Numerical Solution of an Initial Value Problem

Euler's Method

Consider the first-order ordinary differential equation (ODE):

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

The exact solution $y(x)$ can be expanded in a Taylor series about $x = x_n$:

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \cdots$$

where $h = x_{n+1} - x_n$.

Since $\frac{dy}{dx} = f(x, y)$, substitute:

$$y'(x_n) = f(x_n, y_n)$$

truncating the series after the first derivative term (ignoring terms $O(h^2)$), we get the Euler formula:

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Geometrically, this corresponds to using the slope at (x_n, y_n) to extrapolate y at the next point x_{n+1}.

Illustrative Examples

Example 1

Solve the initial value problem:

$$\frac{dy}{dx} = 2y, \quad y(0) = 1$$

using Euler's method with step size $h = 0.1$.

The exact solution is:

$$y(x) = e^{2x}$$

Begin with:

$$x_0 = 0, \quad y_0 = 1$$

Calculate:

$$f(x_0, y_0) = 2 \cdot y_0 = 2 \cdot 1 = 2$$

Apply Eulers formula for the first step:

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1 \times 2 = 1.2$$

Next step:

$$f(x_1, y_1) = 2 \cdot 1.2 = 2.4$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.2 + 0.1 \times 2.4 = 1.44$$

Continuing similarly, approximate values are:

$$y_3 = 1.44 + 0.1 \times (2 \times 1.44) = 1.728$$
$$y_4 = 1.728 + 0.1 \times (2 \times 1.728) = 2.0736$$
$$y_5 = 2.0736 + 0.1 \times (2 \times 2.0736) = 2.4883$$

Compare these with exact solution values:

$$y(0.5) = e^1 \approx 2.7183$$

Eulers method underestimates but approaches the exact solution as h decreases.

Example 2

Solve

$$\frac{dy}{dt} = y, \quad y(0) = 1$$

using Euler's method with step size $h = 0.1$ for $t \in [0, 0.3]$.

Given: $\dfrac{dy}{dt} = y, \quad y_0 = 1, \quad h = 0.1$

Euler's method formula: $y_{n+1} = y_n + hf(t_n, y_n) = y_n + hy_n = y_n(1 + h)$

Iterations:

$t_0 = 0, \quad y_0 = 1$

$t_1 = 0.1, \quad y_1 = y_0(1 + 0.1) = 1 \times 1.1 = 1.1$

$t_2 = 0.2, \quad y_2 = y_1(1 + 0.1) = 1.1 \times 1.1 = 1.21$

$t_3 = 0.3, \quad y_3 = y_2(1 + 0.1) = 1.21 \times 1.1 = 1.331$

Approximate solution values:

$y(0) = 1, \quad y(0.1) \approx 1.1, \quad y(0.2) \approx 1.21, \quad y(0.3) \approx 1.331$

Example 3:

Solve

$$\frac{dy}{dt} = t + y, \quad y(0) = 1$$

using Euler's method with step size $h = 0.2$ for $t \in [0, 0.6]$.

Given: $\dfrac{dy}{dt} = t + y, \quad y_0 = 1, \quad h = 0.2$

Euler's method formula: $\quad y_{n+1} = y_n + h(t_n + y_n)$

Iterations:

$t_0 = 0, \quad y_0 = 1$

$t_1 = 0.2, \quad y_1 = y_0 + 0.2(0 + 1) = 1 + 0.2 = 1.2$

$t_2 = 0.4, \quad y_2 = y_1 + 0.2(0.2 + 1.2) = 1.2 + 0.2 \times 1.4 = 1.2 + 0.28 = 1.48$

$t_3 = 0.6, \quad y_3 = y_2 + 0.2(0.4 + 1.48) = 1.48 + 0.2 \times 1.88 = 1.48 + 0.376 = 1.856$

Approximate solution values:

$y(0) = 1, \quad y(0.2) \approx 1.2, \quad y(0.4) \approx 1.48, \quad y(0.6) \approx 1.856$

Example 4

Solve

$$\frac{dy}{dt} = y - t^2 + 1, \quad y(0) = 0.5$$

using Euler's method with step size $h = 0.1$ for $t \in [0, 0.3]$.

Given: $\dfrac{dy}{dt} = y - t^2 + 1, \quad y_0 = 0.5, \quad h = 0.1$

Euler's method formula: $\quad y_{n+1} = y_n + h(y_n - t_n^2 + 1)$

Iterations:

$t_0 = 0, \quad y_0 = 0.5$

$t_1 = 0.1, \quad y_1 = y_0 + 0.1(0.5 - 0^2 + 1) = 0.5 + 0.1 \times 1.5 = 0.65$

$t_2 = 0.2, \quad y_2 = y_1 + 0.1(0.65 - 0.1^2 + 1) = 0.65 + 0.1(0.65 - 0.01 + 1) = 0.65 + 0.1 \times 1.64 = 0.814$

$t_3 = 0.3, \quad y_3 = y_2 + 0.1(0.814 - 0.2^2 + 1) = 0.814 + 0.1(0.814 - 0.04 + 1) = 0.814 + 0.1 \times 1.774 = 0.9914$

Approximate solution values:

$y(0) = 0.5, \quad y(0.1) \approx 0.65, \quad y(0.2) \approx 0.814, \quad y(0.3) \approx 0.9914$

Derivation of Global Error Bound for Euler's Method

Consider the initial value problem (IVP):

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

Suppose f satisfies a Lipschitz condition in y:

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

for some constant $L > 0$.

Let $y(t)$ be the exact solution, and y_n be the Euler approximation at $t_n = t_0 + nh$, where h is the step size.

Local truncation error (LTE):

From Taylor expansion,

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2} y''(\xi_n), \quad \xi_n \in [t_n, t_{n+1}].$$

The Euler method gives:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

Hence the local truncation error

$$\tau_{n+1} = y(t_{n+1}) - y_{n+1} = \frac{h^2}{2} y''(\xi_n).$$

$$\therefore |\tau_{n+1}| \leq \frac{h^2}{2} M,$$

where $M = \max_{t_0 \leq t \leq T} |y''(t)|$.

Global Truncation Error

Define $e_n = y(t_n) - y_n$. Then

$$e_{n+1} = y(t_{n+1}) - y_{n+1} = e_n + h\left(f(t_n, y(t_n)) - f(t_n, y_n)\right) + \frac{h^2}{2} y''(\xi_n).$$

Taking absolute values and using the Lipschitz condition:

$$|e_{n+1}| \leq |e_n| + hL|e_n| + \frac{h^2}{2} M = (1 + hL)|e_n| + \frac{h^2}{2} M,$$

where

$$M = \max_{t_0 \leq t \leq T} |y''(t)|$$

Applying this recursively:

$$|e_n| \leq (1 + hL)^n |e_0| + \frac{h^2}{2} M \sum_{k=0}^{n-1} (1 + hL)^k.$$

Since initial error $e_0 = 0$,

$$|e_n| \leq \frac{h^2}{2} M \sum_{k=0}^{n-1} (1 + hL)^k$$

Using the formula for geometric series:

$$\sum_{k=0}^{n-1} (1 + hL)^k = \frac{(1 + hL)^n - 1}{hL}$$

Thus,

$$|e_n| \leq \frac{h^2}{2} M \cdot \frac{(1+hL)^n - 1}{hL} = \frac{hM}{2L}\left((1+hL)^n - 1\right)$$

Using the inequality $(1+x)^m \leq e^{mx}$,

$$|e_n| \leq \frac{hM}{2L}\left(e^{L(t_n - t_0)} - 1\right).$$

Hence the **global error bound for Euler's method is**

$$\boxed{|y(t_n) - y_n| \leq \frac{hM}{2L}\left(e^{L(t_n - t_0)} - 1\right).}$$

This shows that the global error grows proportional to the step size h, M and the length of the interval.

Global Error Bounds: Example

General Error Bound Formula

For the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

the global error bound at x_n is given by

$$|y(x_n) - y_n| \leq \frac{hM}{2L}\left(e^{L(x_n - x_0)} - 1\right),$$

where

- h is the step size,
- L is the Lipschitz constant for f in y,
- M is a bound on the second derivative of the exact solution.

Numerical Example with Step Sizes

Given the initial value problem

$$y' + 2y = x^3 e^{-2x}, \qquad y(0) = 1,$$

with the exact solution found to be

$$y(x) = \frac{e^{-2x}}{4}(x^4 + 4),$$

applying Eulers method for step sizes $h = 0.1$, 0.05, 0.025, we observe the error at selected values of x decreases as h decreases.

x	Exact $y(x)$	Euler ($h = 0.1$)	Error ($h = 0.1$)	Euler ($h = 0.05$)	Error ($h = 0.05$)
0.0	1.0000	1.0000	0.0000	1.0000	0.0000
0.5	0.4724	0.4546	0.0178	0.4635	0.0089
1.0	0.1517	0.1378	0.0139	0.1447	0.0070

Table 1: Comparison of exact and Euler's values with global errors for different step sizes.

Effect of Halving Step Size

For any smooth initial value problem, halving the step size h will approximately halve the global error, demonstrating the first-order convergence of Euler's method:

$$\text{Global Error} = \mathcal{O}(h)$$

Determining Step Size for Error Tolerance

To ensure that the global error is less than a given tolerance ε over an interval $[x_0, x_n]$, solve

$$\frac{hM}{2L}\left(e^{L(x_n - x_0)} - 1\right) \leq \varepsilon$$

for h:

$$h \leq \frac{2L\varepsilon}{M\left(e^{L(x_n - x_0)} - 1\right)}$$

Stability Analysis of Euler's Method

A stiff equation is a type of differential equation for which explicit numerical methods (like Euler's or standard Runge-Kutta) become unstable unless extremely small step sizes are used, even when the solution itself is varying smoothly over time.

Consider the simple linear test equation:

$$\frac{dy}{dt} = \lambda y, \quad y(0) = y_0,$$

where $\lambda \in \mathbb{C}$ is a constant. The exact solution is

$$y(t) = y_0 e^{\lambda t}.$$

Applying the explicit Euler method with step size h, the numerical solution is

$$y_{n+1} = y_n + h\lambda y_n = (1 + \lambda h)y_n.$$

By induction, after n steps,

$$y_n = (1 + \lambda h)^n y_0.$$

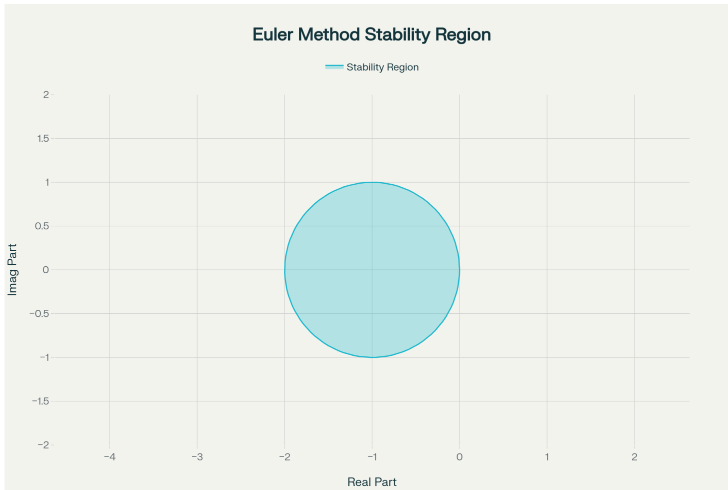

Figure 1:

For the numerical solution to be stable (i.e., bounded as $n \to \infty$), the growth factor must satisfy

$$|1 + \lambda h| \leq 1.$$

This defines the **stability region** of the explicit Euler method:

$$D = \{z \in \mathbb{C} : |1 + z| \leq 1\} \quad \text{where } z = \lambda h.$$

For λ real and negative (typical of stable continuous systems), this condition reduces to

$$-2 < \lambda h < 0 \quad \Rightarrow \quad 0 < h < \frac{-2}{\lambda}.$$

Thus, Euler's method is **conditionally stable**, meaning the step size h must be sufficiently small relative to λ.

If h is too large, the factor $|1 + \lambda h|$ exceeds 1, causing the numerical solution to grow unbounded or oscillate, even when the exact solution decays.

This limitation is particularly important for stiff differential equations where some eigenvalues λ have large negative real parts, forcing very small h for stability.

Summary:

$$\begin{cases} \text{Numerical solution growth factor:} & 1 + \lambda h \\ \text{Stability criterion:} & |1 + \lambda h| \leq 1 \\ \text{Stability region:} & \{z \in \mathbb{C} : |1 + z| \leq 1\} \\ \text{Step size restriction (real negative } \lambda\text{) :} & h < \frac{-2}{\lambda} \end{cases}$$

Modified Euler's Method

Consider the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

The **modified Euler's method** (also known as the Improved Euler method or Heun's method) gives the solution as follows:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y_n + hf(x_n, y_n)\right)\right].$$

Step 1: Taylor expansion of exact solution
The exact solution expanded about x_n is

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(\xi),\ \xi \in [x_n, x_{n+1}]$$

Step 2: Express derivatives in terms of f
Since $y' = f(x, y)$, by the chain rule,

$$y'' = \frac{d}{dx}y' = \frac{d}{dx}f(x, y) = f_x(x, y) + f_y(x, y)y',$$

and therefore,

$$y'' = f_x + f_y f.$$

Note

$$f\big(x_{n+1}, y_n + hf(x_n, y_n)\big) = f(x_n + h, y_n + hf(x_n, y_n)) = f(x_n, y_n) + hf_x + hf_y f + O(h^2).$$

Step 3: Substitute expansions into the method
The method computes:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n, y_n) + hf_x + hf_y f + O(h^2)\right].$$

Simplifying,

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2}(f_x + f_y f) + O(h^3).$$

Step 4: Compute the local truncation error

The local truncation error (LTE) is

$$y(x_{n+1}) - y_{n+1} = \left[y_n + hf + \frac{h^2}{2}(f_x + f_y f) + \frac{h^3}{6}y'''(\xi)\right] - \left[y_n + hf + \frac{h^2}{2}(f_x + f_y f) + O(h^3)\right].$$

Thus, LTE $= O(h^3)$.

This shows that the local truncation error of the modified Euler method is of order h^3, i.e., it is $\mathcal{O}(h^3)$.

Since the global error (overall accumulated error) of a one-step method is generally one order less than the local truncation error, the modified Euler method is globally second-order accurate:

$$\text{Global error} = O(h^2).$$

Numerical Example: Euler and Modified Euler Methods

Consider the initial value problem

$$y' = y - x^2 + 1, \quad y(0) = 0.5,$$

which has the exact solution

$$y(x) = (x+1)^2 - 0.5e^x.$$

We solve this on the interval $[0, 2]$ using step size $h = 0.2$.

Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n),$$

where $f(x, y) = y - x^2 + 1$.

Modified Euler's Method (Heun's method):

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ y^* &= y_n + hk_1, \\ k_2 &= f(x_{n+1}, y^*), \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2). \end{aligned}$$

Comparison of results:

x	Euler	Modified Euler	Exact	Error Euler	Error Modified Euler
0.0	0.5000	0.5000	0.5000	0.0000	0.0000
0.2	0.8000	0.8260	0.8293	0.0293	0.0033
0.4	1.1520	1.2069	1.2141	0.0621	0.0072
0.6	1.5504	1.6372	1.6489	0.0985	0.0117
0.8	1.9885	2.1102	2.1272	0.1387	0.0170
1.0	2.4582	2.6177	2.6409	0.1827	0.0232
1.2	2.9498	3.1496	3.1799	0.2301	0.0304
1.4	3.4518	3.6937	3.7324	0.2806	0.0387
1.6	3.9501	4.2351	4.2835	0.3334	0.0484
1.8	4.4282	4.7556	4.8152	0.3870	0.0596
2.0	4.8658	5.2331	5.3055	0.4397	0.0724

As we know, Taylors series is a numerical method used for solving differential equations and is limited by the work to be done in finding the derivatives of the higher-order. To overcome this, we can use a new category of numerical methods called Runge-Kutta methods to solve differential equations. These will give us higher accuracy without performing more calculations. These methods coordinate with the solution of Taylors series up to the term in hr, where r varies from method to method, representing the order of that method. One of the most significant advantages of Runge-Kutaa formulae is that it requires the functions values at some specified points.

Before learning about the Runge-Kutta RK4 method, lets have a look at the formulas of the first, second and third-order Runge-Kutta methods.

Consider an ordinary differential equation of the form $dy/dx = f(x, y)$ with initial condition $y(x_0) = y_0$. For this, we can define the formulas for Runge-Kutta methods as follows.

Runge-Kutta Method of Order four: RK4

The formula for the fourth-order Runge-Kutta method is given by:
$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
Here,
$k_1 = hf(x_0, y_0), k_2 = hf[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1], k_3 = hf[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2],$
$k_4 = hf(x_0 + h, y_0 + k_3).$

Example 1:
Find $y(1.05)$ using the RK4 method for the initial value
$\frac{dy}{dx} = x^2 + y^2, y(1) = 1.2.$
So, $f(x, y) = x^2 + y^2$
$x_0 = 1$ and $y_0 = 1.2$; $h = 0.05$.
Calculate the values of k_1, k_2, k_3 and k_4.
$k_1 = hf(x_0, y_0) = (0.05)[(1)2 + (1.2)2] = (0.05)(1 + 1.44) = 0.122.$
$k_2 = hf[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1] = (0.05)[f(1 + 0.025, 1.2 + 0.061)]$
since $\frac{h}{2} = 0.05/2 = 0.025$ and $\frac{k_1}{2} = 0.122/2 = 0.061.$
$= (0.05)[f(1.025, 1.261)] = (0.05)[(1.025)2+(1.261)2] = (0.05)(1.051+1.590)$
$= (0.05)(2.641) = 0.1320$
$k_3 = hf[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2] = (0.05)[f(1 + 0.025, 1.2 + 0.066)]$
since $\frac{h}{2} = 0.05/2 = 0.025$ and $\frac{k_2}{2} = 0.132/2 = 0.066$
$= (0.05)[f(1.025, 1.266)] = (0.05)[(1.025)2+(1.266)2] = (0.05)(1.051+1.602)$
$= (0.05)(2.653) = 0.1326.$
$k_4 = hf(x_0 + h, y_0 + k_3) = (0.05)[f(1 + 0.05, 1.2 + 0.1326)]$
$= (0.05)[f(1.05, 1.3326)]$
$= (0.05)[(1.05)2 + (1.3326)2] = (0.05)(1.1025 + 1.7758)$
$= (0.05)(2.8783) = 0.1439.$
By RK4 method, we have;
$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$y_1 = y(1.05) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
By substituting the values of y_0, k_1, k_2, k_3k_4, we get;
$y(1.05) = 1.2 + \frac{1}{6}[0.122 + 2(0.1320) + 2(0.1326) + 0.1439]$
$= 1.2 + \frac{1}{6}(0.122 + 0.264 + 0.2652 + 0.1439)$
$= 1.2 + \frac{1}{6}(0.7951) = 1.2 + 0.1325 = 1.3325$

$$\frac{dy}{dx} = x + 2y^2, y(0.2) = 1.05, h = 0.1$$

Step 1: $k_1 = 0.1[0.2 + 2(1.05)^2] = 0.1(0.2 + 2.205) = 0.1 \times 2.405 = 0.2405$

Step 2:

$$\begin{aligned} k_2 &= 0.1\left[0.25 + 2(1.17025)^2\right] \\ &= 0.1(0.25 + 2.739) = 0.1 \times 2.989 = 0.2989 \end{aligned}$$

Step 3:

$$\begin{aligned} k_3 &= 0.1\left[0.25 + 2(1.19945)^2\right] \\ &= 0.1(0.25 + 2.877) = 0.1 \times 3.127 = 0.3127 \end{aligned}$$

Step 4:

$$\begin{aligned} k_4 &= 0.1\left[0.3 + 2(1.3627)^2\right] \\ &= 0.1(0.3 + 3.717) = 0.1 \times 4.017 = 0.4017 \end{aligned}$$

Step 5:

$$\begin{aligned} y_1 &= 1.05 + \frac{1}{6}(0.2405 + 2 \times 0.2989 + 2 \times 0.3127 + 0.4017) \\ &= 1.05 + \frac{1}{6}(0.2405 + 0.5978 + 0.6254 + 0.4017) \\ &= 1.05 + \frac{1}{6}(1.8654) \\ &\approx 1.05 + 0.3109 = 1.3609 \end{aligned}$$