More Improper Integrals

For some of these problems you might find L'Hôpital's rule helpful to evaluate limits.

L'Hôpital's Rule

Suppose $f(x)$ and $g(x)$ are functions with either:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or
- $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

These quotients $f(x)/g(x)$ are called "indeterminate quotients" of type $0/0$ and ∞/∞, respectively.

An indeterminate product $f(x)g(x)$ of type $0\cdot\infty$ (where one factor approaches 0 as $x \to a$ and the other factor approaches ∞ as $x \to a$) can be turned into an indeterminate quotient by dividing one factor by the reciprocal of the other.

Problems

1. Determine if the integral converges or diverges. If it converges, find its (limiting) value.

   (a) $\displaystyle\int_{-2}^{3} \frac{1}{x^4}\, dx$

   (b) $\displaystyle\int_{1}^{\infty} xe^{-x}\, dx$

   (c) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\, dx$

   (d) $\displaystyle\int_{0}^{1} x \ln x\, dx$

   (e) $\displaystyle\int_{0}^{9} \frac{1}{\sqrt[3]{x-1}}\, dx$

2. Find the values of a for which the integral $\displaystyle\int_0^\infty e^{ax}\cos x\ dx$ converges. Evaluate the integral for those values of a.

Answers

1a. Diverges

1b. Converges to $2/e$

1c. Converges to 0

1d. Converges to $-1/4$

1e. Converges to $9/2$

2. Converges to $\dfrac{-a}{1+a^2}$ if $a < 0$. Does not converge if $a \geq 0$.