Integration of Rational Functions, Part 2

We saw in exploration 17 that any proper rational function $p(x)/q(x)$ can be expressed as a sum of "partial fractions" where the denominators of those fractions are built from the linear and irreducible quadratic factors of $q(x)$. In writing the partial fraction decomposition of $p(x)/q(x)$, we are left with a large number of undetermined constants that must be found. In this exploration we will learn how to find the values of these constants.

Suppose we have decomposed the proper rational function $p(x)/q(x)$ into the sum of its partial fractions

$$\frac{p(x)}{q(x)} = (\text{sum of partial fractions})$$

Multiplying both sides of this equation by $q(x)$ gives

$$p(x) = (\text{sum of partial fractions})\, q(x)$$

Because of the nature of the partial fraction decomposition of $p(x)/q(x)$, the expression on the right hand side of this equation is a polynomial $r(x)$. The polynomial $p(x)$ equals the polynomial $r(x)$ if and only if all the coefficients of the different powers of x in $p(x)$ match the coefficients of the corresponding powers of x in $r(x)$.

By equating the coefficients of corresponding powers in the polynomials $p(x)$ and $r(x)$, we get a system of linear equations in the undetermined constants.[1] Solving this system determines the values of these constants, thereby determining the partial fraction decomposition of $p(x)/q(x)$.

Example

Consider the partial fraction decomposition of the proper rational function

$$\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

where A, B, and C are the undetermined constants.

Multiplying through by $(x-1)(x^2+1)$ gives

$$\begin{aligned} x^2 &= \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1}\right)(x-1)(x^2+1) \\ &= A(x^2+1) + (Bx+C)(x-1) \\ &= Ax^2 + A + Bx^2 - Bx + Cx - C \\ &= (A+B)x^2 + (C-B)x + (A-C) \end{aligned}$$

[1] The partial fraction decomposition of $p(x)/q(x)$ guarantees that there will be the same number of linearly independent equations as there are undetermined constants, thereby insuring a unique solution of the system.

Equating the coefficients of x^2, x, and the constants on both sides of the equation gives us a system of three linear equations in the three undetermined constants A, B, and C

$$A + B = 1 \tag{1}$$
$$C - B = 0 \implies C = B \tag{2}$$
$$A - C = 0 \implies A = C \tag{3}$$

From equations (2) and (3) we see that $A = B = C$. Therefore, equation (1) implies that $2A = 1$, hence $A = B = C = 1/2$.[2] Thus, we have determined that

$$\frac{x^2}{(x-1)(x^2+1)} = \frac{1}{2} \cdot \frac{1}{x-1} + \frac{1}{2} \cdot \frac{x+1}{x^2+1}$$

Problem

1. Evaluate $\displaystyle\int \frac{x^2}{(x-1)(x^2+1)}\,dx$

Problems

Find the partial fraction decomposition of each of the following proper rational functions. Solve a system of linear equations to find the values of the undetermined constants as in the previous example.

2. $\dfrac{x-1}{x^2-9}$
3. $\dfrac{x-7}{x^2-2x-3}$
4. $\dfrac{6x+2}{4x^2+4x+1}$
5. $\dfrac{3x^3-8}{x^4+4x^2}$

Problems

Integrate each of the rational functions in problems (2)–(5).

[2]There are methods in linear algebra using matrices that makes solving a system of n equations in n unknowns a simple mechanical procedure.