Arclength and the Arclength function

Arclength

Imagine a car moving at a *constant* speed $\|\boldsymbol{v}(t)\|$ along a curvy road. The distance it travels during a time interval Δt is $\|\boldsymbol{v}(t)\|\Delta t$. This distance measures the length of road Δs travelled during the time interval Δt.

Now imagine the car moving at a *continuously varying* speed $\|\boldsymbol{v}(t)\|$ over a time interval $a \le t \le b$. Over a very small time interval dt at some particular time t within this interval, the car's speed is the (approximately) constant speed $\|\boldsymbol{v}(t)\|$. Therefore, the distance ds it travels over this time interval is $\|\boldsymbol{v}(t)\|dt$. Summing up (integrating) all these distances over the whole time interval $[a, b]$, we get the length of road travelled over this time interval.

$$s = \int_a^b \|\boldsymbol{v}(t)\| \, dt$$

where s is the length of road (arc-length of the curve) travelled over the time interval $[a, b]$.

Problems

1. Let $\boldsymbol{r}(t) = \left\langle t, -\frac{3}{4}t + 3 \right\rangle$, for $0 \le t \le 4$ be the straight line segment L from $(0, 3)$ to $(4, 0)$.
   (a) Using geometry, find the length of L.
   (b) Recalling that $\boldsymbol{v}(t) = \boldsymbol{r}'(t)$, find the length of L using the arclength integral above. Do the two answers agree?

2. The same straight line segment L from $(0, 3)$ to $(4, 0)$ can also be described by the vector-valued function $\boldsymbol{r}(t) = \left\langle t^2/4, -3t^2/16 + 3 \right\rangle$, for $0 \le t \le 4$ (how can you verify this?) With this parameterization for L, what is the value of the arclength integral? Does this answer agree with previous answers for the length of L? If not, why not?

3. The same straight line segment L from $(0, 3)$ to $(4, 0)$ can also be described by the vector-valued function $\boldsymbol{r}(t) = \langle 4 \sin t, -3 \sin t + 3 \rangle$, for $0 \le t \le \pi/2$ (how can you verify this?) With this parameterization for L, what is the value of the arclength integral? Does this answer agree with previous answers for the length of L? If not, why not?

4. The same straight line segment L from $(0, 3)$ to $(4, 0)$ can also be described by the vector-valued function $\boldsymbol{r}(t) = \langle 4 \sin t, -3 \sin t + 3 \rangle$, for $0 \le t \le \pi$ (how can you verify this?) With this parameterization for L, what is the value of the arclength integral? Does this answer agree with previous answers for the length of L? If not, why not?

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Arclength function

Let $\boldsymbol{r}(t)$ for $a \leq t \leq b$ define a curve C. We define the *arclength function* $s(t)$, $a \leq t \leq b$ by the integral

$$s(t) = \int_a^t \|\boldsymbol{r}'(\tau)\| \, d\tau$$

This function gives us the distance travelled along the curve C (the length of the curve) over the time interval $[a, t]$.

Problems

1. Let $\boldsymbol{r}(t) = \langle t, -3t/4 + 3 \rangle$, for $0 \leq t \leq 4$ be the straight line segment L from $(0, 3)$ to $(4, 0)$. Find $s(t)$, then evaluate $s(1)$, $s(2)$, $s(3)$, and $s(4)$.

2. The same straight line segment L from $(0, 3)$ to $(4, 0)$ can also be described by the vector-valued function $\boldsymbol{r}(t) = \left\langle t^2/4, -3t^2/16 + 3 \right\rangle$, for $0 \leq t \leq 4$. Find $s(t)$, then evaluate $s(1)$, $s(2)$, $s(3)$, and $s(4)$. How do these distances compare with the distances in the previous problem at the same times?