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ROLL NO 60
BATCH S13

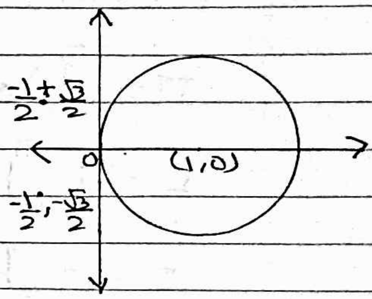
Complex Integration

Q1 Evaluate $\oint_C \frac{1}{(z^3-1)^2} dz$ where C is the circle $|z-1|=1$.

A1 $|z-1|=1$ is a circle with centre at $(1,0)$ and radius 1.

$z^3-1=(z-1)(z^2+z+1)=0$ gives $z=1$ or $z=\frac{-1\pm\sqrt{3}i}{2}$

The point $z=1$ lies inside the circle and the points $z=\frac{-1\pm\sqrt{3}i}{2}$ lie outside the circle.



$$\therefore \oint_C \frac{dz}{(z^3-1)^2} = \oint_C \frac{1/(z^2+z+1)^2}{(z-1)^2} dz$$

$z=1$ is repeated twice

$$\therefore \int_C \frac{f(z)}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \int_C \frac{dz}{(z^3-1)^2} = \int_C \frac{1/(z^2+z+1)^2}{(z-1)^2} dz$$

$$= 2\pi i \left[\frac{d}{dz}\left(\frac{1}{z^2+z+1}\right)^2\right]_{z=1} = 2\pi i \left[\frac{-2(2z+z)}{(z^2+z+1)^2}\right]_{z=1}$$

$$= 2\pi i \left[\frac{-2(3)}{3^3}\right] = -\frac{4\pi i}{9}.$$

Q2 show that $\int_c \log z \, dz = 2\pi i$ where c is a unit circle in z plane.

A2 $z = re^{i\theta}$ $\because r = 1$

$\therefore z = e^{i\theta}$ $dz = ie^{i\theta} d\theta$

θ varies from 0 to 2π

$\therefore I = \int_0^{2\pi} \log e^{i\theta} . ie^{i\theta} d\theta$

$I = i \int_0^{2\pi} i\theta . e^{i\theta} d\theta$

$I = -\int_0^{2\pi} \theta e^{i\theta} d\theta$

$I = -\left[\frac{\theta . e^{i\theta}}{i} - \frac{e^{i\theta}}{-1}\right]_0^{2\pi}$

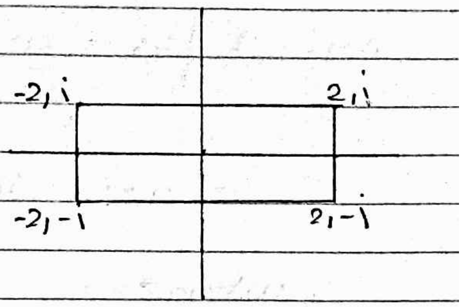
$I = -\left[[-i 2\pi e^{2\pi i} + e^{2\pi . i}] - [0+1]\right]$

$= \left[e^{2\pi i}[i 2\pi + 1] + 1\right]$

$I = 2\pi i$

Q3 Evaluate $\int_C \frac{\cos \pi z}{(z^2-1)}\, dz$ where c is a rectangle whose vertices are $2 \pm i, -2 \pm i$

$$\frac{\cos(\pi z)}{z^2-1} = \frac{h(z)}{z-1}, \quad h_z = \frac{\cos(\pi z)}{z+1}$$



$$\text{At } z=1, \; h(1) = \frac{\cos \pi}{2} = \frac{-1}{2}$$

$$\frac{\cos(\pi z)}{z^2-1} = \frac{k(z)}{z+1} \; ; \; k(z) = \frac{\cos(\pi z)}{(z-1)}$$

$$\text{At } z=1, \; k(1) = \frac{\cos \pi}{-2} = \frac{1}{2}$$

$$\int_C f(z)\, dz = \int_C \left[\frac{-1/2}{z-1} + \frac{1/2}{z+1} \right] dz$$

$$= 2\pi i \left[\frac{-1}{2} + \frac{1}{2} \right]$$

$$= \underline{0}.$$

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Q4 Expand $f(z) = \frac{1}{z^2(z-1)(z+2)}$ about $z=0$ for i) $|z|<1$, iii) $|z|>2$
ii) $1<|z|<2$

A4 Let $f(z) = \frac{a}{z} + \frac{b}{z^2} + \frac{c}{z-1} + \frac{d}{z+2}$

$\therefore 1 = az(z-1)(z+2) + b(z-1)(z+2) + cz^2(z+2) + dz^2(z-1)$

when $z=0$ $\quad 1 = -2b \therefore b = -1/2$
when $z=1$ $\quad 1 = 3c \quad \therefore c = 1/3$
when $z=-2$ $\quad 1 = -12c \quad \therefore d = -1/12$

Equating power of z^3, $0 = a+c+d \therefore a = -\frac{1}{3} + \frac{1}{12} = -\frac{1}{4}$

$\therefore f(z) = -\frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{3(z-1)} - \frac{1}{12(z+2)}$

When $0<|z|<1$

$f(z) = -\frac{1}{4z} - \frac{1}{2z^2} \cancel{+} -\frac{1}{3}[1-z]^{-1} - \frac{1}{24}\left[1+\frac{z}{2}\right]^{-1}$

$= -\frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{3\cancel{z}}\left[1+z+z^2+z^3+\dots\right] - \frac{1}{24}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\dots\right]$

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when $1<|z|<2$

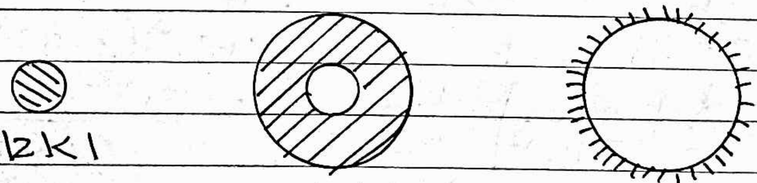
$$f(z) = -\frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{3z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{24}\left[1+\frac{z}{2}\right]^{-1}$$

$$f(z) = -\frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{3z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right] - \frac{1}{24}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\dots\right]$$

when $|z|>2$

$$f(z) = -\frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{3z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{12z}\left[1+\frac{2}{z}\right]^{-1}$$

$$= -\frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{3z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right] - \frac{1}{12z}\left[1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\dots\right]$$



$0<|z|<1$ $\quad$ $1<|z|<2$ $\quad$ $|z|>2$

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Q5 Using C.R.T evaluate $\oint \frac{e^z}{(z^2+\pi^2)^2} dz$ where C is $|z|=4$.

The poles of $f(z)$ are given by $(z^2+\pi^2)^2=0$

$\therefore (z-\pi i)^2 (z+\pi i)^2 = 0$

$\therefore z = \pi i, \pi i, -\pi i, -\pi i$

Residue at $z=\pi i$

$$= \frac{1}{1!} \lim_{z\to\pi i} \frac{d}{dz}\left[(z-\pi i)^2 \cdot \frac{e^z}{(z-\pi i)^2 (z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left[\frac{e^z}{(z+\pi i)^2}\right] = \lim_{z\to\pi i}\left[\frac{(z+\pi i)^2 \cdot e^z - e^z (z+\pi i) \cdot 2}{(z+\pi i)^4}\right]$$

$$= \lim_{z\to\pi i} \frac{e^z (z+\pi i-2)}{(z+\pi i)^3} = e^{\pi i} \cdot \frac{2(\pi i-1)}{(2\pi i)^3}$$

$$= e^{\pi i} \frac{2i(\pi+i)}{-8\pi^3 i} = -\frac{e^{\pi i}(\pi+i)}{4\pi^3} = \frac{(\pi+i)}{4\pi^3}$$

Residue at $z=-\pi i$

$$= \frac{1}{1!} \lim_{z\to-\pi i} \frac{d}{dz}\left[(z+\pi i)^2 \cdot \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}\right]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right] = \lim_{z\to-\pi i}\left[\frac{(z-\pi i)^2 e^z - e^z (z-\pi i) \cdot 2}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to-\pi i}\left[\frac{e^z (z-\pi i-2)}{(z-\pi i)^3}\right] = e^{-\pi i}\frac{(-2\pi i-2)}{(-2\pi i)^3}$$

$$= e^{-\pi i}\frac{(-2i)(\pi-i)}{8\pi^3 i} = e^{-\pi i}\frac{(\pi-i)}{4\pi^3} = \frac{\pi-i}{4\pi^3}$$

Integral is $= 2\pi i\left[\frac{\pi+i}{4\pi^3} + \frac{\pi-i}{4\pi^3}\right] = \frac{i}{\pi}$