we do about it? There are systematic procedures for wriggling out of this quandary.

The first procedure examines the nature of the statements and modifies them in such a way as to exorcise the paradox. The basic point is that apparently grammatical, meaningful statements need have no

logical content. At first sight this appears quite repellent, an eminently unsatisfying constraint. But recall that one of the prices of a consistent arithmetic is an injunction against dividing by zero. The "grammatical" expression $1/0$ is outlawed. Freedom is not license.

A point we wish to emphasize is the ease with which such "paradoxes," or at least ambiguities, can enter into ordinary speech. It is not at all easy to say precisely what one means, nor, conversely, to point out what is wrong in an argument by someone else, leading to an obviously incorrect conclusion.

11. JOKE OF THE HIGHEST ORDER

It is a joke of the highest order that logical theories should contain so much that is illogical, and it is a joke of just the right type. As has been noted before, the world was not made for consciousness. It would seem in the face of paradox and puzzle, mischief and mystery, that only sense of humor allows us to preserve our sanity. Too rigid insistence on logic must be a symptom of, or produce, insanity.

12. RUSSELL'S THEORY OF TYPES

One of the ways of avoiding the paradox mentioned above is to introduce the idea of levels or types. Thus, "The house is green" is on a different level from the sentence, "'The house is green'" is a sentence containing four words." In other words, we must recognize the existence of different levels. A statement about a statement is on a different level from a statement itself.

Unfortunately, this technique allows us to avoid certain paradoxes, but many other paradoxes exist. It seems to be the case that paradoxes are inherent. A discussion would take us too far afield, and we give references for the reader who wants to pursue this fascinating topic.

13. DISCUSSION

We have indicated briefly how a logical device can be used to generate jokes by systematically violating logic. Naturally, a human is needed, at this stage, to determine which are good jokes and which are not.

This violation of logic has given us an opportunity to discuss some of the paradoxes that are inherent in logic. What is important then is the fact that the use of the computer as a logic device is automatically limited. How far it is limited and in what way would be a mathematical discussion.

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CHAPTER TEN

local logics

"When I use a word," Humpty Dumpty said, in rather a scornful tone, "it means just what I choose it to mean—neither more nor less."

"The question is," said Alice, "whether you can make words mean so many different things."

"The question is," said Humpty Dumpty, "which is to be master—that's all."

—*Through the Looking Glass* (Lewis Carroll)

1. INTRODUCTION

Let us make some further remarks about logic. In the last chapter, we discussed how logic produced humor. Here, we are interested in decision making.

The point that we stress is that it would be highly desirable to have one logic that would cover all situations. What we find is that the variety of situations is so great that different situations require different logics.

2. LOGIC AND DECISION MAKING

We shall begin with some brief remarks concerning logic.

Let us regard logic as a synthesis of experience. By this we mean that what is, determines the mathematical structure of what should be. Over untold centuries we observed the real world and gradually saw certain regularities. These regularities became "common sense" and sometimes are dignified by the term "laws of nature." Let us observe that all advances are made by not taking laws of nature too seriously. They are the results of some common observations, but not all.

Let us recall that the bumble bee theoretically cannot fly because its wings are too small. Furthermore, Helmholtz proved by neglecting the viscosity of the air that lighter-than-air flight was impossible. Fortunately, these proofs were not taken too seriously by the pioneers.

What is common sense in one world would be fantasy in another,

and conversely. We often wonder what kind of logics we would have if the world were all water and there were no stars visible. The common acceptance of the existence of fixed points in space and time is a considerable help. We are influenced more than we think by the real world.

3. AXIOMS

It was recognized about twenty-five hundred years ago that axioms were necessary to provide a firm mathematical foundation for geometry. As we know, many axiomatic foundations are possible for geometry. There are many equivalent formulations. Which ones we choose is a matter of personal aesthetics and also dictated by the problems we want to tackle. It would be nice to have a meta-theory that selects one axiom system over another.

This use of the concept of axioms was a bold and imaginative step. It is not obvious that we need axioms at all. But it is one way of cutting the Gordian knot and beginning with a basis. We are so used to the method that we don't think enough about it.

4. ADAPTIVE AXIOMS

It would be highly desirable to have several theories that did not use axioms or any general premise. It is not clear that any such theory exists, nor are we aware of any formal proof. Any such theory would necessarily be an adaptive theory and the mathematical formulation of adaptive theories is not yet well based. It is probable that an adaptive theory would have to use a theory of types argument. By this we mean that we would have to stratify arguments and axioms, but any detailed discussion would take us too far afield.

5. INFLUENCE OF AXIOMATIC METHODS

This method based on axioms was widely imitated in other fields. We may cite Newton, the Declaration of Independence, and the founders of modern probability theory.

6. REALITY

The burden is on the user, *caveat emptor*. The user must demonstrate that the axiom system is a good realization of reality. Often it is not, but it is all that he has.

Not much has been done by mathematicians on these novel problems of approximation. What do we mean by "approximation," such as

a "good approximation of reality"? What measures do we use of a good fit?

The universe is strange and complex. What we call reality is constantly surprising us. As scientists we would prefer in some ways the universe to be simple. We would like the observed regularities to cover all cases.

As has been pointed out, not only is the universe stranger than we think, it may be stranger than we can think. Many things are implied by this statement. Let us make the following heretical statement: it may well be that certain issues are beyond our comprehension forever.

7. COMPLEXITY

One result of this complexity is that the physicist, who must deal with the real world, has at least three logics. These are naturally all different, but all agree in part with experiment. One logic for ordinary state variables, classical mechanics; one for the microscopic, quantum mechanics; and one for the macroscopic, relativity theory. He probably uses more than these, but these have been discussed at great length. Where they overlap, which is everywhere, there is naturally some difficulty. Various types of reconciliations have been built up in different regions. He only meets the real difficulty when applications are involved.

8. CONSISTENCY

Thus, a desired consistency gives way to utility. By this we mean that any consistent view of nature would seem to be too complex to use. Consequently, we simplify in a local area. We develop methods that can be readily applied. If these work we try to make them exact. This means that we introduce mathematics and logic. The contradictions could probably be resolved. But, as pointed out before, the user must always be sure that the axioms are satisfied. Since they never are in the real world, a bit of hand-waving is entailed. We pick and choose the experiments to be explained. This means also that there is more fashion in science than we would desire.

It is possible, and even probable, that one theory would handle all three cases, but that such a theory would be too complex to use. Consequently, a physicist must decide on the basis of experiment and theory which logic he wants to use. Always, there is some difficulty as one logic merges with another, or, equivalently, as the state variables change.

9. UNCERTAINTY

Let us now discuss some aspects of uncertainty. This phenomenon is part of the future. Indeed, we may define the future as what is uncertain. Where something is certain, we can call it the past. Let us point out that both the past and the future are determined by the present, by the model that we create of the processes. This is one way of defining past and future.

We definitely need more logics to deal with uncertainty. Right now we have one logic which forms a basis for one theory.

The classical theory of probability is essentially a frequency theory. Attempts have been made to found the theory on the frequency idea. Although these seem to be intuitive, they do not seem to be rigorous. At the present time, all frequency results are theorems and not good axioms. By this we mean that any such axiom is rather complicated, both to state and to use. It seems rather difficult to use a frequency idea.

Frequency theory works well in dealing with uncertainty in many cases, poorly in many others. Naturally, those cases in which the theory works well are based on frequency ideas. We are thinking of card games, gambling in general, insurance, and stock market operations. Even in these, fluctuations cause a great deal of trouble.

10. FUZZY SYSTEMS

The recent theory of Lotfi Zadeh, fuzzy systems, seems quite promising. This theory deals with uncertainty and applies different axioms. These are not based on a frequency approach. This theory handles many cases of uncertainty that the classical theory does not handle well.

There are a number of interesting consequences of the foregoing remarks as far as the use of computers and artificial intelligence are concerned. Every important process should have its own technique for solution.

11. GENERALITY

The generality of mathematics is very important, but special problems need special methods.

Let us first observe the consoling fact that human intelligence will always be required. This means that humans must observe the process and use its special features with experience playing an important role.

About twenty-five years ago when computers were first being introduced, it was thought that the human mind would be supplanted by the digital computer. For that reason, interactive systems were not used then as they should have been in the design of computers.

Computers will become more powerful and the use of interactive computers will increase but computers will always be an adjunct to the mind. With the aid of computers we can speed up time. This means that computers and simulation will play a significant part in education and in on-the-job training. But we cannot leave decision making to computers alone, particularly where people are involved.

12. DECISION MAKING

Decision making concerning people must be done by people. This decision making will use computers, but the final decision must be made by the human mind. Again, this means that a theory of types argument is involved. We don't know much about decision making in real systems yet.

13. DISCUSSION

In the foregoing pages, we have made some comments concerning the use of logic in the operation of systems. What we have found is that different systems require different logics. The usual logic is very useful for simple systems, but is not applicable when the system becomes too complicated.

Unfortunately, it does not agree with experiment. In other words, what we observe in some systems is not the case in others.

The whole area is relatively unexplored.

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Section 1. As a professional mathematician, Lewis Carroll (Charles Dodgson) was very much interested in logic. Many of the remarks in his books *Alice in Wonderland* and *Through the Looking Glass* can only be interpreted from this viewpoint.

For further discussion of these matters, see

R. Bellman and L. Zadeh, "Local and Fuzzy Logics," *Modern Uses of Valued Logics*, ed. D. Epstein. D. Reidel, Dordrecht, Germany, to appear 1977.

CHAPTER ELEVEN

mathematical models of the mind

Mathematics is the language of science.

—Galileo

1. INTRODUCTION

A couplet well known to generations of college classes in philosophy reads

*What is matter? Never mind.
What is mind? No matter.*

This succinct description of a problem that has perturbed philosophers for thousands of years furnishes the theme of this chapter. Perhaps a simpler way of putting it is the following. Consider that we are alternately astonished and amused at the reports of talking porpoises. How then should we view the prospect of a group of atoms that is astonished and amused?

Yet, of course, this is what a human being is, a complex structure composed of atoms arranged in different fashions that perform specialized tasks. That the phenomenon called life should emerge from inanimate stuff is remarkable enough. That superimposed upon a co-operating group of organs is consciousness is astounding.

The challenge to the mathematician, or more properly, the mathematical physicist, is plain. Can he explain and interpret the many varieties of behavior associated with consciousness using the concepts and methodology that have served so admirably to create the image of a well-ordered inanimate universe? The task is by no means an easy one. As a matter of fact, it is not clear that any totally satisfying answers will ever be forthcoming. We will discuss some reasons for this pessimistic belief later.

In any case, there are a multitude of questions that warrant exploration, both as a matter of intellectual curiosity, and because of their

importance in both the mathematical, medical, and engineering fields.

As always in the scientific world, progress is a consequence of precise delineation of a problem area. We shall therefore devote a considerable amount of time to analyzing what constitutes a reasonable question. This is just as well since, as will be clear as we proceed, we do not have too much to say about answers.

We shall cover in mathematical terms some ground that has been covered before. We feel that there is no harm in a certain amount of repetition, particularly if the repetition is done in different ways.

2. BRAIN VERSUS MIND

It is tempting to introduce some terminology. Let the word brain be used to describe the purely physiological apparatus enclosed in the skull. We assume that this mechanism operates according to well-established chemical and physical laws and there is no particular reason to suspect otherwise. We need not belabor this point, however, since it plays no part in our subsequent discussion.

Let the word mind, on the other hand, be employed to encompass the superimposed psychological operations of the brain. In particular, the intellectual curiosity that impels us to explore these matters is, according to our definition, an attribute of the mind, not the brain.

The trouble with this neat dichotomy is that it begs the question. Although it serves some intuitive purpose, it does not provide us with any useful way of distinguishing between a physiological and a psychological process involving the brain.

This is hardly a new problem. The time has, however, at last arrived when we have some reasonable hope of substantial progress towards its elucidation. This is due to a fortunate conjunction of an increase in physiological knowledge and mathematical sophistication and the development of the digital computer. We can even begin to handle "fuzzy problems" in a precise fashion.

Biology is the last scientific frontier, with the brain-mind constituting the last wilderness. What are some of the specific goals in our exploration?

3. MOTIVATION

There are a number of precise objectives, aside from the obvious one of understanding, arising from both medicine and mathematics.

A great deal can be done in the field of medicine with regard to repairing the human machine when it breaks down. The methods of

mathematical physics can be used with considerable success to provide understanding of physiological mechanisms in chemotherapy, cardiology, respiratory control, neurophysiology, radiation dosimetry, and in many other areas.

Yet psychological factors must also be added in when treating a patient, psychosomatic factors. There is no easy way of separating the psychological and physiological effects in dealing with numerous ailments ranging from allergy to cardiology.

Furthermore, in apparently purely bioengineering areas, such as those of prosthetics and orthotics, as well as in the treatment of eye and ear disorders, the interaction of the brain and the mind must be examined and understood.

There are still other types of stimuli responsible for the activity of so many mathematicians in this area. The mathematician wants to use the digital computer to obtain the effective numerical solution of many different kinds of equations. He is painfully limited in applying known methods by the severe limitations of the computer as a device for the storage, retrieval, and processing of data, as we have discussed.

Considering the fantastic change in the ability of the human to perform arithmetic in the last thirty years, and taking account of what is in the wings, we might be fairly smug and philosophic about contemporary computers were it not for the invidious comparison with the mind in other activities.

Take, for example, the task of controlling large-scale systems. It is not difficult to convince ourselves, using simple estimates of time required for conventional storage and retrieval and data processing, that certain basic operations, such as pattern recognition, language translation, and decision making in general, are impossible for the human—if known methods are employed.

What saves us from committing mistakes analogous to the classical “proof” by Helmholtz that lighter-than-air flight was impossible is the fact that we observe humans and other animate systems performing tasks of unbelievable complexity every day.

From the mathematical point of view, these are existence proofs. If we cannot duplicate these achievements using methods now available, we conclude that there exist feasible methods for treating the associated information processing and decision making based upon entirely different principles from those now known. This is a very exciting thought to a mathematician.

An impetus to the detailed study of biological phenomena is the

strong possibility that the examination of operational data-processing systems will furnish valuable clues to the understanding of the basic mechanisms. With the aid of these new techniques, we could presumably build efficient computers that would then be used to solve many more types of scientific problems. It may well be that the shortest route to effective computational tools for the solution of outstanding questions in mathematical physics is by way of neurophysiology.

4. A DIAMETRIC APPROACH

Despite the obvious importance of the approach from the physiological side, there are some valid reasons for the mathematician to pursue a different path, or at least a parallel path. Neurophysiology is not an easy field. What is known is not readily absorbed by a mathematician, and not all is known. Consequently, it is not obvious that direct use of current knowledge in the neurophysiological domain to construct mathematical models will yield immediate results.

This is the usual difficulty in research. *A posteriori*, concepts and methods are simple and logical; *a priori*, it is not even plausible that anything can be done.

It is tempting then for the mathematician, particularly one directly concerned with the utilization of digital computers, to concentrate solely upon various phenomena without direct reference to the underlying mechanisms. In the previous chapters, we posed the general problem of using classical mathematical concepts and conventional general-purpose digital computers to replicate various activities associated with the human brain-mind. We have in mind such phenomena as memory, decision making, and learning.

This is the line we wish to pursue here, well aware at the outset of its dangers and limitations. In return it may well be that in connection with the use of a computer in certain areas, artificial methods are superior to human methods. As an analogy it may be pointed out that the use of a jet engine is superior to the use of wings in powered flight and that there are no helicopterlike birds.

5. MEMORY

The digital computer is concerned with the storage, retrieval, and processing of data. It is fair to state that we possess reasonably efficient techniques for processing information, based upon mathematical analysis and technology, and very meager means for storing and retrieving data.

What we call fast memory is essentially a woodbin for numbers, and it is much more appropriate to avoid anthropomorphisms and designate it as rapid-access storage.

On the other hand, we have many reasons for thinking that the human memory (or memories) employs superior techniques for storing and retrieving data. It is an understatement to say that it would be worthwhile to know what these are. As mathematicians, we can make some uneducated guesses (uneducated, that is, as far as neurophysiology is concerned) that are quite important as far as use of the computer is concerned.

To begin with, if we are required to remember sets of data with structure as opposed, say, to automobile licenses, we can simplify the storage aspects by remembering methods for reconstructing the data rather than the data itself. In other words, we store algorithms rather than numbers. This point is important as far as the use of the digital computer is concerned, since rapid-access storage is limited.

A basic question is that of determining the proper combination of an algorithm that is effective for the required task and an algorithm that is readily stored. This gets into the area of associative memories, one of major importance as far as the storage and retrieval of data are concerned. If we understood the association techniques of the mind, we could do a great deal here.

6. PATTERN RECOGNITION

Closely associated with the foregoing question of memory is the problem of pattern recognition. How does the brain-mind recognize? Does it remember the entire pattern, or does it decompose a picture into subpatterns and use some multistage process to assemble the entire picture? In any case, a set of problems of intense mathematical interest and formidable difficulty arises from attempts to use digital and analog computers to recognize numbers, handwriting, X-rays, cardiograms, and so forth. A point perhaps of some significance is that humans make mistakes and are subject to illusions, which could mean that techniques distinct from those of classical mathematics are employed.

Related to the problems of pattern recognition is that of language translation by computer, either for complete translation or for psychological and psychiatric analysis of taped interviews. Again there are formidable difficulties.

7. HIERARCHIES IN THINKING

Considering the variety of phenomena encompassed by the term thinking, it is clear that any serious study must begin by narrowing the class of processes under discussion. Let us agree for the moment to restrict our attention to decision making as one obvious manifestation of thinking, as we have done in previous chapters. Even here, as we shall see, there are considerable obstacles to any unified theory.

The experienced mathematician knows from a study of other areas of mathematics that the key to a successful entry into a field is generally the construction of levels of problems. He knows better than to study all decision making processes at the same time.

Let us cite some examples, chosen from different parts of mathematics, that illustrate this point. To begin with, consider the integration of functions. Fairly soon after the invention of calculus, it began to be realized that not all elementary functions could be integrated in elementary terms. Thus, for example, the integrals

$$\int \exp(x^2)dx, \quad \int \frac{\exp(x)}{x}dx \quad (11.1)$$

defy all the tricks of the trade.

It requires a certain degree of sophistication to think of attempting a nonexistence proof, and still further sophistication to think of how to go about it. The problem was attacked successfully by Liouville and represents one of the most elegant, if little-known, parts of analysis.

The major problem that confronts us in this analysis, quite analogous to what we have discussed earlier, is that of giving a precise definition of the word "elementary." The basic idea is not difficult when perceived, but a rigorous solution requires a considerable amount of care in its execution.

The idea is well worth exploring, since it furnishes the key to the construction of hierarchies in many other situations. Let us briefly sketch it. Take as the basic elementary function a rational function of x and e^x , where x can assume complex values, or a rational function of the inverse of a function of this type. Call these functions of level one. A function of level two is the result of the application of a level-one function to a level-one function.

Continuing in this fashion, we can construct functions of all integral levels. It remains to show that there are functions of all levels. For

example, we wish to show that $\log(\log x)$ is not a function of level one. Since levels can collapse, e.g.,

$$\log(e^x) = x, \quad (11.2)$$

this is not a trivial matter.

Once this hierarchy of functions has been constructed, it is a relatively simple matter to show, by means of an induction, that neither of the indefinite integrals in 11.1 can be functions of level N for any integer N . This constitutes a rigorous proof of the impossibility of integration in elementary terms.

The same general idea was employed by Cantor in constructing levels of infinity. For the first level we take that of the infinite sequence of integers, denumerable sets. As a second level, we use the set of points in an interval. Immediately, two problems arise. Is the second set really nondenumerable (a question readily answered in the affirmative), and, far more difficult, is there a level intermediate between these two intuitive levels of infinity? Ignoring this, there are systematic techniques for constructing arbitrarily high levels of infinity. With the aid of these levels, we can speak in precise terms of large classes of infinite sets.

Perhaps closest to the kind of question we wish to consider is the Russell theory of types, a theory specifically designed to study the possibility of providing logical answers to reasonable questions. Possibly the most important contribution of this classic work, designed to avoid certain classical paradoxes, is the understanding of the fact that is it not meaningful to consider all possible statements simultaneously, as we have discussed in a previous chapter.

To summarize the essential features of the foregoing discussion, we may say that previous mathematical analysis in a number of areas has, on one hand, shown us the futility of attempting to consider all aspects of a class of problems simultaneously and, on the other, has demonstrated the utility of constructing theories based upon levels of problems, or as we shall occasionally say, hierarchies.

8. DECISION MAKING OF LEVEL ONE

The question of what distinguishes rational behavior is not an easy one. Much has been written on this subject and, as we shall see later, it can never be a closed topic.

From the standpoint of the mathematician, the simplest case is that

where rational behavior is equivalent to maximizing a prescribed scalar function. This is a criterion function that enables the rational individual to single out the most profitable types of behavior.

Let us use the symbol p to denote the set of data available to the decision maker, the state variable, and let q denote the decision variable. Take $g(p,q)$ to be the scalar utility function that is to be maximized by an appropriate choice of q . Let $a=T(p)$ designate the desired maximizing value. We shall call this function $T(p)$ a policy. It determines the decision that is made, or the action that is taken, given the input vector p .

In different terms, we can say that when a stimulus p exists, the response is $q=T(p)$. Conceptually there is no difficulty in using a computer to carry out this type of decision making. At present, we see control techniques of this nature in action in the operation of chemical refineries, supply depots, and so forth.

There are operational difficulties, however, which we have already discussed. As in other areas, it is the close examination of the feasibility of a given theoretical structure that leads both to more sophisticated theoretical structures and to more efficient application.

We can, if we wish, identify this programmed behavior, $q=T(p)$, with an aspect of the human and animal called instinct, as we have pointed out. We observe behavior of this general type in organisms ranging from insects to man.

This is probably a useful identification if we do not take it too seriously and begin to equate our simple mathematical model with the far more complex processes involved in what is loosely called instinct. It is probably better to avoid the use of the term instinct completely in any rigorous discussion, and to use an alternative expression, such as programmed behavior.

Working backward from an observation of $T(p)$ in a species, we can set up a number of hypothetical utility functions $g(p,q)$ which are maximized by this policy. This is the standard technique in mathematical physics. For example, we might wish to show that a particular type of programmed behavior can be interpreted as an attempt to maximize a probability of survival.

A justification for this type of approach lies in the observation that the organisms that survived *la lutte pour la vie* must have operated in some ways that were more effective than the ways used by those that did not survive. There are obvious flaws in this concept since it omits any consideration of violent cataclysms of nature, such as a shift in the

earth's axis, volcanic eruption, intense changes in radiation, and so forth, which can eliminate a species or its competitors.

Let us agree to call programmed behavior *decision making of level one*.

9. DECISION MAKING UNDER UNCERTAINTY

The determination of optimal decisions is very much more involved when there are uncertainties in the observation of p , the evaluation of $g(p,q)$, and the effect of the policy $T(p)$, as we have discussed in Chapter 5. It is safe to say that there is a considerable amount of well-justified debate currently over what constitutes rational behavior in the face of uncertainty. We shall explore this a bit further later.

The mathematician has developed a number of ways of approaching decision making under uncertainty. To begin with, he can suppose that it is essential to guard against the worst possible outcome. Let $g(p,q,r)$ denote the scalar criterion function where p and q are, as before, the state and decision variables, respectively, and r represents the variable corresponding to the unknown factors. We can then proceed to do the best in the face of the worst by choosing q so as to maximize the function

$$g_1(p,q) = \min g(p,q,r). \quad (11.3)$$

This fear response is a healthy one. When carried to extremes, however, it becomes paranoia. The problem we face constantly is that it is impossible to use this min-max technique as an operational procedure. The assumption that the universe is completely hostile is not a feasible one, since it is too expensive.

A more reasonable assumption in many cases is that the universe is indifferent. We can interpret this indifference by means of the theory of probability. Take r to be a random variable with a given probability distribution $dG(r)$. Then we agree to evaluate outcomes of decisions in some average sense.

This means that we replace $g(p,q,r)$ by the criterion function

$$g_2(p,q) = \int g(p,q,r) dG(r). \quad (11.4)$$

and proceed to choose q so as to maximize the new criterion function.

The conceptual and analytic bases are now very much more compli-

cated than in the deterministic case. Nonetheless, at the end of the optimization process we come out with a policy function $T(p)$ that can once again be given to a computer. Hence, there is no difficulty in using a digital computer to do decision making in the face of uncertainty of the type just described. It is still level-one decision making.

Two comments are appropriate at this point. In the first place, we see that what we call rational behavior depends crucially upon the way we choose to set up a criterion function. It is certainly not an absolute concept. Second, it is conceivable that the approach we use, either one based upon min-max or upon averaging, is again instinctive, a result of several hundred million years of evolution.

10. OPERATIONAL DIFFICULTIES

It is important here to focus attention on some of the fundamental difficulties involved in carrying out decision making of the type described in the foregoing. The difficulties we are about to enumerate are the principal reasons why the computer is doomed to low-level decision making for as far as we can see into the future.

In realistic situations requiring decision making, we encounter the following, alone or in combination.

(a) There is no specific way of evaluating the effect of different policies, i.e., criterion functions.

(b) The criterion function exists, but is vector valued.

(c) The criterion function is scalar, but some of the elements of the policy vector are chosen by individuals who are wholly or partially opposed to our interests.

(d) The probability distribution $dG(r)$ is not known.

(e) The dimension of p is so great that the process of optimization cannot be carried out by means of current mathematical algorithms and digital computers.

(f) The decisions must be made in a time too short to permit the use of available mathematical techniques and digital computers; on-line decision making.

In the face of either of the first two effects, we are as far away now as ever from arriving at some definitive way of determining rational behavior. The third difficulty has given rise to a mathematical theory of decision making in situations involving conflict or competition, the theory of games of von Neumann and Borel, which contains many interesting mathematical ideas. It has to date proved of limited applicability, essentially because of points *a* and *b* and the additional

fact that different individuals may possess different scalar criterion functions that are only partially related.

The fifth point has also produced mathematical theories, such as dynamic programming, specifically devoted to overcoming the dimensionality barrier. However, many problems of current importance escape these techniques completely because of the other five effects mentioned. The sixth point has so far been little studied, but the outlook in terms of current mathematical methods is not promising.

Let us focus our attention on point *d*, since it brings us into contact with a psychological phenomenon of great importance, learning.

11. LEARNING

That we do not know certain features of the system requiring control or, equivalently, decision making, is not an insuperable barrier to the construction of a theory of rational behavior.

What we want to do is to construct some precise theories concerning the way we can use previous experience to improve our performance over time. This is one aspect of what we call learning in the human being. The caveat is the observation that there seem to be many different types of learning behavior and that how the human carries out these learning processes is not known.

Once we take the position of determining various efficient techniques for improving performance on the basis of past experience, it is not at all difficult to construct algorithms, as we have discussed in Chapter 7. Let us suppose that the criterion function is $g(p, q, r)$, where p is the state variable, q the decision variable, and r a variable representing the unknown aspects. We consider a multistage decision process where at any particular time we observe the outcome of a combination of a sequence of state vectors p_1, p_2, \dots, p_N , and decision vectors q_1, q_2, \dots, q_N , where q_i is the decision made in state p_i . We may or may not directly observe the r_i . We wish to determine a policy function

$$q_{N+1} = T(p_1, p_2, \dots, p_N; q_1, q_2, \dots, q_N). \quad (11.5)$$

Let us agree to call this *decision making of level two*.

12. INSTINCT ONCE AGAIN

It is reasonable to suppose that certain aspects of animal and human learning are instinctive. By this we mean that the methods used in these types of learning are preprogrammed. A reasonable explanation

for this is, as before, that what we observe is a result of some combination of survival of the fittest with accidental events. This is also entering the domain of what is commonly called adaptation.

13. HIGHER LEVELS OF DECISION MAKING

It is now easy to see how we can construct an infinite hierarchy of decision making. So far we have considered policies as well as policies for determining policies. We can also speak about policies for determining policies about policies, and so on, or learning about learning, and so on.

In no way is this hierarchy to be considered definitive. It is merely one such that can be constructed. That there are decision making processes that lie outside any such hierarchy seems rather clear. We need merely ask where the problem of determining the level of a certain decision making process fits.

An interesting speculation, which is not mathematical, is that of ascertaining the nature of the “free will” that we possess. We have seen that certain types of decision making, level one, can be considered as instinctive, which is to say preprogrammed. The way that we learn from experience to improve over level-one decision making may also be preprogrammed, higher-level instinctive. Could it be that all of the methods that we call rational are also preprogrammed? Can the very way we proceed to break traditional concepts itself be traditional?

It is amusing to see the old problem of free will versus predestination creeping into intellectual thought again. The hierarchical methods described earlier constitute a way of making a discussion of this type precise, if we wish.

14. THE DEFECTS OF MATHEMATICS

Mathematics is an excellent tool in certain idealized situations. Fortunately or unfortunately, depending on one's point of view, the real world escapes the simple axiomatic structure needed for the application of mathematics. Essentially, the principal difficulty of mathematics is that it always insists upon supplying more information than is needed, answers to questions that are never asked.

We are generally not interested in all continuous functions, nor all differential equations, nor all situations in which the statement “A implies B, B implies C, therefore A implies C” is valid. What is desired are sets of rules for working with various subclasses of functions, differential equations, and logical statements. In the parlance of

mathematical physics, we need closure. We require ways of modifying general methods to handle specific questions.

Another way of putting this is that we require methods for extracting the essential information from the sea of data in which we float. This is an extraordinarily difficult problem, or class of problems. We can assert that the ability of the human being that is remarkable is not so much the ability to handle large amounts of data as the ability to discard large amounts and to concentrate upon the essentials.

It is quite probable that the methods used by the brain-mind are entirely different from those of classical mathematics. One way of seeing this, as we said above, is to consider the fact that humans make mistakes in recognition and are susceptible to illusions of all types. Analysis of the kinds of mistakes and the possible causes of these errors would be extremely valuable in connection with the understanding of human processes.

In any case, it seems safe to say that at present the mathematical methods we now possess are not adapted to study significant decision making processes.

Using the technique of considering levels of behavior, we can formulate models of creativity and of emotional reaction. However, for the reasons just mentioned, we do not arrive at any sophisticated theory.

15. DISCUSSION

In this chapter, we have tried to discuss some of the questions we raised in previous chapters, in a more methodical fashion. It is clear that a great deal of work has to be done in order to understand what we mean by "rational behavior."

Again, we want to emphasize that the mind appears to use quite different methods from those of classical mathematics and that is very intriguing to the mathematician.

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CHAPTER TWELVE

communication and ambiguity

Arma virumque cano . . .

—Virgil

1. INTRODUCTION

Despite profound and sustained study by many generations of scholars, human communication and understanding remain mysterious processes, perhaps more so than ever. What is thinking and how and where do we do it? How do we translate ephemeral thoughts into small symbols and large actions: How do we speak, read and hear? Since civilization is based on communication, these are crucial questions which concern everyone in many ways.

Much to our frustration in some areas and delight and relief in other areas, definitive answers seem far off in the distance. It may well be the case that there are neither definitive answers, nor even definitive questions. There are, nevertheless, many hypotheses that can plausibly be examined. In particular, there is a wide range of fascinating mathematical questions associated with these matters, much important research that should be done and many significant applications.

We wish here to examine communication from the standpoint of decision and control theory. It turns out that constraints of time, resources, uncertainty and complexity combine to produce various types of ambiguity so far as understanding is concerned. This lack of precision becomes a serious handicap when we attempt to use some mix of mathematical theory, computer technology and human abilities to operate the systems of society.

Far more significant is the ambiguity that arises from the consideration of human values in a human society. It turns out, however, that

this ambiguity that so bedevils us actually furnishes the path to our salvation. Why this is the case will be discussed below.

It is in the best ironic tradition that our success will be the consequence of our failure.

2. COMMUNICATION AS A CONTROL PROCESS

We begin with the observation that we communicate with others for various purposes, to signal, to teach, to learn, to entertain. In general, we are attempting to transmit information in various guises, both facts and ideas. Within this framework, it becomes a reasonably routine scientific affair to use such mathematical theories as probability theory, dynamic programming and the calculus of variations to discuss the vague and much-abused concept of "information" in precise terms. The so-called "theory of information" of Shannon emerges as a special case.

The intellectual device we employ is that of imbedding. Specifically, we imbed a communication process within a decision or control process. This approach is, of course, most effective when the enveloping decision process can itself be treated by means of existing conceptual and mathematical methods. Although this is not always possible, the overall approach is always useful.

As we take account of the cost in time and effort involved in preparing a message, as well as that required for transmission, interpretation and utilization, it is readily apparent that in any particular instance we wish to consider the advantages gained from the successful delivery of the message in relation to the various costs incurred in transmission.

Compromise is clearly necessary. Risks of misunderstanding must be equated with the expenditure of different types of resources, including time required for understanding. Many kinds of allocation processes arise in this fashion at all levels. This balancing of costs is characteristic of modern control theory. The fact that this balancing is not possible within a classical framework in most situations, i.e., the existence of incommensurables, is one of the major factors leading to ambiguity.

What is ironic is that what saves us ultimately, and allows us to make vital decisions, is not so much the conscious use of superior logic and sophisticated mathematics, but rather the proto-logic of an animal mind developed by evolution.

3. ALGORITHMS VS. DATA

As an example of this allocation of effort, let us describe a basic problem encountered in the storage and retrieval of data.

In using a digital computer, no matter what its size, we are constantly being forced to play the intriguing game of trading time for rapid-access storage capacity. Time is unlimited in extent, but rather expensive; rapid-access storage is free, but severely limited, even in the largest of modern computers, as we have pointed out before. This constant attempt to overcome the constraints of time and storage leads to many important, novel, and entertaining types of mathematical investigations.

Nevertheless, we believe that it can safely be said that the greatest single factor holding back effective use of digital computers in engineering and science over the last twenty years was the lack of recognition by the major computer manufacturers of the critical nature of rapid-access storage.

Consider the function e^t , the solution of the differential equation

$$u' = u, \quad u(0) = 1. \quad (12.1)$$

We feel at ease with this familiar function because of the existence of extensive tables of values prepared by indigent mathematicians employed by the WPA during the Depression.

Suppose, however, we are assigned the task of solving the equation

$$v' + v + v^2 = e^t, \quad v(0) = 1, \quad (12.2)$$

and required to furnish a determination of $v(t)$ at intervals of 10^{-6} over $[0,1]$. The task of storing the values of e^t required for this calculation, using any of a number of standard methods, is onerous and could well make the calculation impossible on a small computer.

We can circumvent the obstacle of storage in this case in a simple fashion. What we do is calculate the values of e^t as we need them. Thus, in place of 12.2 we consider the system

$$v' + v + v^2 = w, \quad v(0) = 1, \quad w' = w, \quad w(0) = 1. \quad (12.3)$$

At the cost of a minor increase in the complexity of instructions and in execution time, the strain in storage has been vastly reduced.

There are extensive applications of this idea in numerical analysis. Let us mention specifically the use of successive approximation and the computational solution of differential-difference equations.

4. THE EDUCATIONAL PROCESS

All of the foregoing is part of a far more extensive domain of problems. The task of achieving a suitable balance between ideas and facts, between algorithms and data, is basic to the educational process. Since teachers are given a limited time within which to train and influence the student, they constantly face allocation processes: what to teach, when, and for how long.

Consider, for example, the role of theory. In one sense, it is useful only because of our limited memories. If we could retain all of the facts that are available, who would need the groupings and categorizations that are often called "theories"? On the other hand, we feel that existing theory can serve as a springboard to further theory, leading to undreamt-of developments. There are also aesthetic considerations which cannot be evaluated solely on straightforward utilitarian grounds. This type of considerations will be examined again below.

Ultimately, of course, we must grapple with the crucial question of understanding and the significance of understanding. Can we comprehend facts without theory, or theory without facts, and in any case what is the proper balance, and how does it vary from person to person?

The problems of education are further complicated by the mushrooming of both data and theories. Hence, we must not only teach what is, but in addition the ability to comprehend and use what will be. We need therefore not only algorithms, but algorithms for constructing and interpreting algorithms. Students must learn to recognize, understand and use adaptive control processes.

To discuss hierarchies of algorithms in a meaningful fashion, one must have recourse to concepts of modern logic; in particular, to the theory of types. Russell and others pointed out that logical theories of this nature are required for any careful discussion of communication; Gödel and Tarski indicated the limitations on understanding.

It has finally been recognized that running a library, or any system requiring the extensive storage and retrieval of facts and ideas, is a complex mathematical process. In the same fashion we must acknowledge the intricate mathematical structure of the educational process,

deeply interwoven with psychological aspects, and rescue it, and the students, and thus all of us, from the stultifying dogmas of the contemporary schools of education.

What follows would seem to indicate that we must combine the ancient technique of teaching by parable, as is so successful in parts of law and medicine, with the linear logical presentation.

5. IMPOSSIBILITY OF EXACT MESSAGES

In the foregoing we have pointed out that common-sense considerations of time and money may limit the accuracy of a message. Below we shall discuss some of the effects of limited human intellectual powers on both the quantity and quality of data. First, however, let us examine the possibility of the existence of intrinsic limitations imposed by the particular language that is being used.

Consider, for example, the task of assigning an arithmetic value to a familiar object, the hypotenuse of an isosceles right triangle whose sides are of unit length.

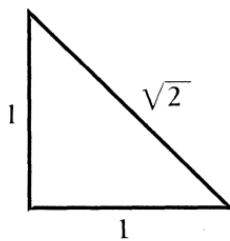


FIGURE 12.1

Pythagoras' theorem asserts that the length of the hypotenuse is $\sqrt{2}$, a quantity whose decimal expansion is $1.414\dots$. Truncating at various stages, we obtain a sequence of better and better approximations, 1.4 , 1.41 , 1.414 , \dots . We know, however, that $\sqrt{2}$ is irrational, which means that the decimal expansion does not terminate.

Much more important is the fact that this irrationality means $\sqrt{2}$ cannot be expressed as a ratio of two integers. This means that it cannot be formed by a finite combination of the fundamental operations of arithmetic, addition, subtraction, multiplication, and division, which means that it cannot be calculated exactly by a digital computer. This was a shattering result twenty-five hundred years ago, and it has not diminished in significance. It is a portent of the basic paradoxes and logical difficulties that frustrate any completely rational approach to life.

6. TRANSLATION

The foregoing observations lead to some further comments. With many modes of communication at our disposal, e.g., different human languages, it is essential that we be able to translate meaning from one language to another. What we find, however, is that quite simple concepts in one language frequently cannot be translated into any ideas in another language.

Consider, for example, the two mathematical languages of arithmetic and geometry. We see that the concept of the hypotenuse of the isosceles right triangle (one of the interpretations of $\sqrt{2}$) can be conveyed by means of elementary geometrical language, but not at all by elementary arithmetical language. Similarly, the length of the circumference of a circle of radius 1, although a simple basic geometrical concept, is not even encompassed by algebra since π is transcendental. The trisection of the general angle is impossible in the geometry of the unmarked ruler and compass, but of only moderate complexity in algebra. The solution of the general fifth-degree equation cannot be effected algebraically, but can be accomplished by means of elliptic functions, and so on, and so on.

Phenomena of this type are not at all surprising when we examine the problems of translation from one human language to another. As a result of extensive research by philologists, linguists, anthropologists, historians, et al., we know that the culture of a society affects its language, and conversely, that the language affects the culture.

The translation of concepts from one language to another, from one culture or subculture to another, is a basic problem of our times. Indeed, one can accept the idea that our very survival depends upon how well others understand what we mean by such terms as "freedom," "dignity," "peace," what importance we assign to these concepts, and what we are prepared to sacrifice for them. Yet the area remains curiously unexplored.

It is not even clear how to discuss questions of this nature, whether one can employ either of the original languages or whether a third metalanguage, or perhaps ur-language, is required.

7. PATTERN RECOGNITION

The great advance in human communication was the invention of the alphabet. This was done in stages, first by pictorial representation,

een by the use of stylized symbols combined first to form words and sentences, and then many different ideas.

At one time, there was a movement to carry this use of symbols to a logical extreme. All words and sounds were to be converted to strings of 0's and 1's which could be stored and processed by digital computers. It began to be apparent several years ago, however, that the process had been carried too far. The drawback was that the human mind can neither readily grasp the significance of a large set of numbers nor use them readily to recreate the original structure that the numbers purported to describe. It is far better to use pictures directly to convey ideas in many situations. Hence, the popularity of computer graphics for many purposes.

The two-dimensional, or multi-dimensional, pattern or structure is violently upset by arrangement in a linear array of numbers. It is interesting to see that the intuitive notion of dimension is meaningful. Questions of this type become critical in the automation of pattern recognition required for the operation of large economic, engineering, and medical systems.

It appears that new languages will have to be created, specifically designed for various decision processes. The all-purpose language of arithmetic is much too crude.

8. COMMUNICATION BY APPROXIMATION

For a number of reasons, then, we are forced to communicate by means of approximation, approximations in both quantity and quality. Sometimes, one level of approximation may be satisfactory. But, generally, as more facts are produced, more observations are made, as more complex phenomena are studied, an existing set of concepts or theory becomes more and more inadequate to convey meaning.

Consider, for example, the task of describing the motion of the planets. Newton's theory provided an explanation (in some senses) which fit the known data of Kepler and Brahe with an amazing degree of accuracy. Furthermore, the inverse square law of gravitation was easy to grasp conceptually.

As more and more observations of more accurate nature were made, discrepancies between theory and observation inevitably began to appear. After many attempts to patch up the older theory, reluctantly a new theory was superimposed, Einstein's theory of relativity. Since this theory does not eliminate all discrepancies, new and more complex theories were devised.

Has the increase in complexity improved overall understanding of physics and astronomy? It is certainly not clear that this is the case. One has an uneasy feeling that in some ways we have actually regressed. We have improved our curve-fitting, but at the expense of our intuition. It is pertinent to ask, in the spirit of an allocation process, whether this state of affairs justifies huge amounts of money being spent to probe the "secrets of nature."

9. GENTLEMAN'S AGREEMENT

In order to communicate, some common language must be employed. Much more, however, is needed. There must be agreement as to the meaning and use of the terms employed to solve problems according to a particular methodology, and indeed what the problems are.

Thus, for example, if we wish to use mathematical methods to operate a system, we must have explicit recognition of the axioms we are employing, and the logic that will be used. Further, we require state variables to designate the significant characteristics of the system, cause-and-effect relations to tell how these state variables will change over time, and criteria with which to judge the behavior of the system.

Even if all of these conditions are met, we are in no condition to deal fully with the complexity of real systems. Some of the reasons have been only hinted at in the preceding pages, since any detailed discussion would involve us in the intricacies of mathematical analysis. We shall avoid this morass with the observation that, in any case, these are not the principal difficulties encountered in a rational approach to the operation of human systems.

The required unanimity of ground rules does not at present exist, nor is it clear that it ever will. We possess no language that enables us to describe a political, economic, or social system with any degree of precision, or even to specify its objectives. Such frequently used terms as "liberty," "democracy," "prosperity," "justice," remain fuzzy. There have been recent attempts by Zadeh to introduce a mathematical theory of fuzzy systems, a new approach to uncertainty, which seems quite promising, but much remains to be done.

At the moment there is no mathematical approach that enables us to dissipate the fog of vagueness that hangs over much of life. Most discussions then are gentleman's agreements in which we temporarily suspend disbelief and assume politely that there is some foundation of meaning and that in some magic fashion we are communicating. The

amazing thing is that sometimes we do, as evidenced by the existence of cooperative action.

10. AMBIGUITY AND THE HUMAN MIND

It follows that, for a variety of reasons, we inhabit a world of ambiguity. The mathematical and logical methods that yield such dazzling results in celestial mechanics and nuclear physics cannot be used to treat most of the problems of daily life.

Yet we cope, and many obviously do very well: businessmen, politicians, medical diagnosticians, all perform mathematically impossible tasks. This means that the human mind possesses abilities that enable it to make reasonable decisions in many situations despite vagueness, inconsistency, and incompleteness. We don't know what these resources are, but we know they exist and we want to make use of them.

People can exercise judgment—a completely fuzzy concept. They resolve the ambiguity culturally, different cultures in different ways at different times.

How to train individuals in the use of judgment is a serious difficulty. We do, however, possess the powerful tool of simulation, discussed in Chapter 6.

11. THE DILEMMA OF LARGE SYSTEMS

We face an interesting puzzle in the operation of large human systems. Clearly, we cannot handle a great mass of detail in reasonable time without the aid of computers. But the computer requires the simplicity of mathematics. Indeed, its demands are non-negotiable. Thus, it cannot deal directly with the complexity of the decision making process involved in human behavior. We fortunately, however, have available one tool that can handle this complexity—the human mind.

Consequently, to make a large and complex society feasible we need combinations of man and machine in various forms. Some machines will direct and monitor some men, but mostly it will be men directing and monitoring machines.

12. DECENTRALIZATION

Humans have limited intellectual power in the sense that they can do only a small number of things at the same time, and continue in the same activity for only a relatively short time. A person can supervise only a certain number of machines; a doctor can examine only a cer-

tain number of patients per day. Humans get too tired, and bored before they get tired.

It follows that large systems will only work when hierarchical control is exercised. This involves a decentralization with a large number of groupings of relatively small components in series-parallel structure.

A system of this nature will clearly require a large number of people to ensure reasonable operation. Communication will be a major factor in its success, a communication only possible with humans. This is the salvation we referred to above. In the first place, there will be jobs enough for all. In the second place, it will be recognized that human experience is extremely valuable in dealing with human problems, and indeed essential. This will change our present attitude towards old age and the elderly. In the third place, systems of this type will have a tremendous amount of inertia, i.e., political stability. They will be hard to change and hard to control. Thus, they will be effective brakes against centralized control, the principal threat of our times.

13. FEASIBILITY AND TOLERANCE

We must begin to accept the fact that the major problem inherent in large systems is that of making them work at all. For many reasons they are intrinsically unstable without the controlling human influence of the human. Feasibility then is the key word, not efficiency.

This in turn directs us to take a more tolerant view of humanity in the large and in the small. We are struggling to handle impossible problems of decision making. Hence, we must learn to design human systems for human behavior, and make effective use of human abilities.

14. DISCUSSION

In the foregoing pages, we have made some remarks about communication and how it affects the use of computers.

Obviously, much more should be said. What we have pointed out is that the problem has not been well studied. In addition, we have pointed out the importance of this problem of communication.

In any discussion of thinking, communication plays a major role. We must communicate with ourselves and with others. The techniques that we use on the outside seem to be quite different from the techniques we use on the inside.

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Logic was discussed to a slight extent in chapters 9 and 10.

Section 5. Considerations of this type show that it is impossible to write a law without loopholes. The best we can hope for is a series of laws that close up the loopholes as they are disclosed.

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CHAPTER THIRTEEN

can computers really think?

Man is the measure of all things.

—Protagoras

1. INTRODUCTION

Let us review where we are. The problem we posed was: can computers think? Since we are interested in studying this problem in a scientific fashion, not just in a philosophical fashion, we have tried to make the problem precise by making it operational. We could, of course, try to define what do we mean by "think." Instead of doing that, we say: Let us give some examples of thinking. We have tried to answer the original question by showing that we can perform with mathematics and a computer certain processes that we accept as representative of thinking.

We may still say: We have not answered the original question. We said that thinking may mean decision making, that thinking may mean learning, that thinking may mean a number of other things. We then gave examples of the computer being able to carry out decision making processes, learning processes and so on. Does that mean that computers can really think?

To this we retort: We do not know what the word "think" really means. All we want to do is to have the computer perform a number of actions that the human being does when we say he thinks.

If we continue to ask what does one mean by human thinking, the answer is simple: nobody knows what human thinking is.

We have thus answered the original question in a very peculiar fashion: we have carried out certain kinds of processes, one of which is decision making. After we have studied some aspects of decision mak-

ing, we have studied learning, and we have studied learning for a purpose.

2. DECISION MAKING

When we talk about decision making, we are not talking in general terms, about general decision making. We say: Let us show kinds of decision making that we can carry out by means of the computer and let us, in particular, talk about multistage decision processes.

Our objective then has been to take a large number of problems and to show that they can be considered to be multistage decision processes. We might ask why we do not try to change from a multistage decision process to a single-stage decision process, which appears to be a simpler problem. The answer is we have a theory for handling multistage decision processes and it is much easier to do research in the following way: first we get a good method, then we look for problems.

In addition, we have shown that questions of time and storage play a dominant role in the mathematical theory we employ.

The hard way to do the research is to get a good problem and look for methods. Of course, originally, at some point in our life we are in a situation where we have a problem and we look for methods. Then, if we are lucky, we find a few methods. After that point on, for quite a while, we go looking around for problems that fit these methods. It is like finding a key and then going up and down the street trying all the different doors. If we are lucky we find some doors that open; if we are unlucky we find some doors that do not open. One never knows what is behind the closed door. Curiosity keeps us searching.

3. THE FUTURE OF INTELLIGENT COMPUTERS

It would be highly desirable to possess classes of computer programs suitable for such activities as language translation, medical diagnosis of tissue smears, interviewing of candidates, and so forth. After almost forty years of work in this direction, the real obstacles in these areas have begun to be recognized. In the beginning a great deal of the work was carried on by enthusiastic amateurs untrained in mathematics. Anthropomorphism was rampant, a great deal of nonsense was promulgated, and a number of extravagant claims were made.

A professional mathematician is often painfully aware that there is often little connection between the simplicity of formulation of a problem and the complexity of its solution. Amateurs believe that the task of using a computer to play chess is simpler than that of solving a

nonlinear partial differential equation because the latter involves partial derivatives. This is not at all true. Fortunately, time takes care of scientific nonsense. There are simple criteria for deciding whether or not a language translation program works or not, or whether pattern recognition can be carried out.

Currently, the professional mathematicians dominate this field and a good deal of sound work is being done. It is, however, being done slowly, with no breakthroughs in sight. At the present time, it is not clear whether computers can ever transcend the relatively simple types of decision making described earlier in connection with levels one and two. This type of activity is extremely valuable as far as society is concerned. It will serve to release human energies for more interesting and difficult activities, but it is not startling, and not on any high conceptual level.

4. THE FUTURE OF HUMAN INTELLIGENCE

Frequently, mathematical physics is stymied by the task of describing common phenomena, such as turbulence, stress and strain, elastic creep, and so on. In these cases, we use either extensive experimentation or an analog computer of some type to carry out the process of obtaining numerical answers. We have emphasized in the foregoing that mathematics and digital computers are presently incapable of handling the problem of carrying out decision making in situations where there is insufficient information along classical lines. This means that if we wish to obtain numerical results in processes of this nature we must employ an analog computer of some type. The human being is this analog computer, and it is the only one we possess.

Consequently, the role of the human as the principal decision maker in economic, political, and sociological processes is secure for as far as we can see into the future.

Many alarmists have spoken and written about a future society in which humans will be molded to the needs of machines. The real situation is just the opposite. What is needed for complex decision making is a variety of special-purpose computers designed around the specific psychology of the users. Current research and design in the field of computers is along these lines, emphasizing the ability to communicate back and forth with the computer.

Over the last hundred years, we have frequently seen the statements in many different guises that Copernicus dethroned man as the center of the universe, that Darwin removed his claim to uniqueness, that

Freud dispossessed him as master in his own body. Nowadays, the resigned attitude prevails that computers have removed the necessity for his very existence as far as thinking is concerned.

We maintain that the sum total of the mathematical analyses of the processes of control and decision making carried out over the last forty years conclusively refutes this pessimism. It makes evident that the human mind is an instrument of fantastic power and subtlety, far beyond our abilities to comprehend. Its powers are barely tapped, and it is impossible to predict its growth over the coming centuries, aided and reinforced by technical aids such as computers. Whatever happens, the human mind will remain dominant and in the central position. The human spirit remains far above anything that can be mechanized.

The human mind and spirit have already been "mechanized" by evolution. The problem is how to understand how all these "mechanisms" (or procedures or strategies or algorithms) work towards the enhancement of the quality of survival.

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