1
The Ancients

A genie (as the story is told) lights a candle at a minute before midnight. After half of the minute has passed, the genie extinguishes the flame. Fifteen seconds later, she relights the candle, and again, halfway to midnight, she puts the flame out. This continues as midnight approaches, the time always divided in two, the flame soon leaping up and vanishing faster than we can see.

Now the genie asks you, "At midnight, will the flame be lit or out?"

Leaving aside the issue of *when* this question is asked, you are still left with some bewildering possibilities. The candle is neither lit nor out? The candle is both lit *and* out? We never *get* to midnight?

But of course we *get* to midnight; there has yet to be a midnight that we have failed to get to. There is a midnight right now that is approaching. Or are *we* approaching *it*? Which is staying still? Which is the arrow and which is the target?

One thing we have learned during the story of physics (in 1632) is that nothing sits still; you may see a passenger on a boat and a bird perched on the mast over her head as 'moving', but from their joint point of view, you are the one who is moving. Later in the story (in 1905), we learned that one observer may experience time as running more slowly than does another observer.

Questions like these, that explore the nature of reality, are ancient, and the pursuit of the answers has led thinkers down paths that we have named: physics (Can the genie's candle exist? Is time infinitely divisible?), philosophy (If our shared experience of time does not reflect its true nature, then what does that say about hope? knowledge? ethics?), and mathematics (Is it possible to make sense of all this?). This book approaches calculus as the culmination of a journey that began when people asked questions like these.

1.1 Zeno holds a mirror to the infinite

The question-asker **Zeno** lived (according to our linear experience of time) 2500 years 'ago' in Greece. One of his thought experiments questioned motion itself. We may phrase it like so: suppose an arrow, aimed at a target, is fired. Its tip must travel halfway to the target, then halfway again, and so on, reminiscent of the genie's candle, only with distance in place of time. This halving process continues indefinitely; thus, the arrow does not reach the target.

You are reading these words thanks to light bouncing off a surface and traveling to your eye. Why does the light not fall prey to Zeno's paradox? Can every distance, no matter how small, be divided in two?

Distance is tangible; we see it, we move through it. It feels infinitely divisible, just as time feels as though it is flowing into the future. It is tempting to discard reality and recast the paradox in terms of abstract objects that *are* always divisible: numbers. In this language, Zeno's tale becomes an infinite sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

that equals 1 if the arrow reaches the target and never quite gets to 1 if the paradox holds.

There are several ways to argue that this sum never *exceeds* 1. But what if it never reaches 1? The sum, as we go along, is always growing; so, if the sum never reaches 1, it seems clear that it reaches something smaller than 1. This is a little like moving the target a bit closer to the archer and then asking Zeno what he thinks now. But we can show that no matter what target we choose smaller than 1, the infinite sum eventually slips past it into the gap between the target and 1.

For what if we stop the sum at some point and calculate the total thus far? After two terms, the total is $1/2 + 1/4 = 3/4$. (This number is called a *partial sum* of the infinite sum.) The next partial sum is $7/8$. In general, the partial sum after we add the first n terms is

$$1 - \frac{1}{2^n}\ . \tag{1.1}$$

The fraction $1/2^n$ represents how far from 1 the partial sum is when we stop adding at the nth term. So, no matter where we set the target, we can always choose n large enough (thus making $1/2^n$ small enough) to slip by.

When we say that the partial sums become "arbitrarily close" to 1, this is what we mean: there is no number smaller than 1 to which the partial sums approach. No, the partial sums approach 1. What conclusion can we draw other than

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \tag{1.2}$$

in the face of this argument?

If you believe that (1.2) uncomfortably stretches the notion of 'equals', then you keep good company. Nevertheless, it is true, especially in the story of calculus, that mathematics often advances thanks to people who are willing to take uncomfortable risks. Consider, for example, this approach to proving that (1.2) is true. If we assume that the infinite sum in (1.2) adds to some number that we call S, then we have

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \tag{1.3}$$

and can multiply both sides by $1/2$, yielding

$$\frac{1}{2}S = \quad \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots .$$

Subtracting the new equation from (1.3), we have

$$S - \frac{1}{2}S = \frac{1}{2} . \tag{1.4}$$

The vexatious infinite, as depicted by the three dots in (1.3), vanishes thanks to our ingenuity. Leaving aside the issue of how one might multiply an infinite number of terms by anything, we can conclude that $S = 1$. Figure 1.1 offers visual support for this conclusion. Might we leave aside this issue permanently, and treat infinite sums just as we do normal sums? That is an option, but if someone finds a troublesome example, we should pay attention. Infinite sums are often troublesome.

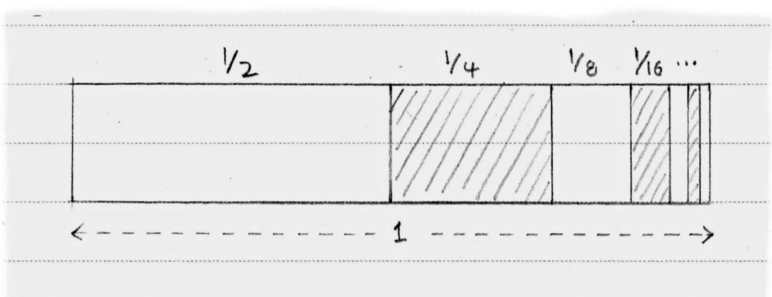

Figure 1.1. This figure speaks without words about the mathematics underlying Zeno's paradox of the arrow.

For instance, in the infinite sum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \tag{1.5}$$

each term is smaller than the last, so we might guess that this sum, too, adds to something. But it does not. Name any large number you want, and the partial sums will eventually pass it by. There is no cap on how large this sum grows, but over a thousand years passed after Zeno's life before someone could persuasively argue why.

By simply alternating the signs in (1.5), however, we get the infinite sum

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

that approaches a finite number (approximately 0.693). Two thousand years after Zeno, someone identified this number. Puzzles that take centuries to solve warn us of deep mystery.

Because we usually think of adding when we say 'sum', we can use a more general term than 'infinite sum' that allows subtraction: *series*. The series

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

sparked much debate. What might it equal? By grouping pairs of terms like so,

$$\begin{aligned} S &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 0 + 0 + 0 + \cdots, \end{aligned}$$

we conclude that S equals 0. Skipping the initial 1 and *then* grouping pairs like so,

$$\begin{aligned} S &= 1 - (1-1) - (1-1) - \cdots \\ &= 1 - 0 - 0 - \cdots, \end{aligned}$$

we discover that the sum is 1. Grouping, applied in slightly different ways, leads to two different answers; now *this* is troublesome.

It gets better; the grouping

$$S = 1 - (1 - 1 + 1 - 1 + 1 - \cdots)$$

reveals the original series to *contain itself*, so

$$S = 1 - S,$$

leading to the answer $S = 1/2$. We again find a new answer simply by using parentheses (one of which is somehow flung infinitely far to the right). There are, in fact, even more ways to sum the series, a signal that the infinite contains mysteries that do not succumb to ordinary arithmetic.

None of this deterred mathematicians from tinkering; in fact, such mysteries probably provoked them. There is no harm in exploring, and these explorations put humankind on a path that led to the discovery of a new branch of mathematics.

1.2 The 'infinitely small'

Archimedes (Greece, born *c.* 287 BCE) investigated the infinite fearlessly. An inventor and astronomer, Archimedes seems to have taken his greatest joy in pure mathematics. Many Greek mathematicians found the study of geometry among the purest of pursuits; after all, where on Earth can we find a square or an equilateral triangle or a circle? These shapes are unattainable generalizations of what we see in the world, made completely theoretical by the very purity that makes them elegant. No one can draw a circle; one can only draw things that *look* like circles. But we can *imagine* circles.

A geometer, pondering a shape, immediately asks about its area. Imagine slicing a circle like a pizza (or the ancient Greek equivalent thereof) and standing the slices together with their tips up as in Figure 1.2. Because the circumference of the circle

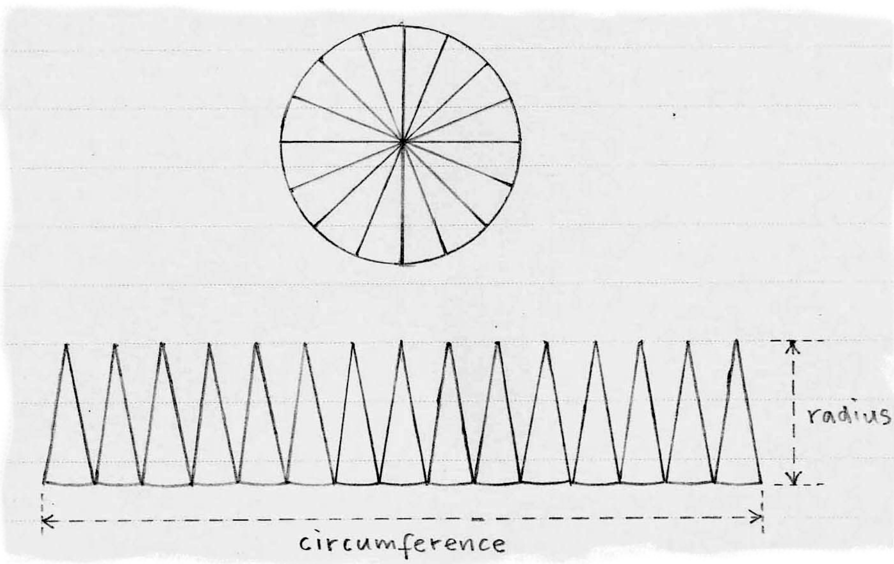

Figure 1.2. We rearrange the pieces of a circle to reveal the connection between its area, circumference, and radius.

has now been laid out in (virtually) a straight line along the bottom, the total area of these (pseudo) triangles sums to

$$\frac{1}{2}(\text{radius})(\text{circumference}) \ .$$

But this is the area of the original circle as well, so we suspect that the area A of a circle is related to the circumference C and radius r by

$$A = \frac{1}{2} r C \ . \tag{1.6}$$

This depends on how willing we are to believe that the thin slices are acting like triangles. Are you persuaded that the thinner the slices get, the more they behave like triangles? What if someone pointed out that as each slice gets thinner, its area becomes arbitrarily close to zero — in other words, the slice vanishes — so that if you want the slices to *be* triangles, they will first have to disappear?

This dispute notwithstanding, our argument leads to the correct conclusion: formula (1.6) tells the truth about circles. This blend of the infinitely many (the slices becoming more numerous) and the infinitely small (each slice on its way to vanishing) is at the heart of the story of calculus.

1.3 Archimedes exhausts a parabolic segment

Archimedes investigated the areas of circles (see exercise 1.1) and many other shapes, as well as the volumes of solids like cones and spheres. One of his efforts in particular carries us back to infinite sums. A *parabola* is, among other things, the path traveled by a ball thrown into the air, and Archimedes calculated the area within the *segment* created when a line cuts a parabola at points A and B, as in Figure 1.3.

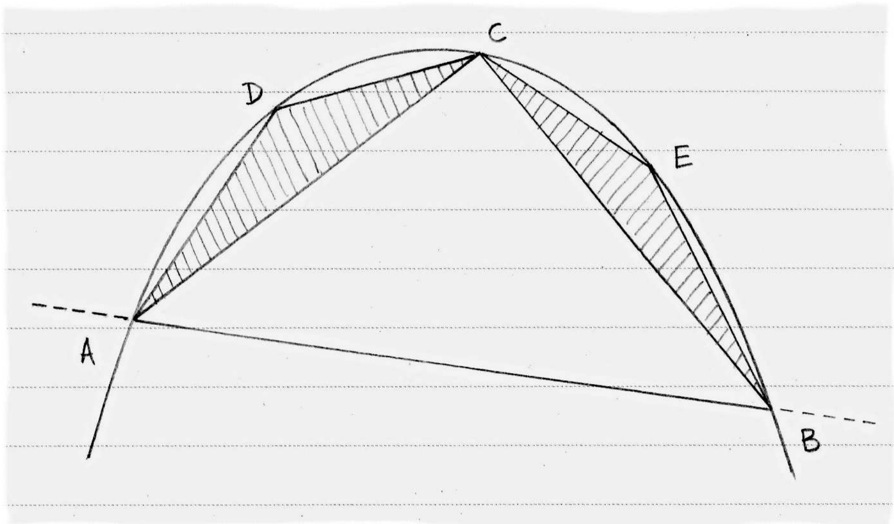

Figure 1.3. Archimedes exhausted the parabolic segment with successively smaller triangles.

Archimedes located[1] point C on the arc AB so that the line tangent to the parabola at C is parallel to AB; this creates triangle ABC. Letting $\triangle$ denote the area of this triangle, we proceed as he did, and create triangles ADC and CEB so that the tangents at D and E are parallel to AC and CB respectively. Archimedes showed that the areas inside these two new triangles totaled exactly one-fourth of the area of the first triangle ABC.

Now the area within the segment is pretty well exhausted by the three triangles; in fact, this method is called just that: *exhaustion*. How might he cope with the four small unfilled areas between the triangles and the parabola, however? Archimedes continued filling the unfilled areas with triangles, doubling the number each time, and proving that each new set of triangles totals one-fourth the area of the previous set. This process, continued indefinitely, produces the equation

$$\text{area within segment} = \triangle + \frac{1}{4}\triangle + \frac{1}{4}\left(\frac{1}{4}\triangle\right) + \frac{1}{4}\left(\frac{1}{4}\left(\frac{1}{4}\triangle\right)\right) + \cdots .$$

The equals sign is justified in the same way we discussed earlier: the partial sums grow arbitrarily close to the area within the segment as the host of triangles gradually exhaust that area. Factoring $\triangle$ from the right side yields

$$\text{area within segment} = \triangle + \triangle\left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right),$$

[1]Although the arguments that underlie the geometry in this paragraph are missing, the reader should rest assured that such omissions are rare in this text.

which contains a sum much like (1.3). The first few partial sums are

$$\begin{aligned}
\frac{1}{4}+\frac{1}{4^2} &= \frac{4+1}{4^2} &&= \frac{5}{16} &&= 0.3125,\\
\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3} &= \frac{4^2+4+1}{4^3} &&= \frac{21}{64} &&\approx 0.3281,\\
\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^4} &= \frac{4^3+4^2+4+1}{4^4} &&= \frac{85}{256} &&\approx 0.3320.
\end{aligned}$$

Further calculations suggest that the infinite sum equals 1/3, prompting us to express the partial sums like so:

$$\begin{aligned}
\frac{5}{16} &= \frac{1}{3}-\frac{1}{3\cdot 4^2},\\
\frac{21}{64} &= \frac{1}{3}-\frac{1}{3\cdot 4^3},\\
\frac{85}{256} &= \frac{1}{3}-\frac{1}{3\cdot 4^4}.
\end{aligned} \tag{1.7}$$

What we are deducting from 1/3 in (1.7) is vanishing the further we go, so

$$\text{area within segment} = \triangle + \triangle\left(\frac{1}{3}\right) = \frac{4}{3}\triangle\ . \tag{1.8}$$

Because $\triangle$, the area of ABC, is simple to calculate, Archimedes had, in his own words, "shown that every segment bounded by a straight line and [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment."

Despite his clever handling of ever-shrinking quantities, Archimedes carefully stated that he did not believe in numbers so small that they behaved like zero. (Specifically, he claimed that every positive number, no matter how tiny, may be added to itself enough times to create arbitrarily large sums.) As one of history's most accomplished mathematicians, Archimedes could tell when he was playing with fire, as we did when we generated (1.4). Centuries passed before anyone truly understood what he was being careful about.

1.4 Patterns

What do you see when you look at the drawing on the left in Figure 1.4? What if someone told you that this picture proved something lovely about odd integers? Pictures like this appear in documents that survive from several ancient cultures — Greece, India, China, Japan — despite the fact that such knowledge is unlikely to have been shared between these peoples. This leaves little doubt that mathematics belongs in the same discussion as music, poetry, and art when it comes to what pursuits are innate in humans. In fact, Figure 1.4 seems to speak as to how mathematics and art can befriend one another. As another example, the truth of the identity

$$\frac{1}{3}=\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^4}+\cdots$$

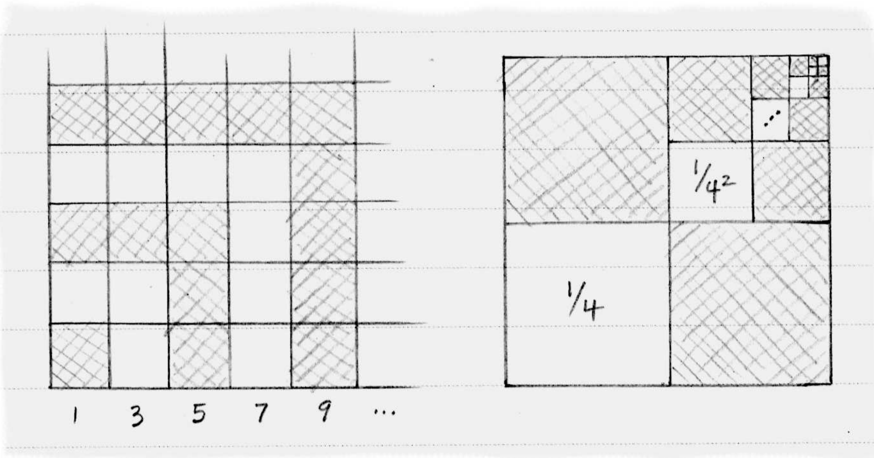

Figure 1.4. Each figure reveals a truth about a sum.

is revealed by the drawing on the right in Figure 1.4. The white boxes occupy areas equal to the terms on the right-hand side of the identity, but together they occupy one-third of the entire square. The drawing provides a more appealing argument, for some, than does our arithmetic approach in (1.7).

The image of a mathematical pattern often unlocks a secret. Consider the following pattern, known to all of the cultures mentioned above; starting at 1, add the positive integers consecutively, stopping at each partial sum:

$$\begin{aligned} 1 + 2 &= 3, \\ 1 + 2 + 3 &= 6, \\ 1 + 2 + 3 + 4 &= 10, \\ 1 + 2 + 3 + 4 + 5 &= 15, \end{aligned}$$

and so on. Is there any pattern to the partial sums 3, 6, 10, 15, ... that we can use to predict something as difficult, say, as the sum of the first 1000 integers?

The partial sums are called *triangular numbers* thanks to the depiction in Figure 1.5. The nth triangular number T_n equals the sum of the first n natural numbers. We have the beginnings of an image that will help us find the 1000th triangular number without actually adding the first 1000 integers. Observe that any of the figures used to represent a triangular number can be copied and flipped, as in Figure 1.5. In each case, the nth triangular number has been copied to create an n by $n + 1$ rectangle. The number of squares in such a rectangle is simply $n(n + 1)$. Because T_n accounts for half the area of the rectangle, we are led to believe that

$$T_n = \frac{1}{2}n(n + 1)\,. \tag{1.9}$$

For example, T_4 is 10, and

$$1 + 2 + 3 + 4 = \frac{1}{2}(4)(5) = 10.$$

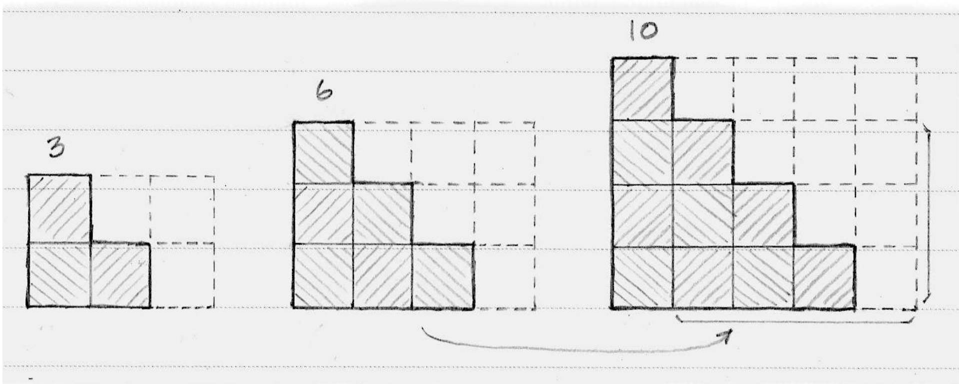

Figure 1.5. Each row of shaded squares represents an integer, and the triangular stacks represent their sums.

Using (1.9) is much simpler than adding the first thousand integers to find that $T_{1000} = 500500$.

This clever shortcut will appear obvious to many who have followed this argument, but, for the skeptics, we can argue the same point using language. The figures themselves suggest the path we will take. Look inside Figure 1.5 again. Hidden within the figure for $T_4 = 10$ is the figure for the previous case $T_3 = 6$, as indicated by the arrow. What if we knew that $1 + 2 + 3$ equaled half of 3 times the next integer 4; could we use this knowledge to show that $1 + 2 + 3 + 4$ equals half of 4 times the next integer 5?

Well, we *do* know that $1 + 2 + 3 = \frac{1}{2}(3)(4)$, because it is clearly true; just do the arithmetic. In fact, we can verify cases like this to our heart's content. But because there are rather too many cases to check in the long run, we will have to stop somewhere. Suppose we stop checking by hand after the first n integers, so that we know

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1) \ . \tag{1.10}$$

If we can argue, in general, that adding the next integer $n+1$ preserves the pattern, will this convince you? Consider:

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \frac{1}{2}n(n+1) + (n+1) \\ &= (n+1)(\frac{1}{2}n + 1) \\ &= \frac{1}{2}(n+1)(n+2) \ . \end{aligned}$$

The first equality is justified by what we know from (1.10). The rest is factoring. Are you convinced that the sum of the first $n+1$ integers is half of the integer $n+1$ times the next integer $n+2$? If so, then you are agreeing to believe in a method called *proof by mathematical induction*. This form of proof allows us to accept the evidence shown in the figures we have seen. Although ancient cultures used induction proofs implicitly, only in the 1600s would mathematicians formalize such arguments.

1.5 The evolution of notation

Underlying the patterns we have studied thus far are series, arguably the most important tool in the calculus kit. As we follow the story of calculus, we will see scholars grow adept at tackling ever more complicated problems, and the series they use will keep pace in complexity.

Thus, we pause to introduce *summation notation*, despite the fact that it was not invented until many centuries after Archimedes. (In fact, we have already transgressed in exactly this way by our use of signs like '+' and '−'. A Greek who wished to convey (1.10) would do so in words, and would not think of the numbers as quantities but as lengths of line segments or areas of regions.)

The capital Greek letter 'sigma' indicates a sum, as in

$$\sum_{k=1}^{n} k = \frac{1}{2} n(n+1) ,$$

which translates (1.10) into summation notation. The variable k is a placeholder that appears in the formula of the sum and increases from the number below Σ to the number above Σ by ones. We may write infinite series like (1.3) as

$$S = \sum_{k=1}^{\infty} \frac{1}{2^k} ,$$

and express our observations in (1.7) as

$$\sum_{k=1}^{n} \frac{1}{4^k} = \frac{1}{3} - \frac{1}{3 \cdot 4^n} .$$

The symbol ∞ denotes 'infinity,' indicating that the series never terminates. Again we rely on a symbol ∞ that entered general use centuries after Archimedes. We will use commonly known symbols for the sake of clarity, and summation notation thanks to its tie to series, but we will save the notation of calculus proper until the historical clock reads what it should.

1.6 Furthermore

1.1 **Archimedes estimates π.** Formula (1.6) is one of the tools needed to prove the well-known formula

$$A = \pi r^2 \tag{1.11}$$

for the area A of a circle in terms of its radius r and the constant π.

Quite a few ancient cultures knew that π was a number slightly larger than 3. Archimedes pursued this number by sandwiching it between two other numbers that he calculated using geometry. Taking a circle of radius 1 (and therefore of area π), he inscribed a regular hexagon within it (*regular* means that all of the angles are equal, and all of the sides are equal, as in Figure 1.6). Whatever the area of this hexagon, he could see that it equaled a number smaller than π.

(a) Find the area of the hexagon. (It is possible to do so using the Pythagorean Theorem, if you add a few lines to Figure 1.6.)

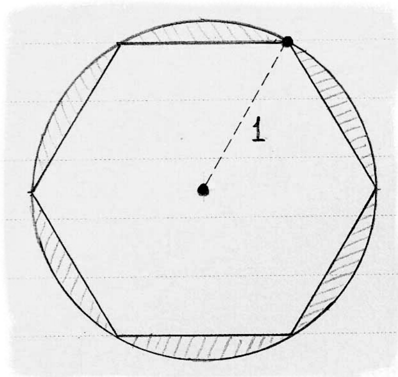

Figure 1.6. Archimedes began his approximation of π by inscribing a regular hexagon inside a unit circle.

(b) Archimedes also swapped the shapes, inscribing the circle of radius 1 in a larger regular hexagon. Find the area of this new hexagon. (This result is therefore larger than π.)

(c) To improve his estimates of π, Archimedes repeated both exercises using regular 12-sided polygons, which exhaust (and surround) the circle more completely. Find the area of such a 12-sided polygon that is inscribed in a circle of radius 1.

	# of red sides showing							
# of sticks tossed	0	1	2	3	4	5	6	7
3	1	3	3	1	-	-	-	-
4								
5								
6								
7								

Table 1. Each cell of the table contains the number of ways of tossing a certain number of sticks and getting a certain number of red sides.

1.2 **Binomial coefficients.** The boardgames Senet (Egypt) and The Royal Game of Ur (Mesopotamia) predate Zeno and Archimedes by well over a thousand years. In some versions of these games, the players tossed sticks rather than dice; each stick was two-sided, like a small popsicle stick, one side painted red, the other painted white. The number of red sides showing indicated the number of pieces the player could move.

For example, say that a player tosses three sticks. With the sticks labeled A, B, and C for convenience, we observe that there are three ways that the sticks will allow a player to move two pieces: the red sides can show on sticks AB, sticks AC, or sticks BC. Following this reasoning, we can fill in the first row of Table 1.

(a) Fill in the rest of the table.

(b) As you do so, look for any patterns you see, and express your observation in words.

(c) Argue on behalf of your observation in 1.2(b). One way to do this is to consider the fate of one of the sticks: is it white, or is it red? This breaks your counting problem into two cases.

Historical note. The patterned numbers that appear in Table 1 were known, at least in part, to such scholars as **Omar Khayyám** (Persia, born 1048) and **Chu Shih-Chieh** (China, born *c.* 1260), to name just two. **Blaise Pascal**, who we will encounter again (exercise 6.1), revealed many truths hidden in these numbers, which are commonly named after him. These particular numbers have been of interest to scholars for many centuries; we return to them in exercise 3.4.

1.3 **The sum of the first n cubes.** A triangular number T_n is the sum of the first n integers and is calculated using formula (1.9). In this exercise, we explore the surprising connection between triangular numbers, cubed numbers, and squares.

(a) In particular, we seek a formula for the sum of the first n cubed numbers

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 \ .$$

Calculate the sum for $n = 1, 2, 3, \ldots$ until you see a pattern in the sums that uses triangular numbers.

(b) Use a proof by induction to establish the truth of your pattern.

(c) In Figure 1.4, each image is designed to reveal a truth about numbers. Create an image that illustrates the connection that you have discovered about the sum of the first n cubes and the corresponding triangular number T_n. One such approach considers a cubed number as the product of that number times its square (or $m^3 = m \cdot m^2$ in symbols), and the result is a two-dimensional image.

1.4 **The sum of the first n squares.** Here we investigate another formula known to early cultures, including the Greeks. Write out the details of each step of this approach.

(a) We seek a formula for

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 \; .$$

Calculate the sum for $n = 1, 2, 3, 4$. Because the formula we seek is a bit more complicated than most we have encountered in this chapter, it is not likely that these examples alone will suggest a pattern to you.

(b) Show that

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1 \; . \tag{1.12}$$

(c) Substitute $1, 2, 3, \ldots, n$ for k in (1.12) and add all n of these equations together. (Why substitute for k, you may wonder? We are using k simply as a place holder; it is n that we care about, as we are seeking to add the first n squared numbers.)

(d) If we algebraically manipulate the result from part (c), we can arrive at the formula

$$\begin{aligned} \sum_{k=1}^{n} k^2 &= \frac{1}{6} n(n+1)(2n+1) \\ &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \; . \end{aligned}$$

Show this work.