The curvature κ of a curve

We have seen that the position $\boldsymbol{r}$ on a curve C can be described in terms of the time t travelling along the curve as $\boldsymbol{r}(t)$ or in terms of the distance s moving on the curve as $\boldsymbol{r}(s)$. We can convert between these different descriptions of position using the relation

$$s = \int_a^t \|\boldsymbol{r}'(\tau)\|\ d\tau$$

if we can solve it for t in terms of s (which is usually difficult if not impossible).

In addition to position along a curve, we would like to know how "bendy" the curve is at different places on the curve. We can get a measure of this "bendiness" (the curvature κ of the curve) by comparing the change in the direction of a tangent vector to the curve as we move a small distance along the curve; The more the direction of the tangent vector changes for a given small change in distance, the more bendy is the curve.

This change in the direction of the tangent vector can be measured as the change in the angle $\Delta\theta$ of the tangent vector over a small change in distance Δs moved along the curve. We define the curvature κ of the curve at a point $\boldsymbol{r}(s)$ on the curve as

$$\kappa = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds} \tag{1}$$

We showed in class that if position $\boldsymbol{r}$ on the curve is parameterized in terms of the distance s moved along the curve then we can calculate κ by

$$\kappa = \|\boldsymbol{r}''(s)\| \tag{2}$$

This is fine as long as we can reparameterize a curve $\boldsymbol{r}(t)$ in terms of s, but (as mentioned above) this can be difficult if not impossible. Fortunately, there are formulas for κ using a time parameterization of $\boldsymbol{r}$.

$$\kappa = \frac{\|\hat{\mathbf{T}}'(t)\|}{\|\boldsymbol{r}'(t)\|} \quad \text{or} \tag{3}$$

$$\kappa = \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3} \tag{4}$$

Problems

1. Using formula (1) find κ for a circle of radius a.
2. A circular helix of radius a with axis along the z axis is parameterized by $\boldsymbol{r}(t) = \langle a\cos t, a\sin t, bt\rangle$, where b determines the distance between consecutive coils of the helix. Use formula (4) to find the curvature κ of the helix in terms of a and b.
3. A parameterization of the parabola $y = x^2$ is $\boldsymbol{r}(t) = \left\langle t, t^2 \right\rangle$. It is difficult (possibly impossible) to reparameterize this curve in terms of s. Use formula (4) to find the curvature κ of the parabola. (Note: in order to calculate the cross product, you will have to include a zero z component in $\boldsymbol{r}(t)$.)