CV 4. část 1. SNV

Př. 1)

$$F(x) = \begin{cases} 0 & x \leq 0 \\ c x^2 & 0 < x \leq 1 \\ 1 & 1 < x \end{cases}$$

$$f(x) = \frac{\partial F(x)}{\partial x}$$

$$f(x) = \begin{cases} 0 & x \leq 0 \vee x > 1 \\ \underline{c \cdot 2x} & \underline{0 < x \leq 1} \end{cases}$$

$$(c \cdot x^2)' = c \cdot 2x$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_0^1 c \cdot 2x \, dx = 1$$

$$\left[c \cdot x^2 \right]_0^1 = 1$$

$$c \cdot 1 - c \cdot 0 = 1$$

$$c \cdot 1 = 1$$

$$\underline{\underline{c = 1}}$$

$\wedge$ F(x)

ν 0 $\quad cx^2 = 0 \qquad \underline{c \cdot 0 = 0}$

ν 1 $\quad cx^2 = 1 \qquad \underline{c \cdot 1 = 1} \Rightarrow \underline{\underline{c = 1}}$

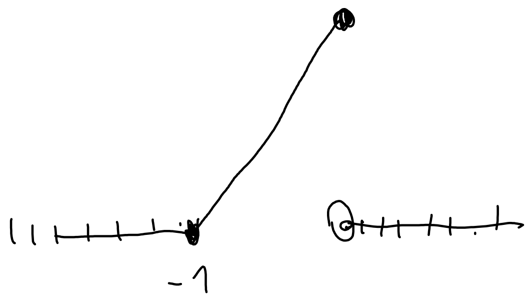
$$F(x) = \begin{cases} 0 & \ldots \\ x^2 & \ldots \\ 1 & \ldots \end{cases}$$

Př. 2

$$f(x) = \begin{cases} 2x+2 & x \in \langle -1, 0 \rangle \\ 0 & x \notin \langle -1, 0 \rangle \end{cases}$$

a) $F(x) = ?$

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$



$$F(t) = \begin{cases} 0 & t < -1 \\ & t \in \langle -1, 0 \rangle \\ 1 & t > 0 \end{cases}$$

$$F(t) = \underbrace{\int_{-\infty}^{-1} f(x)\,dx}_{0} + \int_{-1}^{t} f(x)\,dx \;\ldots\; t < 0$$

$$F(t) = \int_{-1}^{t} 2x+2 \, dx = \left[x^2+2x\right]_{-1}^{t} =$$

$$= t^2+2t-(1-2) = \underline{t^2+2t+1}$$

$$F(t) = \begin{cases} 0 & t < -1 \\ t^2+2t+1 \quad \sim (t+1)^2 & t \in \langle -1, 0 \rangle \\ 1 & t > 0 \end{cases}$$

b)

$$P(-2 \leq X \leq -0{,}5) = \int_{-2}^{-0{,}5} f(x) \, dx$$

$$= \int_{-1}^{-0{,}5} 2x+2 \, dx = \left[x^2+2x\right]_{-1}^{-0{,}5} = \frac{1}{4} - 1 - (1-2)$$

$$= \underline{\underline{\frac{1}{4}}}$$

$$P(-2 \leq X \leq -1) = \int_{-2}^{-1} f(x)\,dx = 0$$

$$P(X > 0{,}5) = \int_{0{,}5}^{\infty} f(x)\,dx = 0$$

$$P(X = \underline{\underline{0{,}3}}) = 0$$

c) $$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$$

$$D(X) = E(X^2) - E(X)^2$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{-1}^{0} x \cdot (2x+2)\,dx$$

$$\int_{-\infty}^{-1} \ldots = 0 \qquad \int_{-1}^{0} \ldots = ? \qquad \int_{0}^{\infty} \ldots = 0$$

$$= \int_{-1}^{0} 2x^2 + 2x \, dx = \left[\frac{2}{3}x^3 + x^2 \right]_{-1}^{0} = 0 - \left(-\frac{2}{3} + 1\right)$$

$$= \underline{\underline{-\frac{1}{3}}}$$

$$E(x^2) = \int_{-1}^{0} x^2 \cdot (2x+2) \, dx = \int_{-1}^{0} 2x^3 + 2x^2 \, dx =$$

$$= \left[\frac{1}{2}x^4 + \frac{2}{3}x^3 \right]_{-1}^{0} = 0 - \left(\frac{1}{2} - \frac{2}{3} \right) = \underline{\underline{\frac{1}{6}}}$$

$$D(X) = E(X^2) - E(X)^2 = \frac{1}{6} - \left(-\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9}$$

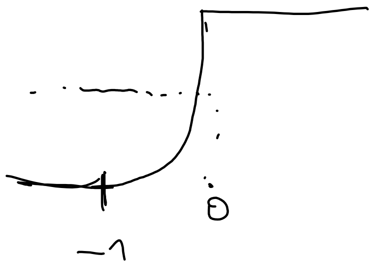
$$= \frac{3}{18} - \frac{2}{18} = \underline{\underline{\frac{1}{18}}}$$

$$\sigma(X) = \sqrt{\frac{1}{18}} \doteq 0{,}2357$$

d) $\underset{x}{\operatorname{argmax}} \, f(x) = 0 = \hat{x}$

e) $P(X < x_{0,5}) = 0,5$

$$\underline{F(x_{0,5}) = 0,5}$$



$x_{0,5} \in \langle -1, 0 \rangle$

$F(x) = x^2 + 2x + 1 \qquad x \in \langle -1, 0 \rangle$

$\Downarrow$

$x^2 + 2x + 1 = 0,5 \qquad x = x_{0,5}$

$x^2 + 2x + 0,5 = 0$

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 0,5}}{2} = \frac{-2 \pm \sqrt{2}}{2}$$

$$= -1 \pm \frac{1}{\sqrt{2}}$$

$$x_{0,5} = \underline{\underline{-1 + \frac{1}{\sqrt{2}}}} \in \langle -1, 0 \rangle$$

Pr 3.

$$Y = 3X + 1$$

a) $F_Y(y) = P(Y < y) = P(3X + 1 < y)$

$$\boxed{P(X < \alpha) = F_X(\alpha)}$$

$$= P(3X < y - 1) = P\left(X < \frac{y-1}{3}\right) = F_X\left(\frac{y-1}{3}\right)$$

$$F_Y(y) = \begin{cases} 0 & \frac{y-1}{3} < -1 \\ \left(\left(\frac{y-1}{3}\right) + 1\right)^2 & \frac{y-1}{3} \in \langle -1, 0 \rangle \\ 1 & \frac{y-1}{3} > 0 \end{cases}$$

$$\frac{1}{9} \cdot (y - 1 + 3)^2 = \frac{1}{9} \cdot (y + 2)^2$$

$$F_Y(y) = \begin{cases} 0 & y < -2 \\ \frac{1}{4}(y+2)^2 & y \in \langle -2, 1) \\ 1 & y > 1 \end{cases}$$

b) $$f_Y(y) = \frac{\partial F_Y(y)}{\partial y}$$

$$f_Y(y) = \begin{cases} 0 & y < -2 \vee y > 1 \\ \frac{2}{4}(y+2) & y \in \langle -2, 1) \end{cases}$$

c) $$E(aX+b) = a \cdot E(X) + b$$

$$D(aX+b) = a^2 \cdot D(X)$$

$$E(Y) = 3 \cdot E(X) + 1 = 0$$

$$D(Y) = 9 \cdot D(X) = 0{,}5$$

$$\sigma(Y) = \sqrt{D(Y)} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \doteq 0{,}707$$

PŘ 4:

$$f(x) = \begin{cases} 0 & x < 0 \\ 3e^{-3x} & x \geq 0 \end{cases}$$

spočítejte w takové, aby náhodná veličina X byla s pravděpodobností $0{,}3$ větší než w.

$$P(X > w) = 0{,}3$$

$$\boxed{F(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-3t} & t \geq 0 \end{cases}}$$

$$t \geq 0 \qquad F(t) = \int_0^t 3 \cdot e^{-3x} \, dx =$$

$$\boxed{\begin{array}{l} \left(e^{ax}\right)' = a \cdot e^{ax} \\ \int e^{ax} dx = \frac{1}{a} e^{ax} \end{array}}$$

$$= \left[-e^{-3x} \right]_0^t =$$

$$= -e^{-3t} - (-1) = 1 - e^{-3t}$$

$$P(\underbrace{X > w}_{\shortparallel}) = 0{,}3$$

$$1 - P(X \leq w) = 0{,}3$$

$$1 - \underbrace{P(X < w)} = 0{,}3$$

$$1 - F(w) = 0{,}3 \quad \backslash -1$$

$$-F(w) = -0{,}7 \quad | \cdot (-1)$$

$$F(w) = 0{,}7$$

$$\boxed{1 - e^{-3x} = 0{,}7} \qquad x = w$$

$$-e^{-3x} = -0{,}3 \quad | \cdot (-1)$$

$$e^{-3x} = 0{,}3 \quad | \log(\cdot)$$

$$-3x = \log(0{,}3)$$

$$x = -\frac{1}{3} \log(0{,}3)$$

Část 2. Náhodný vektor

$X\backslash Y$	1	2	3	4	$P_X(x)$	$x \cdot P_X$
3	0,01	0,02	0,03	0,25	0,31	0,93
5	0,04	0,16	0,18	0,05	0,43	2,15
7	0,12	0,07	0,06	0,01	0,26	1,82
$P_Y(y)$	0,17	0,25	0,27	0,31		$E(X)=4,9$
$y \cdot P_Y$	0,17	0,5	0,81	1,24	$E(Y)=2,72$	
$y^2 \cdot P_Y$	..	..	..	.	$E(Y^2)$	

$$F(2,8;\ 7,1) = 0,42$$

$$Y < 2,8 \wedge X < 7,1$$

$$P_X(x) = P(X = x,\ Y \in \mathbb{R})$$

d) $P(Y > 2,1 \mid X < 5,3) =$

$$= \frac{P(Y > 2{,}1,\ X < 5{,}3)}{\underbrace{P(X < 5{,}3)}} = \frac{0{,}51}{0{,}74} \doteq \underline{\underline{0{,}69}}$$

$$P(X = 5 \mid Y = 1) = \frac{P(X=5,\ Y=1)}{P(Y=1)} =$$

$$\frac{p(1,5)}{P_Y(1)} \doteq \underline{\underline{0{,}24}}$$

$\underline{p(x \mid y)}$

$x \backslash y$	1	2	3	4
3	α			
5				
7			μ	

$$\alpha = P(X = 3 \mid Y = 1)$$

$$\mu = P(X = 7 \mid Y = 3)$$

f) $E(X \mid Y=2)$

$E(X) = \sum x_i \cdot P_X(x_i)$

$E(X \mid Y=2) = \sum x_i \cdot \underline{P(x_i \mid Y=2)}$

$= 3 \cdot 0{,}08 + 5 \cdot 0{,}64 + 7 \cdot 0{,}28 =$

$= 5{,}4$

g) $\operatorname{cov}(x,y) = \underline{E(X \cdot Y)} - \underline{E(X)} \cdot \underline{E(Y)}$

$E(X \cdot Y) = \sum_{y_i} \sum_{x_i} \underline{P(y_i, x_i)} \cdot \underbrace{x_i \cdot y_i}$

$= 12{,}28$

$\operatorname{cov}(X,Y) = 12{,}28 - 2{,}27 \cdot 4{,}9 = -1{,}08$

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{D(X) \cdot D(Y)}} \doteq -0{,}645$$

$X = -2, -1, 0, 1, 2$

$Y = X^2$