Trigonometric Substitution

In this exploration we consider integrals whose integrands contain expressions of the form $a^2 - x^2$, $x^2 - a^2$ or $x^2 + a^2$ (especially when these expressions are under a square root), where a is a constant. We will see that these integrands can sometimes be simplified by substituting various trigonometric functions for the x variable in these expressions.

The trig. functions useful for this purpose are:

- $a \sin\theta$
- $a \sec\theta$
- $a \tan\theta$

We will also need use of the trig. identities:

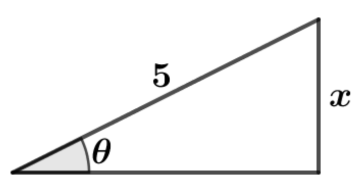
- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$

Integrand contains $a^2 - x^2$

Consider the integral $\int \sqrt{25 - x^2}\ dx$. Make the substitution $x = 5\sin\theta$ in the integral (don't forget to also substitute out the differential dx). You should now be able to simplify the resulting expression under the radical using one of the trig. identities above and end up with a radical free integrand. Complete the integration, to get an antiderivative in terms of the variable θ.

Our final step is to get our θ-antiderivative back to the original x variable. The connection between these two variables is our original substitution $x = 5\sin\theta$. From this equation we can solve for θ in terms of x and we can get any of the trig. functions of θ in terms of x.

Solving for θ gives $\theta = \arcsin(x/5)$. Solving for $\sin\theta$, we get $\sin\theta = \frac{x}{5}$. We can use this trig. relation between x and θ to draw a right angled triangle (see the figure below) and use it to find any of the other trig. functions of θ in terms of x and use these to get our antiderivative expression in terms of x. Carrying out the above



process, you should end up with:

$$\int \sqrt{25 - x^2}\ dx = \frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{x\sqrt{25 - x^2}}{2} + C$$

Integrand contains $x^2 + a^2$

Consider the integral $\int \sqrt{x^2+25}\ dx$. Make the substitution $x = 5\tan\theta$ in the integral (don't forget to also substitute out the differential dx). Carry out a process similar to the one above to resolve the integral. Hint: $\int \sec^3\theta\ d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$

You should get:

$$\int \sqrt{x^2+25}\ dx = \frac{25}{2}\left[\frac{x\sqrt{x^2+25}}{25} + \ln\left|\frac{x+\sqrt{x^2+25}}{5}\right|\right] + C$$

Integrand contains $x^2 - a^2$

Consider the integral $\int \sqrt{x^2-25}\ dx$. Make the substitution $x = 5\sec\theta$ in the integral (don't forget to also substitute out the differential dx). Carry out a process similar to the one above to resolve the integral. Hint: $\int \sec^3\theta\ d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$

You should get:

$$\int \sqrt{x^2-25}\ dx = \frac{25}{2}\left[\frac{x\sqrt{x^2-25}}{25} - \ln\left|\frac{x+\sqrt{x^2-25}}{5}\right|\right] + C$$