1 Area, Number, and Limit Concepts in Antiquity

Babylonian and Egyptian Geometry

The historical origins of what we now call mathematical concepts—those that deal with number, magnitude, and form—can be traced to the rise of civilizations in the fertile river valleys of China, Egypt, India, and Mesopotamia. In particular, fairly detailed and reliable information is now available concerning the highly organized cultures of the peoples who lived along the Nile in Egypt and in the "fertile crescent" of the Tigris and Euphrates rivers in Mesopotamia in the early centuries of the second millenium B.C.

The Greeks, whose geometrical investigations provided the foundations for the development of much of modern mathematics (including the calculus), generally assumed that geometry had its origin in Egypt. For example, the Greek historian Herodotus (fifth century B.C.) wrote that agricultural plots along the Nile were taxed according to area, so that when the annual flooding of the river swept away part of a plot and its owner applied for a corresponding reduction in his taxes, it was necessary for surveyors to determine how much land had been lost. Obviously, this would have required the invention of elementary techniques of geometrical measurement.

More direct information is provided by the Egyptian papyri that have been rediscovered in modern times. In regard to Egyptian mathematics, the most important of these is the *Rhind Papyrus* which was copied in about 1650 B.C. by a scribe named Ahmes who states that it derives from a prototype from the "middle kingdom" of about 2000 to 1800 B.C. This papyrus consists mainly of a list of problems and their solutions, about twenty of which relate to the areas of fields and volumes of granaries. Each

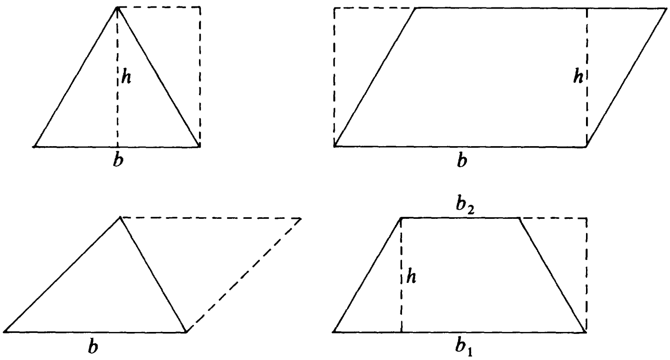

Figure 1

problem is stated in terms of particular numbers (rather than literal variables), and its solution carried out in recipe fashion, without explicitly specifying either the general formula (if any) used or the source or derivation of the method.

Apparently it is taken for granted that the area of a rectangle is the product of its base and height. The area of a triangle is calculated by multiplying half of its base times its height. In one problem the area of an isosceles trapezoid, with bases 4 and 6 and height 20, is calculated by taking half the sum of the bases, "so as to make a rectangle," and multiplying this times the height to obtain the correct area of 100. This and similar examples suggest that Egyptian prescriptions for area computations may have stemmed from elementary *dissection methods* involving the idea of cutting a rectilinear figure into triangles and then rearranging the parts so as to obtain a rectangle.

Exercise 1. Use the dissections suggested by Figure 1 to derive the familiar formulas for the areas of triangles ($\frac{1}{2}bh$), parallelograms (bh), and trapezoids ($\frac{1}{2}(b_1+b_2)h$).

Exercise 2. A later papyrus calculates the area of a quadrilateral (4-sided polygon) by multiplying half the sum of two opposite sides times half the sum of the other two sides. Does this give the correct result for a trapezoid or parallelogram that is not a rectangle?

Exercise 3. (a) In one of the Rhind papyrus problems the area of a circle is calculated by squaring 8/9 of its diameter. Compare this method with the area formula $A = \pi r^2$ to obtain the Egyptian approximation $\pi \cong 3.16$.

(b) This very good approximation to π may have been obtained as follows. Trisect each side of the square circumscribed about a circle of diameter d, and cut off its 4 corners as indicated in Figure 2. Show that the area of the resulting octagon is

$$A = \frac{7}{9}d^2 = \frac{63}{81}d^2 \cong \frac{64}{81}d^2 = \left(\frac{8}{9}d\right)^2.$$

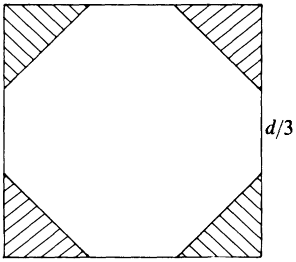

Figure 2

During the past half-century a great many mathematical cuneiform tablets, dating from the Old Babylonian age of the Hammurabi dynasty (ca. 1800–1600 B.C.), have been unearthed and deciphered. It now appears that Babylonian mathematics was considerably more advanced than Egyptian mathematics. For example, the Babylonians were adept at the solution of algebraic problems involving quadratic equations or pairs of equations, either two linear equations in two unknowns or one linear and one quadratic equation. They computed accurate numerical answers using a positional sexagesimal (base 60) system of numeration. For example, they calculated $\sqrt{2}$ as

$$1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3}=1.414213,$$

which differs by less than 0.000001 from the true value.

In regard to geometry, the Babylonians correctly calculated the areas of triangles and trapezoids, and the volumes of cylinders and prisms (as the area of the base times the height). They were also familiar (at least on an empirical basis), well over a millenium before the time of Pythagoras, with the so-called Pythagorean theorem to the effect that the sum of the squares of the legs of a right triangle is equal to the square of its hypotenuse. In a typical Babylonian problem to be solved using this result, a ladder of given length would be standing against a wall, and it would be asked how far the bottom of the ladder slides away from the wall, if its top is lowered by a given distance.

Just as in the case of the Egyptian papyri, the Babylonian tablets mainly present problems solved by means of prescriptions that do not provide the basis for their methods. However, the following exercise presents a derivation of the Pythagorean theorem that would have been well within their range, because they were familiar with the formula $(a+b)^2=a^2+2ab+b^2$.

Exercise 4. Four copies of a right triangle with legs a and b and hypotenuse c, together with a square of edge c, are assembled as in Figure 3 to form a square of edge $a+b$. Explain why the assembled figure *is* a square, and derive the Pythagorean relation by computing its area in two different ways.

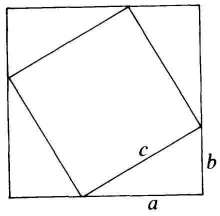

Figure 3

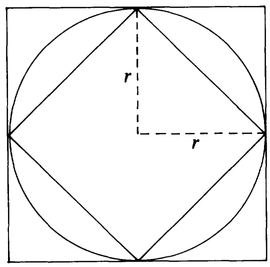

Figure 4

EXERCISE 5. The Babylonians generally used $3r^2$ for the area of a circle of radius r, corresponding to the poor approximation $\pi \cong 3$. Show that this approximation could have been obtained by averaging the areas of the inscribed and circumscribed squares in Figure 4.

EXERCISE 6. The Babylonians generally calculated the volume of a frustum of a cone or pyramid by means of the plausible (?) formula $V = \frac{1}{2}(A_1 + A_2)h$, where h is its height and A_1, A_2 the areas of its top and bottom. Show that this formula is incorrect by calculating the volume of a frustum of height 2, cut from a cone of height 4 and base radius 2 (Fig. 5). Use the (correct) formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a cone.

In summary, the Egyptians and especially the Babylonians acquired a significant accumulation of elementary geometrical facts that they used to solve particular numerical problems. However, their surviving texts include

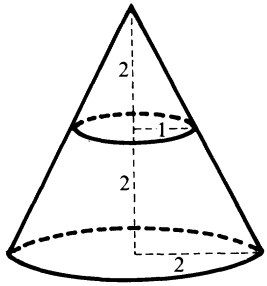

Figure 5

few if any explicit statements of general rules or methods of procedure. They made no clearcut distinctions between exact and approximate results. There is no indication of any emphasis on logical proofs or derivations in Egyptian and Babylonian thought. Thus their mathematics, despite its notable accomplishments, seems not to have been organized into any deductive system of investigation.

More complete accounts of Egyptian and Babylonian mathematics may be found in the books by Boyer [1], Neugebauer [8], and van der Waerden [12] cited in the references at the end of this chapter. Detailed discussions of ancient approximations and computations of the number π may be found in the articles by Seidenberg [9] and Smeur [10].

Early Greek Geometry

The Babylonian and Egyptian lore of number and geometry was assimilated by the Greeks, who contributed to mathematics the consciously logical and explicitly deductive approach that is now its distinguishing feature. The history of Greek mathematics begins in the sixth century B.C. with Thales and Pythagoras, both of whom are said to have traveled to Babylonia and Egypt to acquire the knowledge of those lands.

Thales lived in the first half of the sixth century B.C. On the basis of a late fourth century B.C. history of Greek mathematics that is now lost, the fourth century A.D. philosopher Proclus (in his commentary on the first book of Euclid's *Elements*) states that Thales proved the following theorems:

1. A diameter of a circle divides it into two equal parts.
2. The base angles of an isosceles triangle are equal.
3. The vertical angles formed by two intersecting straight lines are equal.
4. The angle-side-angle congruence theorem for triangles.

In addition, the fact that an angle inscribed in a semi-circle is a right angle is still known as the "theorem of Thales." Whether or not the tools for actual proofs of such theorems existed as early as Thales, it is significant that he is the first human being to whom proofs of specific mathematical results have even been attributed. Proclus adds (as quoted by van der Waerden [12], p. 90) that Thales

> made many discoveries himself, in many other things he showed his successors the road to the principles. Sometimes he treated questions in a more general manner, sometimes in a more intuitive way . . . Pythagoras, who came after him, transformed this science into a free form of education; he examined this discipline from its first principles and he endeavored to study the propositions, without concrete representation, by purely logical thinking.

Pythagoras is thought to have died about 500 B.C. He established a secret society or cult with distinctly mystical aspects that continued after his death. However, the Pythagoreans were actively engaged in the pursuit of learning, including mathematics (which orginally meant "that which is learned"). At their hands the subject gradually assumed an abstract character that distinguished it from the empirical and pragmatic mathematics of the Babylonians and Egyptians. Before the end of the fifth century B.C. they had formulated and proved on a rational basis the common theorems dealing with relations between triangles and other rectilinear plane figures and their areas.

"All is number" is quoted as the motto of the Pythagorean school. The Greeks used the word number to mean a "whole" number, a positive integer. In Greek theoretical mathematics (as distinguished from practical or commercial arithmetic) a fraction that we would write as a/b was not regarded as a number, as a single entity, but as a relationship or ratio $a : b$ between the (whole) numbers a and b. Thus the ratio $a : b$ was, in modern terms, simply an ordered pair, rather than a rational number.

Two ratios were said to be *proportional*, $a : b = c : d$, if (with the obvious meaning) a is the same part or parts or multiple of b as c is of d. For example, $6 : 9 = 10 : 15$ because 6 is two of the three parts of 9, as 10 is two of the three parts of 15. More formally, $a : b = c : d$ provided that there exist integers p, q, m, n such that $a = mp$, $b = mq$, $c = np$, $d = nq$ (so a/b and c/d are both integral multiples of p/q). On this basis the early Pythagoreans developed an elementary theory of proportionality.

EXERCISE 7. Establish the following implications.

(i) $a : b = c : d \;\Rightarrow\; a : c = b : d \;\Rightarrow\; ad = bc$
(ii) $a : c = b : c \;\Rightarrow\; a = b$
(iii) $a : b = c : d \;\Rightarrow\; (a+b) : b = (c+d) : d$
(iv) $a : b = c : d \;\Rightarrow\; (a-b) : b = (c-d) : d$ if $a > b$.

This *discrete* view of number or size was also applied to geometrical magnitudes—lengths, areas, and volumes. In particular, it was believed by the early Pythagoreans that any two line segments are *commensurable*, that is, are multiples of a common unit. On this assumption, the theory of integer ratios and proportions readily extends so as to apply to lengths and

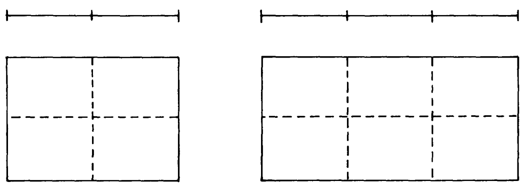

Figure 6

areas of simple figures such as line segments and rectangles. For example, the ratio $a : b$ of the lengths of the two line segments in Figure 6 is equal to the ratio $2 : 3$ of integers, while the ratio $A : B$ of areas of the two rectangles is equal to $4 : 6$. Thus we can talk about proportions $a : b = A : B = 2 : 3$ between ratios of magnitudes of different types—numbers, lengths, and areas.

For simple geometric figures with commensurable dimensions, the usual results involving area relationships are then easily established. For example, given two rectangles R and S with commensurable bases a and b and equal height h, the ratio $A : B$ of their areas is equal to the ratio $a : b$ of their bases. For if $a = mc$ and $b = nc$ where m and n are integers, then R consists of m subrectangles with base c and height h, while S consists of n such subrectangles. Hence $A : B = m : n = a : b$.

Exercise 8. Suppose that two rectangles are *similar*, meaning that the ratio of their bases is proportional to the ratio of their heights. If their bases and heights are commensurable, prove that the ratio of their areas is proportional to the ratio of the *squares* of (or on) their bases. By taking halves, the same result obtains for similar triangles (why?).

Exercise 9. A *regular* polygon is one with equal sides and equal angles. Define similarity for regular polygons. Then prove that the ratio of the areas of two similar regular polygons (with commensurable sides) is proportional to the ratio of the squares of their respective sides. *Hint*: By joining its vertices to its center, any regular polygon can be dissected into congruent isosceles triangles.

According to a fragment from the lost history of Eudemus that was allegedly copied verbatim in the sixth century A.D. by the Aristotelian commentator Simplicius, Hippocrates of Chios (ca. 430 B.C.; not to be confused with the physician Hippocrates of Cos) proved that the ratio of the areas of two circles is equal to the ratio of the squares of their diameters (or radii). Presumably he deduced (if not rigorously proved) this result by inscribing in the two circles similar regular polygons, and then "exhausting" the areas of the circles by increasing indefinitely the number of sides of the polygons (Fig. 7). Since, at each stage, the ratio of the areas of the two inscribed polygons is equal to the ratio of the squares of the radii of the two circles (as a consequence of Exercise 9), it would seem to follow "in the limit" that the same is true of the areas of the circles. However, Hippocrates probably had no limit concept sufficient to "clinch" this essentially infinitesimal argument.

Although it appears that the area of a circle can be approximated arbitrarily closely by the area of an inscribed regular polygon with sufficiently many sides, the area of the circle is not precisely equal to that of any inscribed polygon. The quadrature or "squaring of the circle"—the problem of finding a square with area precisely equal to that of a given circle—was one of the three classical problems of antiquity (together with

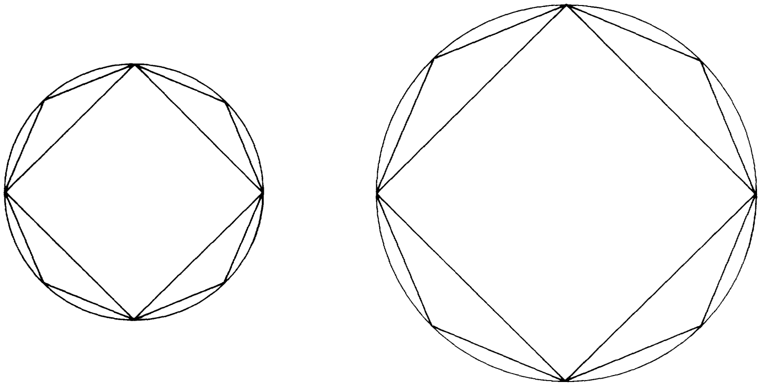

Figure 7

the duplication of a cube and the trisection of an angle). This is an example of a problem, involving a distinction between approximate and exact computation, that is unlike any considered by the Babylonians and Egyptians.

EXERCISE 10. Hippocrates applied his result on areas of circles to obtain the quadrature of a certain "lune." Consider a semicircle circumscribed about an isosceles right triangle ABC (Fig. 8). Let $ADBE$ be a circular segment on the base (hypotenuse) that is similar to the circular segments on the legs of the right triangle. Use the fact that similar circular segments are in area as the squares of their bases (why?), and the Pythagorean theorem applied to the right triangle ABC, to show that the area of the lune $ADBC$ between the circular arcs is equal to the area of the triangle ABC, and hence to half of the area of the square on AB.

According to the introduction to Archimedes' treatise *The Method*, Democritus (ca. 460 B.C.–ca. 370 B.C.) was the first to discover the fact that the volume of a pyramid (or cone) is one-third that of a prism (or cylinder) with the same base and height, but he did not rigorously prove it. A possible indication of Democritus' approach is indicated by the following question attributed to him by Plutarch (quoted by van der Waerden [12], p. 138):

> If a cone is cut by surfaces [i.e. planes] parallel to the base, then how are the sections, equal or unequal? If they were unequal then [i.e. thinking of

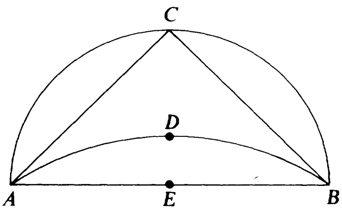

Figure 8

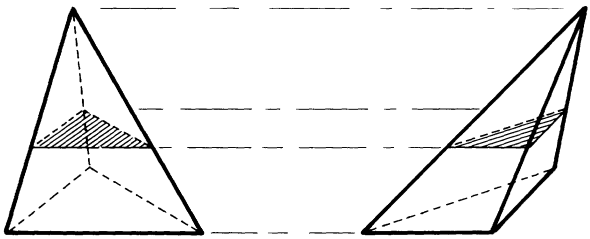

Figure 9

> the slices as cylinders] the cone would have the shape of a staircase; but if they were equal, then all sections will be equal, and the cone will look like a cylinder, made up of equal circles; but this is entirely nonsensical.

Here Democritus is thinking of a solid as being composed of sections parallel to its base. From this idea it is plausible to conclude that two solids composed of equal parallel sections at equal distances from their bases should have equal volumes (Fig. 9). This fact was exploited extensively by Cavalieri in the early seventeenth century, and now bears his name. It implies that triangular pyramids with the same height and bases of equal areas will have equal volumes.

EXERCISE 11. If the bases of the two pyramids in Figure 9 have equal areas, why does it follow that corresponding sections parallel to their bases have equal areas?

Given a triangular pyramid *ABCE*, it can be "completed" to form a prism of the same base and height (Fig. 10). But then the pyramids *ABCE*, *DEFC*, *ADEC* have equal volumes, because the first and second have equal bases and heights, as do the first and third. Consequently the volume of the pyramid *ABCE* is one-third that of the prism. Since any pyramid with polygonal base can be dissected into triangular pyramids, the same result obtains for arbitrary polygonal pyramids.

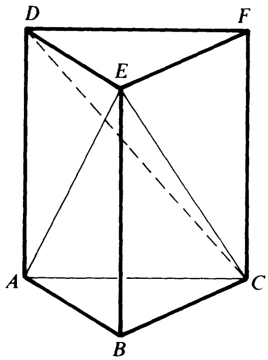

Figure 10

Just as Hippocrates exhausted a circle with inscribed regular polygons, a circular cone can similarly be exhausted with pyramids over regular polygons inscribed in its circular base. It seems to follow "in the limit" that the volume of the cone will be one-third that of the cylinder with the same base and height.

We will see that these infinitesimal plausibility arguments of the late fifth century B.C. were, about a half century later, converted into rigorous proofs by Eudoxus.

Incommensurable Magnitudes and Geometric Algebra

The Pythagorean geometry of the fifth century B.C. was based on the discrete number concept and theory of proportions discussed in the previous section. During the latter part of that century it was discovered that there exist pairs of line segments, such as the edge and diagonal of a square, that are not commensurable—they cannot be subdivided as integral multiples of segments of the same length, and hence the ratio of their lengths is not equal to the ratio of two integers. For example, the Pythagorean theorem says that the square on the diagonal of the unit square has area 2, whereas (in modern terms) $\sqrt{2}$ is not a rational number. The chronology of this discovery is discussed in detail by Knorr [6], Chapter II.

The existence of incommensurable geometric magnitudes (lengths, areas, volumes) necessitated a thorough reexamination and recasting of the foundations of mathematics, a task that occupied much of the fourth century B.C. During this period Greek algebra and geometry assumed the highly organized and rigorously deductive form that is set forth in the 13 books of the *Elements* that Euclid wrote about 300 B.C. This systematic exposition of the Greek mathematical accomplishments of the preceding three centuries is the earliest major Greek mathematical text that is now available to us (due perhaps to the extent to which the *Elements* subsumed previous expositions).

Today we simply say that $\sqrt{2}$ is an irrational number. However, for the Greeks, the discovery of incommensurability meant that there existed geometric magnitudes that could not be measured by numbers! For, as we have seen, their conception of numbers as integers alone was *discrete* in character, whereas the phenomenon of incommensurable lengths implied that geometric magnitudes have some sort of inherently (and unavoidable) *continuous* character. It followed that geometric magnitudes could not be manipulated without hesitation in algebraic computations just as though they were numbers. Although it was obvious that lengths or areas could be added by taking unions of sets, what, for example, would be meant by the product or quotient of two lengths or areas?

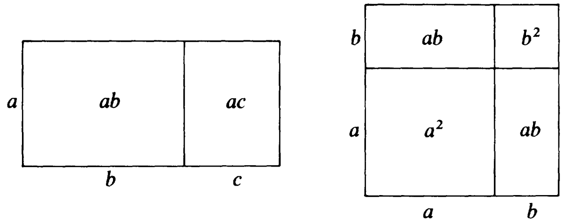

Figure 11

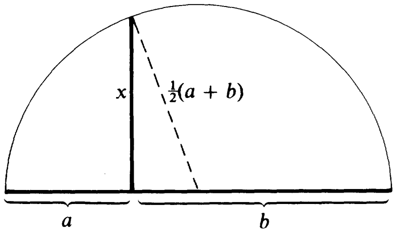

Figure 12

The Greek answer to such fundamental questions is presented in the *geometric algebra* of Books II and VI of Euclid's *Elements*. The product of two lengths a and b is not a third length, but rather the area of a rectangle with sides a and b. Algebraic identities, such as $a(b+c)=ab+ac$ and $(a+b)^2=a^2+2ab+b^2$, are interpreted as the geometric propositions whose proofs are indicated in Figure 11.

Whereas such a simple equation as $x^2=2$ has no solution in the domain of (Greek, rational) numbers, the equation $x^2=ab$ where a and b are given lengths can be solved geometrically by constructing a square with edge x whose area is equal to that of the rectangle with sides a and b. This is the real point (not always understood) to the "ruler and compass" constructions of the *Elements*—the solution of algebraic equations in terms of geometric magnitudes.

Exercise 12. If x is the chord in Figure 12 of a semicircle of diameter $a+b$, apply the Pythagorean theorem to show that $x^2=ab$.

The principal Greek technique for the geometric solution of algebraic equations was based on the "application of areas." For example, given a segment AB of length a, the construction in Proposition I.44 (Prop. 44 of Book I) of the *Elements*, of a rectangle with base AB and area equal to that of a given square of edge b (Fig. 13), provides a solution of the equation $ax=b^2$. This corresponds to geometric division, and we say that the given area b^2 has been *applied* to the given segment AB.

Proposition VI.28 of the *Elements* shows how to apply a given area b^2 to a given segment of length a, but "deficient" by a square. That is, a

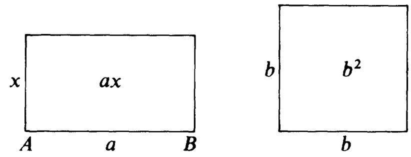

Figure 13

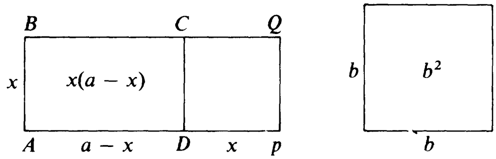

Figure 14

rectangle $ABCD$ of area b^2 is constructed with its base lying along the given segment AP (of length a), but falling short (of the rectangle on the whole segment AP) by a square ($CDPQ$). This construction (Fig. 14) provides a geometric solution of the quadratic equation $ax - x^2 = b^2$.

Exercise 13. Proposition VI.29 of Euclid's *Elements* shows how to apply a given area b^2 to a line segment of length a, but "in excess" by a square. That is, the base AB of the constructed rectangle with area b^2 extends beyond the given segment AP of length a, with the "excess" part of this rectangle being a square. Draw the indicated figure, and interpret the construction as a geometric solution of the quadratic equation $ax + x^2 = b^2$.

The Greeks used these admittedly cumbersome techniques of geometric algebra to handle with power and precision the staple fare of today's high school algebra, but without assuming the existence of irrational numbers. They were well aware of the existence of geometric magnitudes that we call "irrational," but simply did not think of them as numbers. This was not a lack of sophistication on their part, but rather a direct result of their unyielding insistence on logical rigor. In this connection, it is instructive to examine Book X of Euclid's *Elements*, which devotes 115 propositions and over 250 pages (in Heath's annotated translation [2]) to a comprehensive classification of irrational magnitudes of the forms $a \pm \sqrt{b}$, $\sqrt{a} \pm \sqrt{b}$, $\sqrt{a \pm \sqrt{b}}$, and $\sqrt{\sqrt{a} \pm \sqrt{b}}$, where a and b are commensurable lengths.

Eudoxus and Geometric Proportions

Any two line segments can be compared (by ruler and compass methods if one insists) to determine which has the greater length, and two lengths can be added by placing two line segments end-to-end to form a third one. The application of areas technique made possible the same operations with areas, because any rectilinear plane figure could be transformed into a rectangle with the same area and with a preassigned height. The areas of two rectangles with the same height could then be compared by comparing the lengths of their bases, and could be added by placing the two rectangles side-by-side to form a third one. By repeated addition, a geometric magnitude (length, area, or volume) can be multiplied by a positive integer.

However, the discovery of incommensurables made the Pythagorean theory of integral proportions useless for the comparison of ratios of geometric magnitudes, and thereby invalidated those geometric proofs that had utilized proportionality concepts. This crisis in the foundations of geometry was resolved by Eudoxus of Cnidus (408?–355? B.C.), a student at Plato's Academy in Athens who became the greatest mathematician of the fourth century B.C.

The key to Eudoxus' accomplishment was (as often happens in mathematics) the proper formulation of a definition—in this case, the definition of proportionality of ratios of geometric magnitudes. Let a and b be geometric magnitudes of the same type (both lengths or areas or volumes). Let c and d be a second pair of geometric magnitudes, both of the same type (but not necessarily the same type as the first pair). Then Eudoxus defines the ratios $a : b$ and $c : d$ to be *proportional*, $a : b = c : d$, provided that, given any two positive integers m and n, it follows that either

$$na > mb \quad \text{and} \quad nc > md, \tag{1}$$

or

$$na = mb \quad \text{and} \quad nc = md, \tag{2}$$

or

$$na < mb \quad \text{and} \quad nc < md. \tag{3}$$

EXERCISE 14. Show that Eudoxus' definition generalizes the familiar notion of proportionality (or equality) of ratios of integers. In particular, if a, b, c, d are integers such that $a/b = c/d$, and m and n are two positive integers, show that (1), (2), or (3) holds, depending on whether m/n is less than, equal to, or greater than $a/b = c/d$.

Thus Eudoxus' definition of proportionality for geometric ratios is simply a necessarily ponderous way of saying what is essentially obvious in the case of proportional ratios of numbers. In addition, it may be noted that, given incommensurable magnitudes a and b, this definition effectively splits the field of rational numbers m/n into two disjoint sets: the set L of those for which (1) holds, or $m : n < a : b$, and the set U of those for which (3) holds, or $m : n > a : b$. A separation of the rational numbers into two disjoint subsets L and U, such that every element of L is less than every element of U, is now called a "Dedekind cut," after Richard Dedekind, who in the nineteenth century defined a *real* number to be precisely such a "cut" of the *rational* numbers. Dedekind thereby established a firm foundation for the real number system by retracing some of Eudoxus' steps of over two thousand years earlier.

The general theory of proportionality that Eudoxus erected on the basis of the above definition is presented in Book V of Euclid's *Elements*. A critical assumption is innocuously included in Definition 4 of Book V, which states that two geometric magnitudes a and b "are said to have a ratio to one another which are capable, when multiplied, of exceeding one another," that is, if there exists an integer n such that $na > b$. The assumption that, given two comparable geometric magnitudes a and b, there exists an integer n such that $na > b$, was first stated explicitly as an axiom by Archimedes, with whose name it is therefore usually associated. We prefer, however, to call it here the "axiom of Eudoxus."

The critical role of this axiom of Eudoxus is illustrated by the proof that

$$a : c = b : c \quad \text{implies} \quad a = b. \tag{4}$$

Suppose to the contrary that $a > b$. Then there exists an integer n such that

$$n(a-b) > c. \tag{5}$$

Let mc be the smallest multiple of c that exceeds nb, so

$$mc > nb \geqslant (m-1)c. \tag{6}$$

Addition of (5) and (6) then gives

$$na > mc, \quad \text{while} \quad nb < mc,$$

which contradicts the definition of the proportionality $a : c = b : c$. It follows that $a = b$, as desired.

As a further example of the extreme care with which Eudoxus framed his theory of proportions, let us apply (4) to show that

$$a : b = c : d \quad \text{implies} \quad ad = bc. \tag{7}$$

First note that

$$a : b = ad : bd$$

because $na > mb$ implies $nad > mbd$, etc. Similarly

$$c : d = bc : bd,$$

so it follows that

$$ad : bd = bc : bd,$$

which by (4) implies that $ad = bc$, as desired.

EXERCISE 15. (Euclid V.16) Show that $a : b = c : d$ implies $a : c = b : d$. *Hints*: First apply the definition of proportionality to show that

$$a : b = c : d \quad \text{implies} \quad na : nb = mc : md$$

for any two integers m and n. Then apply (4) and/or its proof to show that $na > mc$, $na = mc$, $na < mc$ imply $nb > md$, $nb = md$, $nb < md$ respectively. Finally apply the definition of proportionality again.

EXERCISE 16. Consider two rectangles with bases a and b, areas A and B, that have the same height. Apply Eudoxus' definition of proportionality to show that $A : B = a : b$.

The proofs of (4) and (7) above indicate the manner in which the "usual" properties of proportions are demonstrated in Book V of the *Elements*, to an extent that enabled the Greeks to work with ratios of geometric magnitudes in much the same way, and to the same ends, that we today carry out arithmetical computations with real numbers. On this basis Eudoxus proceeded to give rigorous proofs of the results of Hippocrates and Democritus on areas of circles and volumes of pyramids and cones.

These area and volume computations form the content of Book XII of Euclid's *Elements*, and of the remainder of this chapter. In order to spare the reader a heavy burden of geometric algebra and Eudoxian proportions, our exposition will make free use of real numbers and modern algebraic notation. However, in order to preserve the original flavor and spirit as carefully as possible, we will follow closely both the geometrical constructions and the logical sequence of the proofs presented by Euclid.

Before proceeding in this fashion, however, it may be instructive to interpret in quite modern terms the Greek view of geometric magnitudes, taking the case of area of plane figures as an example. Say that two *polygonal* figures "have the same area" if by application of areas techniques they can be transformed to the same rectangle. This is an equivalence relation that separates the class of all polygonal plane figures into a set $\mathscr{Q}$ of equivalence classes. Given a polygonal figure P, denote by $a(P) \in \mathscr{Q}$ the equivalence class containing P, and call $a(P)$ the *area* of P. Then the set $\mathscr{Q}$ of areas is what we might call a "Eudoxian semigroup"—an ordered commutative semigroup satisfying the axiom of Eudoxus. That is, given $a, b, c \in \mathscr{Q}$, it follows that

1. (Associativity) $a + (b + c) = (a + b) + c$
2. (Commutativity) $a + b = b + a$
3. $a > b$ implies $a + c > b + c$
4. There exists an integer n such that $na > b$.

This interpretation emphasizes the fact that it is not necessary to think of areas as numbers (as the Greeks did not).

Area and the Method of Exhaustion

The Greeks assumed on intuitive grounds that simple curvilinear figures, such as circles or ellipses, have areas that are geometric magnitudes of the same type as areas of polygonal figures, and that these areas enjoy the following two natural properties.

(i) (Monotonicity) If S is contained in T, then $a(S) \leqslant a(T)$.
(ii) (Additivity) If R is the union of the non-overlapping figures S and T, then $a(R) = a(S) + a(T)$.

Given a curvilinear figure S, they attempted to determine its area $a(S)$ by means of a sequence $P_1, P_2, P_3, \ldots,$ of polygons that fill up or "exhaust" S, analogous to Hippocrates' sequence of regular polygons inscribed in a circle. The so-called method of exhaustion was devised, apparently by Eudoxus, to provide a rigorous alternative to merely taking a vague and unexplained limit of $a(P_n)$ as $n \to \infty$. Indeed, the Greeks studiously avoided "taking the limit" explicitly, and this virtual "horror of the infinite" is probably responsible for the logical clarity of the method of exhaustion.

In any event, the crux of the matter consists of showing that the area $a(S - P_n)$, of the difference between the figure S and the inscribed polygon P_n, can be made as small as desired by choosing n sufficiently large. For this purpose the following consequence (Euclid X.1) of the Archimedes-Eudoxus axiom is repeatedly applied.

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

This result, which we will call "Eudoxus' principle," may be phrased as follows. Let M_0 and ϵ be the two given magnitudes, and $M_1, M_2, M_3, \ldots,$ a sequence such that $M_1 < \frac{1}{2}M_0$, $M_2 < \frac{1}{2}M_1$, $M_3 < \frac{1}{2}M_2$, etc. Then we want to conclude that $M_n < \epsilon$ for some n. To see that this is so, choose an integer N such that

$$(N+1)\epsilon > M_0.$$

Then ϵ is at most half of $(N+1)\epsilon$, so it follows that

$$N\epsilon \geqslant \tfrac{1}{2}M_0 > M_1.$$

Similarly, ϵ is at most half of $N\epsilon$, so

$$(N-1)\epsilon \geqslant \tfrac{1}{2}M_1 > M_2.$$

Proceeding in this way, at each step subtracting ϵ (which is at most half) from the left-hand-side and halving the right-hand-side, we arrive in N steps at the desired inequality

$$\epsilon > M_N. \qquad \square$$

EXERCISE 17. Conclude from Eudoxus' principle that, if M, ϵ, and $r < \frac{1}{2}$ are given positive numbers, then $Mr^n < \epsilon$ for n sufficiently large. Is it necessary that r be at most $\frac{1}{2}$?

We first apply Eudoxus' principle to describe precisely the manner in which the area of a circle can be exhausted by means of inscribed polygons.

Given a circle C and a number $\epsilon > 0$, *there exists a regular polygon P inscribed in C such that*

$$a(C) - a(P) < \epsilon. \tag{8}$$

PROOF. We start with a square $P_0 = EFGH$ inscribed in the circle C, and write $M_0 = a(C) - a(P_0)$. Doubling the number of sides, we obtain a regular octagon P_1 inscribed in C (Fig. 15).

Continuing in this fashion, we obtain a sequence $P_0, P_1, P_2, \ldots, P_n, \ldots,$ with P_n having 2^{n+2} sides. Writing

$$M_n = a(C) - a(P_n),$$

we want to show that

$$M_n - M_{n+1} > \tfrac{1}{2} M_n. \tag{9}$$

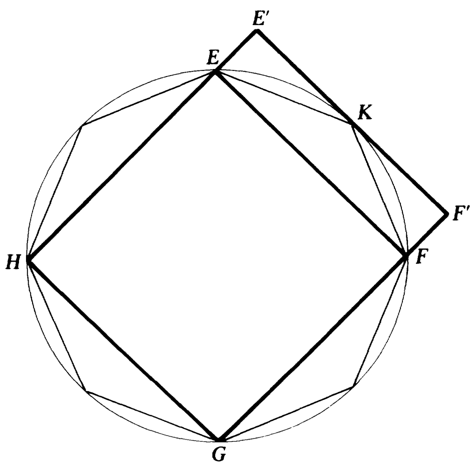

Figure 15

It will then follow from Eudoxus' principle that $M_n < \epsilon$ for n sufficiently large, and we will be finished.

The proof of (9) is essentially the same for all n, so we consider the case $n = 0$ illustrated by Fig. 15. Then

$$\begin{aligned} M_0 - M_1 &= a(P_1) - a(P_0) \\ &= 4a(\Delta EFK) \\ &= 2a(EFF'E') \\ &> 2a(\widehat{EKF}) \\ &= \tfrac{1}{2}\cdot 4a(\widehat{EKF}) \\ &= \tfrac{1}{2}[a(C) - a(P_0)] \\ M_0 - M_1 &> \tfrac{1}{2}M_0, \end{aligned}$$

where we denote by $\widehat{EFK}$ the circular segment cut off the circle by the side EF of the square P_0. In the general case, we obtain

$$\begin{aligned} M_n - M_{n+1} &= a(P_{n+1}) - a(P_n) \\ &> \tfrac{1}{2}[a(C) - a(P_n)] = \tfrac{1}{2}M_n, \end{aligned}$$

where $a(C) - a(P_n)$ is the sum of the areas of the 2^{n+1} circular segments cut off by the edges of P_n. □

The above lemma provides the basis for a rigorous proof of the theorem on areas of circles (Euclid XII.2).

If C_1 and C_2 are circles with radii r_1 and r_2, then

$$\frac{a(C_1)}{a(C_2)} = \frac{r_1^2}{r_2^2}. \tag{10}$$

PROOF. If $A_1 = a(C_1)$, $A_2 = a(C_2)$, then either

$$\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2} \quad \text{or} \quad \frac{A_1}{A_2} < \frac{r_1^2}{r_2^2} \quad \text{or} \quad \frac{A_1}{A_2} > \frac{r_1^2}{r_2^2}.$$

The proof is a double *reductio ad absurdum* argument, characteristic of Greek geometry, in which we show that the assumption of either of the inequalities leads to a contradiction.

Suppose first that

$$\frac{A_1}{A_2} < \frac{r_1^2}{r_2^2}, \quad \text{or} \quad A_2 > \frac{A_1 r_2^2}{r_1^2} = S,$$

and let $\epsilon = A_2 - S > 0$. Then, by the lemma, there exists a polygon P_2 inscribed in C_2 such that

$$A_2 - a(P_2) < \epsilon = A_2 - S,$$

so $a(P_2) > S$. But

$$\frac{a(P_1)}{a(P_2)} = \frac{r_1^2}{r_2^2} = \frac{A_1}{S}, \quad \text{(Exercise 9)}$$

where P_1 is the similar regular polygon inscribed in C_1. It follows that

$$\frac{S}{a(P_2)} = \frac{A_1}{a(P_1)} = \frac{a(C_1)}{a(P_1)} > 1$$

so $S > a(P_2)$, which is a contradiction. Hence the assumption $A_1/A_2 < r_1^2/r_2^2$ is false.

By interchanging the roles of the two circles, we find similarly that the assumption

$$\frac{A_1}{A_2} > \frac{r_1^2}{r_2^2} \quad \text{or} \quad \frac{A_2}{A_1} < \frac{r_2^2}{r_1^2}$$

is also false. We therefore conclude that (10) holds, as desired. □

If we rewrite (10) as

$$\frac{a(C_1)}{r_1^2} = \frac{a(C_2)}{r_2^2}, \tag{11}$$

and denote by π the common value of these two ratios, then we obtain the familiar formula $A = \pi r^2$ for the area of a circle. In fact, however, the Greeks could not do this, because for them (11) was a proportion between ratios of areas, rather than a numerical equality. Hence the number π does not appear in this connection in Greek mathematics.

Exercise 18. Apply the lemma on the exhaustion of a circle by inscribed polygons, together with the fact that the volume of a prism is the product of its height and the area of its base, to give a double *reductio ad absurdum* proof that the volume of a circular cylinder is equal to the product of its height and the area of its base. Given a polygon P inscribed in the base circle, consider the prism Q with base P and height equal to that of the cylinder. Then the cylinder can be exhausted by prisms like Q.

Volumes of Cones and Pyramids

If P is either a triangular pyramid or a circular cone, then its volume is given by

$$v(P) = \tfrac{1}{3}Ah, \tag{12}$$

where h is its height and A the area of its base. According to Archimedes, the two results described by this formula were discovered by Democritus,

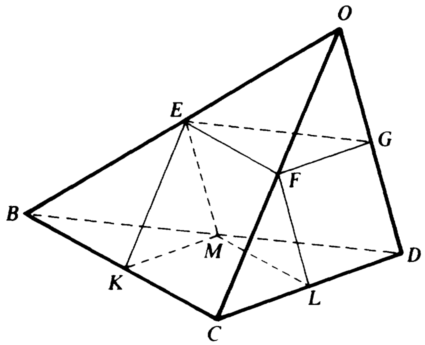

Figure 16

but were first proved by Eudoxus. In this section we discuss their treatment by Euclid in Book XII of the *Elements*.

The calculation of the volume of a pyramid is based on the dissection of an arbitrary pyramid with triangular base into two prisms and two similar pyramids, as indicated in Figure 16. The points E, F, G, K, L, M are the midpoints of the six edges of the pyramid $OBCD$. It is clear that the pyramids $OEFG$ and $EBKM$ are similar to $OBCD$ and are congruent to each other. The crucial fact about this dissection is that the sum of the volumes of the two prisms

$$EKMFCL \quad \text{and} \quad MLDEFG$$

is *greater than half* the volume of the original pyramid $OBCD$. This is true because

$$v(OEFG) = v(FKCL) < v(EKMFCL)$$

and

$$v(EBKM) = v(GMLD) < v(MLDEFG).$$

If we denote by h the height and by A the area of the base BCD of the pyramid $OBCD$, then

$$v(MLDEFG) = \tfrac{1}{8}Ah$$

because the height of this prism is $\frac{1}{2}h$ and the area of its base MLD is $\frac{1}{4}A$. Also,

$$v(EKMFCL) = \tfrac{1}{8}Ah$$

because the area of the parallelogram $KCLM$ is $\frac{1}{2}A$, and the prism $EKMFCL$ is half of a parallelepiped with base $KCLM$ and height $\frac{1}{2}h$. Consequently the sum of the volumes of the two smaller prisms is $\frac{1}{4}Ah$.

Now let us similarly dissect each of the two pyramids $OEFG$ and $EBKM$ into two smaller pyramids and two prisms. The sum of the volumes of the four resulting smaller prisms is then greater than half of the sum of the volumes of the pyramids $OEFG$ and $EBKM$. Because these two latter pyramids both have height $h/2$ and base area $A/4$, it follows that the sum of the volumes of the four smaller prisms is

$$2 \cdot \frac{1}{4} \cdot \frac{A}{4} \cdot \frac{h}{2} = \frac{Ah}{4^2}.$$

After n steps of this sort, we obtain an *n-step-dissection* of the original pyramid. At the kth step we have 2^k subdivided small pyramids, and hence 2^k pairs of smaller prisms. Each of the 2^k small pyramids has height $h/2^k$ and base area $A/4^k$, so the sum of the volumes of the 2^k pairs of smaller prisms is

$$2^k \cdot \frac{1}{4} \cdot \frac{A}{4^k} \cdot \frac{h}{2^k} = \frac{Ah}{4^{k+1}}.$$

Finally, if P denotes the union of all the prisms obtained in all the steps of this n-step-dissection, it follows that

$$v(P) = Ah\left(\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n+1}}\right). \tag{13}$$

Furthermore, because at each step the sum of the volumes of the prisms is greater than half the sum of the pyramids obtained in the previous step, Eudoxus' principle implies that, given $\epsilon > 0$,

$$V - v(P) < \epsilon \tag{14}$$

if n is sufficiently large, and $V = v(OBCD)$. This construction is the basis for Euclid's proof of Proposition XII.5.

Given two triangular pyramids with the same height and with base areas A_1 and A_2, the ratio of their volumes V_1 and V_2 is equal to that of their base areas,

$$\frac{V_1}{V_2} = \frac{A_1}{A_2}. \tag{15}$$

PROOF. The demonstration of (15) is a double *reductio ad absurdum* argument almost identical to that used in the proof of the theorem on areas of circles. Suppose first that

$$\frac{V_1}{V_2} < \frac{A_1}{A_2}, \quad \text{or} \quad V_2 > \frac{V_1 A_2}{A_1} = S,$$

and let $\epsilon = V_2 - S$. Denote by P_2 the union of all the prisms obtained in an n-step-dissection of the second pyramid, with n sufficiently large that

$$V_2 - v(P_2) < \epsilon = V_2 - S,$$

so $v(P_2) > S$. It then follows from (13) that, if P_1 is the similar union of prisms obtained in an n-step-dissection of the first pyramid, then

$$\frac{v(P_1)}{v(P_2)} = \frac{A_1}{A_2} = \frac{V_1}{S}.$$

Hence

$$\frac{S}{v(P_2)} = \frac{V_1}{v(P_1)} > 1$$

because P_1 is properly contained in the first pyramid. But $S > v(P_2)$ is a contradiction, so the assumption $V_1/V_2 < A_1/A_2$ is false.

By interchanging the roles of the two pyramids, we find that the assumption $V_1/V_2 > A_1/A_2$ is also false. It therefore follows that $V_1/V_2 = A_1/A_2$, as desired. □

We have previously seen that the formula $V = \frac{1}{3}Ah$, for the volume of a triangular pyramid, follows from the fact that two pyramids with equal heights and base areas must have the same volumes. For any given pyramid is one of three pyramids with equal volumes, whose union is a prism with height and base area equal to those of the given pyramid (see Figure 10). We assume here (as in the above construction) the elementary fact that the volume of a prism is the product of its height and its base area.

Alternatively, it is interesting to derive the formula $V = \frac{1}{3}Ah$ directly by using the sum of the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}.$$

Given $\epsilon > 0$, we see from (13) that

$$V - Ah\left(\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n+1}}\right) < \epsilon$$

if n is sufficiently large. It follows that the volume of the pyramid is

$$V = \frac{Ah}{4} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{Ah}{4} \cdot \frac{4}{3} = \frac{1}{3}Ah.$$

Although the Greeks knew how to sum a finite geometric progression, they used *reductio ad absurdum* arguments to avoid the formal summation of an infinite series.

EXERCISE 19. Show that the volume formula $V = \frac{1}{3}Ah$ holds for a pyramid whose base is an arbitrary convex polygon (that can be dissected into triangles, thereby dissecting the pyramid into triangular pyramids for which the formula is already known).

EXERCISE 20. Show that the ratio of the volumes of two similar pyramids is equal to the ratio of the cubes of corresponding edges.

In Proposition XII.10 Euclid uses inscribed pyramids to exhaust a circular cone so as to establish the volume formula $V = \frac{1}{3}Ah$ for cones. To outline this proof, let T be a cone with vertex O, base circle C, and height h. Let

$$P_0, P_1, P_2, \ldots, P_n, \ldots$$

be the sequence of inscribed regular polygons previously used to exhaust the circle, with P_n having 2^{n+2} sides. If T_n denotes the pyramid with vertex O and base P_n, then

$$T_0, T_1, T_2, \ldots, T_n, \ldots$$

is a sequence of pyramids inscribed in the cone T, and $v(T_n) = \frac{1}{3}a(P_n)h$, where h is the height of T.

Recall that we proved that, if $M_n = a(C) - a(P_n)$, then $M_n - M_{n+1} > \frac{1}{2}M_n$. By joining every polygon involved with the vertex O, we can similarly prove that, if

$$\overline{M_n} = v(T) - v(T_n),$$

then

$$\overline{M_n} - \overline{M_{n+1}} > \tfrac{1}{2}\overline{M_n}.$$

Eudoxus' principle therefore implies that, given $\epsilon > 0$,

$$\overline{M_n} = v(T) - v(T_n) < \epsilon \tag{16}$$

if n is sufficiently large. Also, if Q is the cylinder with base C and height h, and Q_n is the inscribed prism with base P_n and height h, then

$$v(Q) - v(Q_n) < \epsilon$$

for n sufficiently large (why?).

We are now ready for the *reductio ad absurdum* proof that

$$v(T) = \tfrac{1}{3}v(Q) = \tfrac{1}{3}Ah. \tag{17}$$

Otherwise, either $v(T) < \frac{1}{3}v(Q)$ or $v(T) > \frac{1}{3}v(Q)$.

Assuming that $v(T) < \frac{1}{3}v(Q)$, choose n sufficiently large that

$$v(Q) - v(Q_n) < v(Q) - 3v(T).$$

Then $v(Q_n) > 3v(T) > 3v(T_n)$ because the pyramid T_n is inscribed in the cone T. But the conclusion that $v(T_n) < \frac{1}{3}v(Q_n)$ is a contradiction, because the pyramid T_n and the prism Q_n have the same base and height, so we know (Exercise 19) that $v(T_n) = \frac{1}{3}v(Q_n)$.

Assuming that $v(T) > \frac{1}{3}v(Q)$, choose n sufficiently large that

$$v(T) - v(T_n) < v(T) - \tfrac{1}{3}v(Q).$$

Then $v(T_n) > \frac{1}{3}v(Q) > \frac{1}{3}v(Q_n)$ because the prism Q_n is inscribed in the cylinder Q. But this is a contradiction for the same reason as before, so we conclude that $v(T) = \frac{1}{3}v(Q)$ as desired. □

Volumes of Spheres

The final result in Book XII of the *Elements* is Proposition 18, to the effect that the volume of a sphere is proportional to the cube of its radius. Euclid proves this in the following form.

If S_1 and S_2 are two spheres with radii r_1 and r_2 and volumes V_1 and V_2, then

$$\frac{V_1}{V_2} = \frac{r_1^3}{r_2^3}. \tag{18}$$

As a preliminary lemma (XII.17) he shows that, given two concentric spheres S and S' with S' interior to S, there exists a polyhedral solid P inscribed in S that contains S' in its interior. The polyhedral solid P is a union of finitely many pyramids, each of which has the common center O of the two spheres as its vertex, with its base being a polygon inscribed in the outer sphere S (Fig. 17).

In his proof of Proposition 18, Euclid assumes without proof that, given a sphere S with volume V and $V' < V$, there exists a concentric sphere S' with $v(S') = V'$. We will repair this minor gap by using the slightly simpler

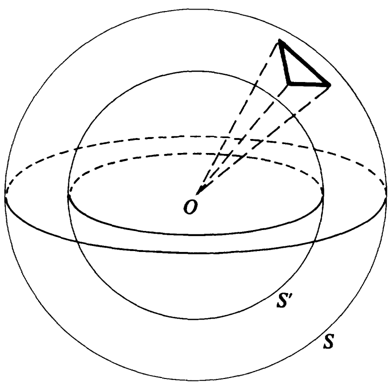

Figure 17

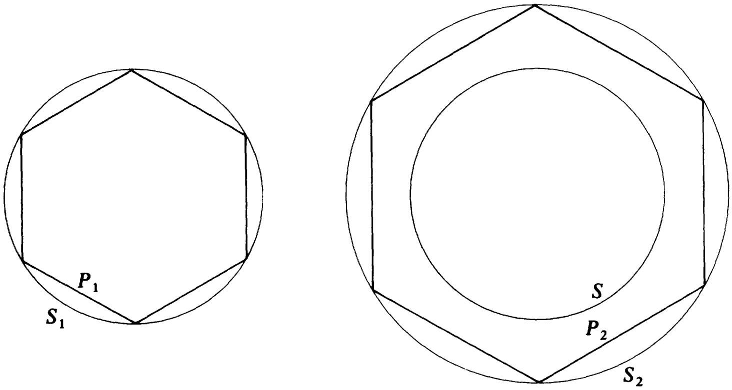

Figure 18

fact that there exists a concentric sphere S' with $V' < v(S') < V$ (Exercise 21 below).

Assuming that $V_1/V_2 < r_1^3/r_2^3$, let

$$\epsilon = V_2 - \frac{r_2^3 V_1}{r_1^3},$$

and let S be a sphere interior to and concentric with S_2 (see Fig. 18) such that

$$v(S) = V > V_2 - \epsilon = \frac{r_2^3 V_1}{r_1^3}. \tag{19}$$

Now let P_2 be a polyhedral solid inscribed in S_2 that contains S in its interior. If P_1 is the similar polyhedral solid inscribed in S_1, then

$$\frac{v(P_1)}{v(P_2)} = \frac{V_1'}{V_2'} = \frac{r_1^3}{r_2^3} \tag{20}$$

by Exercise 20, because P_1 and P_2 are made up of pairwise similar pyramids with corresponding edges r_1 and r_2. Hence

$$V_2' > V > \frac{r_2^3 V_1}{r_1^3}$$

by (19), so

$$\frac{V_1}{V_2'} < \frac{r_1^3}{r_2^3} = \frac{V_1'}{V_2'}$$

by (20). Thus $v(S_1) = V_1 < V_1' = v(P_1)$. But this is a contradiction, because S_1 contains P_1.

Interchanging the roles of the two spheres S_1 and S_2, the assumption that $V_1/V_2 > r_1^3/r_2^3$ leads similarly to a contradiction. Consequently we conclude that $V_1/V_2 = r_1^3/r_2^3$, as desired. □

Exercise 21. Let S be a sphere of radius r and equatorial circle C, and $\epsilon > 0$. Show as follows, without using the formula for the volume of a sphere, that there is a sphere $\bar{S}$ with $v(S) - \epsilon < v(\bar{S}) < v(S)$. First choose $\delta > 0$ such that

$$\pi(r+\delta)^2 - \pi(r-\delta)^2 = 4\pi r\delta < \frac{\epsilon}{4\pi r}.$$

If $r - \delta < \bar{r} < r$, then the annular ring A bounded by C and the concentric circle $\bar{C}$ of radius $\bar{r}$ can be covered by non-overlapping rectangles $R_1, R_2, \ldots, R_n$ such that

$$\sum_{i=1}^{n} a(R_i) < \frac{\epsilon}{4\pi r}. \qquad \text{(Why?)}$$

If T_i is the cylinder-with-hole obtained by revolving the rectangle R_i about a horizontal axis through the center of the circles, then the sets $T_1, T_2, \ldots, T_n$ cover the spherical shell between the sphere S and the sphere $\bar{S}$ of radius $\bar{r}$. Now apply the formula for the volume of a cylinder to show that

$$\sum_{i=1}^{n} v(T_i) < \epsilon.$$

Why does this imply that $v(S) - \epsilon < v(\bar{S}) < v(S)$?

Let S_1 be an arbitrary sphere with radius r and volume V, and denote by α the volume of a sphere S_2 with unit radius. Then Equation (18) yields the volume formula

$$V = \alpha r^3, \tag{21}$$

according to which the volume of a sphere is proportional to the cube of its radius. There is no indication that Euclid or his predecessors knew the relationship between α and π; it was Archimedes who discovered that $\alpha = 4\pi/3$ (see Chapter 2).

It is instructive to examine the common pattern of the proofs of the five basic results from Euclid XII that we have discussed in this and the preceding two sections. Each of these theorems compares the areas or volumes of two sets A and B that are either

1. two circles,
2. two cylinders with the same height,
3. two pyramids with the same height,
4. a cone and a cylinder with the same height, or
5. two spheres.

In particular, we want to prove that

$$v(B) = kv(A), \tag{22}$$

where the proportionality constant k is equal in the five cases, respectively, to

1. the ratio of the squares of their radii.
2. the ratio of the squares of their radii.
3. the ratio of their base areas.
4. one-third.
5. the ratio of the cubes of their radii.

The first step in each proof is the construction of two sequences of polygonal or polyhedral figures, $\{P_n\}_1^\infty$ inscribed in A and $\{Q_n\}_1^\infty$ inscribed in B, such that

$$v(Q_n) = kv(P_n)$$

for all n. Eudoxus' principle is applied to the construction to show that, given $\epsilon > 0$,

$$v(A) - v(P_n) < \epsilon \quad \text{and} \quad v(B) - v(Q_n) < \epsilon$$

if n is sufficiently large.

In terms of the modern limit concept we would complete the proof by simply noting that

$$\begin{aligned} v(B) &= \lim_{n\to\infty} v(Q_n) \\ &= \lim_{n\to\infty} kv(P_n) \\ &= k \lim_{n\to\infty} v(P_n) \\ v(B) &= kv(A). \end{aligned}$$

In essence, the Greeks avoided this explicit use of limits by completing the proof by means of the double *reductio ad absurdum* argument. Assuming that

$$v(B) > kv(A),$$

and writing $\epsilon = v(B) - kv(A)$, we choose inscribed figures P in A and Q in B such that

$$v(Q) = kv(P) \quad \text{and} \quad v(Q) > v(B) - \epsilon = kv(A).$$

But this is a contradiction, since $v(P) \leqslant v(A)$ because A contains P. Reversing the roles of A and B, the assumption $v(A) > v(B)/k$ leads similarly to a contradiction, so we must conclude that $v(B) = kv(A)$ as desired. □

A logically complete indirect proof is thereby obtained without explicit reference to limits. The mystery which the Greeks attached to the infinite and, in particular, to what we call the limit concept, is absorbed (if not obviated) in Eudoxus' principle. In this connection, Aristotle remarked that mathematicians make no use of magnitudes infinitely large or small, but content themselves with magnitudes that can be made as large or as small as they please (quoted by Heath [3], p. 272).

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