The Calculus of Vector Valued Functions

Let $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^n$ be a vector valued function defined as

$$\boldsymbol{r}(t) = \langle x_1(t), x_2(t), \dots, x_n(t) \rangle, \quad \text{for } a \le t \le b$$

where $x_k(t)$, $k = 1, \dots, n$ are differentiable and integrable scalar valued functions of t over the interval $[a, b]$. If we interpret t as time and $\boldsymbol{r}(t)$ as a position vector to the point with coordinates $(x_1(t), x_2(t), \dots, x_n(t))$, we can interpret the function $\boldsymbol{r}(t)$ as the motion of an object along a "space curve" in $\mathbb{R}^n$.

We showed previously that the derivative $\boldsymbol{r}'$ is obtained by differentiating each component function of $\boldsymbol{r}$

$$\boldsymbol{r}'(t) = \left\langle x_1'(t), x_2'(t), \dots, x_n'(t) \right\rangle$$

Using the interpretation of $\boldsymbol{r}(t)$ above, we also showed that the derivative $\boldsymbol{r}'$ represents the *velocity* vector of the moving object. It is a vector tangent to the path of motion (in the direction of motion) with magnitude equal to the speed of the object.

Similarly, we can obtain the integral of $\boldsymbol{r}$ by integrating each component function of $\boldsymbol{r}$

$$\int \boldsymbol{r}(t)\, dt = \left\langle \int x_1(t)\, dt, \int x_2(t)\, dt, \dots, \int x_n(t)\, dt \right\rangle + \boldsymbol{C}$$

where $\boldsymbol{C} \in \mathbb{R}^n$ is a constant vector.

Problems

1. Let $\boldsymbol{r}(t) = \left\langle t^2, -2t, t+1 \right\rangle$. Find $\boldsymbol{r}'(t)$ and $\int \boldsymbol{r}(t)\, dt$.

2. Let $\boldsymbol{r}(t) = \langle \cos t, \sin t, t \rangle$. Describe the curve $\boldsymbol{r}(t)$. Find the velocity vector $\boldsymbol{r}'(\pi/2)$ and use it to find the symmetric equation of the tangent line to the curve at time $t = \pi/2$. Note: The "tangent line" to a curve $\boldsymbol{r}(t)$ at some point $\boldsymbol{r}(t_0)$ is a line through that point with direction given by $\boldsymbol{r}'(t_0)$.

3. Let $\boldsymbol{r}_1(t) = \left\langle t, 1-t, 3+t^2 \right\rangle$ and $\boldsymbol{r}_2(t) = \langle 3-t, t-2, t \rangle$ be two space curves in $\mathbb{R}^3$. Find the angle between the curves at the points where they intersect. The angle between the curves is the angle between tangent vectors to the curves at the point of intersection.