Nonlinear Dynamics and Chaos

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MSc in Fundamental Physics

Yachay Tech University - 2025

Historical Overview of Non-linear Dynamics

Dynamics — A Capsule History

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff	Complex behavior in Hamiltonian mechanics
	Kolmogorov	
	Arnol'd	
	Moser	
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

Reference: Strogatz Book

Historical Overview of Non-linear Dynamics

Brahe & Kepler:

In the early 1600's, Kepler's laws of planetary motion are reported:

First law (Astronomia nova, 1609):

Each planet moves in an elliptical orbit with the Sun at one of the two foci:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad e = \sqrt{1 - \frac{b^2}{a^2}}$$

a (semi-major axis), b (semi-minor axis), e (eccentricity).

Second law (Astronomia nova, 1609):

The position line joining the planet and the Sun sweeps out equal areas during equal intervals of time:

$$\frac{dA}{dt} = \text{constant}$$

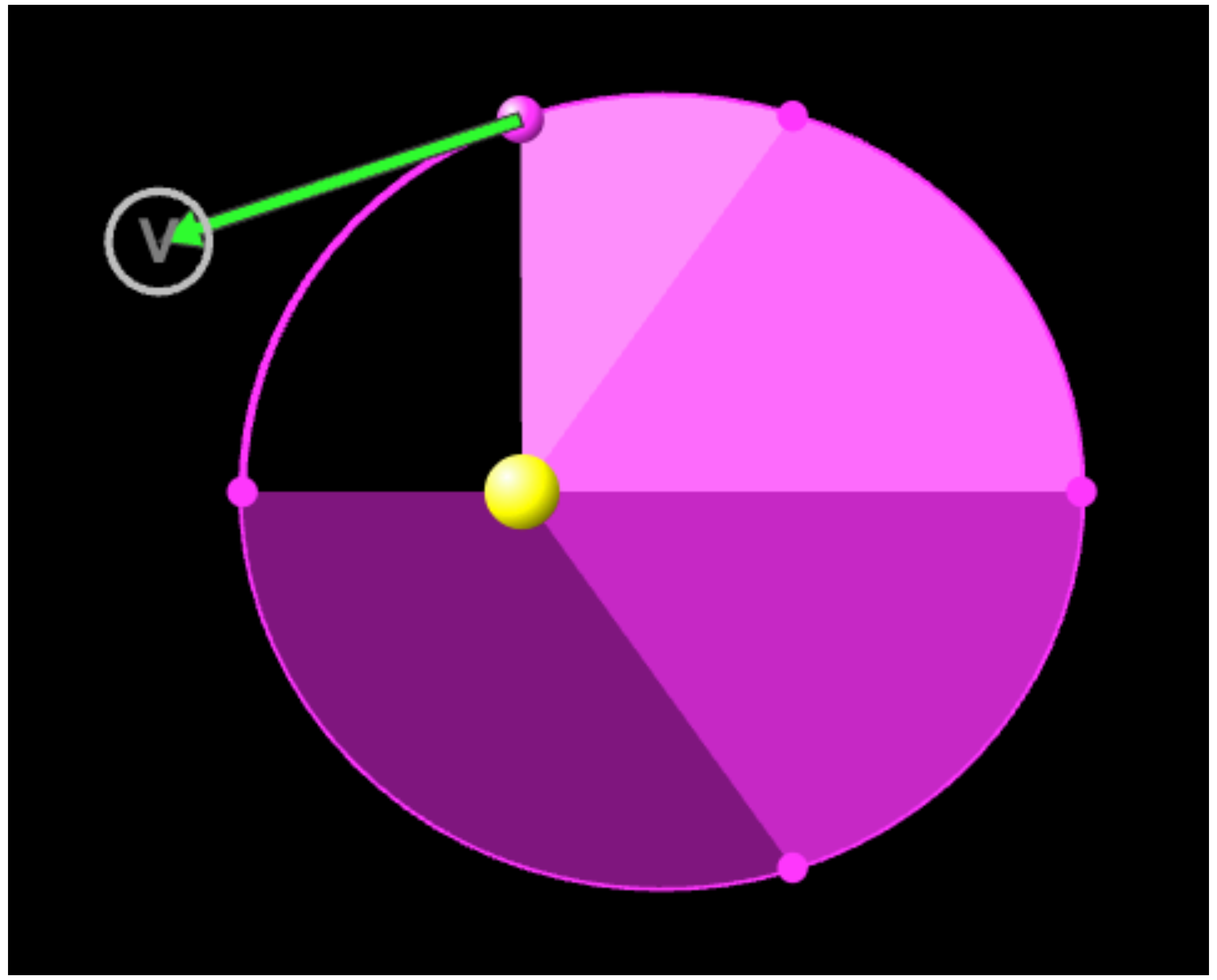
Historical Overview of Non-linear Dynamics

Third law (Harmonices Mundi, 1619):

The square of the orbital period T of a planet is proportional to the cube of the semi-major axis a of its orbit:

$$T^2 \propto a^3 \qquad T^2 = \frac{4\pi^2}{GM_\odot} a^3$$

G (gravitational constant), $M_\odot$ (mass of the Sun).



Applet: https://phet.colorado.edu/en/simulations/keplers-laws

Historical Overview of Non-linear Dynamics

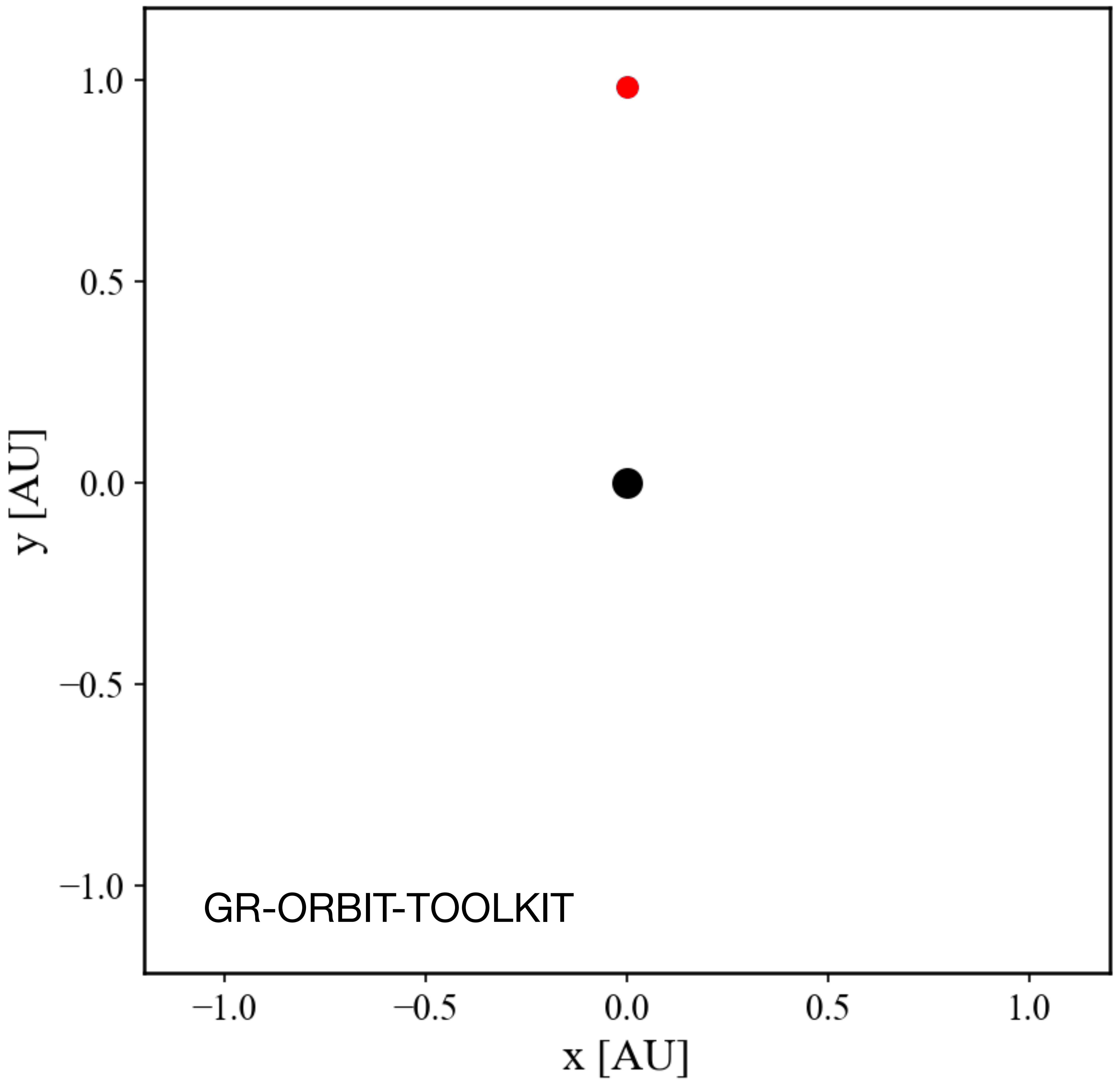
Newton:

In the 1600's Newton invents calculus / ordinary differential equations.

Using calculus he explains planetary dynamics (**Kepler's laws**).

Newton's solves the so-called **two-body problem.**

Newton's cannot solve the **three-body problem.**



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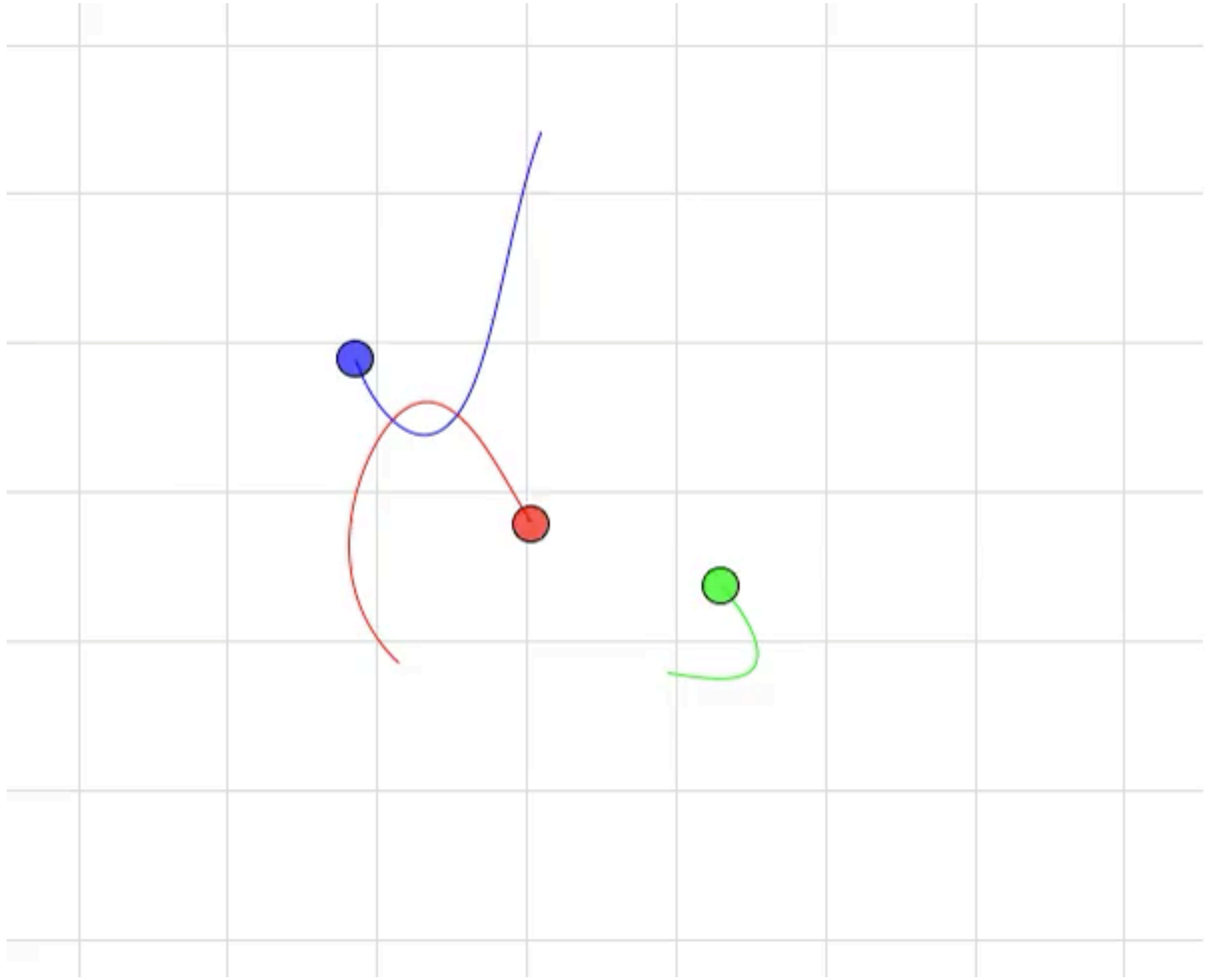
Poincaré:

Poincaré uses geometry and ‘demonstrates’ the 3-body problem cannot be solved!

In the late 1800’s, Poincaré “discovers” chaos.



What is chaos?

Evolution depends on the initial conditions.

It occurs in deterministic systems (no stochasticity.)

It is characterised by aperiodic seemingly unpredictable behaviour.

Credits GIF: https://www.reddit.com/r/physicsgifs/comments/2507k7/threebody_problem_simulation_2d/

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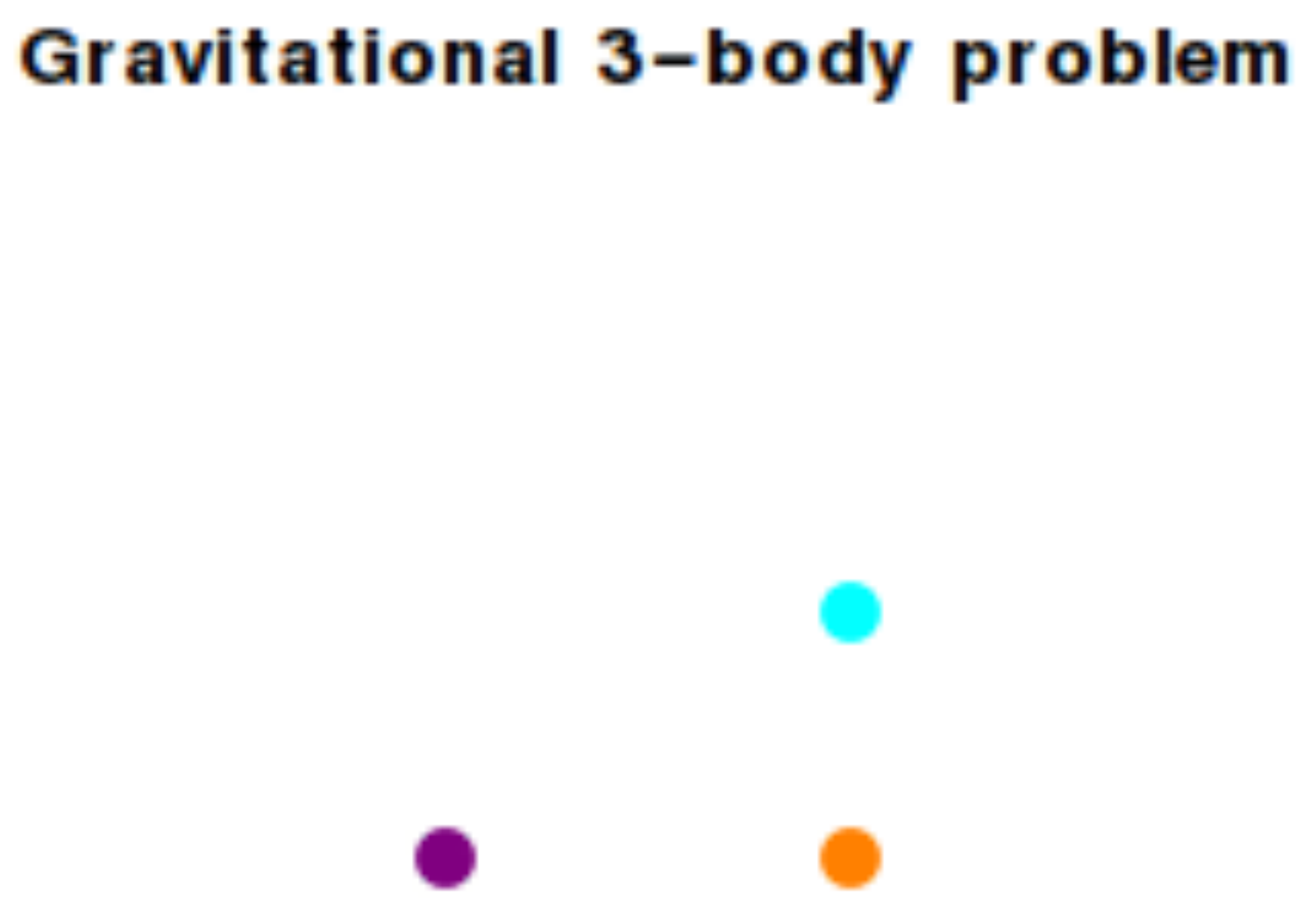
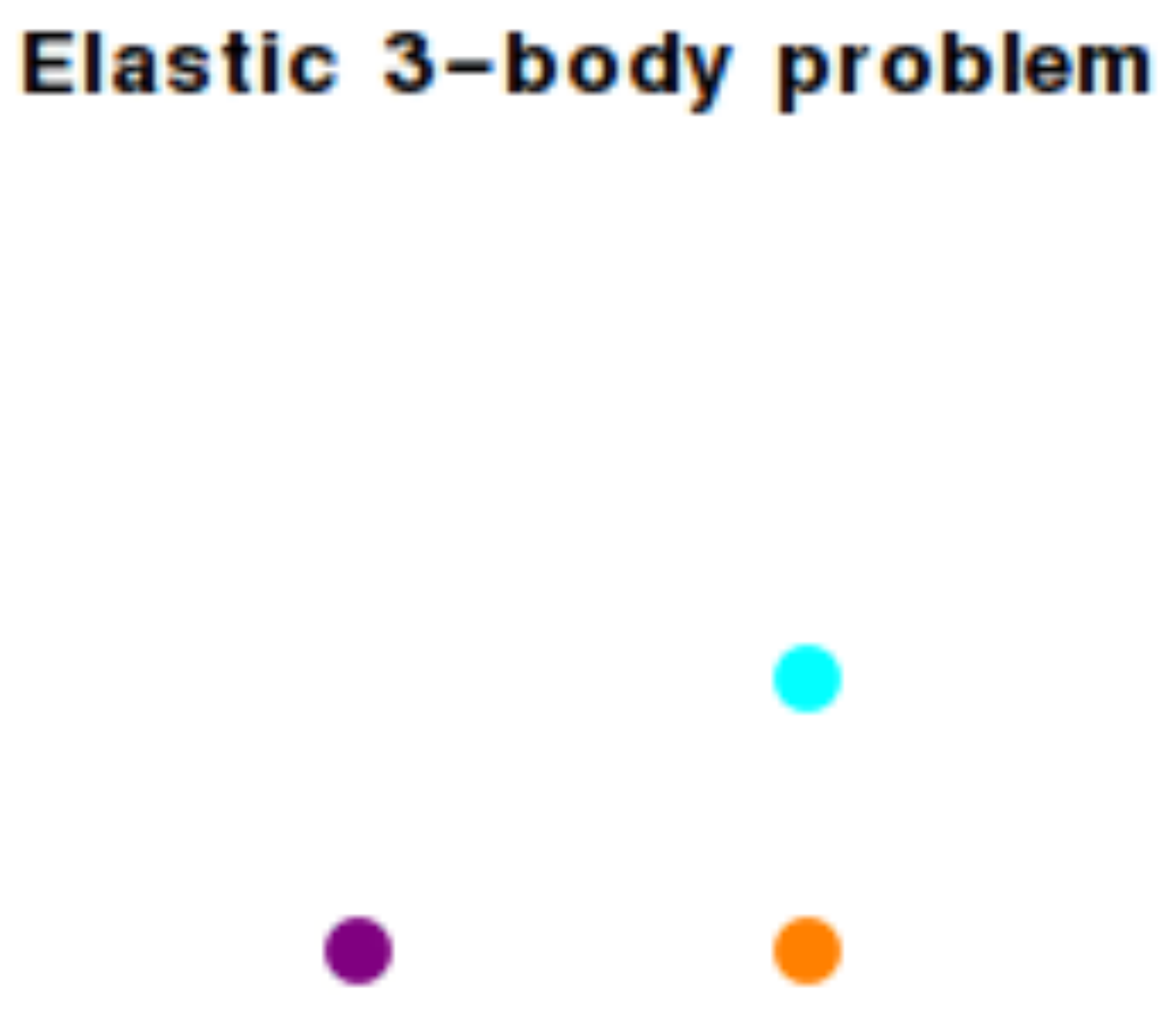
Non-linear oscillators:

In the first half of the 1900’s, non-linear oscillators are widely studied.

Behaviour is not governed by a simple, linear relationship between cause and effect.

Engineering: radio, radar, phase-locked loops, lasers (non-linear oscillations)





Credits GIF: https://commons.wikimedia.org/wiki/File:3bodyproblem.gif

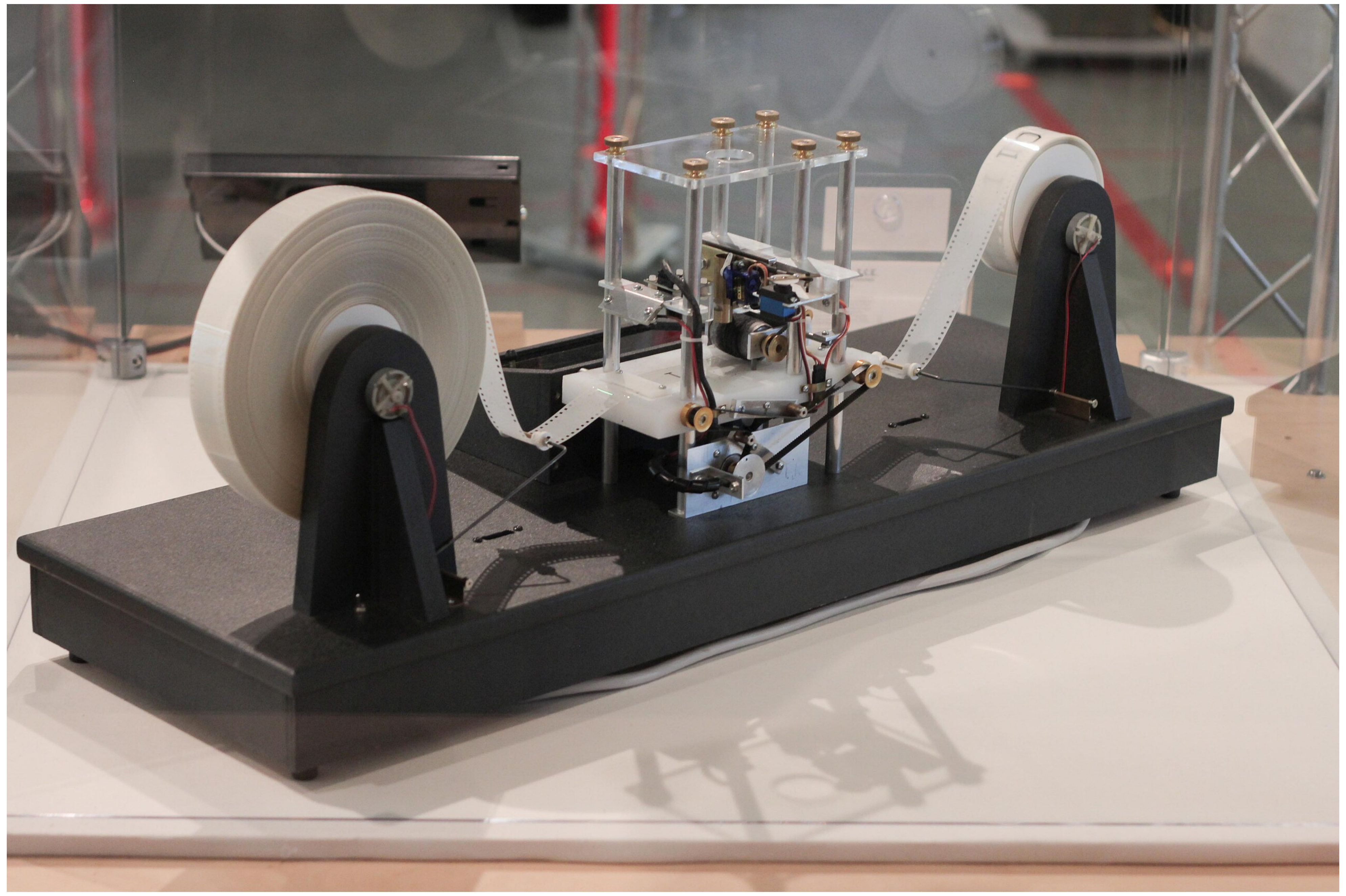
Historical Overview of Non-linear Dynamics

The advent of computers:

In the 1930's - 1950's the first computers are devised and invented.

Alan Turing described an abstract "universal machine" that could simulate human logic and sets the basis of modern computer science.

The Turing machine can manipulate symbols on a strip of tape according to a table of rules.



https://en.wikipedia.org/wiki/Turing_machine

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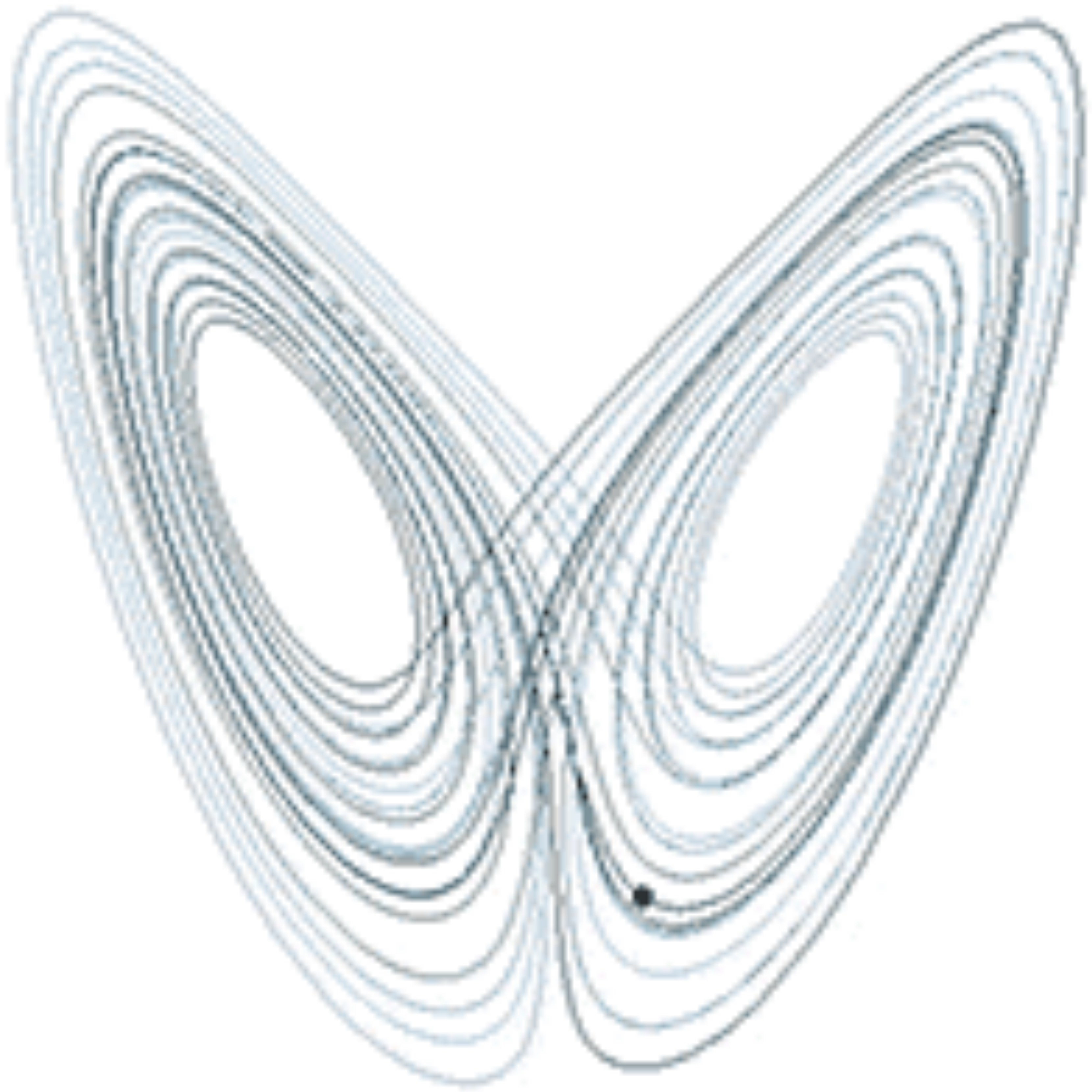
Lorenz

In the 1960's, Lorenz works on atmospheric convection and chaos.

Lorenz used a computer model to simulate weather patterns. A tiny change in the initial data led to a **completely different long-term weather forecast (1963).**

He described this phenomenon as the "butterfly effect".



Paper: https://journals.ametsoc.org/view/journals/atsc/20/2/1520-0469_1963_020_0130_dnf_2_0_co_2.xml

Historical Overview of Non-linear Dynamics

Li & Yorke:

In the 1970's, formally introduced the term in their paper "Period Three Implies Chaos" (https://www.its.caltech.edu/~matilde/LiYorke.pdf)

May:

In the 1970's, chaos was studied in **iterated maps.**

May studies the logistic equation in population biology, to model the growth of a population from one generation to the next.

$$x_{n+1} = f(x_n) = r\,x_n\,(1 - x_n)$$

where x_n is the population at a given time and r is a growth parameter. For large r the population would begin to oscillate between two values (https://www.nature.com/articles/261459a0).

Historical Overview of Non-linear Dynamics

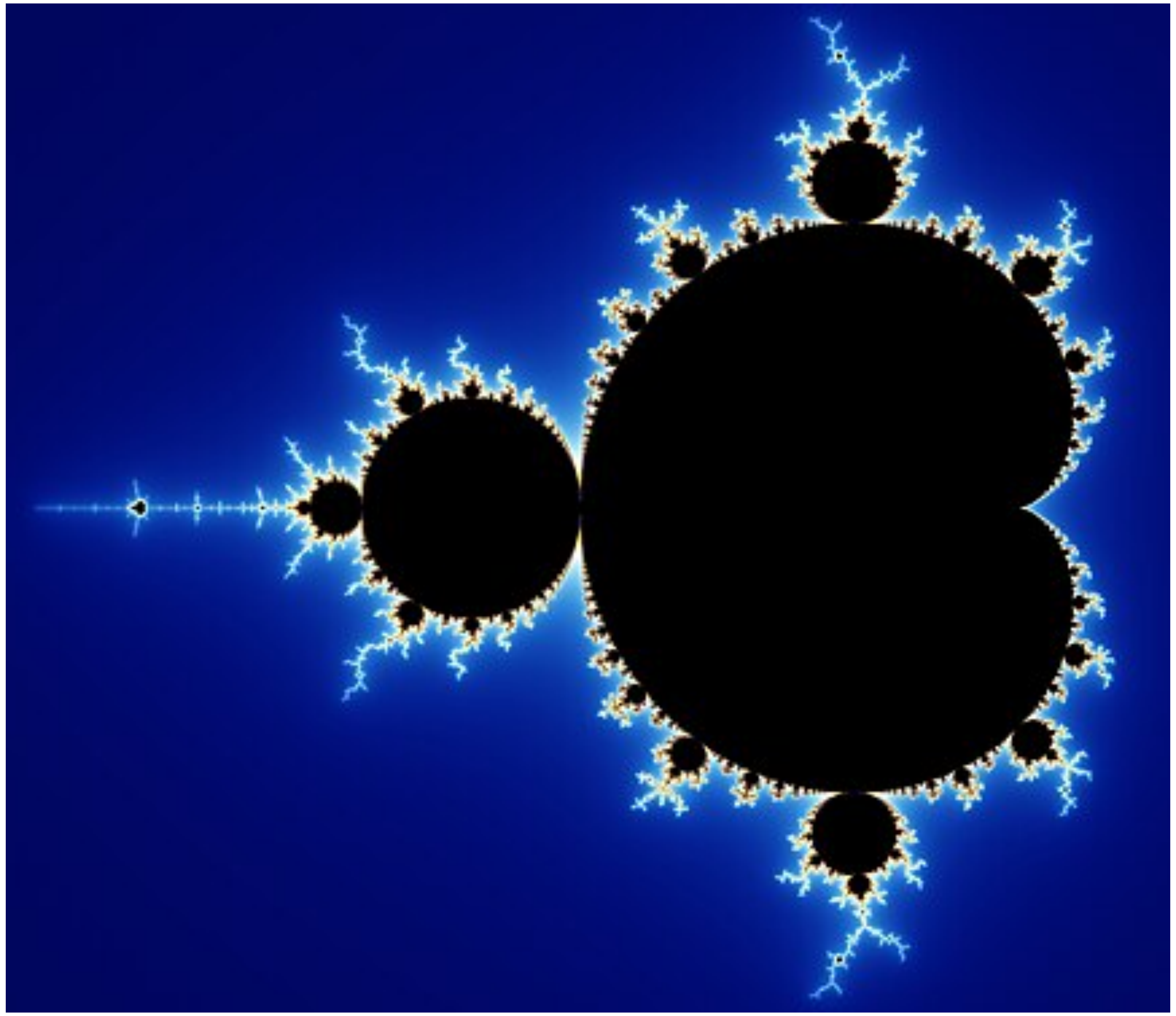
Mandelbrot:

He was interested in nature and geometry.

He introduced **fractal geometry** to describe the irregular and complex shapes found in nature that Euclidean geometry could not.

He coined the term "**fractal**" to refer to objects that exhibit **self-similarity**.

Using an IBM computer, he visualised the **Mandelbrot set.**



Credits: https://en.wikipedia.org/wiki/Mandelbrot_set

Historical Overview of Non-linear Dynamics

Winfree

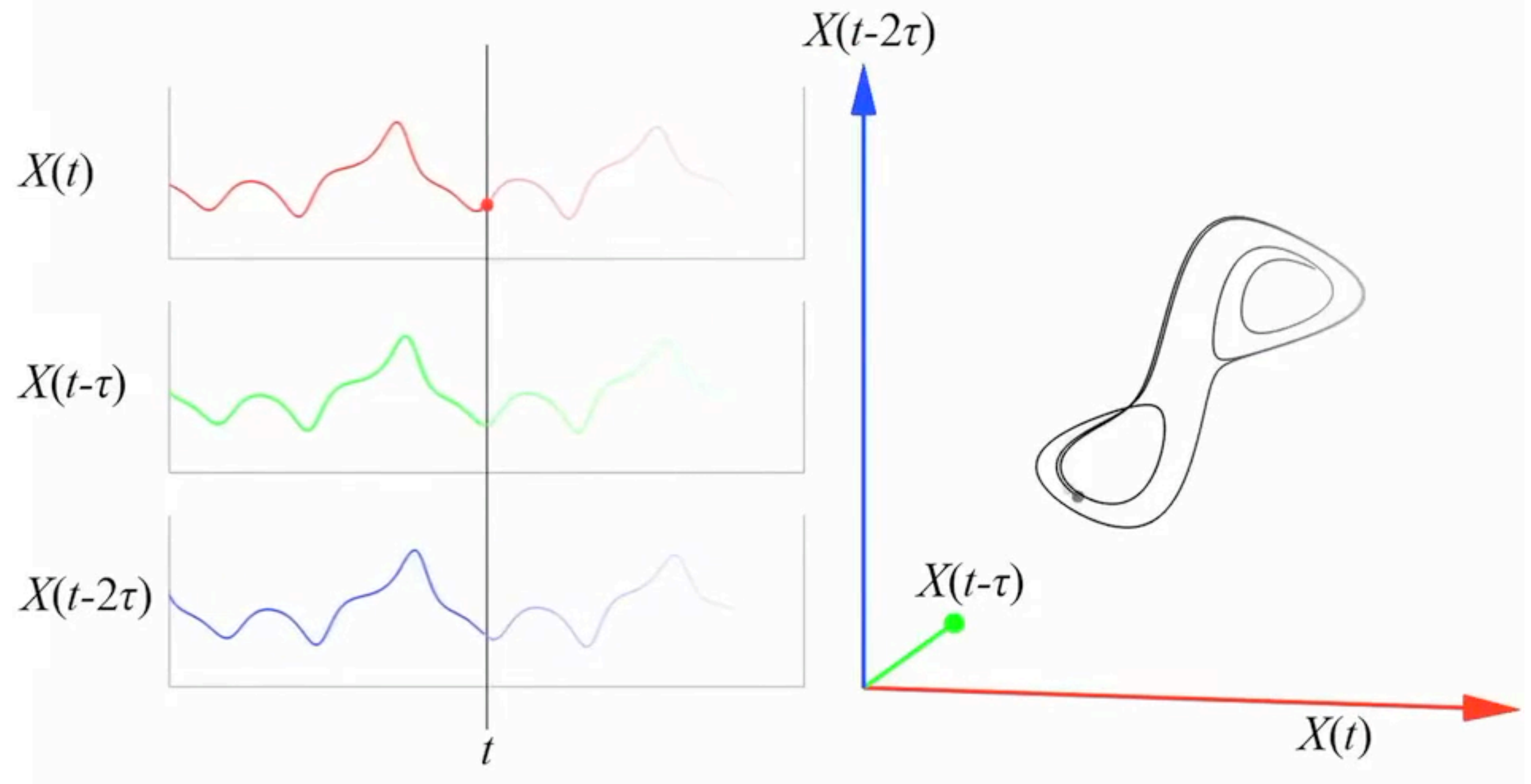
Non-linear oscillators in biology (oscillations and synchronisation in living systems).

His work provided a theoretical framework for understanding complex phenomena like cardiac arrhythmias.

Ruelle and Takens:

The problem of turbulence and chaos.

A smooth, orderly flow that undergoes just a few bifurcations (system's behaviour qualitatively changes) could become chaotic (**strange attractor**).



Historical Overview of Non-linear Dynamics

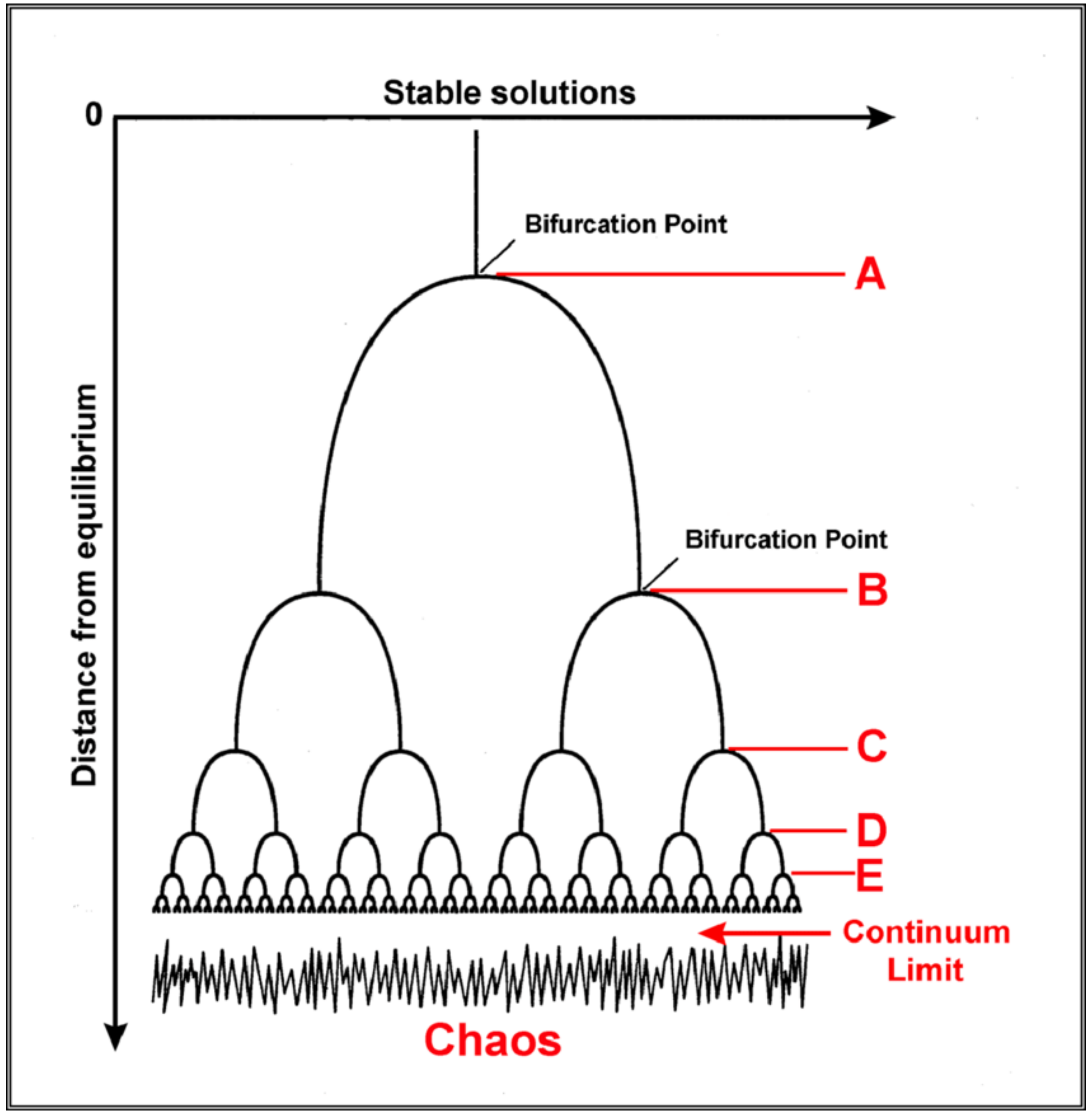
Feigenbaum

Progression into chaos from order (universal route to chaos).

Connection between phase transitions and statistical physics (renormalisation).

He studies the logistic map. By adding a parameter, he observed a predictable progression to chaos through a series of **period-doubling bifurcations.**

The ratio between the distances of these successive bifurcations converged to a constant number.



Historical Overview of Non-linear Dynamics

Boom of Chaos and Fractals:

In the 1980’s, non-linear dynamics becomes popular

Everyone is talking about fractals, chaos.

Experimental confirmation of some of the models.

Complex Systems

In the 1990’s, engineering applications of chaos.

Encoding private communications.

Complex systems emerge (millions of variables).

Historical Overview of Non-linear Dynamics

Network theory

In the 2000’s, complex systems and network theory.

Types of Dynamical Systems

Differential equations describe the evolution of systems in continuous time, whereas iterated maps arise in problems where time is discrete.

Differential equations are used much more widely in science and engineering, and we shall therefore concentrate on them.

We will see that iterated maps can also be very useful, both for providing simple examples of chaos, and also as tools for analysing periodic or chaotic solutions of differential equations.

Mathematical Overview of Non-linear Dynamics

Differential equations

Dynamics relies on differential equations:

$$\dot{x} = \frac{dx}{dt} = f(x)$$

$x \to \mathbb{R}^n$ (Phase space, state space)

$x = (x_1, x_2, \ldots\ldots, x_n)$ (Coordinates)

$f(x)$ (Linear/Non-linear Function)

In components, we have n-coupled ODEs:

$$\dot{x}_1 = f_1(x_1, \ldots\ldots, x_n)$$

$$\ldots\ldots$$

$$\dot{x}_n = f_n(x_1, \ldots\ldots, x_n)$$

Mathematical Overview of Non-linear Dynamics

Linear vs. Non-linear systems:

The system is linear if the RHS x_i appear to the first power

That means it is linear if there are no products, no powers, or $f(x_i)$.

Non-linear terms: $x_1^2, x_2 x_3, \cos(x_4)$.

Autonomous vs Non-autonomous systems:

Autonomous systems show no time dependence on the RHS.

Non-autonomous systems can be converted into autonomous ones.

Mathematical Overview of Non-linear Dynamics

Example of a linear system:

Simple Harmonic Oscillator: $$m\ddot{x} + kx = 0$$

Variable Change:
$$x_1 = x$$
$$x_2 = \dot{x}$$

Linear system in the form of $\dot{x} = f(x)$:

$$\dot{x}_1 = x_2$$

Linear 2nd-order system.

Two 1st-order ODEs coupled.

$$\dot{x}_2 = -k\frac{x_1}{m}$$

Mathematical Overview of Non-linear Dynamics

Example of a non-linear system:

Pendulum:

$$\ddot{x} + \sin(x) = 0$$

Variable Change:

$$x_1 = x$$

$$x_2 = \dot{x}$$

Linear system in the form of $\dot{x} = f(x)$:

$$\dot{x}_1 = x_2$$

Non-linear 2nd-order system.

$$\dot{x}_2 = -\sin(x_1)$$

Mathematical Overview of Non-linear Dynamics

Poincaré's approach:

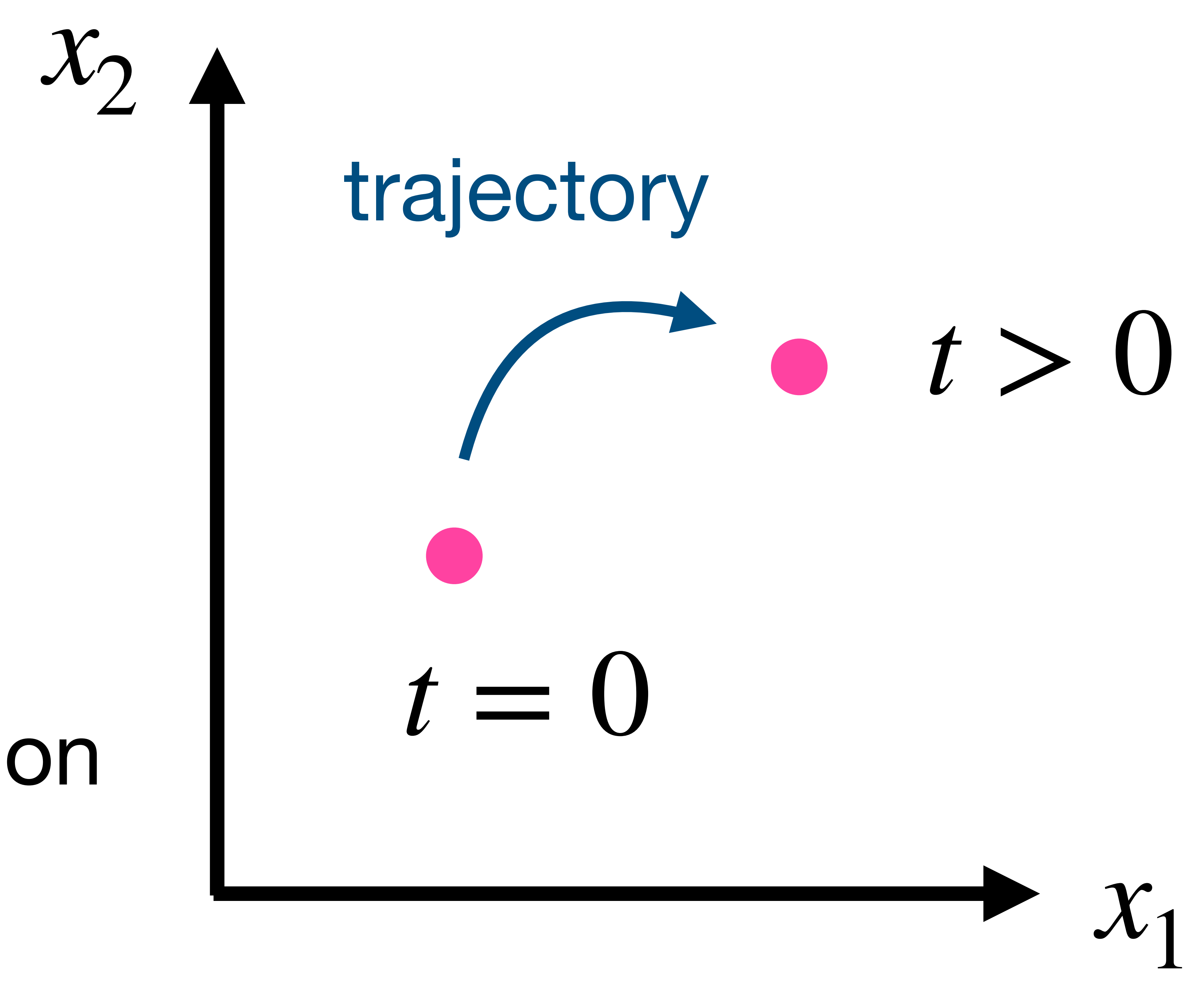
We image a solution: $\left[x_1(t), x_2(t)\right]$

This represents a point in a parametric space which moves along a trajectory.

Poincaré's ideas:

We can figure out the trajectories to gain information about the solutions.

Phase Space



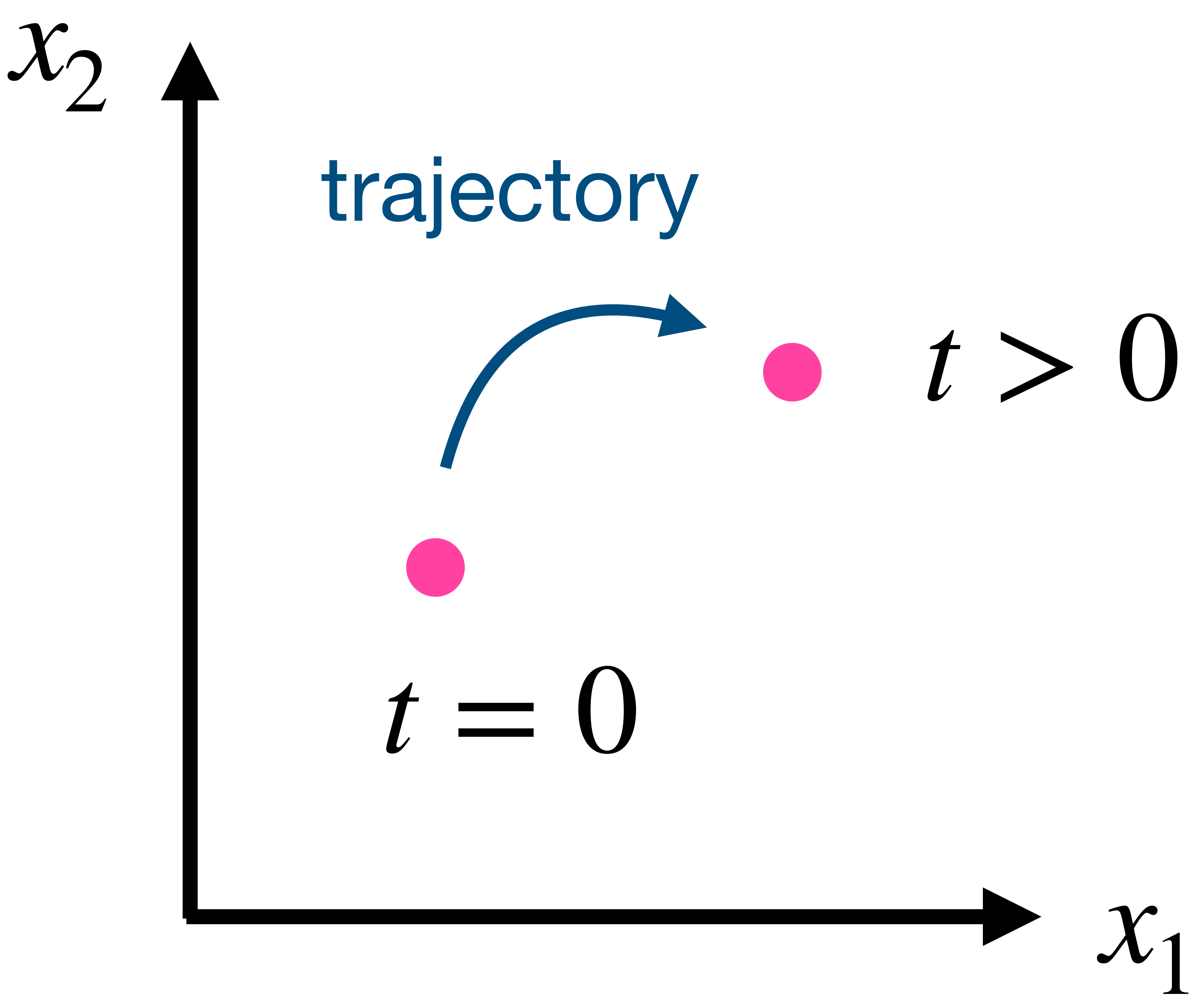
Mathematical Overview of Non-linear Dynamics

Phase Portrait:

It provides a picture of all qualitatively different trajectories.

We will find a solutions without solving the equations analytically.

Phase Space



Number of variables →

Nonlinearity ↓

	$n = 1$	$n = 2$	$n \geq 3$	$n >> 1$	Continuum
Linear	*Growth, decay, or equilibrium* Exponential growth RC circuit Radioactive decay	*Oscillations* Linear oscillator Mass and spring RLC circuit 2-body problem (Kepler, Newton)	Civil engineering, structures Electrical engineering	*Collective phenomena* Coupled harmonic oscillators Solid-state physics Molecular dynamics Equilibrium statistical mechanics	*Waves and patterns* Elasticity Wave equations Electromagnetism (Maxwell) Quantum mechanics (Schrödinger, Heisenberg, Dirac) Heat and diffusion Acoustics Viscous fluids
				The frontier	
Nonlinear	Fixed points Bifurcations Overdamped systems, relaxational dynamics Logistic equation for single species	Pendulum Anharmonic oscillators Limit cycles Biological oscillators (neurons, heart cells) Predator-prey cycles Nonlinear electronics (van der Pol, Josephson)	*Chaos* Strange attractors (Lorenz) 3-body problem (Poincaré) Chemical kinetics Iterated maps (Feigenbaum) Fractals (Mandelbrot) Forced nonlinear oscillators (Levinson, Smale) Practical uses of chaos Quantum chaos ?	Coupled nonlinear oscillators Lasers, nonlinear optics Nonequilibrium statistical mechanics Nonlinear solid-state physics (semiconductors) Josephson arrays Heart cell synchronization Neural networks Immune system Ecosystems Economics	*Spatio-temporal complexity* Nonlinear waves (shocks, solitons) Plasmas Earthquakes General relativity (Einstein) Quantum field theory Reaction-diffusion, biological and chemical waves Fibrillation Epilepsy Turbulent fluids (Navier-Stokes) Life

Reference: Strogatz Book