Nonlinear Dynamics and Chaos

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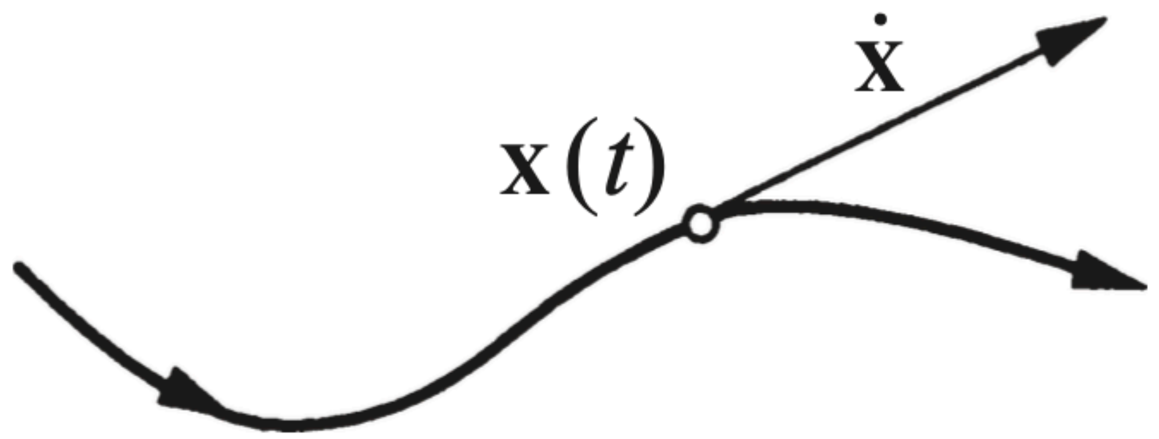
NonLinear Systems in 2D

Phase Portraits

The general form of a vector field on the phase plane is:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \quad \Rightarrow \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$



Here $\mathbf{x}$ represents a point in the phase plane, and $\dot{\mathbf{x}}$ is the velocity vector at that point.

The entire phase plane is filled with trajectories, since each point can play the role of an initial condition

For **nonlinear systems,** there's typically no hope of finding the trajectories analytically. We should determine the **qualitative behaviour of the solutions.**

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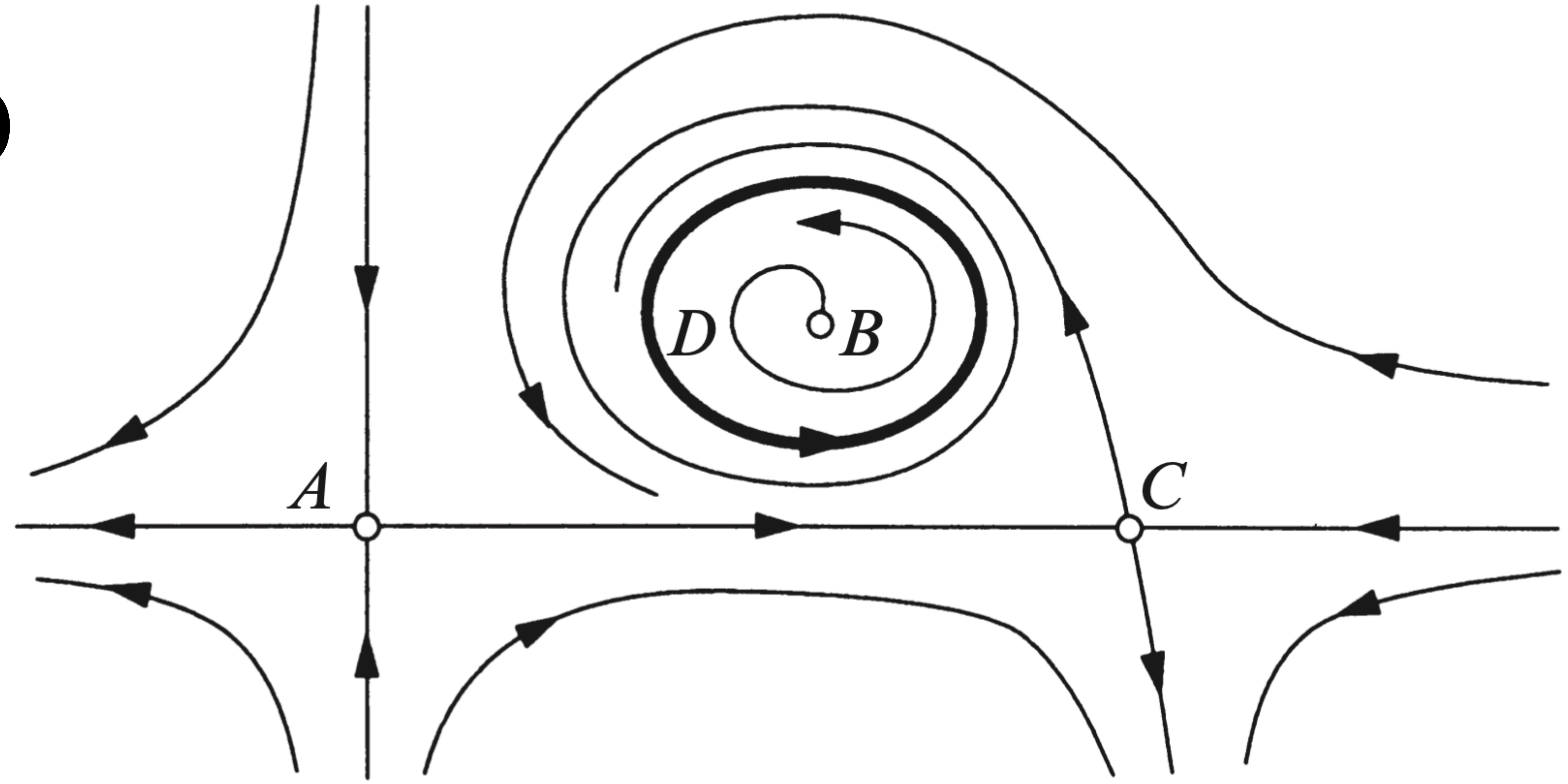
(x_1, x_2) Phase Portrait Example



The **fixed points**, A, B, and C, which satisfy $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, correspond to **steady states or equilibria of the system.**

The **closed orbits** (e.g. D) correspond to periodic solutions, i.e., solutions for which $x(t + T) = x(t)$ for all t, for some $T > 0$.

The flow pattern near *A* and *C* is similar, and different from that near *B*. The **fixed points *A*, *B*, and *C* are unstable**, because nearby trajectories tend to move away from them, whereas **the closed orbit *D* is stable.**

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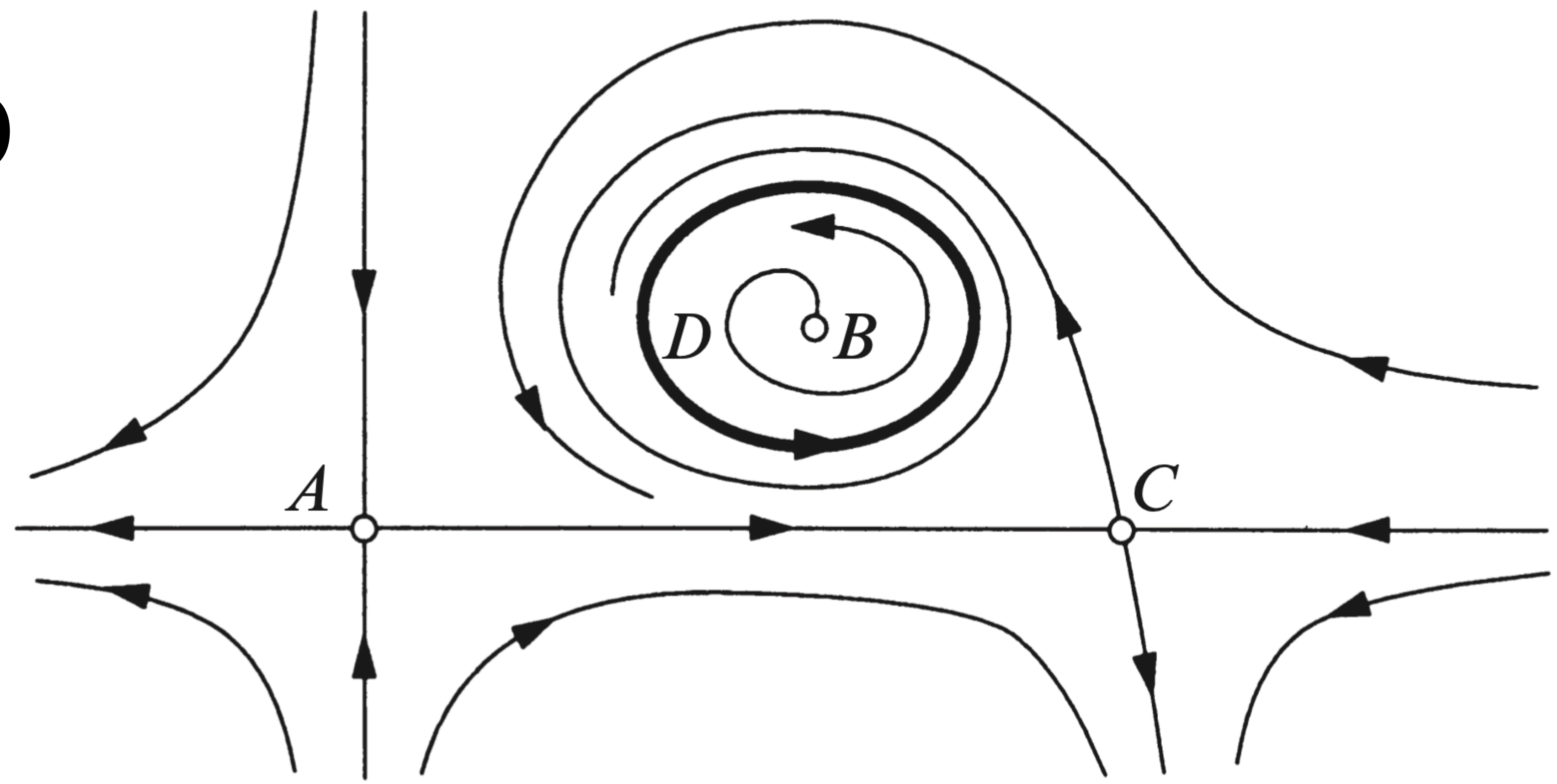
(x_1, x_2) **Phase Portrait Example**



Nullclines: curves where $f_1 = 0$ (x-nullcline) $f_2 = 0$ (y-nullcline). The nullclines indicate where the flow is purely horizontal or vertical.

Separatrices: trajectories that separates regions of qualitatively different behaviour (often stable/unstable manifolds of saddles).

The region around B and D contains a closed orbit (a **limit cycle**, indicated by the thick black loop) surrounding an unstable spiral (B) or a stable spiral (if the flow was reversed).

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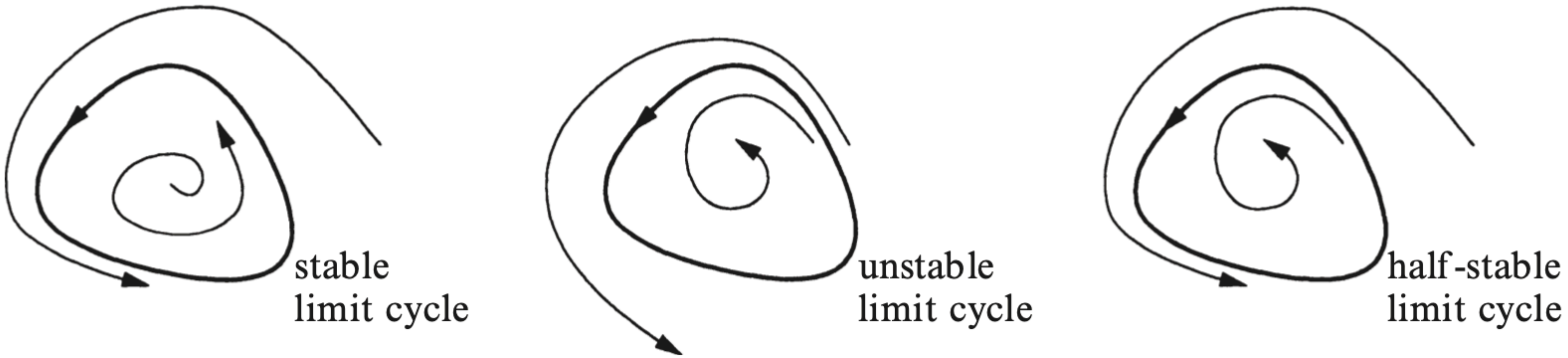
Limit cycles

A **limit cycle** is an isolated closed trajectory. Isolated means that neighbouring trajectories are not closed; they spiral either toward or away from the limit cycle.



If all neighbouring trajectories approach the limit cycle, we say the limit cycle is **stable or attracting**. Otherwise the limit cycle is **unstable,** or in exceptional cases, **half-stable**.

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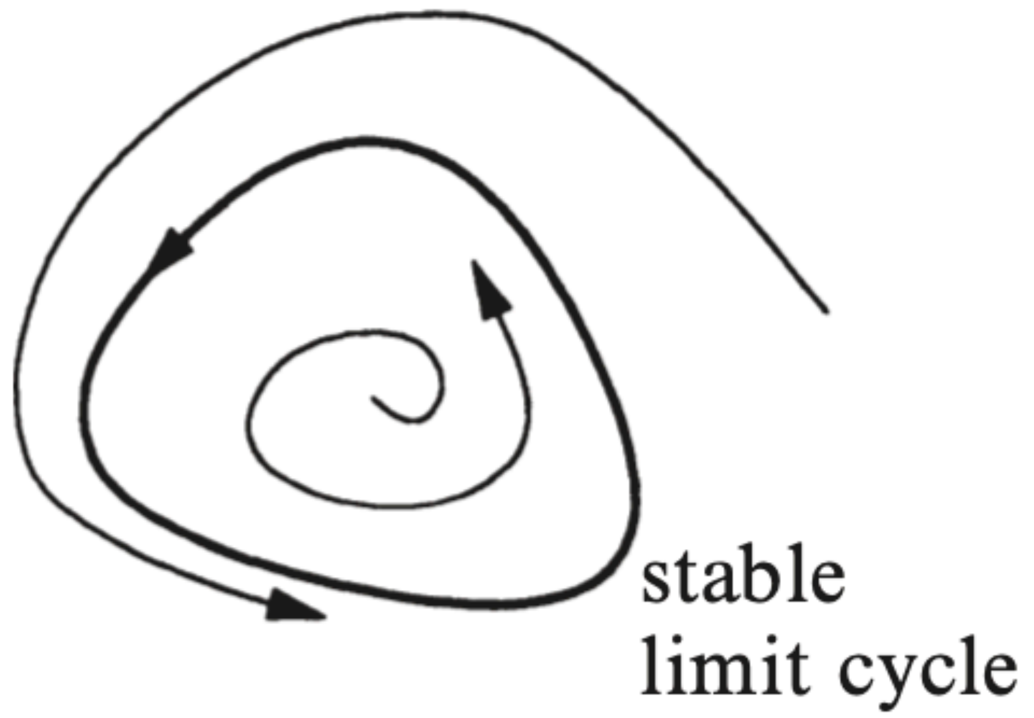


Stable limit cycles are very important scientifically—they model systems that exhibit self-sustained oscillations.

These systems oscillate even in the absence of external periodic forcing.

There is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it always returns to the standard cycle.

Examples:

The beating of a heart.

Daily rhythms in human body temperature and hormone secretion

Chemical reactions that oscillate spontaneously.

Dangerous self-excited vibrations in bridges and airplane wings.

Tutorial:

Consider the system of nonlinear differential equations:

$$\dot{x} = x + e^{-y}$$

$$\dot{y} = -y$$

1. Use qualitative arguments to obtain information about the phase portrait (i.e., find equilibrium points, nullclines, and classify the stability).

2. Using a computer (via Python code), plot the vector field.

3. Use the Runge-Kutta method to compute several trajectories and plot them on the phase plane.

https://github.com/MSc-Fundamental-Physics/nonlinear-dynamics-chaos/blob/main/unit-2/tutorials/17_nonlinear_systems_2D.ipynb

NonLinear Systems in 2D

Existence and uniqueness of solutions of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

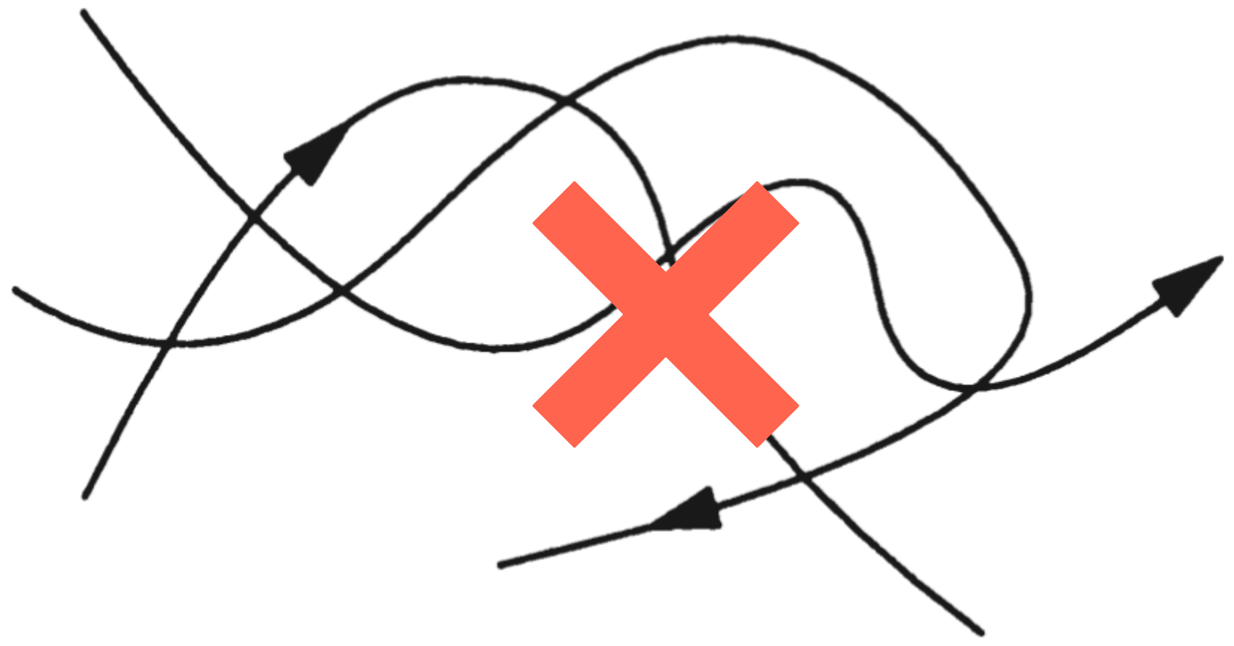
Existence and Uniqueness Theorem: Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \mathbf{x}(0) = \mathbf{x}_0$. Suppose that $\mathbf{f}$ is continuous and that all its partial derivatives $\partial f_i / \partial x_j$, $i, j = 1, \ldots, n$, are continuous for $\mathbf{x}$ in some open connected set $D \subset \mathbf{R}^n$. Then for $\mathbf{x}_0 \in D$, the initial value problem has a solution $\mathbf{x}(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

Existence and uniqueness of solutions are guaranteed if f is continuously differentiable.

Corollaries:

Different trajectories never intersect!

A trajectory can't move in two directions at once.



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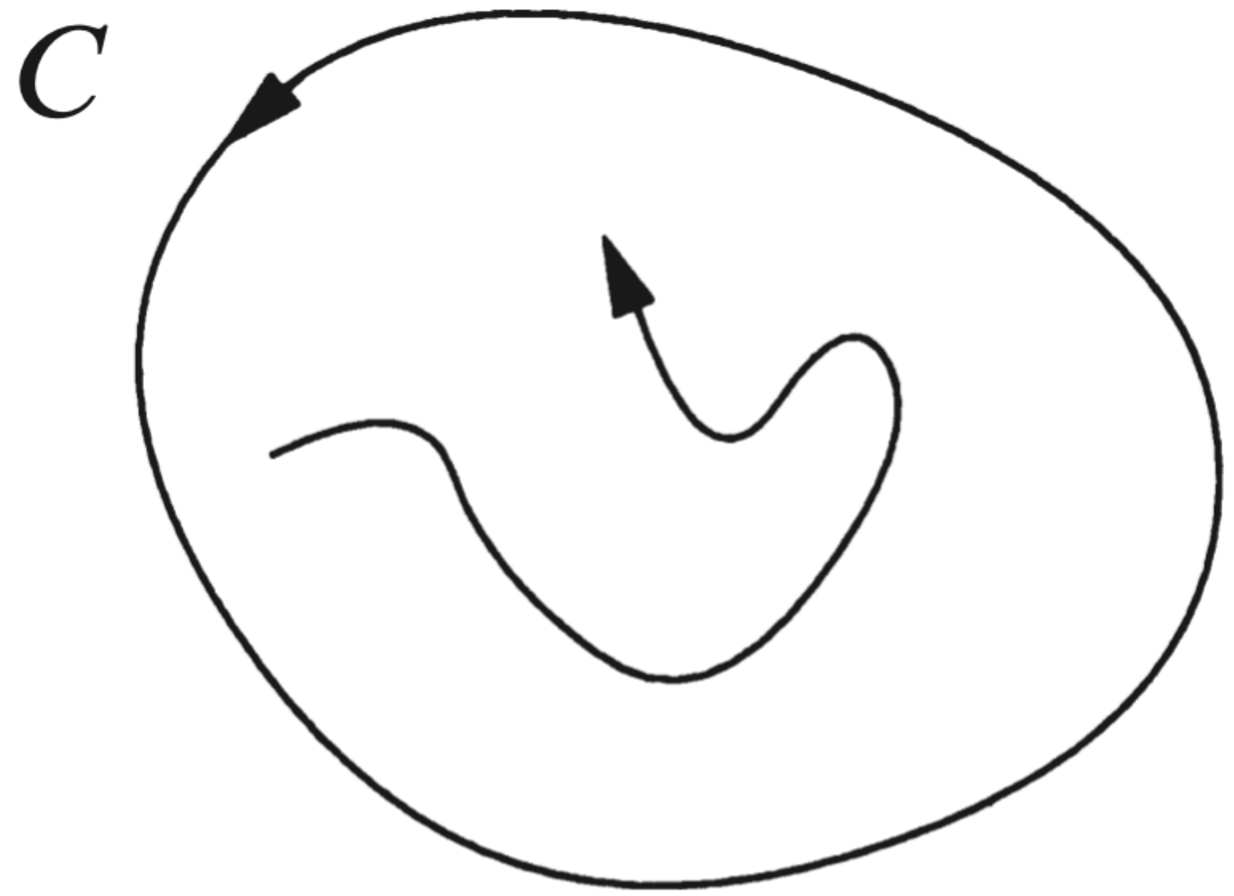
Existence and uniqueness of solutions of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$



Topological consequences:

Suppose there is a closed orbit *C* in the phase plane. Then any trajectory starting inside *C* is trapped in there forever.

What happens if there are no fixed points inside?

The **Poincaré-Bendixson theorem** states that if a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory must eventually approach a closed orbit.

NonLinear Systems in 2D

Fixed Points and Linearisation

We hope that we can approximate the phase portrait near a fixed point by that of a corresponding linear system.

Consider the system with fixed point (x^*, y^*):

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

$$f(x^*, y^*) = 0, \qquad g(x^*, y^*) = 0.$$

Let: $\quad u = x - x^*, \qquad v = y - y^*$

be the components of a small disturbance from the fixed point. To see whether the disturbance grows or decays, we need to derive differential equations for u and v.

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Analysis for u:

$$\dot{u} = \dot{x} \qquad \text{(since } x^* \text{ is a constant)}$$

$$= f(x^* + u, y^* + v) \qquad \text{(by substitution)}$$

$$= f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) \quad \text{(Taylor series expansion)}$$

$$= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) \qquad \text{(since } f(x^*, y^*) = 0\text{).}$$

Partial derivatives are evaluated at the fixed point.

Analysis for v:

$$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + O(u^2, v^2, uv).$$

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The disturbance is:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms.}$$

where we find the Jacobian matrix at the fixed point:

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

By neglecting the quadratic terms, we obtain a **linearised system:**

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

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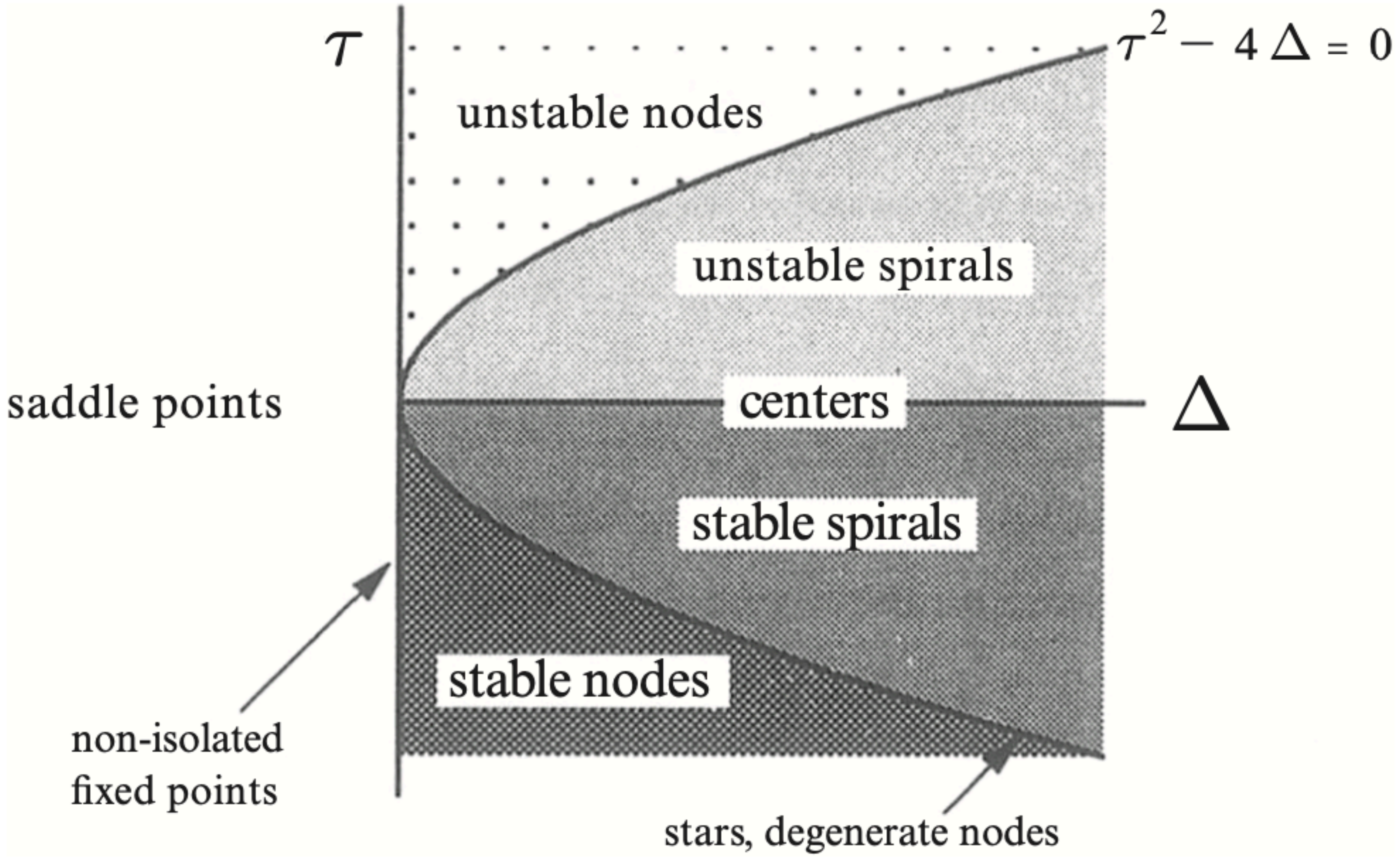
Is it safe to neglect the quadratic terms?

Yes, as long as the fixed point for the linearised system is not one of the borderline cases.

If the linearised system predicts a saddle, node, or a spiral, then the fixed point really is a saddle, node, or spiral for the original nonlinear system.

The borderline cases (centres, degenerate nodes, stars, or non-isolated fixed points) can be altered by small nonlinear terms.



Stars and degenerate nodes can be altered by small nonlinearities, but unlike centres, their stability doesn't change.

Tutorial:

Find all the fixed points of the system:

$$\dot{x} = -x + x^3$$

$$\dot{y} = -2y$$

and use linearisation to classify them.

Then check your conclusions by deriving the phase portrait for the full non-linear system.

NonLinear Systems in 2D

Fixed-point classification (by stability):

Robust cases	**Repellers (sources)**	Both eigenvalues have positive real part.
	Attractors (sinks)	Both eigenvalues have negative real part.
	Saddles	One eigenvalue is positive and one is negative.
Marginal cases $Re(\lambda) = 0$	**Centres**	Both eigenvalues are pure imaginary.
	Higher-order and non-isolated fixed points	At least one eigenvalue is zero

NonLinear Systems in 2D

Fixed-point classification (by stability):

Hyperbolic Fixed points $Re(\lambda) = 0$	Their stability type is unaffected by small nonlinear terms.
Nonhyperbolic fixed points $Re(\lambda) \neq 0$	Fragile points affected by non-linear terms. Analogous to 1D systems: $f'(x^*) \neq 0.$

NonLinear Systems in 2D

Hartman–Grobman theorem:

If eigenvalues have non-zero real parts (hyperbolic fixed point), nonlinear behaviour near it is topologically equivalent to the linear system (**homeomorphism**). The stability type of the fixed point is faithfully captured by the linearisation.

The phase portrait of a **saddle point is structurally stable**, but that of a centre is not: an arbitrarily small amount of damping converts the centre to a spiral.

A phase portrait is **structurally stable** if its topology cannot be changed by an arbitrarily small perturbation to the vector field.

Lotka-Volterra model of competition

We will study the classic Lotka-Volterra model of competition between two species (rabbits and sheep).

x(t) = population of rabbits

y(t) = population of sheep

System of ODEs:

$$\dot{x} = x(3 - x - 2y)$$

$$\dot{y} = y(2 - x - y)$$

Lotka-Volterra model of competition

$$\dot{x} = x(3 - x - 2y)$$

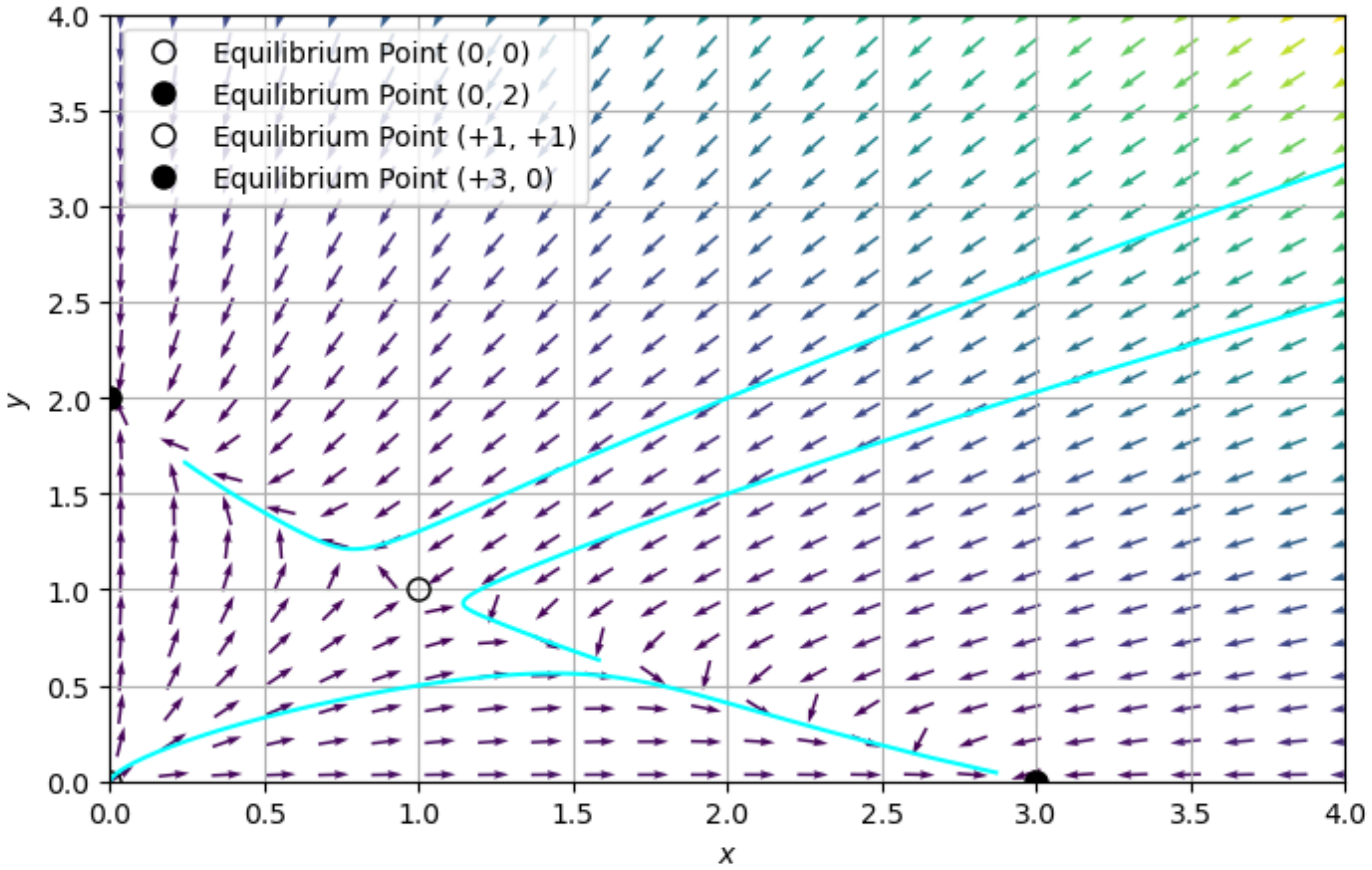
$$\dot{y} = y(2 - x - y)$$

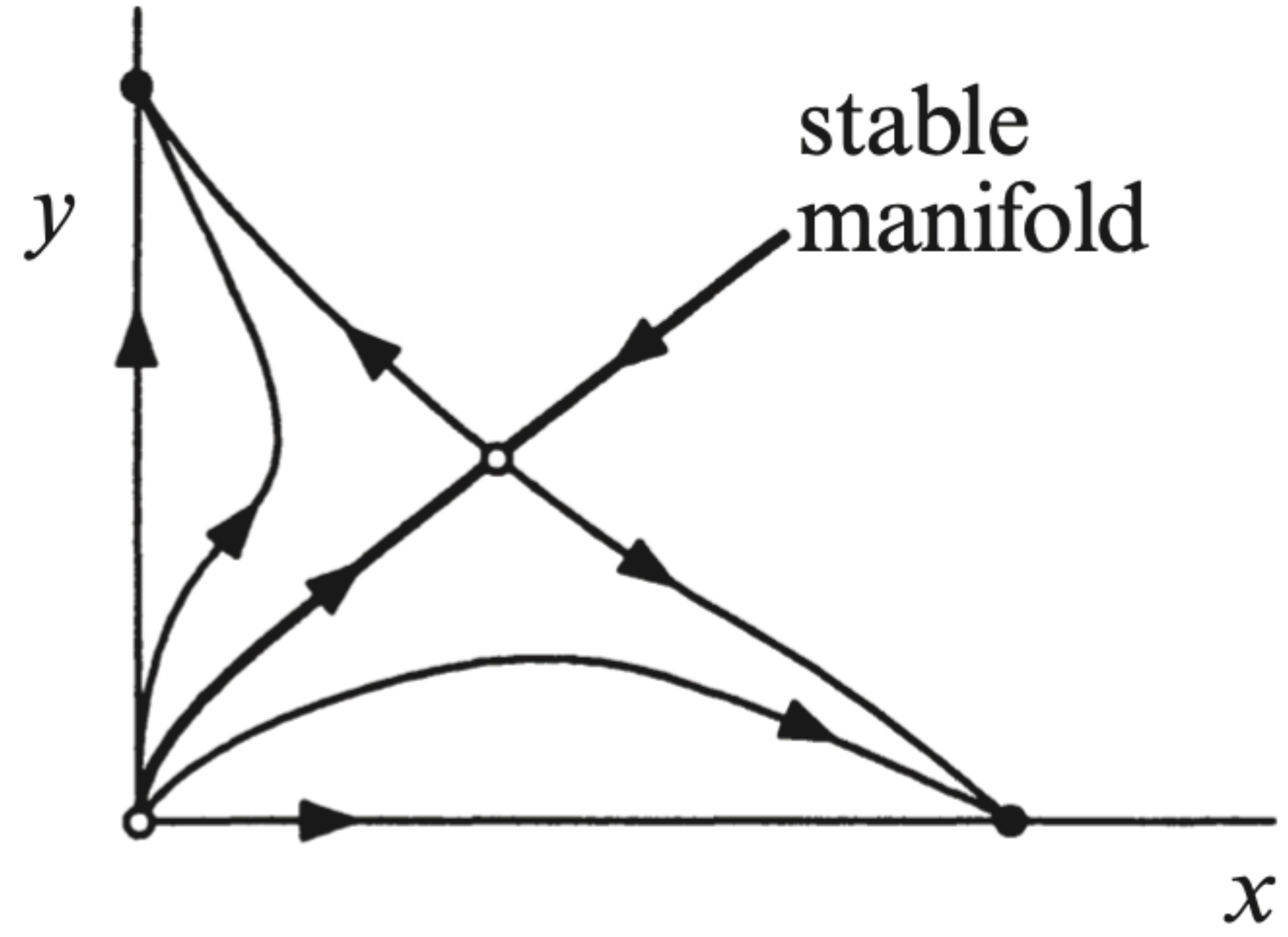
Both species are competing for the same food supply and the amount available is limited. We ignore predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:

1. Each species would grow to its carrying capacity in the absence of the other (logistic growth for each species). Rabbits have a higher intrinsic growth rate.

2. 2. We'll assume that rabbit-sheep conflicts occur at a rate proportional to the size of each population, and we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

Lotka-Volterra model of competition

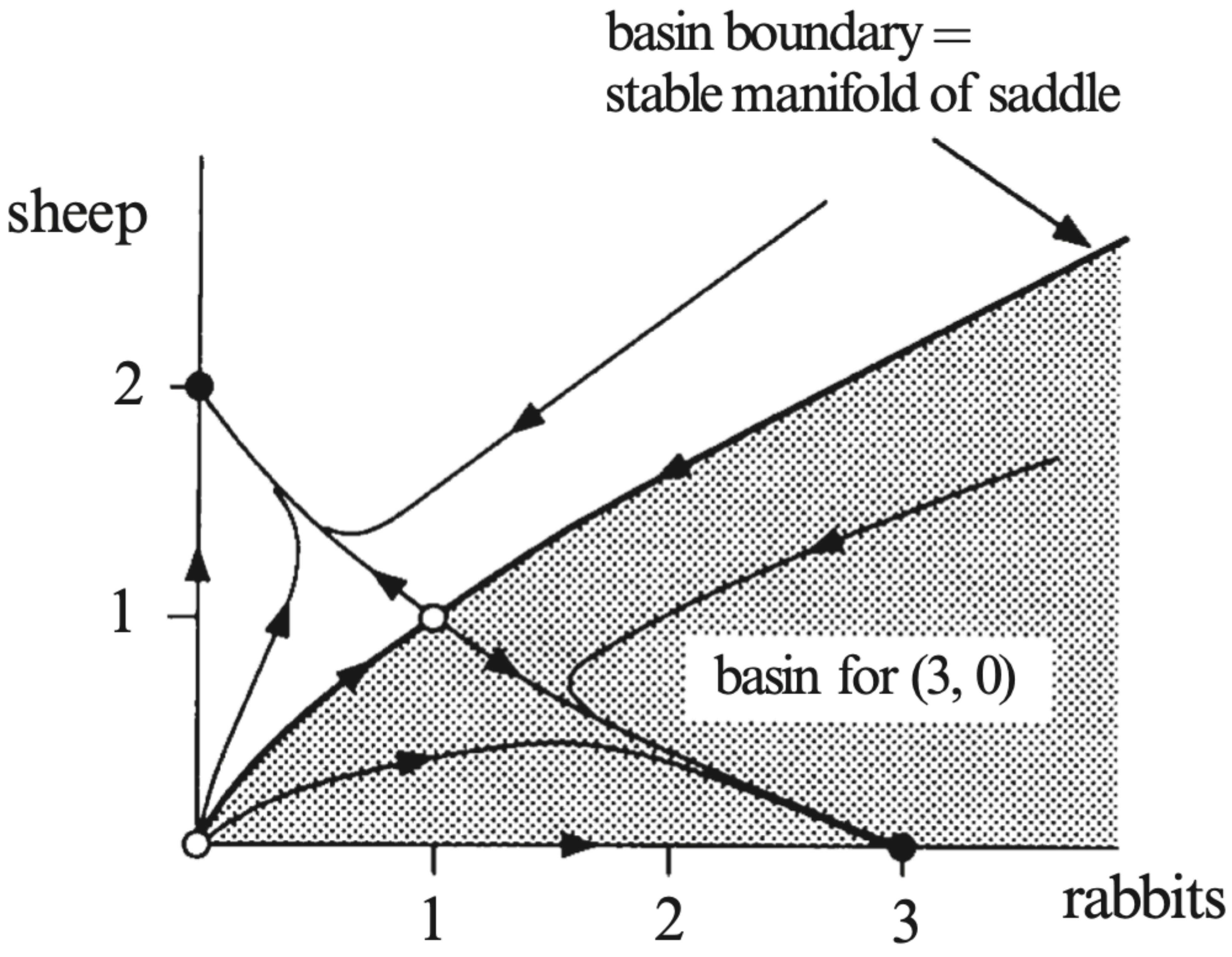




The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction.

Principle of competitive exclusion in biology: two species competing for the same limited resource typically cannot coexist.

Lotka-Volterra model of competition



Given an attracting fixed point $\mathbf{x}^*$, we define its **basin of attraction** to be the set of initial conditions $\mathbf{x_0}$ such that $\mathbf{x(t)} \rightarrow \mathbf{x}^*$as $t \rightarrow \infty$.

Because the stable manifold separates the basins for the two nodes, it is called the **basin boundary**.

The two trajectories that comprise the stable manifold are traditionally called **separatrices**.

Basins and their boundaries are important because they partition the phase space into regions of different **long- term behaviour.**