Limits and derivatives

By HC Verma Sir PGT maths

Saraswati Vidya Mandir Rambagh Basti

Objective

1. *Understand the concept of limits*: Define and explain the concept of limits, including one-sided limits and infinite limits.

2. *Learn how to evaluate limits*: Apply various techniques to evaluate limits, such as direct substitution, factoring, and canceling.

Limits

Let us consider a real-valued function “f” and the real number “c”, the limit is normally defined as

$$\lim_{x \to c} f(x) = L$$

It is read as “the limit of f of x, as x approaches c equals L”. The “lim” shows limit, and fact that function f(x) approaches the limit L as x approaches c is described by the right arrow.

Properties of limits

We assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and c is a constant. Then,

1. $\lim_{x \to a} [c.\, f(x)] = c \lim_{x \to a} f(x)$

You can factor a constant that is multiplicative out of a limit.

2. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

To consider the limit of a sum of difference, select the limits individually and put them back with the corresponding sign. This fact works regardless of number of functions we seperated by "+" or "-".

3. $\lim\limits_{x \to a} [f(x).\, g(x)] = \lim\limits_{x \to a} f(x).\, \lim\limits_{x \to a} g(x)$

Consider the limits of products similar to the limits of sums or differences. Just select the limit of the pieces and put back, and this is not limited to only two functions.

4. $\lim\limits_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ provided $\lim\limits_{x \to a} g(x) \neq 0$

As you can see, we need to bother only if the limit of the denominator is zero when operating the quotient limit. If it were zero, it ends up with a division by zero error.

Example 1:

To Compute $\lim_{x \to -4} (5x^2 + 8x - 3)$

Example 2:

To Compute $\lim_{x \to 6} \left[\frac{(x-3)(x-2)}{x-4} \right]$

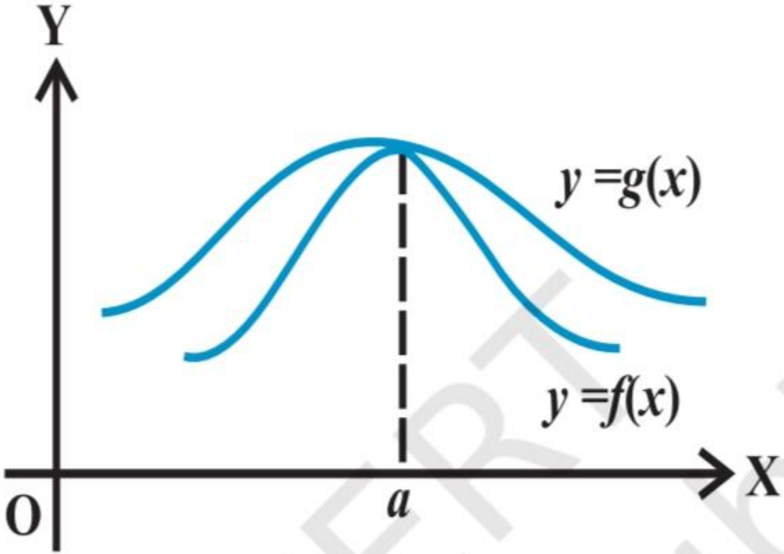
Example 3:

$$\text{Compute } \lim_{\mathbf{x}\to\mathbf{3}} \frac{(\mathbf{x^2-9})}{\mathbf{x-3}}$$

Theorem 2 For any positive integer n,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

Theorem 3 Let f and g be two real valued functions with the same domain such that $f(x) \leq g(x)$ for all x in the domain of definition, For some a, if both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$. This is illustrated in Fig 12.8.



Theorem 4 (Sandwich Theorem) Let f, g and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l$. This is illustrated in Fig 12.9.

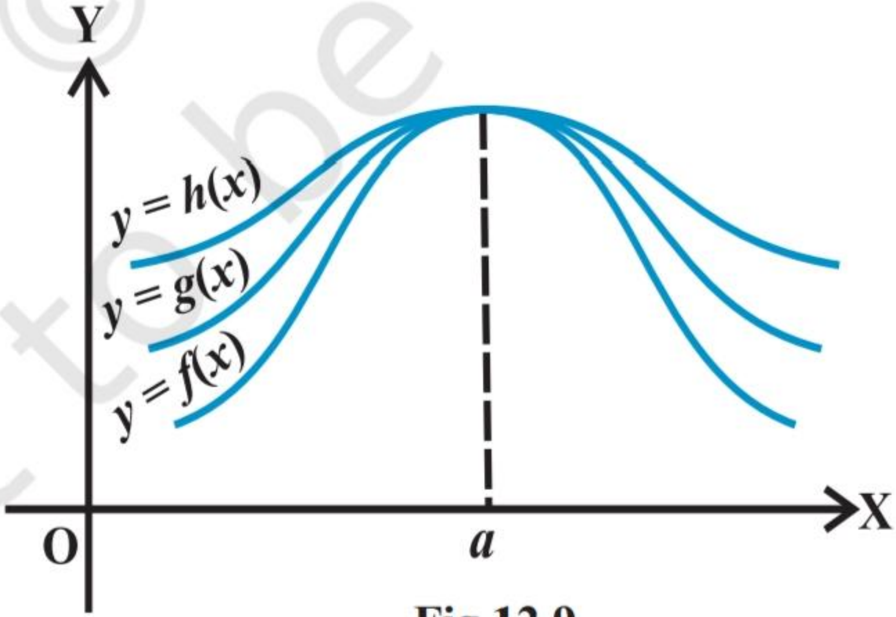


Fig 12.9

Proof of sandwich theorem

Some important trigonometry limits

$$\text{(i)} \lim_{x \to 0} \frac{\sin x}{x} = 1 .$$

$$\text{(ii)} \lim_{x \to 0} \frac{1 - \cos x}{x} = 0 .$$

Example 4 Evaluate: (i) $\lim\limits_{x \to 0} \dfrac{\sin 4x}{\sin 2x}$ (ii) $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$

Derivatives meaning

Derivatives in Maths refers to the instantaneous rate of change of a quantity with respect to the other. It helps to investigate the moment by moment nature of an amount.

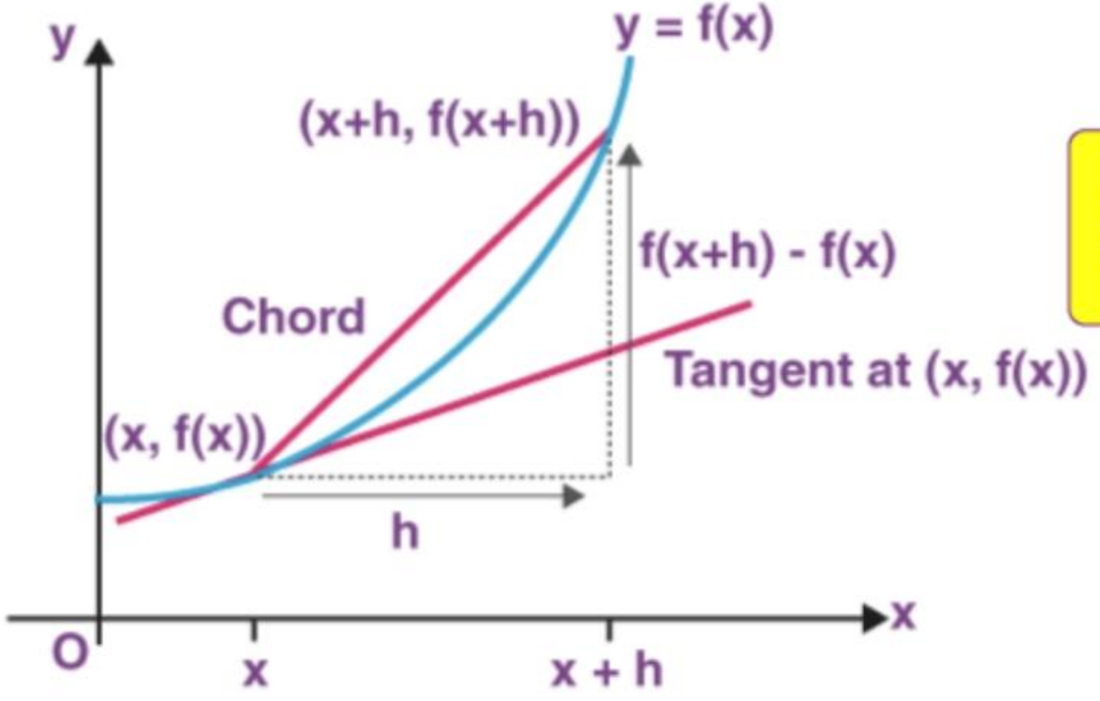
Definition of derivatives

Definition 1 *Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at a is defined by*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. Derivative of $f(x)$ *at a is denoted by* $f'(a)$.

Observe that $f'(a)$ quantifies the change in $f(x)$ at a with respect to x.



$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example 5 Find the derivative at $x = 2$ of the function $f(x) = 3x$.

Solution We have

Example 6 Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also prove that $f'(0) + 3f'(-1) = 0$.

Example 7 Find the derivative of $\sin x$ at $x = 0$.

Example 8 Find the derivative of $f(x) = 3$ at $x = 0$ and at $x = 3$.

Definition 2 *Suppose f is a real valued function, the function defined by*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is also called the first principle of derivative.

Thus $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Clearly the domain of definition of $f'(x)$ is wherever the above limit exists. There are different notations for derivative of a function. Sometimes $f'(x)$ is denoted by $\frac{d}{dx}(f(x))$ or if $y = f(x)$, it is denoted by $\frac{dy}{dx}$. This is referred to as derivative of $f(x)$ or y with respect to x. It is also denoted by D $(f(x))$. Further, derivative of f at $x = a$ is also denoted by $\left.\frac{d}{dx} f(x)\right|_a$ or $\left.\frac{df}{dx}\right|_a$ or even $\left(\frac{df}{dx}\right)_{x=a}$.

Example 9 Find the derivative of $f(x) = 10x$.

Example 10 Find the derivative of $f(x) = x^2$.

Example 11 Find the derivative of the constant function $f(x) = a$ for a fixed real number a.

Example 12 Find the derivative of $f(x) = \dfrac{1}{x}$