Nonlinear Dynamics and Chaos

PHYMSCFUN12

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Lorenz equations

We begin our study of chaos with the **Lorenz equations:**

$$\begin{aligned}
\dot{x} &= \sigma(y-x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz.
\end{aligned}$$

Here σ, r, $b>0$ are parameters.

Ed Lorenz (1963) derived this 3D system from a drastically simplified model of convection rolls in the atmosphere. The same equations also arise in models of lasers and dynamos.

They *exactly* describe the motion of a certain **waterwheel**.

Lorenz equations: a strange attractor and fractal

Lorenz discovered that this simple-looking deterministic system could have extremely erratic dynamics:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

Over a wide range of parameters, the solutions oscillate irregularly, never exactly repeating but always remaining in a bounded region of phase space.

When he plotted the trajectories in three dimensions, he discovered that they settled onto a complicated set, **now called a strange attractor.**

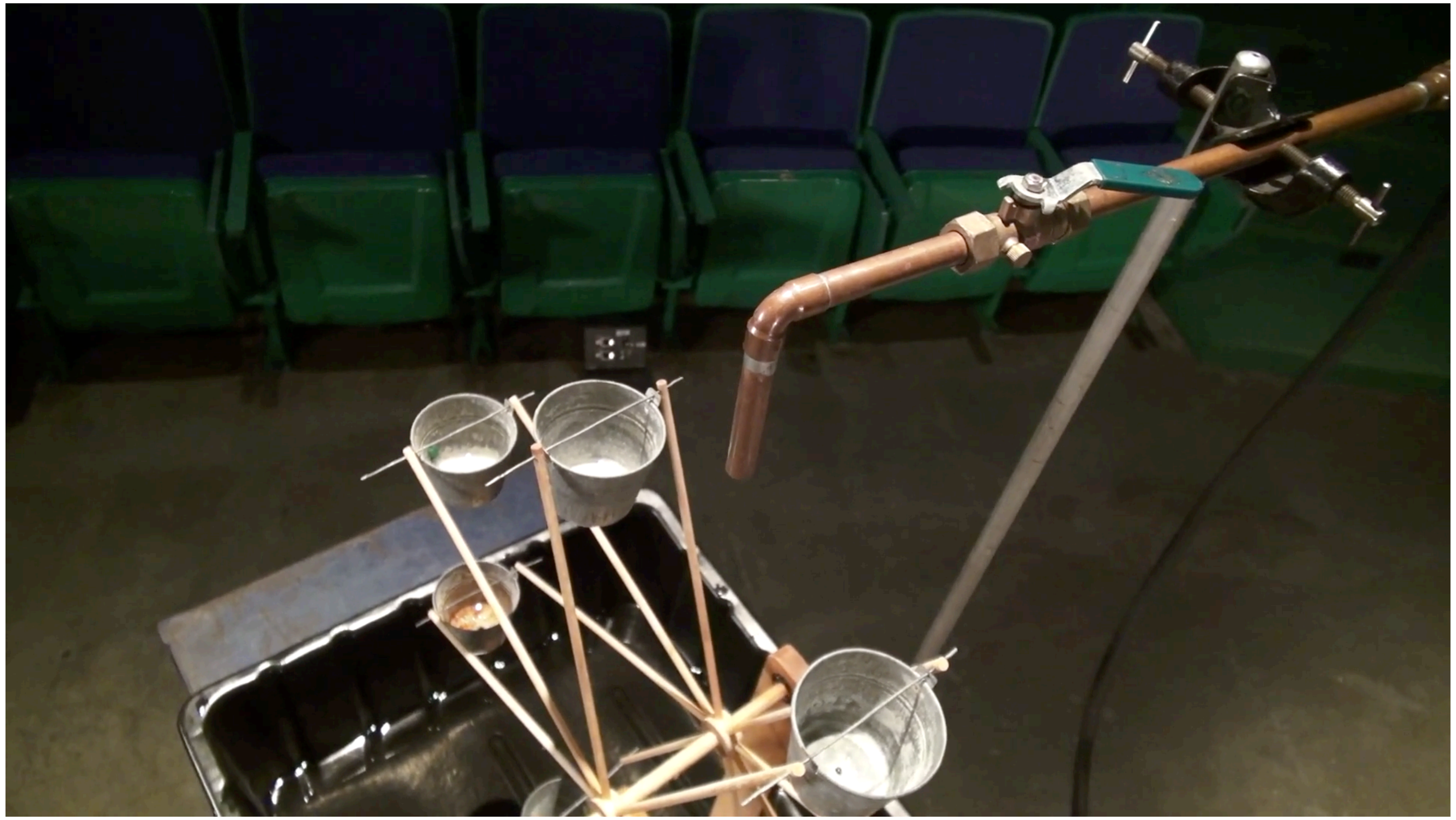
Unlike stable fixed points and limit cycles, the strange attractor is not a point or a curve or even a surface. It’s a **fractal**, with a fractional dimension between 2 and 3.

A Chaotic Waterwheel

A mechanical model of the Lorenz equations was invented by Malkus and Howard at MIT in the 1970s

The simplest version is a toy waterwheel with leaky paper cups suspended from its rim.

A Chaotic Waterwheel



https://www.youtube.com/watch?v=Lx8gMBJBlP8

Water is poured in steadily from the top. If the flow rate is too slow, the top cups never fill up enough to overcome friction, so the wheel remains motionless.

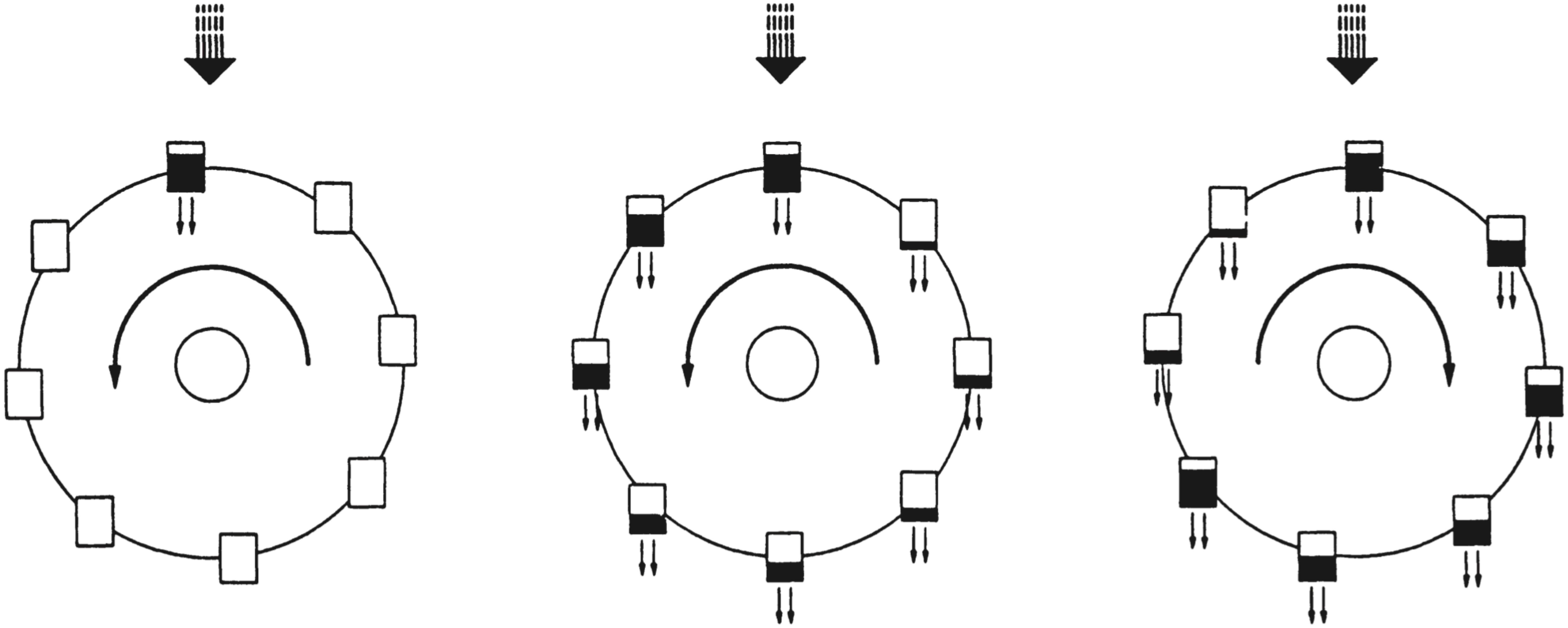
For faster inflow, the top cup gets heavy enough to start the wheel turning.

Eventually the wheel settles into a steady rotation in one direction or the other.

By symmetry, rotation in either direction is equally possible; the outcome depends on the initial conditions.

A Chaotic Waterwheel

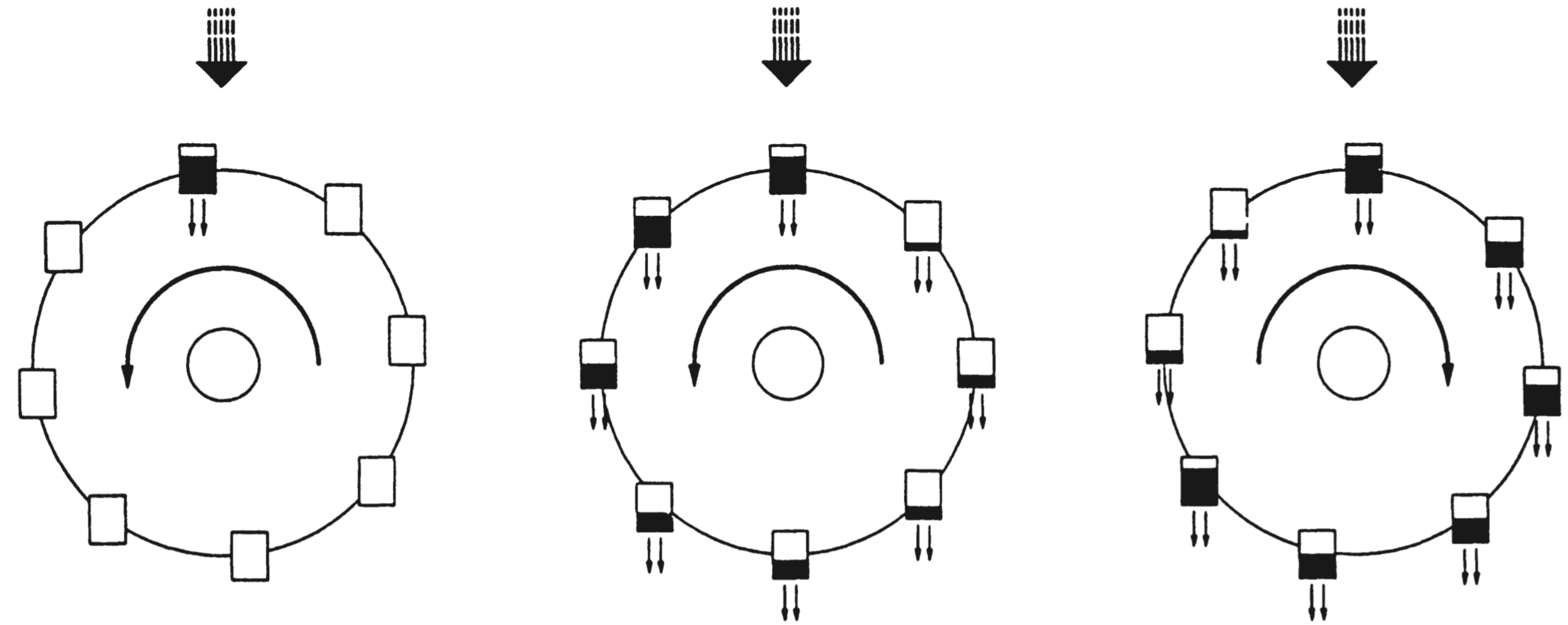
By increasing the flow rate still further, we can destabilise the steady rotation.



(a) (b) (c)

The motion becomes chaotic: the wheel rotates one way for a few turns, then some of the cups get too full and the wheel doesn't have enough inertia to carry them over the top, so the wheel slows down and may even reverse its direction.

A Chaotic Waterwheel



(a) (b) (c)

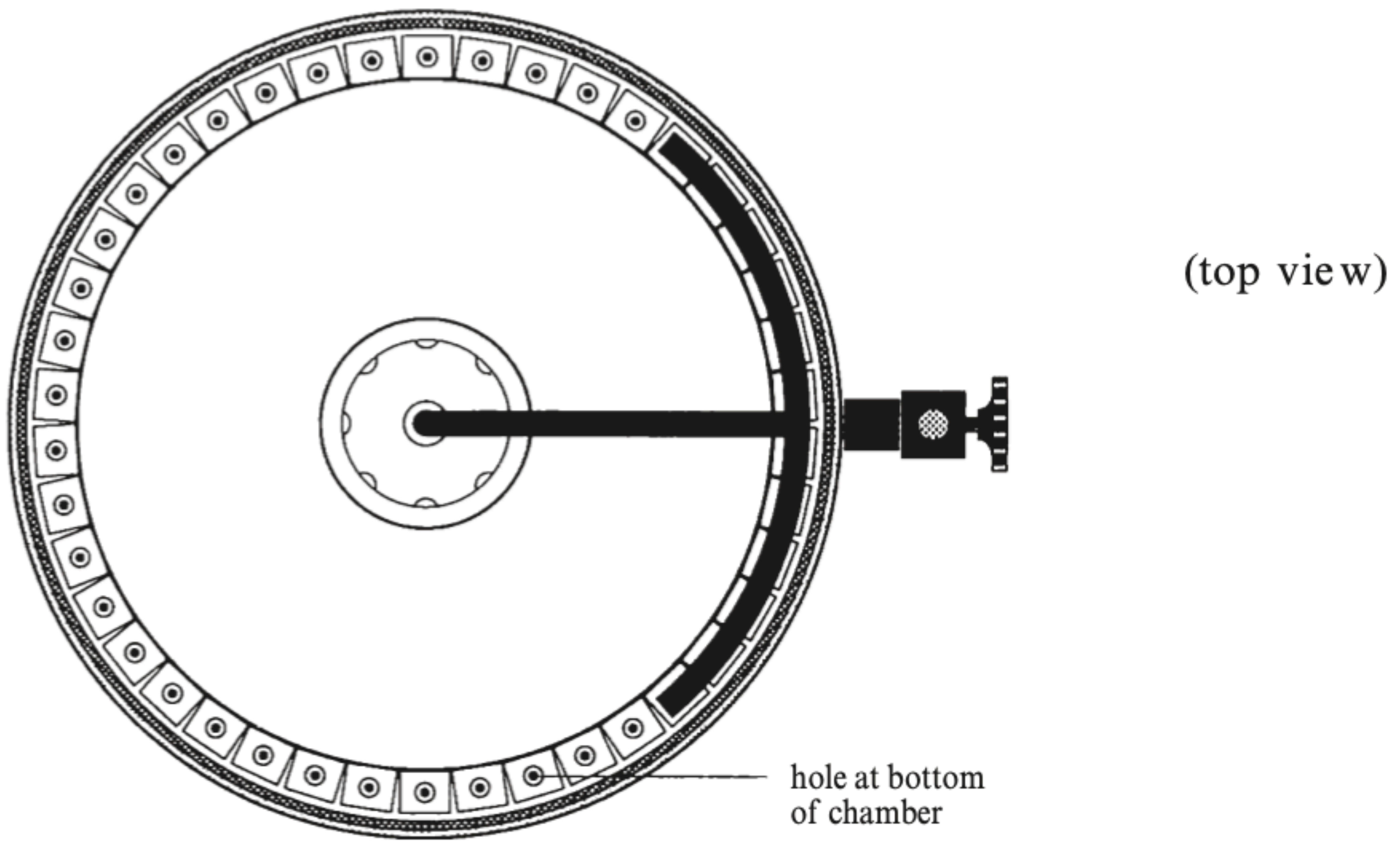
It spins the other way for a while. The wheel keeps changing direction erratically.

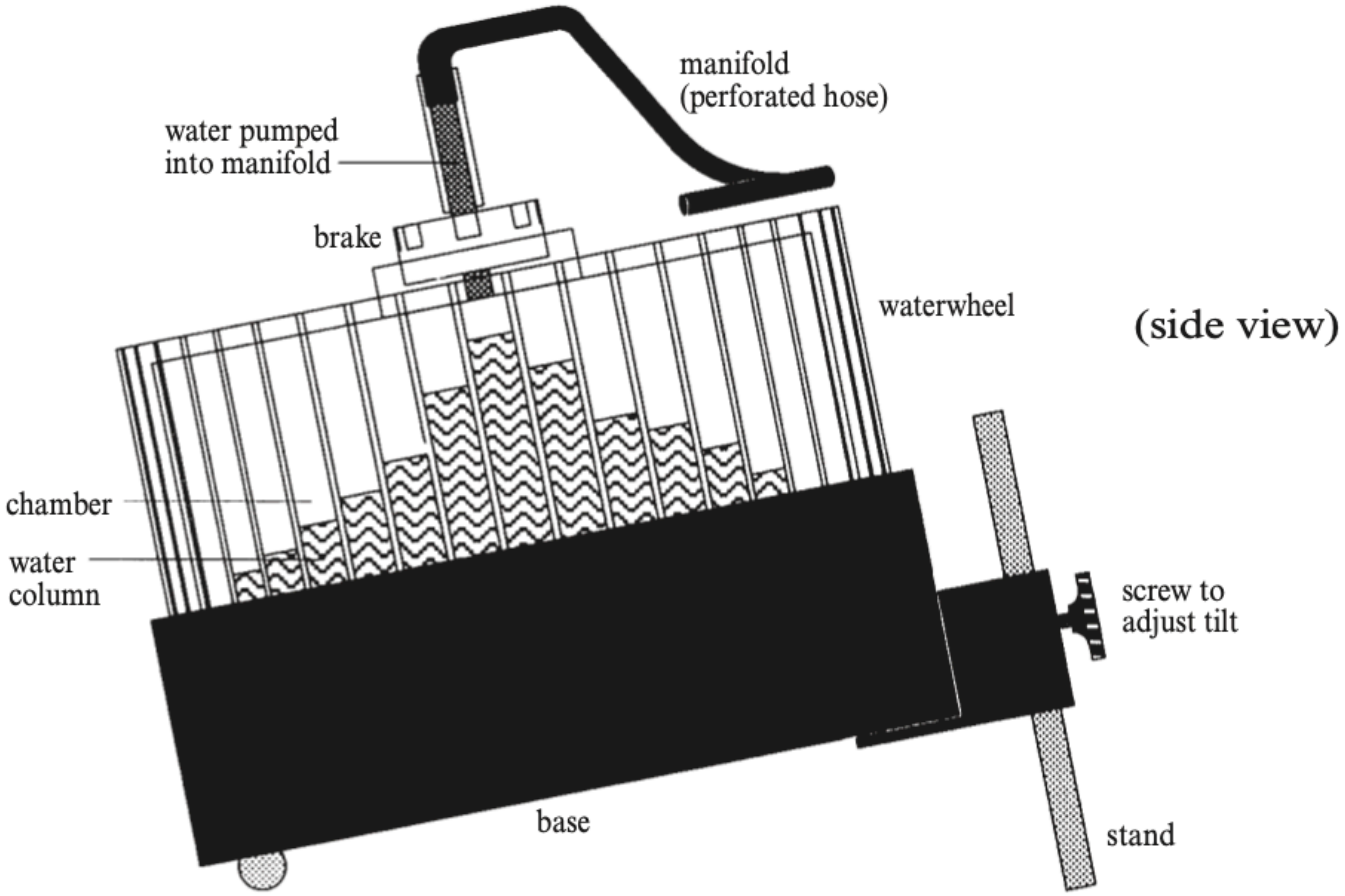
Malkus's Chaotic Waterwheel

The wheel sits on a table top. It rotates in a plane that is tilted slightly from the horizontal.

Water is pumped up into an overhanging manifold and then sprayed out through dozens of small nozzles. The nozzles direct the water into separate chambers around the rim of the wheel.

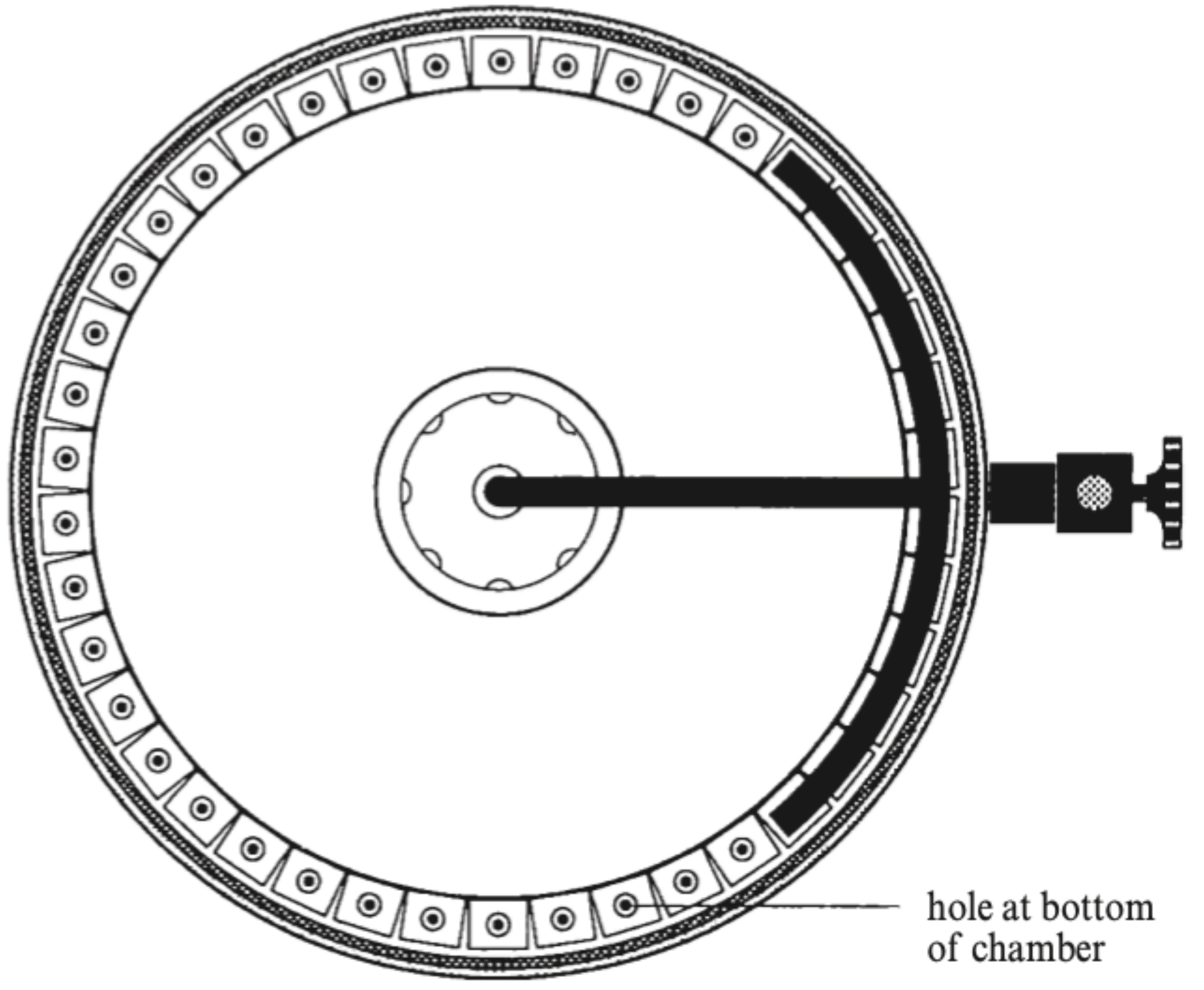
The chambers are transparent, and the water has food colouring in it, so the distribution of water around the rim is easy to see.





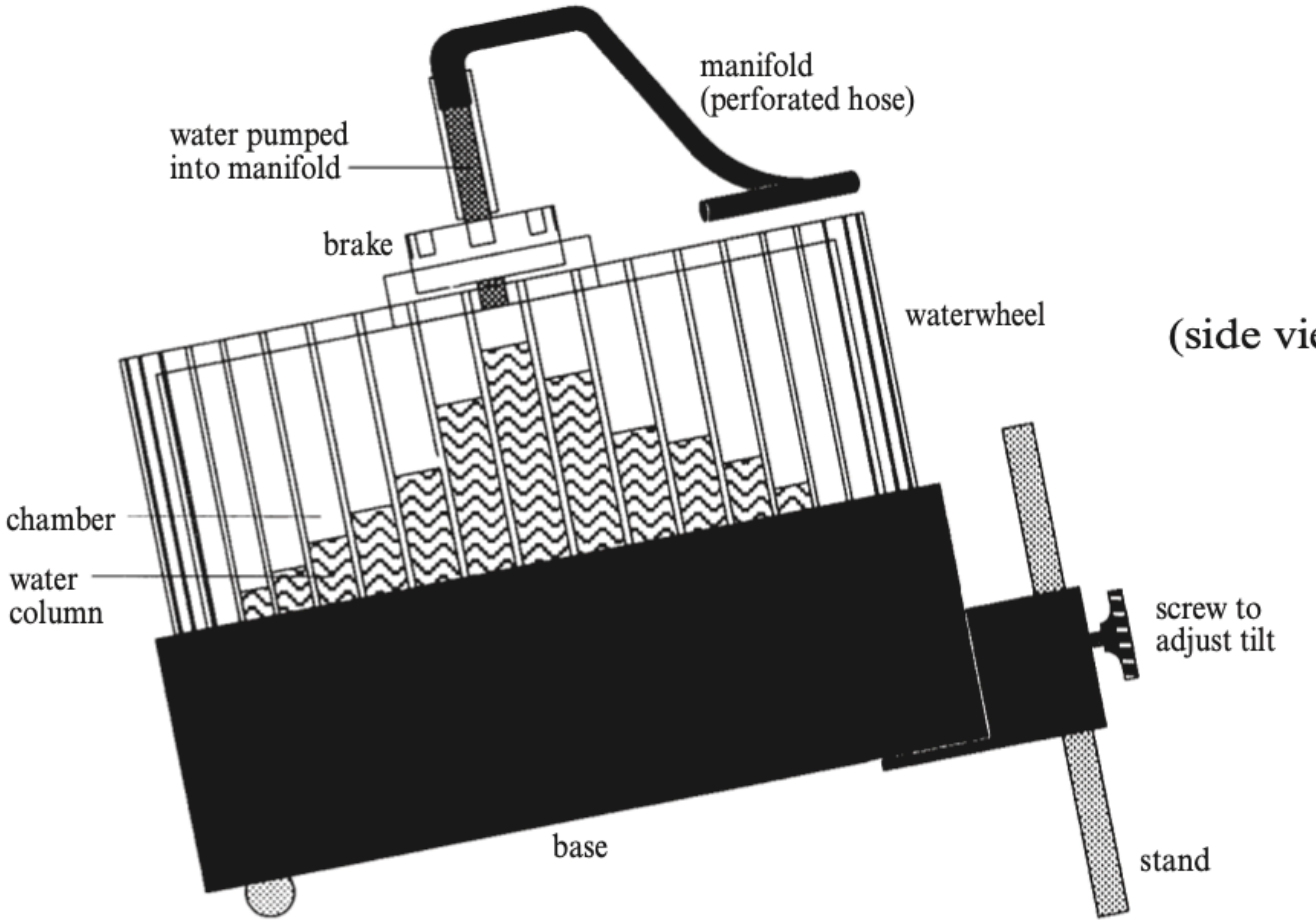
Malkus's Chaotic Waterwheel

The water leaks out through a small hole at the bottom of each chamber, and then collects underneath the wheel, where it is pumped back up through the nozzles. This system provides a steady input of water.



(top view)

The parameters can be changed in two ways. A **brake on the wheel** can be adjusted to add more or less friction. The **tilt of the wheel** can be varied by turning a screw that props the wheel up; this alters the effective strength of gravity.

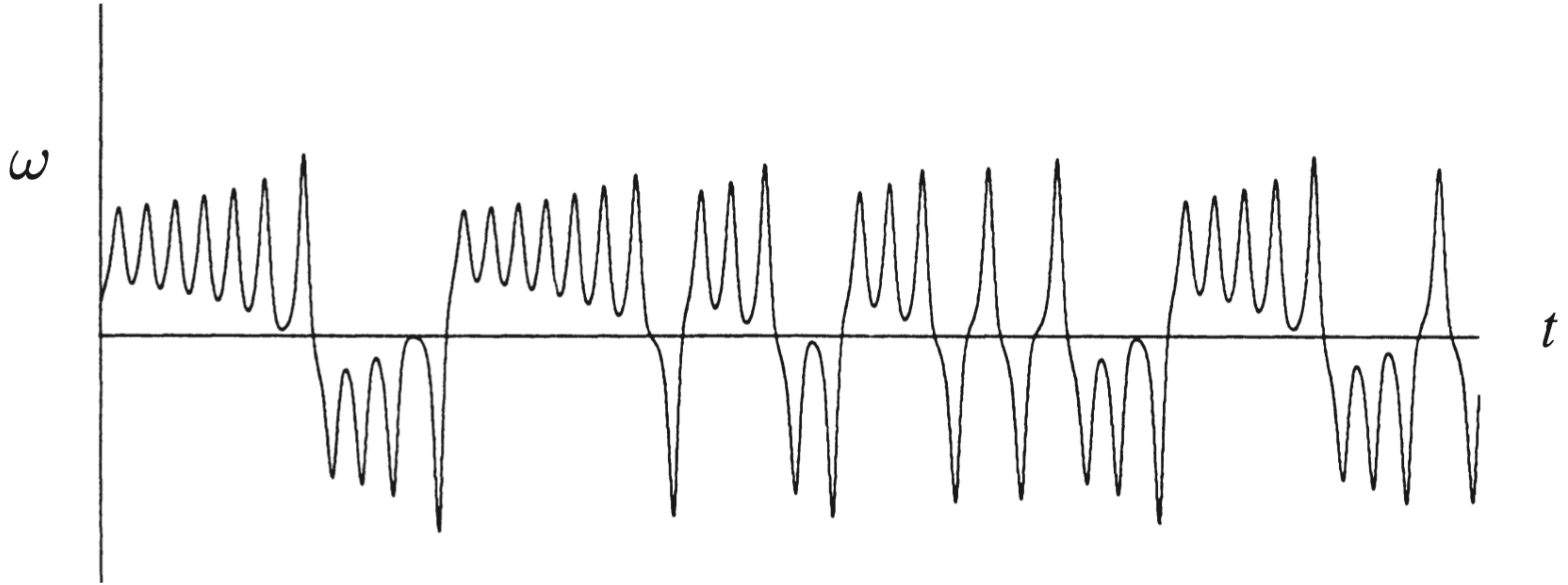


(side view)

Malkus's Chaotic Waterwheel

A sensor measures the wheel's angular velocity $\omega(t)$, and sends the data to a strip chart recorder which then plots $\omega(t)$ in real time.

The Figure on the right shows a record of $\omega(t)$ when the wheel is rotating chaotically. Notice once again the irregular sequence of reversals.



Malkus's Chaotic Waterwheel

We want to explain where this chaos comes from, and to understand the bifurcations that cause the wheel to go from static equilibrium to steady rotation to irregular reversals.

Conservation of mass

Torque balance

Amplitude Equations

Fourier series

The resulting equations are in a closed system:

$$\begin{aligned}\dot{a}_1 &= \omega b_1 - K a_1 \\ \dot{b}_1 &= -\omega a_1 - K b_1 + q_1 \\ \dot{\omega} &= (-v\omega + \pi g r a_1)/I\end{aligned}$$

Equivalent to:

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz.\end{aligned}$$

Malkus's Chaotic Waterwheel

$$\dot{a}_1 = \omega b_1 - K a_1$$
$$\dot{b}_1 = -\omega a_1 - K b_1 + q_1$$
$$\dot{\omega} = (-v\omega + \pi g r a_1)/I$$

$r =$ radius of the wheel
$K =$ leakage rate
$v =$ rotational damping rate
$I =$ moment of inertia of the wheel

a_1: First Fourier coefficient of the **mass distribution** of water around the wheel It measures the **cosine component** of how much water is on the wheel. Physically, how much heavier the right side is compared to the left.

b_1: Second (sine) Fourier coefficient of the **mass distribution** Measures the **sine component** of water imbalance. Physically, how much heavier the *top* is compared to the bottom.

q_1: Fourier component of the **water inflow rate** (captures how water is being poured in at the top).

Chaotic Waterwheel: Fixed points

$$a_1 = \omega b_1 / K$$
$$\omega a_1 = q_1 - K b_1$$
$$a_1 = v\omega / \pi g r.$$

$$\Rightarrow \quad b_1 = \frac{K q_1}{\omega^2 + K^2}$$

$$\omega b_1 / K = v\omega / \pi g r. \quad \Rightarrow \quad \omega = 0, \quad b_1 = K v / \pi g r.$$

There are 2 kinds of fixed points:

1) $\omega = 0$ $\quad a_1 = 0$, $\quad b_1 = q_1 / K.$ $\quad \Rightarrow \quad (a_1^*, b_1^*, \omega^*) = (0,\ q_1/K, 0)$

This corresponds to a state of **no rotation.** The wheel is at rest, with inflow balanced by leakage.

Chaotic Waterwheel: Fixed points

2) $\omega \neq 0$ ➡ $b_1 = Kq_1/(\omega^2 + K^2) = Kv/\pi gr.$

$K \neq 0$ ➡ $q_1/(\omega^2 + K^2) = v/\pi gr.$

➡ $$(\omega^*)^2 = \frac{\pi g r q_1}{v} - K^2$$

If the RHS is positive, there are two solutions: $\pm\omega^*$ corresponding to steady rotation in either direction.

Condition -> Rayleigh number $\dfrac{\pi g r q_1}{K^2 v} > 1$

Chaotic Waterwheel: Rayleigh number

Rayleigh number $$\frac{\pi g r q_1}{K^2 v} > 1$$

Measurement of how hard we're driving the system, relative to the dissipation.

This ratio expresses a competition between g and q_1 (gravity and inflow, which tend to spin the wheel), and K and v (leakage and damping, which tend to stop the wheel).

Steady rotation is possible only if the Rayleigh number is large enough.

Rayleigh number and analogy to convection

The Rayleigh number appears in other parts of **fluid mechanics**, notably convection, in which a layer of fluid is heated from below. It is proportional to the difference in temperature from bottom to top.

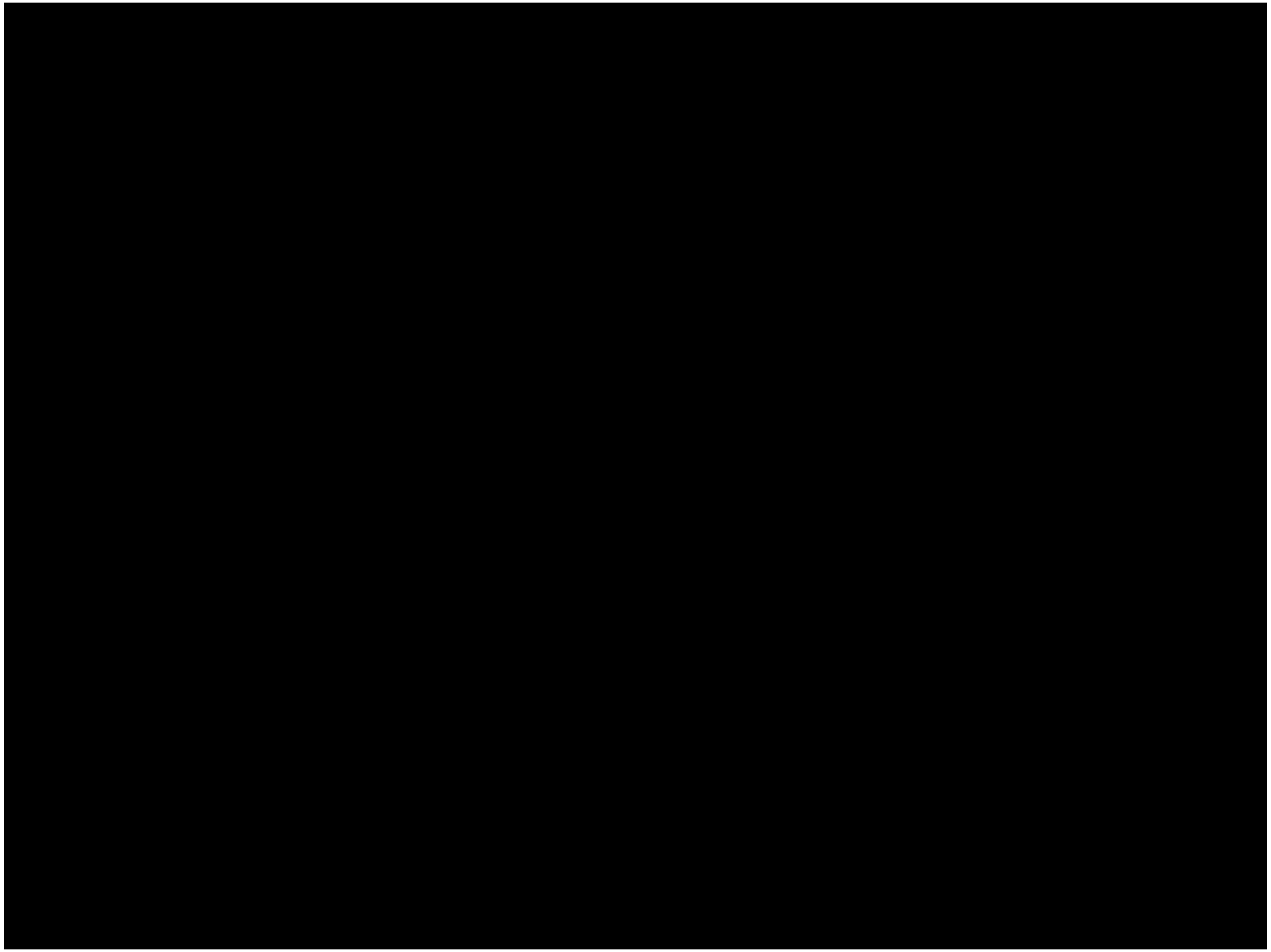
For small temperature gradients, heat is conducted vertically but the fluid remains motionless.

When the Rayleigh number increases past a critical value, an instability occurs: **the hot fluid is less dense and begins to rise, while the cold fluid on top begins to sink.**

This sets up a pattern of **convection rolls**, completely analogous to the steady rotation of our waterwheel.

With further increases of the Rayleigh number, **the rolls become wavy and eventually chaotic.**

Rayleigh number and analogy to convection



The analogy to the waterwheel breaks down at still higher Rayleigh numbers, when **turbulence develops and the convective motion becomes complex** in space as well as time.

The waterwheel settles into a **pendulum-like pattern of reversals**, turning once to the left, then back to the right, and so on indefinitely.

https://www.youtube.com/watch?v=M-yXOEsDivs

Properties of the Lorenz Equations

Lorenz took the analysis as far as possible using standard techniques, but at a certain stage **he found himself confronted with what seemed like a paradox.**

One by one he had eliminated all the known possibilities for the long-term behaviour of his system:

In a certain range of parameters, there could be **no stable fixed points** and **no stable limit cycles**, yet he also proved that **all trajectories remain confined to a bounded region and are eventually attracted to a set of zero volume.**

What could that set be? And how do the trajectories move on it?

That set is the **strange attractor**, and the **motion on it is chaotic.**

Properties of the Lorenz Equations

We will see how Lorenz ruled out the more traditional possibilities.

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

Here σ, r, $b > 0$ are parameters: σ is the ***Prandtl number***, r is the Rayleigh number, and b has no name.

P1) Nonlinearity

There are two nonlinearities, the quadratic terms xy and xz.

Properties of the Lorenz Equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

P2) Symmetry

If we replace $(x, y) \rightarrow (-x, -y)$, the equations stay the same.

If $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$. In other words, all solutions are either symmetric themselves, or have a symmetric partner.

P3) Volume Contraction

The Lorenz system is dissipative: volumes in phase space contract under the flow.

Properties of the Lorenz Equations

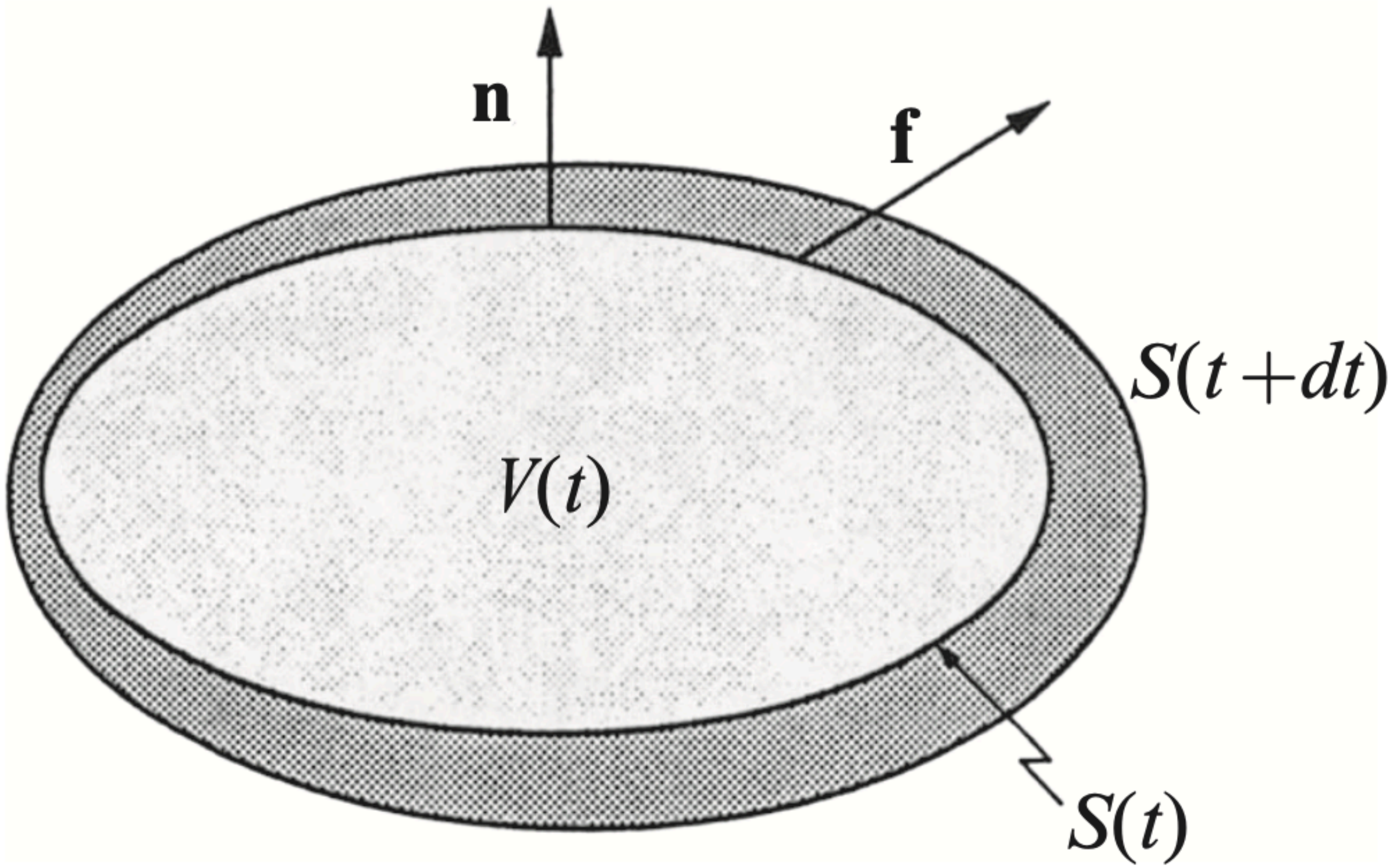
P3) Volume Contraction

The Lorenz system is dissipative: volumes in phase space contract under the flow.

How do volumes evolve?

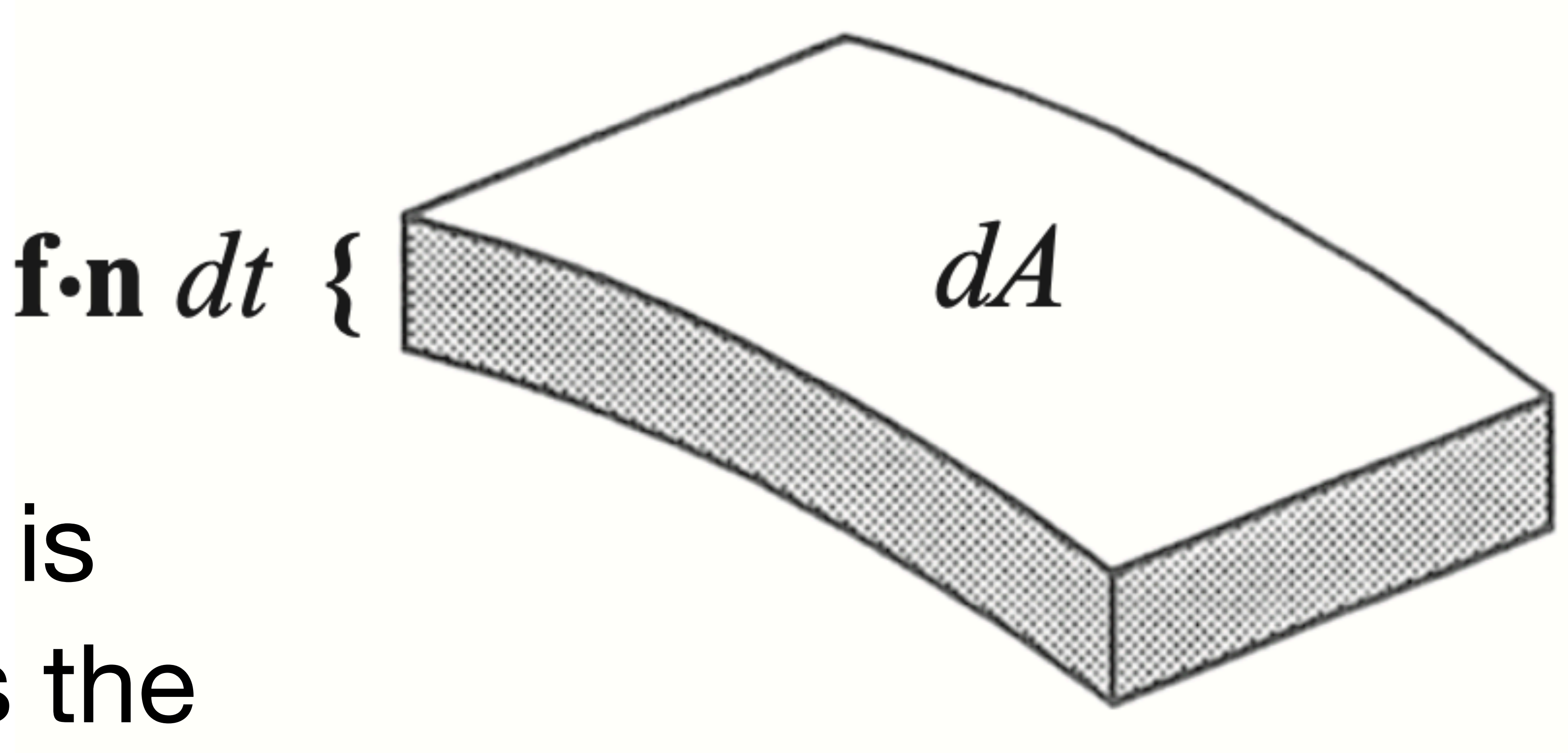
For any 3D system: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, we pick an arbitrary closed surface $S(t)$ of volume $V(t)$ in phase space.

$S(t)$ evolves into a new surface $S(t + dt)$, **what is its volume** $V(t + dt)$**?**



Properties of the Lorenz Equations

P3) Volume Contraction



Let $\mathbf{n}$ denote the outward normal on *S.* Since $\mathbf{f}$ is the instantaneous velocity of the points, $\mathbf{f} \cdot \mathbf{n}$ is the outward normal component of velocity.

Therefore in time dt a patch of area dA sweeps out a volume: $(\mathbf{f} \cdot \mathbf{n}\, dt)\, dA$

$$V(t+dt) = V(t) + \int_S (\mathbf{f} \cdot \mathbf{n}\, dt)\, dA.$$

$$\dot{V} = \frac{V(t+dt) - V(t)}{dt} = \int_S \mathbf{f} \cdot \mathbf{n}\, dA.$$

Properties of the Lorenz Equations

Using the divergence theorem:

$$\dot{V} = \frac{V(t+dt)-V(t)}{dt} = \int_S \mathbf{f}\cdot\mathbf{n}\,dA. \quad \Rightarrow \quad \dot{V} = \int_V \nabla\cdot\mathbf{f}\,dV.$$

For the Lorenz system:

$$\begin{aligned}\nabla\cdot\mathbf{f} &= \frac{\partial}{\partial x}[\sigma(y-x)] + \frac{\partial}{\partial y}[rx-y-xz] + \frac{\partial}{\partial z}[xy-bz] \\ &= -\sigma-1-b<0.\end{aligned}$$

Since the divergence is constant: $\dot{V} = -(\sigma+1+b)V$

Properties of the Lorenz Equations

Since the divergence is constant: $\dot{V} = -(\sigma + 1 + b)V$,

With solutions: $V(t) = V(0)e^{-(\sigma + 1 + b)t}$

This means **volumes in phase space shrink exponentially fast.**

If we start with an enormous solid blob of initial conditions, it eventually shrinks to a limiting set of zero volume.

All trajectories starting in the blob end up somewhere in this limiting set.

Later we'll see it consists of **fixed points, limit cycles,** or for some parameter values, **a strange attractor.**

Volume contraction imposes strong constraints on the possible solutions of the Lorenz equations.

Properties of the Lorenz Equations

P4) No quasi-periodic solutions

If there were a quasi-periodic solution, it would have to lie on the surface of a torus, as discussed, and this torus would be invariant under the flow.

Hence the volume inside the torus would be constant in time. But this contradicts the fact that all volumes shrink exponentially fast.

P5) Neither repelling fixed points nor repelling orbits

Repellers are incompatible with volume contraction because they are sources of volume.

If we encase a repeller with a closed surface of initial conditions nearby in phase space, the surface will expand as the corresponding trajectories are driven away. **Thus the volume inside the surface would increase.**

Fixed points of the Lorenz Equations

We conclude that all fixed points must be sinks or saddles, and closed orbits (if they exist) must be stable or saddle-like.

The Lorenz system has **two types of fixed points:**

1) The origin $(x^*, y^*, z^*) = (0,0,0)$ is a fixed point for all values of the parameters.

2) For $r > 1$, there is also a symmetric pair of fixed points, $C^{\pm}$:

$$x^* = y^* = \pm\sqrt{b(r-1)} \qquad\qquad z^* = r-1$$

They represent left- or right-turning convection rolls (analogous to the steady rotations of the waterwheel). As $r \to 1^+$, C^+ **and C^- coalesce with the origin in a pitchfork bifurcation.**

Linear Stability of the Origin

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

The linearisation at the origin is:

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = rx - y,$$
$$\dot{z} = -bz,$$

The equation for z is decoupled and shows that $z(t) \to 0$ exponentially fast.

The other two directions are governed by the system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Trace: $\tau = -\sigma - 1 < 0$

Determinant: $\Delta = \sigma(1 - r).$

Linear Stability of the Origin

$$\tau = -\sigma - 1 < 0$$

$$\Delta = \sigma(1 - r)$$

If $r > 1$, the origin is a 3D saddle point because $\Delta < 0$.

Including the decaying z-direction, the saddle has **one outgoing and two incoming directions.**

If $r < 1$, all directions are incoming and the **origin is a sink.**

Since: $$\tau^2 - 4\Delta = (\sigma + 1)^2 - 4\sigma(1 - \mathrm{r}) = (\sigma - 1)^2 + 4\sigma r > 0$$

So, **the origin is a stable node for $r < 1$.**

Global Stability of the Origin

For $r < 1$, every trajectory approaches the origin as $t \to \infty$, so **the origin is globally stable.** There can be **no limit cycles or chaos** for $r < 1$.

Proof constructing a Liapunov function

A Liapunov function is a generalisation of an energy function for a classical mechanical system, in the presence of friction or other dissipation, the energy decreases monotonically.

A smooth, positive definite function that decreases along trajectories.

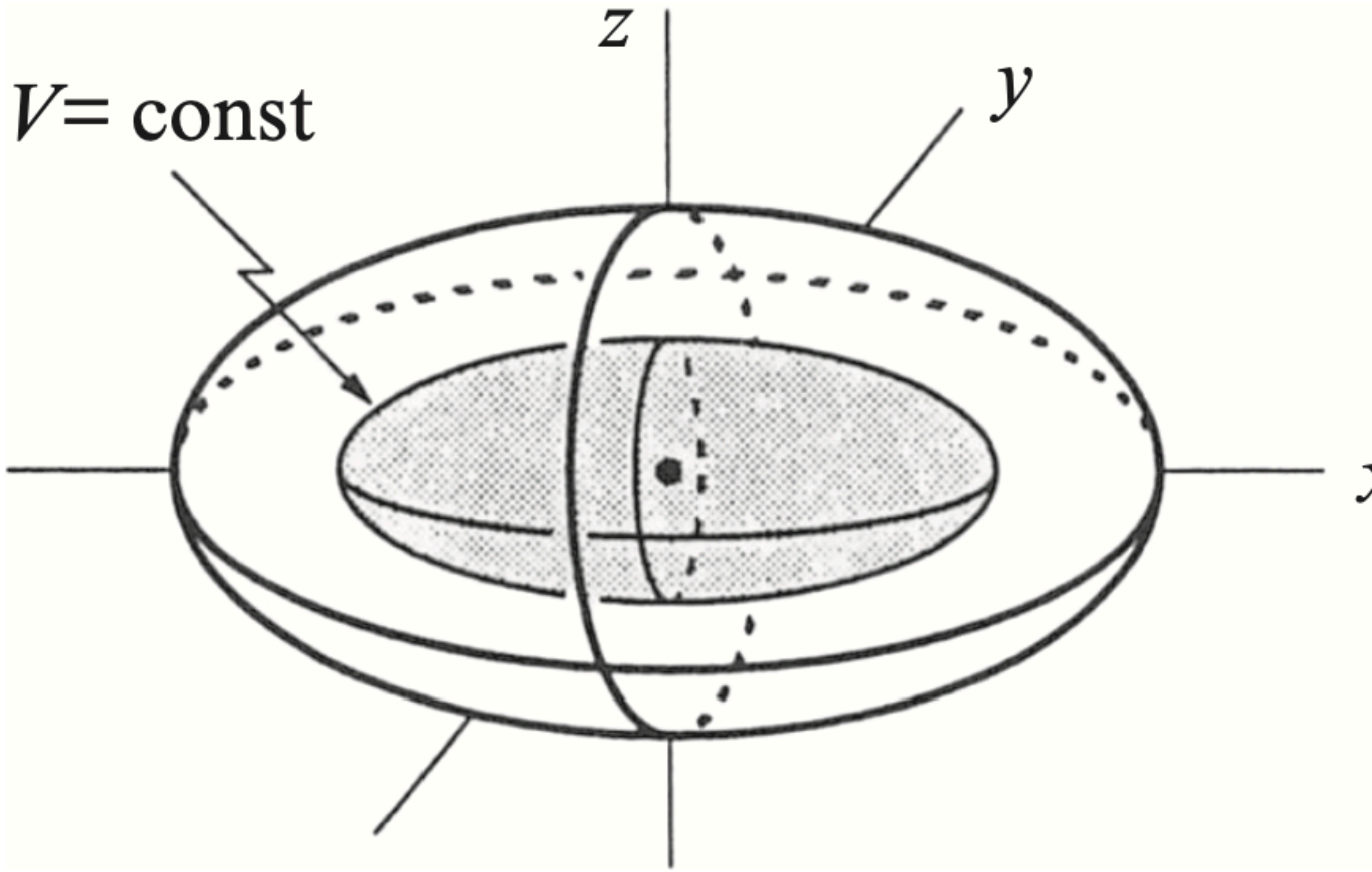
We can try expressions involving sums of squares:

$$V(x, y, z) = \tfrac{1}{\sigma}x^2 + y^2 + z^2$$

Global Stability of the Origin

$$V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$$

The surfaces of constant V are concentric ellipsoids about the origin.



The idea is to show that if $r < 1$ and $(x, y, z) \neq (0,0,0)$, then $\dot{V} < 0$ along trajectories.

The trajectory keeps moving to lower V, and hence penetrates smaller and smaller ellipsoids as $t \to \infty$.

But V is bounded below by 0, $V(\mathbf{x}(t)) \to 0$ and hence $\mathbf{x}(t) \to \mathbf{0}$.

Global Stability of the Origin

$$V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$$

Derivative:

$$\begin{aligned} \tfrac{1}{2}\dot{V} &= \tfrac{1}{\sigma}x\dot{x} + y\dot{y} + z\dot{z} \\ &= (yx - x^2) + (ryx - y^2 - xyz) + (zxy - bz^2) \\ &= (r+1)xy - x^2 - y^2 - bz^2. \end{aligned}$$

$$\tfrac{1}{2}\dot{V} = -\left[x - \tfrac{r+1}{2}y\right]^2 - \left[1 - \left(\tfrac{r+1}{2}\right)^2\right]y^2 - bz^2.$$

The RHS is negative. But could $\dot{V} = 0$?

$\dot{V} = 0$ implies $(x, y, z) = (0,0,0)$, otherwise $\dot{V} < 0$.

Hence the origin in globally stable for $r < 1$.

Stability of C^+ or C^-

Remember: $r > 1$, so that C^+ and C^- exist. They are linearly stable for:

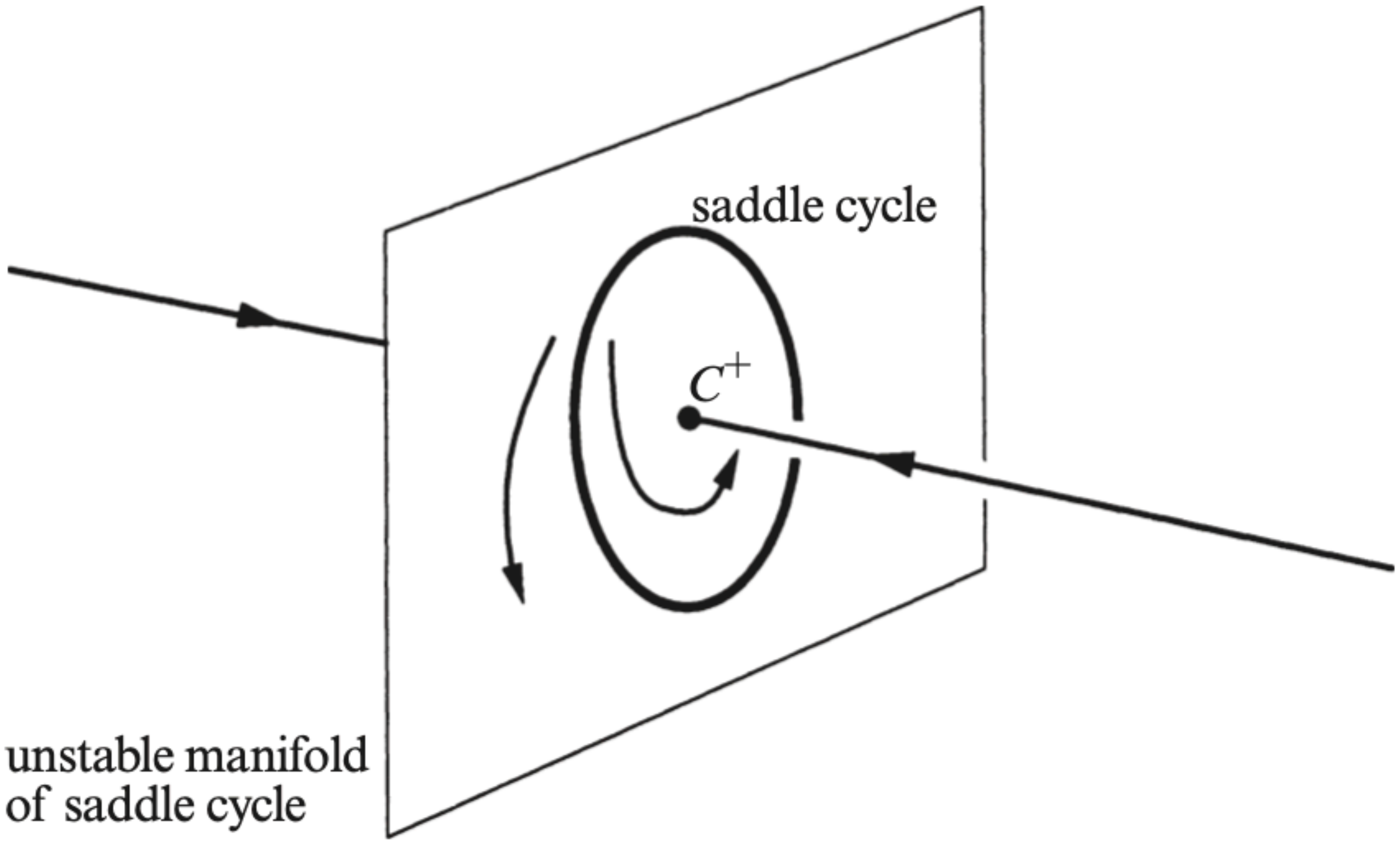
$$1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \qquad \sigma - b - 1 > 0$$

We use a subscript *H* because C^+ and C^- lose stability in a Hopf bifurcation at $r = r_H$.

What happens immediately after the bifurcation, for $r > r_H^+$?

The Hopf bifurcation is subcritical—the limit cycles are unstable and exist only for $r < r_H$.

Stability of C^+ or C^-



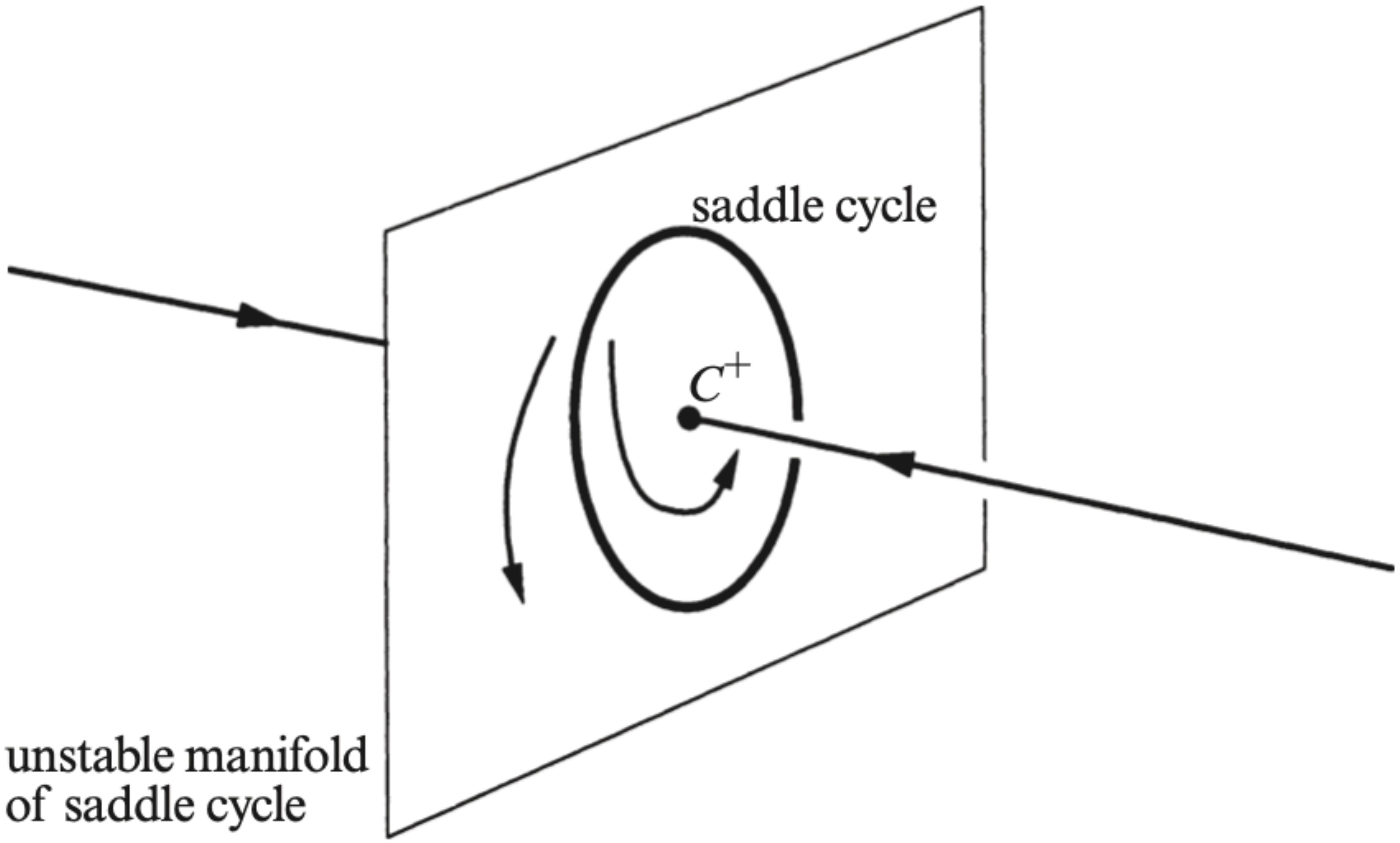
For $r < r_H$, the fixed point is stable.

It is encircled by a **saddle cycle**, a new type of **unstable limit cycle** that is possible only in phase spaces of 3+ dimensions.

The cycle has a 2D unstable manifold (the sheet) and a 2D stable manifold (not shown).

As $r \to r_H$ from below, the cycle shrinks down around the fixed point.

Stability of C^+ or C^-

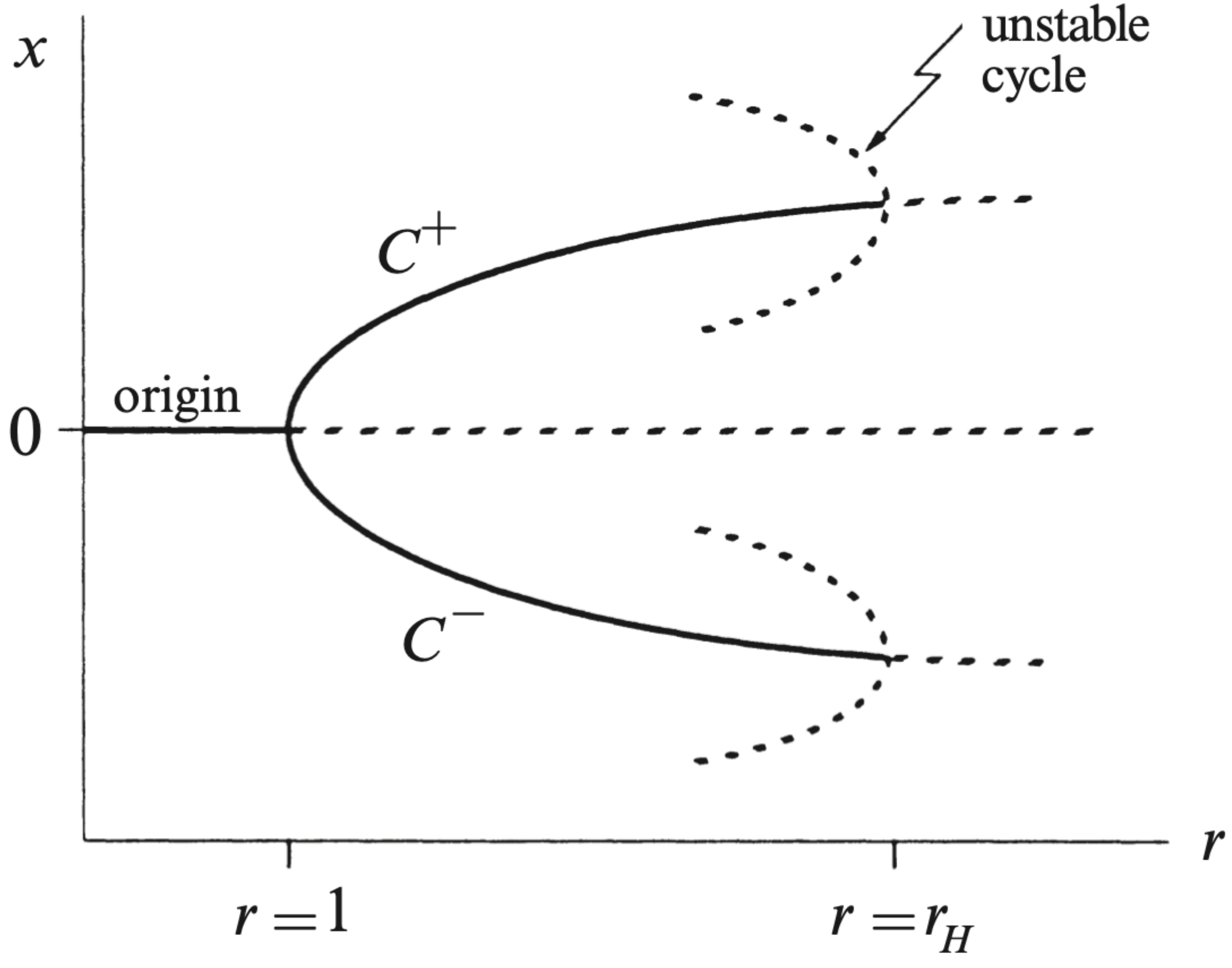


At the Hopf bifurcation, the fixed point absorbs the saddle cycle and changes into a saddle point.

For $r > r_H$ there are no attractors in the neighbourhood, trajectories must fly away to a distant attractor.

But what can it be?

Stability of C^+ or C^-



But what can it be?

A partial bifurcation diagram for the system, based on the results so far, shows no hint of any stable objects for $r > r_H$.

For $r > r_H$ there are no attractors in the neighbourhood, trajectories must fly away to a distant attractor.

Could it be that all trajectories are repelled out to infinity?

No. We can prove that all trajectories eventually enter and remain in a certain large ellipsoid.

Stability of C^+ or C^-

Could there be some stable limit cycles that we're unaware of?

No, Lorenz showed that for r slightly greater than r_H, any limit cycles would have to be unstable.

So the trajectories must have a bizarre kind of long-term behaviour.

Akin to balls in a pinball machine, they are repelled from one unstable object after another.

They are confined to a bounded set of zero volume, yet they manage to move on this set forever without intersecting themselves or others.

Chaos on a Strange Attractor

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

Lorenz used numerical integration to see what the trajectories would do in the long run.

He studied the particular case:

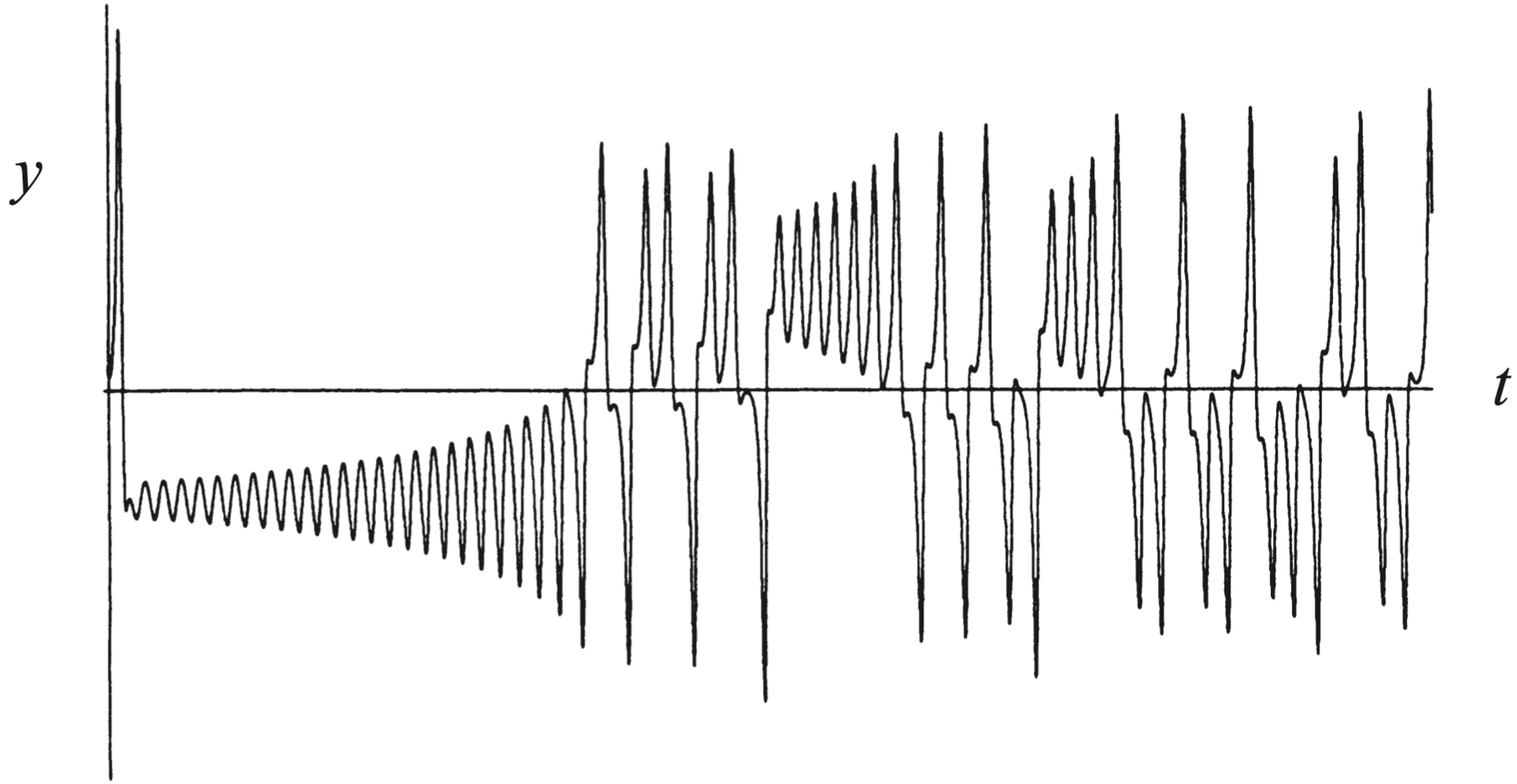
$$\sigma = 10,\ b = \tfrac{8}{3},\ r = 28$$

Hopf bifurcation critical value:

$$r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1) \approx 24.74$$

Chaos on a Strange Attractor

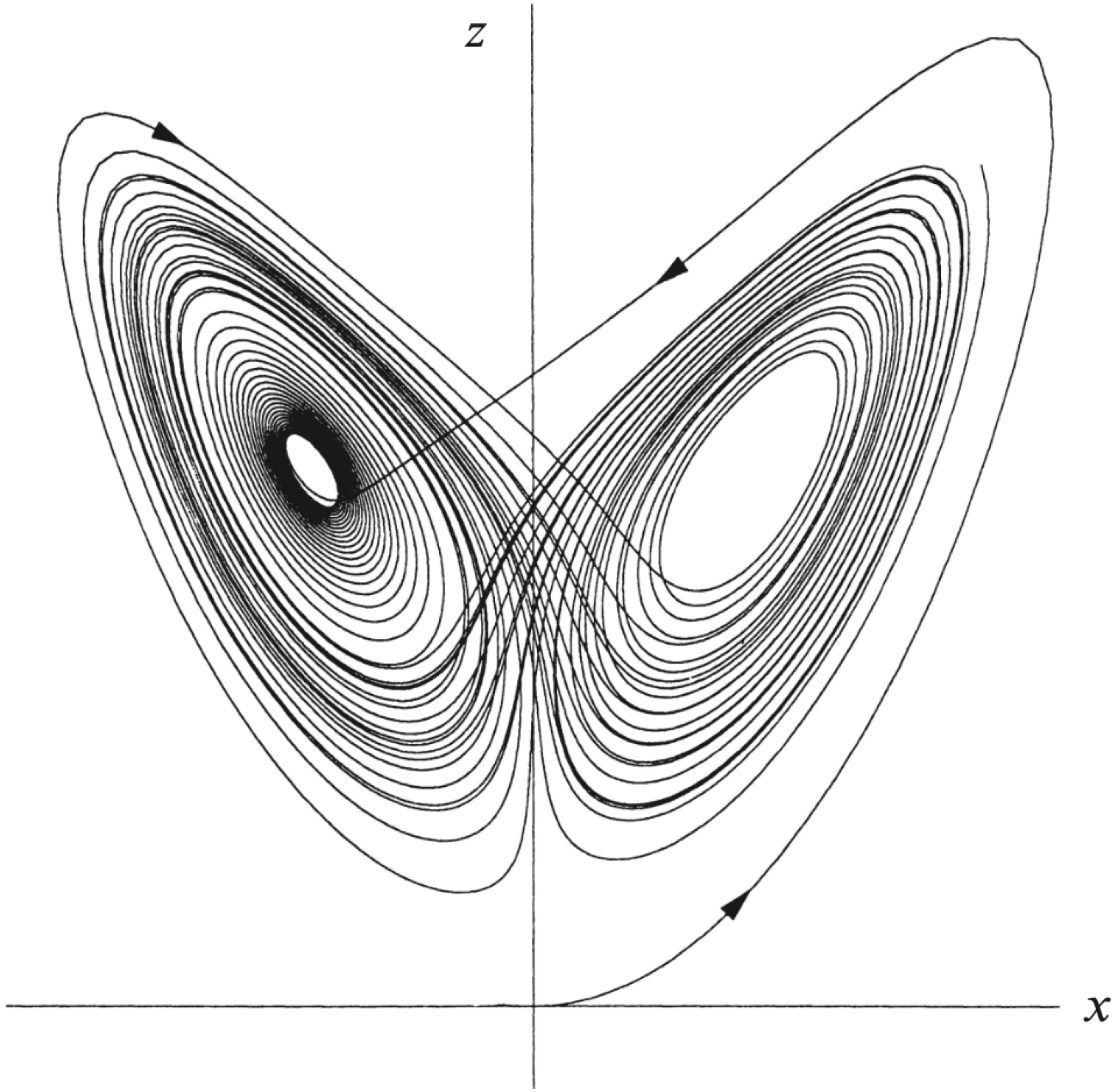
He began integrating from the initial condition (0, 1, 0):



After an initial transient, the solution settles into an irregular oscillation that persists as $t \to \infty$, but never repeats exactly. **The motion is aperiodic.**

Chaos on a Strange Attractor

Butterfly phase plot:



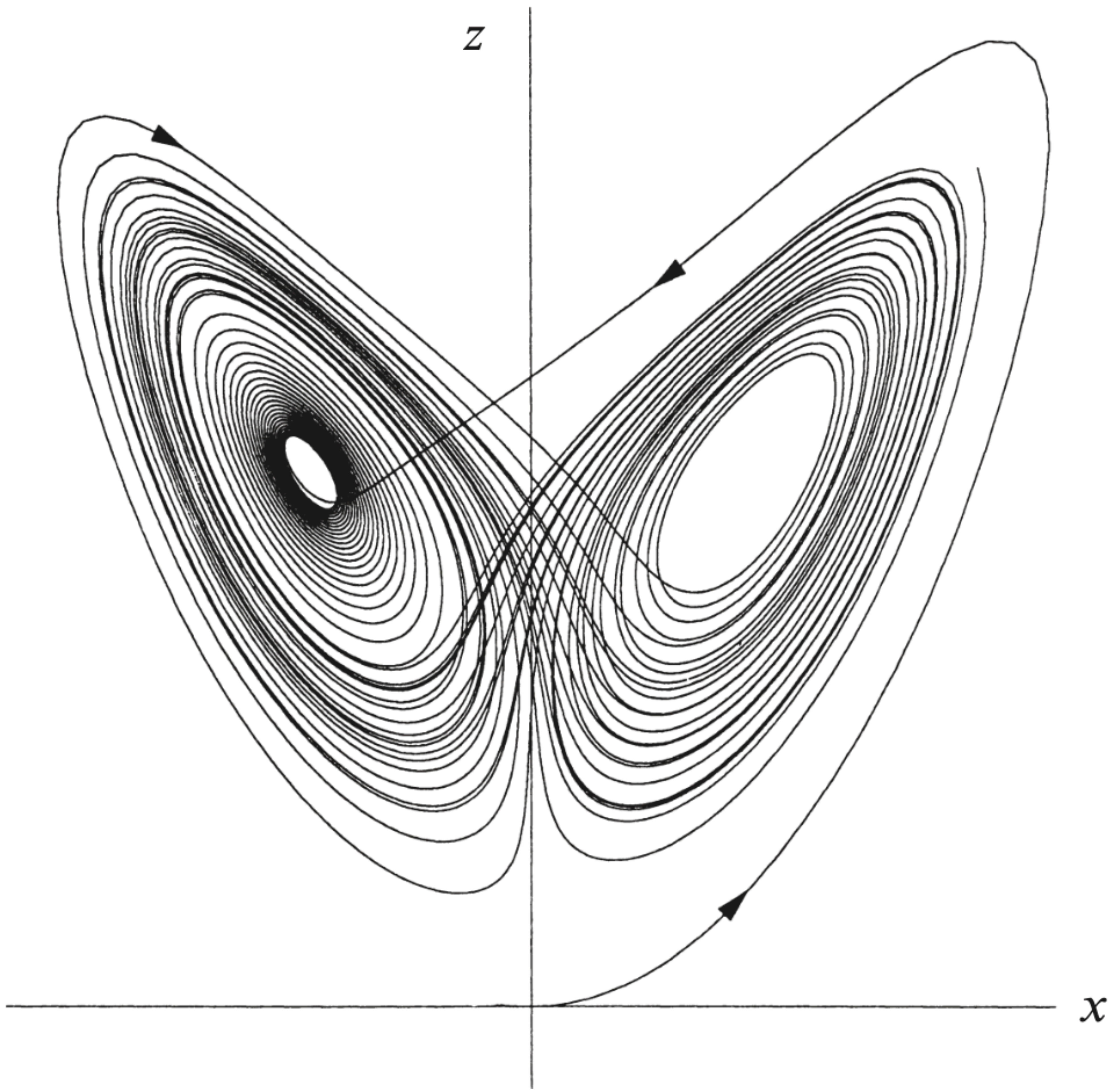
The trajectory appears to cross itself repeatedly, but that's just an artifact of projecting the 3D trajectory onto a 2D plane. In 3D, no self-intersections occur.

The trajectory starts near the origin, then swings to the right, and then dives into the center of a spiral on the left.

After a very slow spiral outward, the trajectory shoots back over to the right side, spirals around a few times, shoots over to the left, spirals around, and so on indefinitely.

Chaos on a Strange Attractor

Butterfly phase plot:



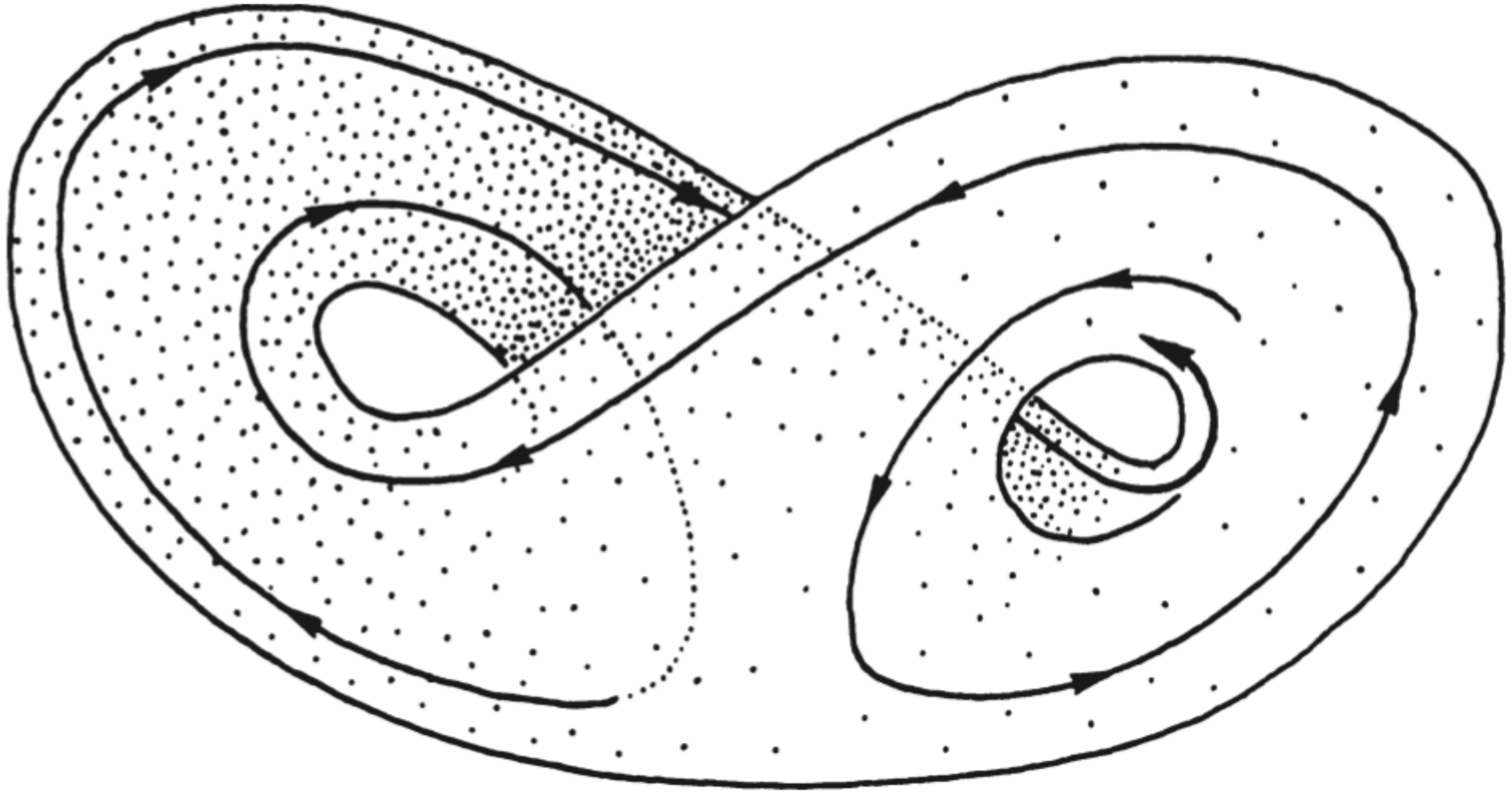
The number of circuits made on either side varies unpredictably from one cycle to the next. The sequence of the number of circuits has many of the characteristics of a **random sequence.**

When the trajectory is viewed in all 3D, rather than in a 2D projection, it appears to settle onto an exquisitely thin set that looks like a pair of butterfly wings.

Strange Attractor (Ruelle and Takens 1971).

Chaos on a Strange Attractor

This limiting set is the attracting set of zero volume.



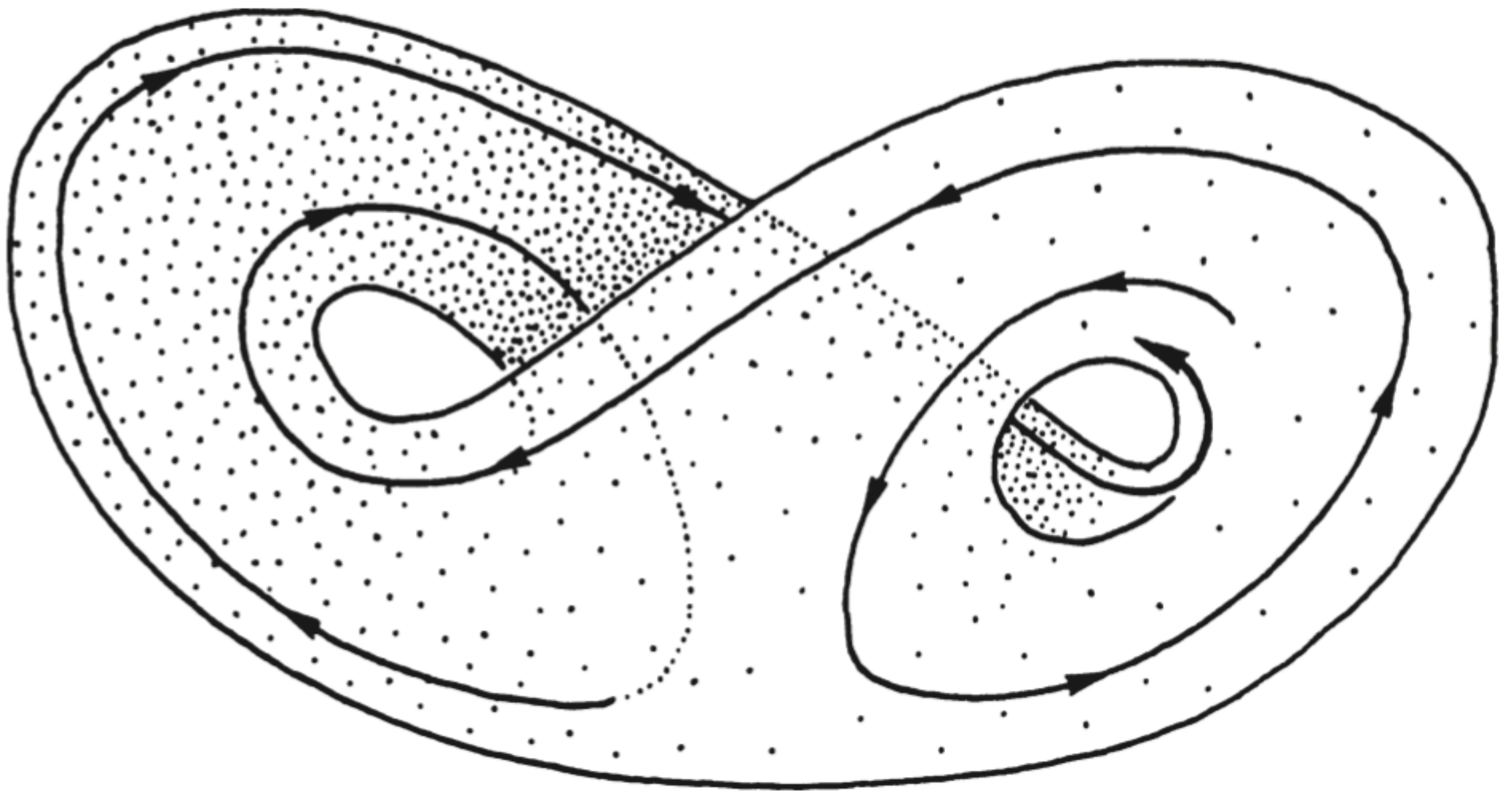
What is the geometrical structure of the strange attractor?

The 2 surfaces only appear to merge. The illusion is caused by the strong volume contraction of the flow, and insufficient numerical resolution.

Following these surfaces along a path parallel to a trajectory, and circling C^+ and C^-, so that, where they appear to merge, there are really 4 surfaces.

Chaos on a Strange Attractor

Continuing this process for another circuit, there are 8 surfaces, etc., so we conclude that there is an infinite complex of surfaces, each extremely close to one or the other of two merging surfaces.



Today this “infinite complex of surfaces” would be called **a fractal.**

A fractal is a set of points with zero volume but infinite surface area.

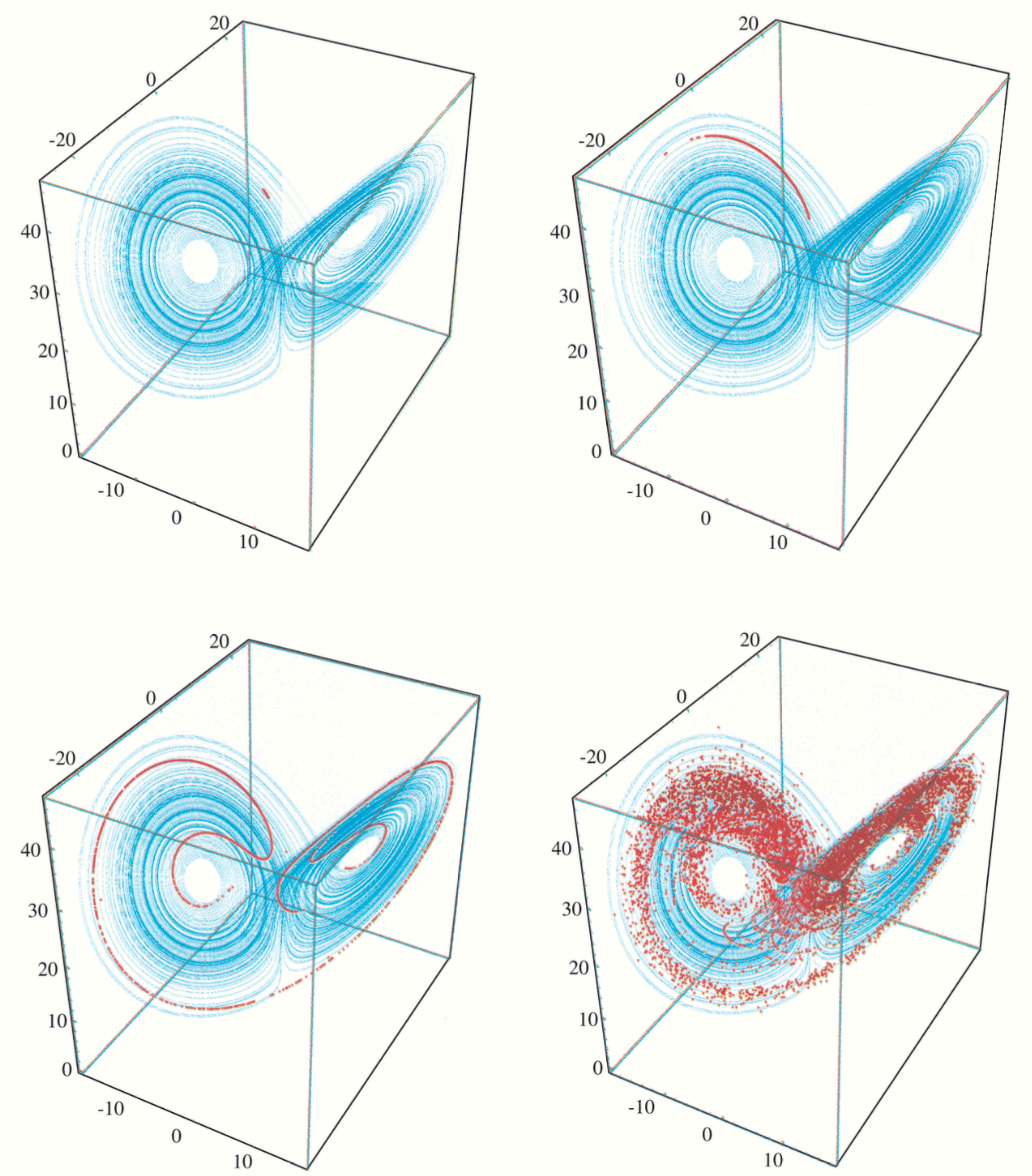
Numerical experiments suggest that it has a dimension of about 2.05!

Chaos: Exponential Divergence of Nearby Trajectories

The motion on the attractor exhibits sensitive dependence on initial conditions.

This means that two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different futures.

The red points show the evolution of a small blob of 10,000 nearby initial conditions, at times 1= 3, 6, 9, and 15.



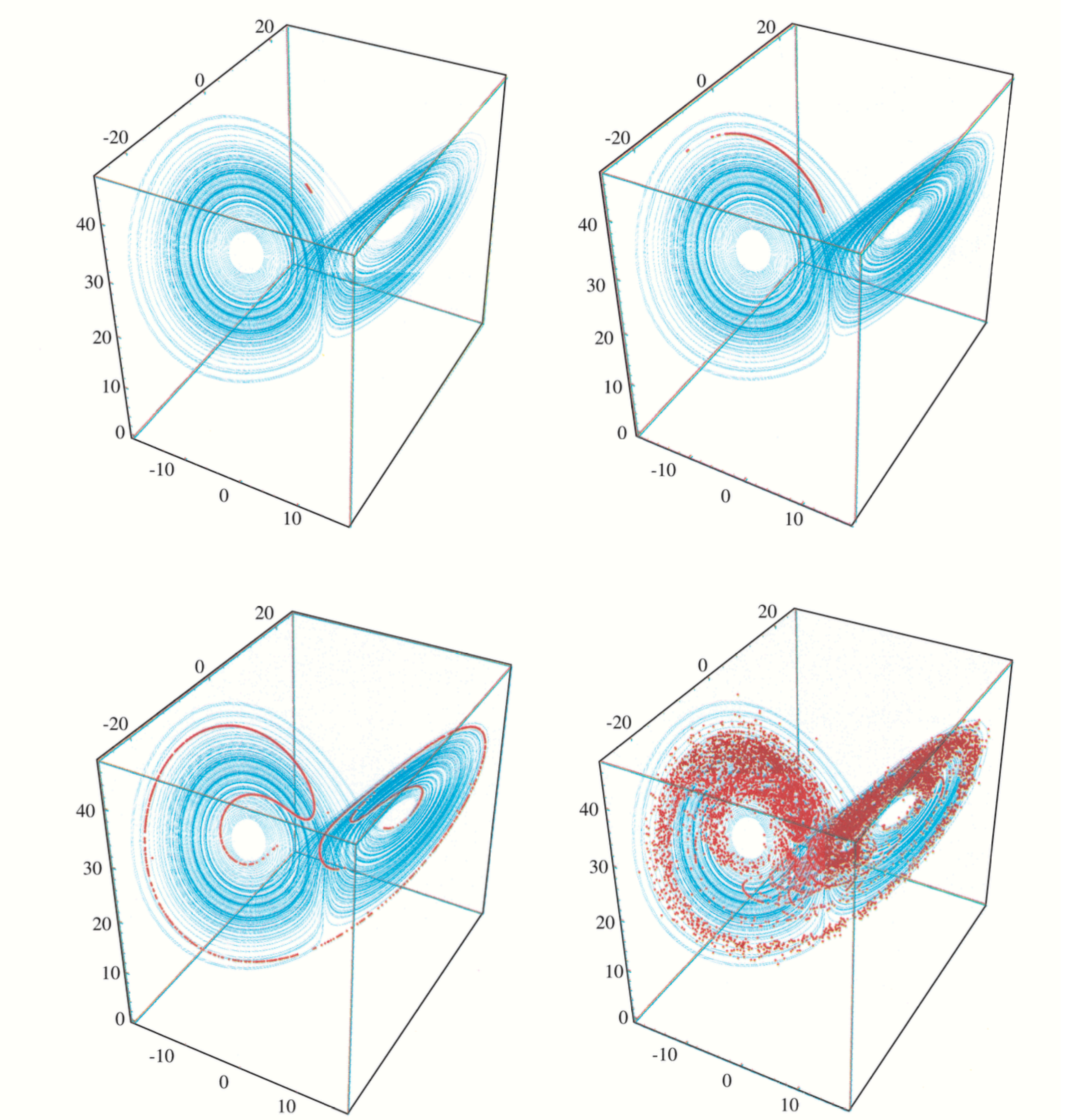
Chaos: Exponential Divergence of Nearby Trajectories

As each point moves according to the Lorenz equations, the blob is stretched into a long thin filament, which then wraps around the attractor.

Ultimately the points spread over much of the attractor, showing that **the final state could be almost anywhere, even though the initial conditions were almost identical.**

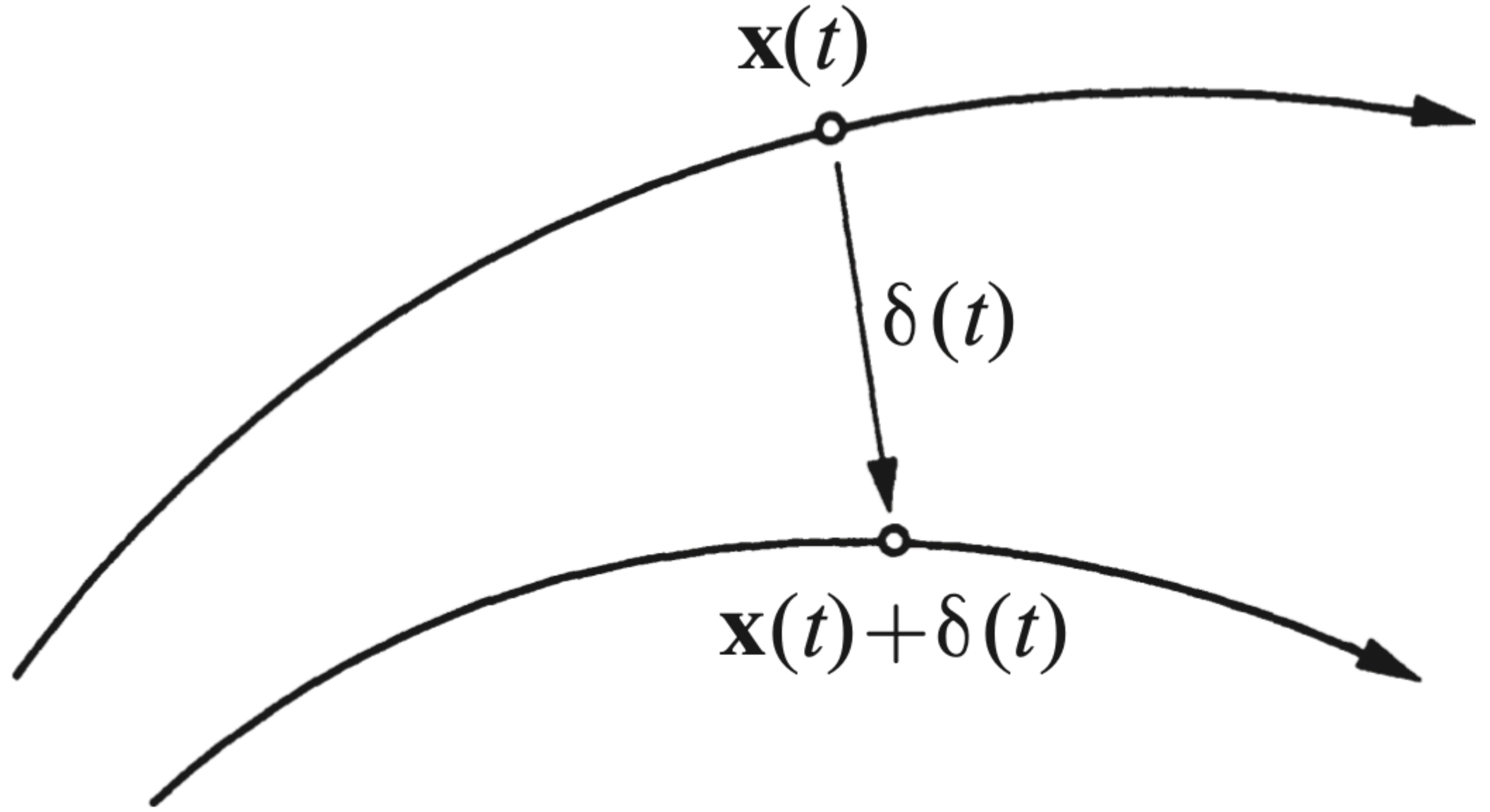
This sensitive dependence on initial condition is characteristic of a chaotic system.

Long-term prediction becomes impossible in a system like this, where small uncertainties are amplified enormously fast.



Chaos: Exponential Divergence of Nearby Trajectories

Suppose $\mathbf{x}(t)$ is a point on the attractor at time t, and consider a nearby point, say $\mathbf{x}(t) + \delta(t)$, where $\delta(t)$ is a tiny separation vector of initial length $||\delta_0|| = 10^{-15}$



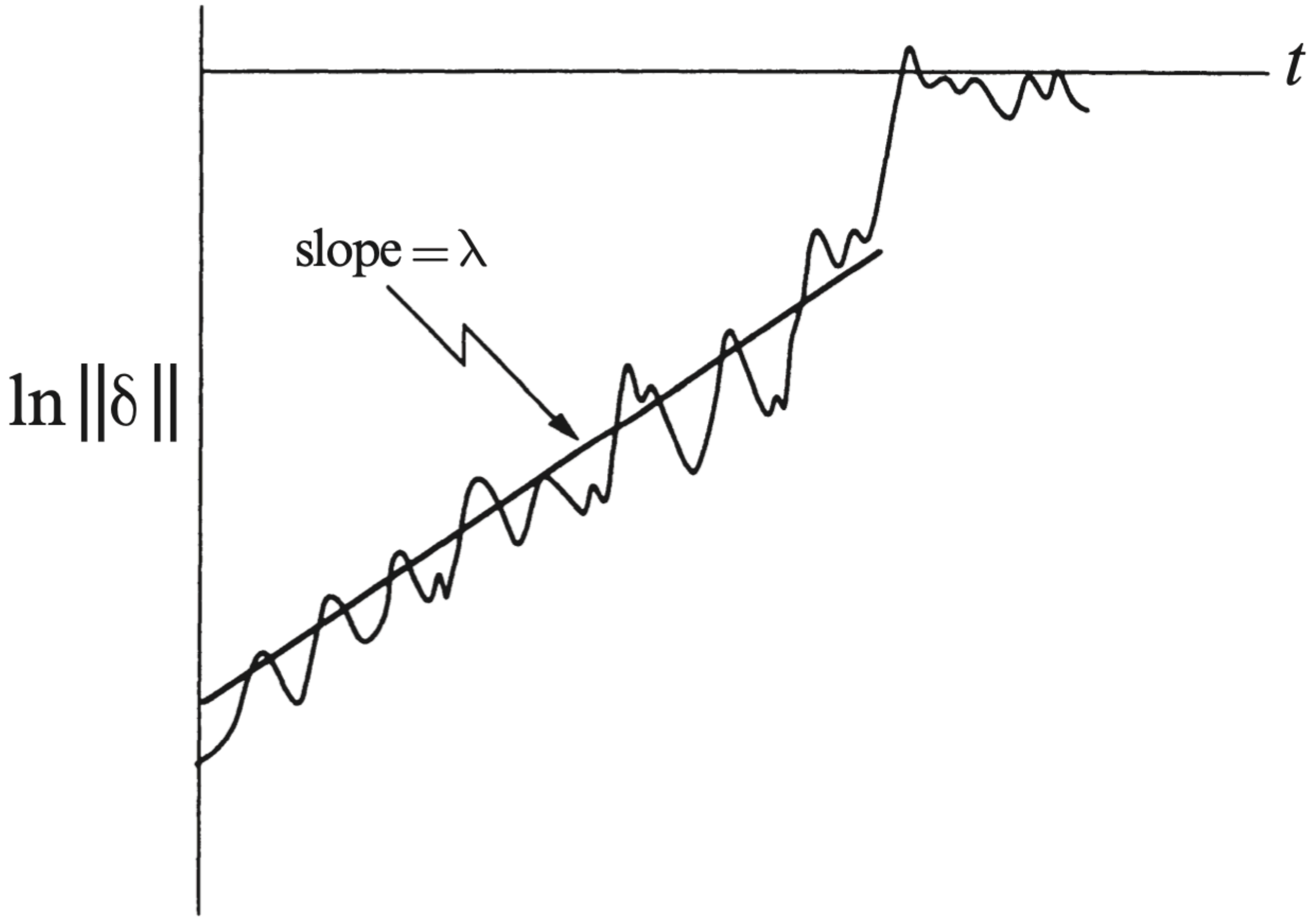
In numerical studies of the Lorenz attractor, one finds that:

$$\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$$

$$\lambda \approx 0.9.$$

Neighbouring trajectories separate exponentially fast.

Chaos: Exponential Divergence of Nearby Trajectories



1. The curve is never exactly straight. It has wiggles because the strength of the exponential divergence varies along the attractor.

2. The exponential divergence must stop when the separation is comparable to the "diameter" of the attractor—the trajectories obviously can't get any farther apart than that. This explains the levelling off or *saturation* of the curve.

3. The number λ is the largest **Liapunov exponent.**

Lyapunov exponents

1. There are n different **Liapunov exponents** for an n-dimensional system.

Consider the evolution of an infinitesimal sphere of perturbed initial conditions. During its evolution, the sphere will become distorted into an infinitesimal ellipsoid.

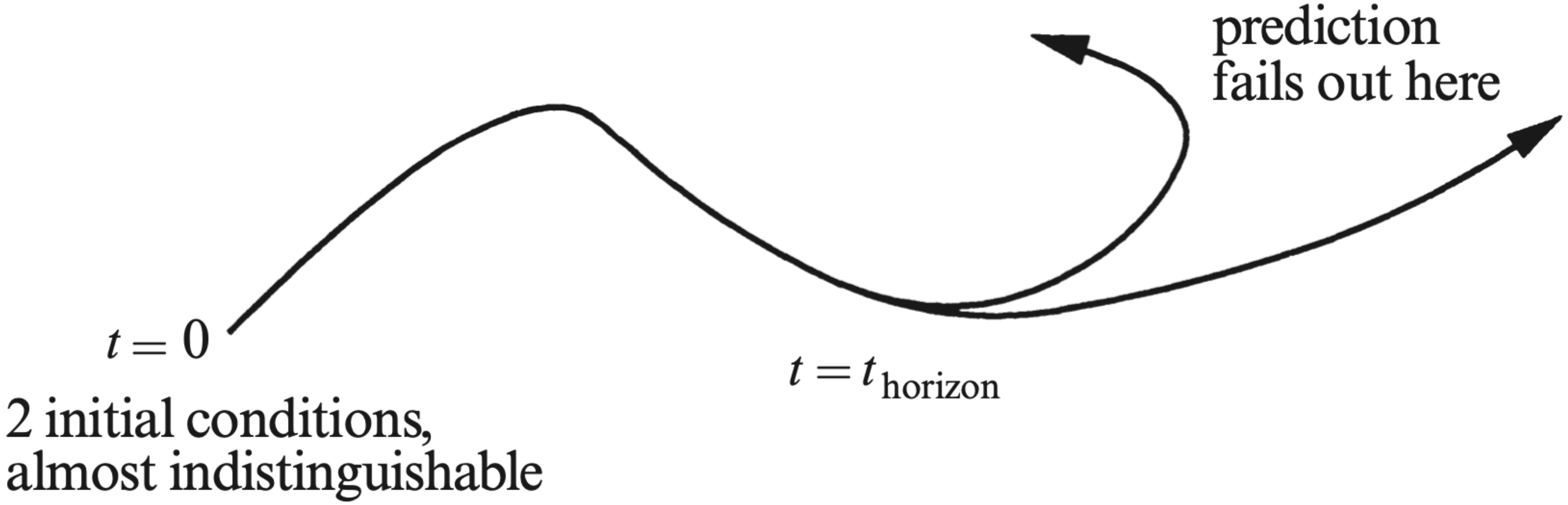
Let $\delta_k(t)$, $k = 1, \ldots, n$, denote the length of the kth principal axis of the ellipsoid. Then $\delta_k(t) \sim \delta_k(0)e^{\lambda_k t}$, where the λ_k are the Liapunov exponents. For large t, the diameter of the ellipsoid is controlled by the most positive λ_k. Thus our λ is actually the *largest* Liapunov exponent.

2. λ depends (slightly) on which trajectory we study. We should average over many different points on the same trajectory to get the true value of λ.

Lyapunov exponents

When a system has a positive Liapunov exponent, there is a time horizon beyond which prediction breaks down.

$$\|\delta_0\|$$



$$\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$$

Let *a* be a measure of our tolerance, i.e., if a prediction is within *a* of the true state, we consider it acceptable. Then our prediction becomes intolerable when

$$t_{\text{horizon}} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{\|\delta_0\|}\right)$$

Tutorial

Suppose we're trying to predict the future state of a chaotic system to within a tolerance of $a = 10^{-3}$. Given that our estimate of the initial state is uncertain to within $\|\delta_0\| = 10^{-7}$, for about how long can we predict the state of the system, while remaining within the tolerance? Now suppose we buy the finest instrumentation, recruit the best graduate students, etc., and somehow manage to measure the initial state a *million* times better, i.e., we improve our initial error to $\|\delta_0\| = 10^{-13}$. How much longer can we predict?

Chaos

Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.

1. Aperiodic long-term behaviour: there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as $t \to \infty$

There is an open set of initial conditions leading to aperiodic trajectories, or perhaps that such trajectories should occur with nonzero probability, given a random initial condition.

2. Deterministic: the system has no random or noisy inputs or parameters. The irregular behaviour arises from the system's nonlinearity, rather than from noisy driving forces.

3. Sensitive dependence on initial conditions: nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent.

Attractor and Strange Attractor

Attractor: set to which all neighbouring trajectories converge. Stable fixed points and stable limit cycles are examples. More precisely, we define an ***attractor*** to be a closed set A with the following properties:

1. A is an *invariant set:* any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.

2. A *attracts an open set of initial conditions*: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction* of A.

3. A is *minimal:* there is no proper subset of A that satisfies conditions 1 and 2.

Attractor and Strange Attractor

Strange attractor

This is an attractor that exhibits sensitive dependence on initial conditions.

Strange attractors were originally called strange because they are often fractal sets.

Nowadays this geometric property is regarded as less important than the dynamical property of sensitive dependence on initial conditions.

The terms chaotic attractor and fractal attractor are used when one wishes to emphasise one or the other of those aspects.

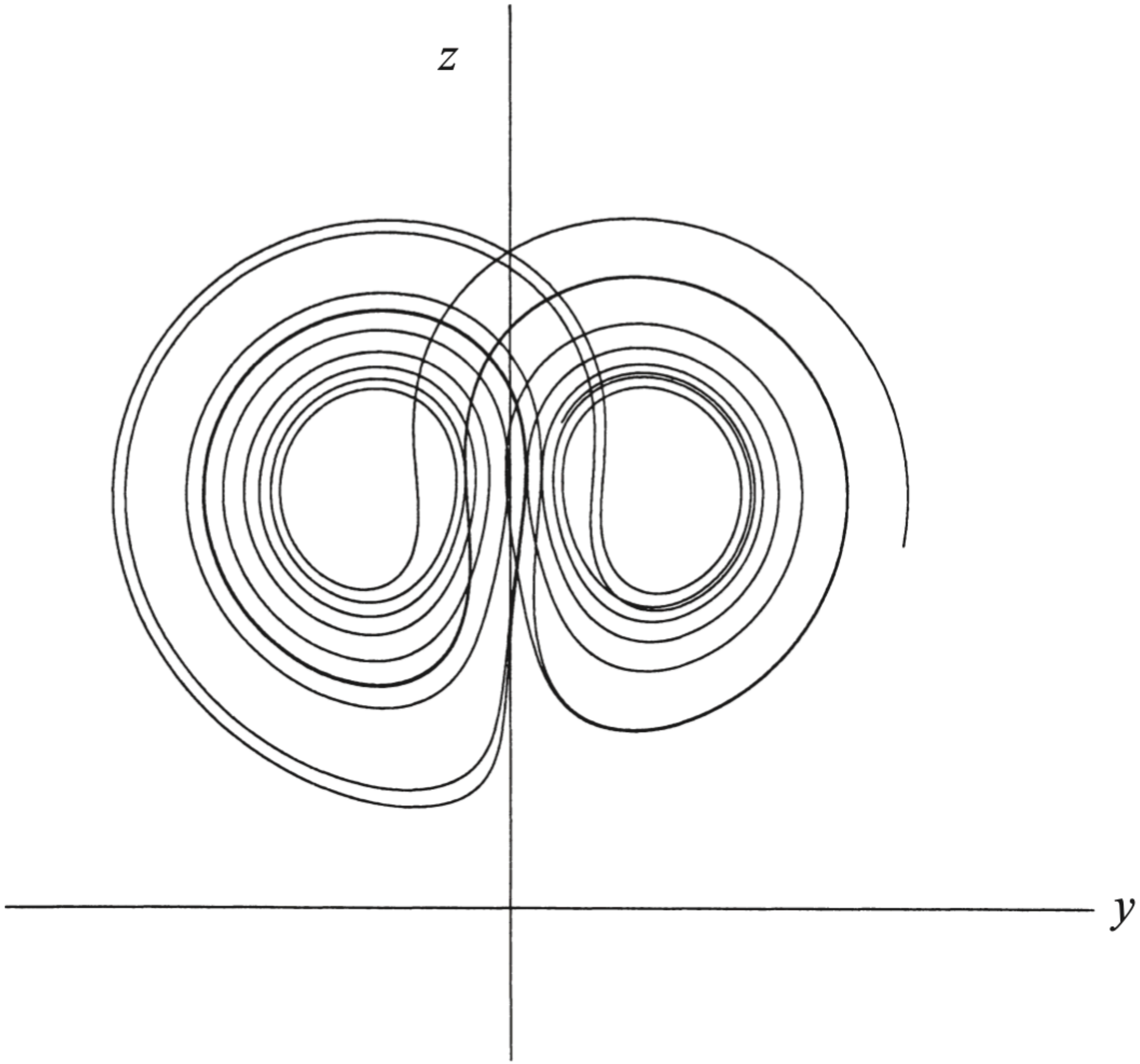
Tutorial

Solve the Lorenz equations numerically using Python.

1. Study the case where $r < 1$, $r > 1$

2. Study the case where $r < r_H$, $r > r_H$

Lorenz Map

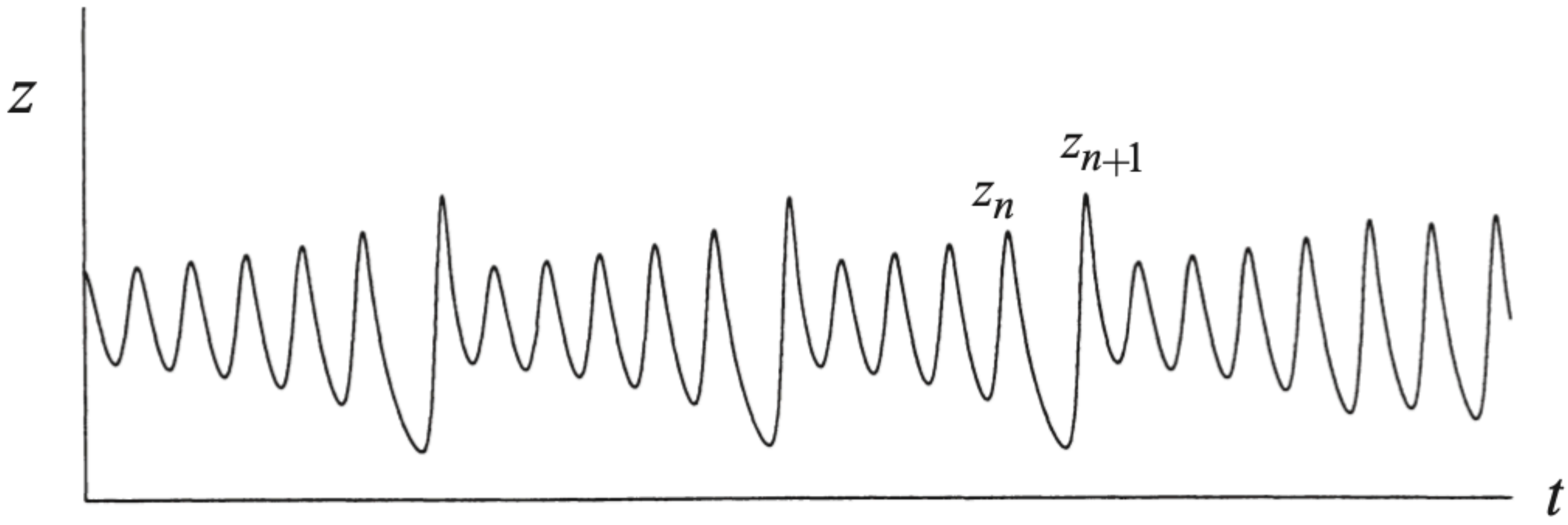


The trajectory apparently leaves one spiral only after exceeding some critical distance from the center.

The extent to which this distance is exceeded appears to **determine the point at which the next spiral is entered**; this in turn seems to d**etermine the number of circuits to be executed** before changing spirals again.

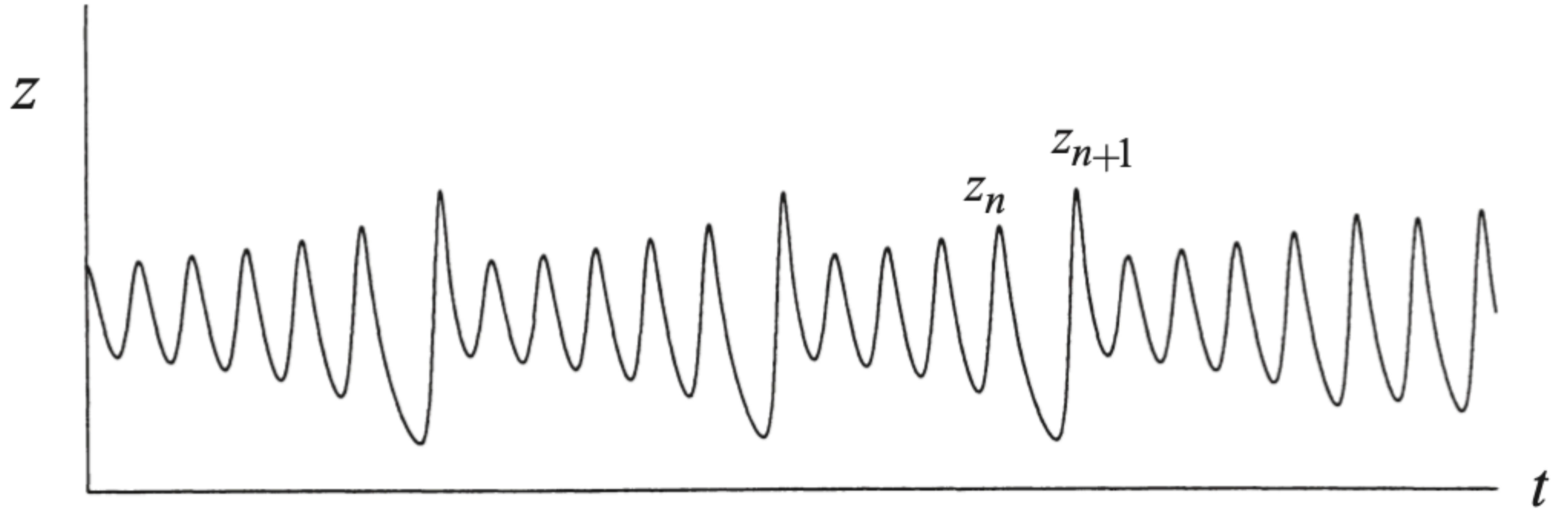
It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit.

Lorenz Map

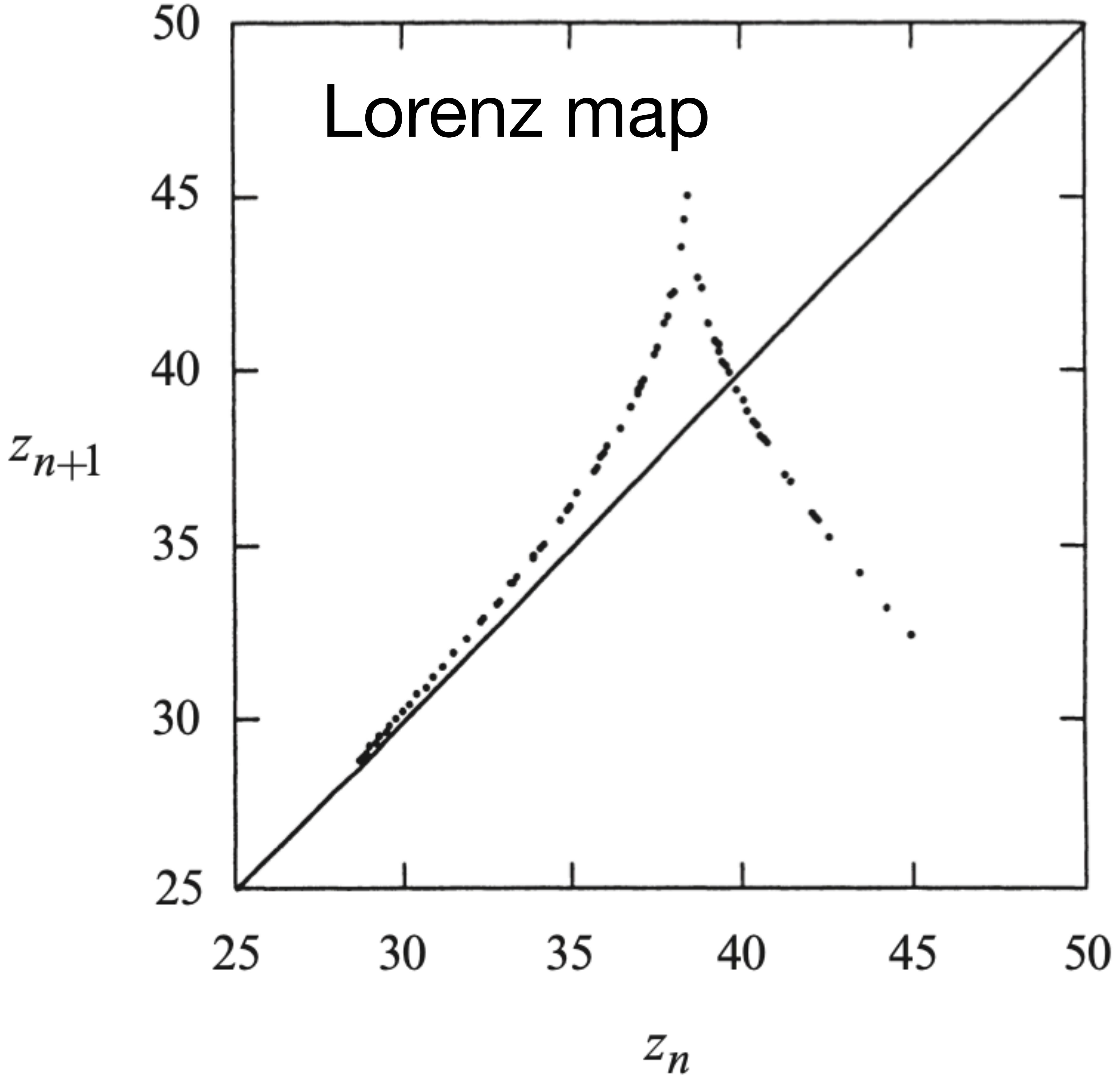


We should focus our attention on a single feature z_n, the nth local maximum of $z(t)$.

Lorenz Map

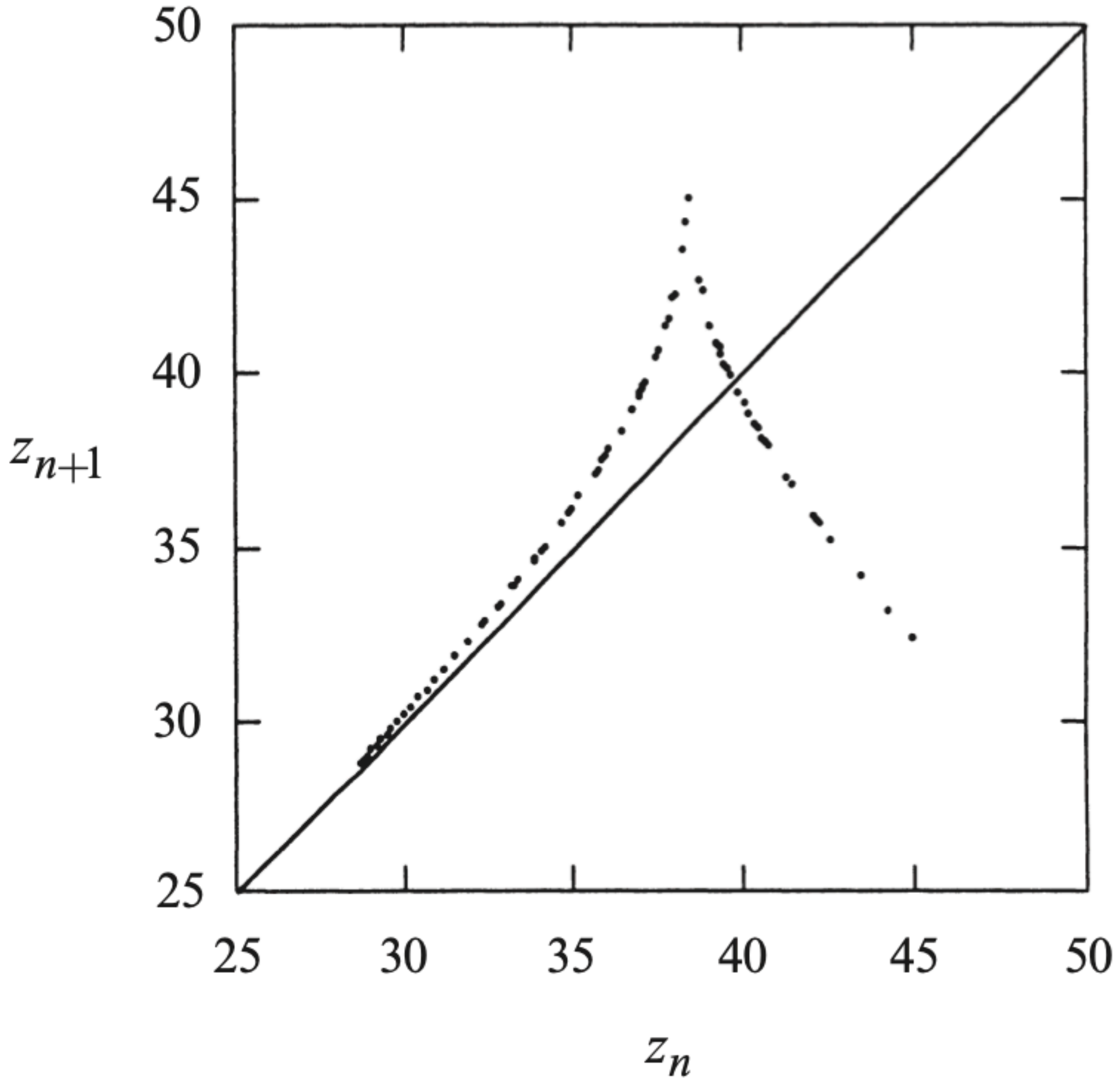


Lorenz's idea is that z_n should predict z_{n+1}. He numerically integrated the equations for a long time, then measured the local maxima of *z*(*t*), and finally plotted z_{n+1}vs. z_n.



The data from the chaotic time series appear to fall neatly on a curve—there is almost no "thickness" to the graph!

Lorenz Map



We can extract order from chaos.

Strictly speaking, $f(z)$ is not a well-defined function, because there can be more than one output z_{n+1} for a given input z_n.

The thickness is so small, and there is so much to be gained by treating the graph as a curve, that we will simply make this approximation.

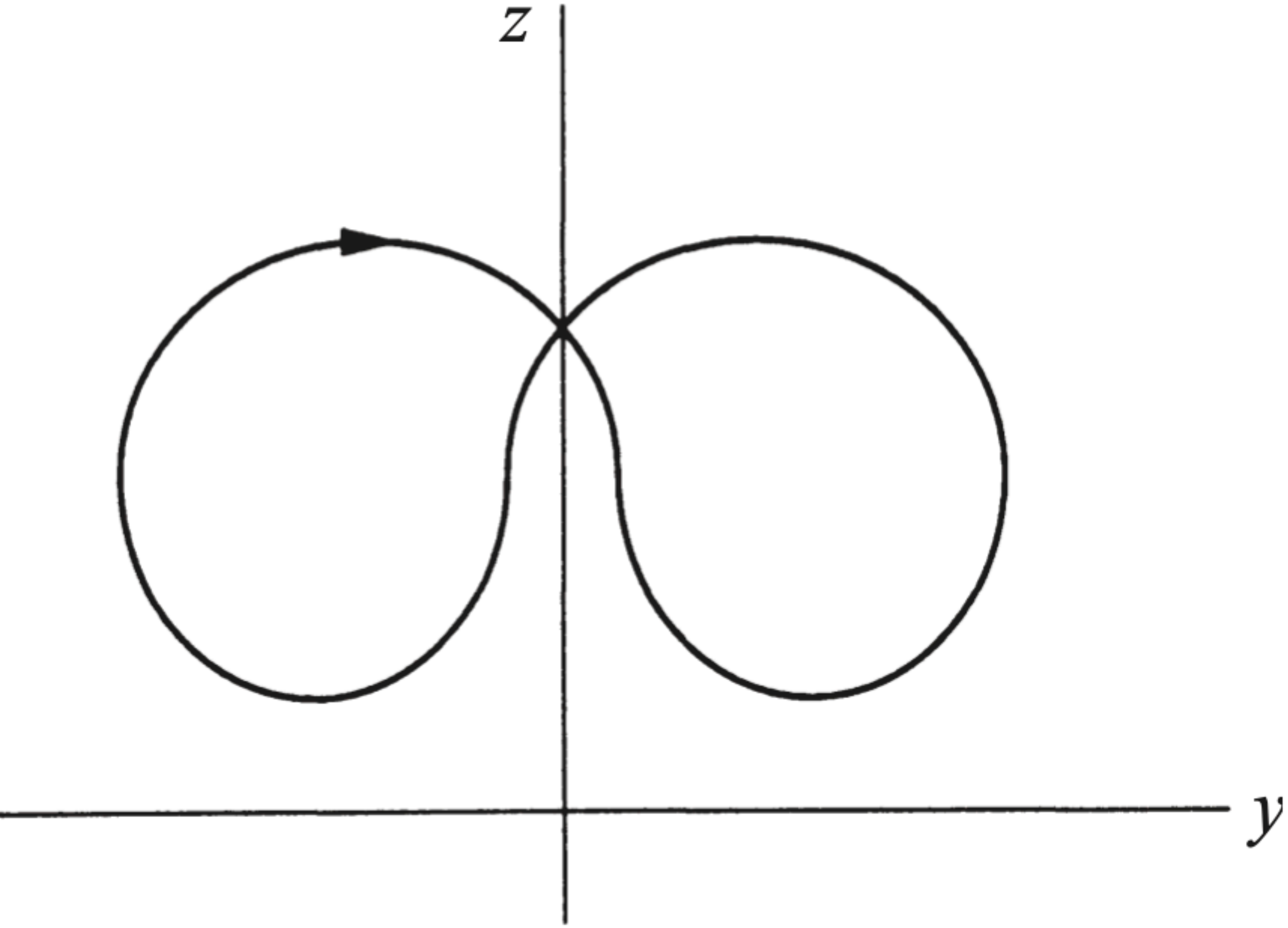
Stable Limit Cycles can be ruled out. If any limit cycles exist, they are necessarily unstable.

$$|f'(z)| > 1$$

Lorenz Map

$$|f'(z)| > 1$$

Why?



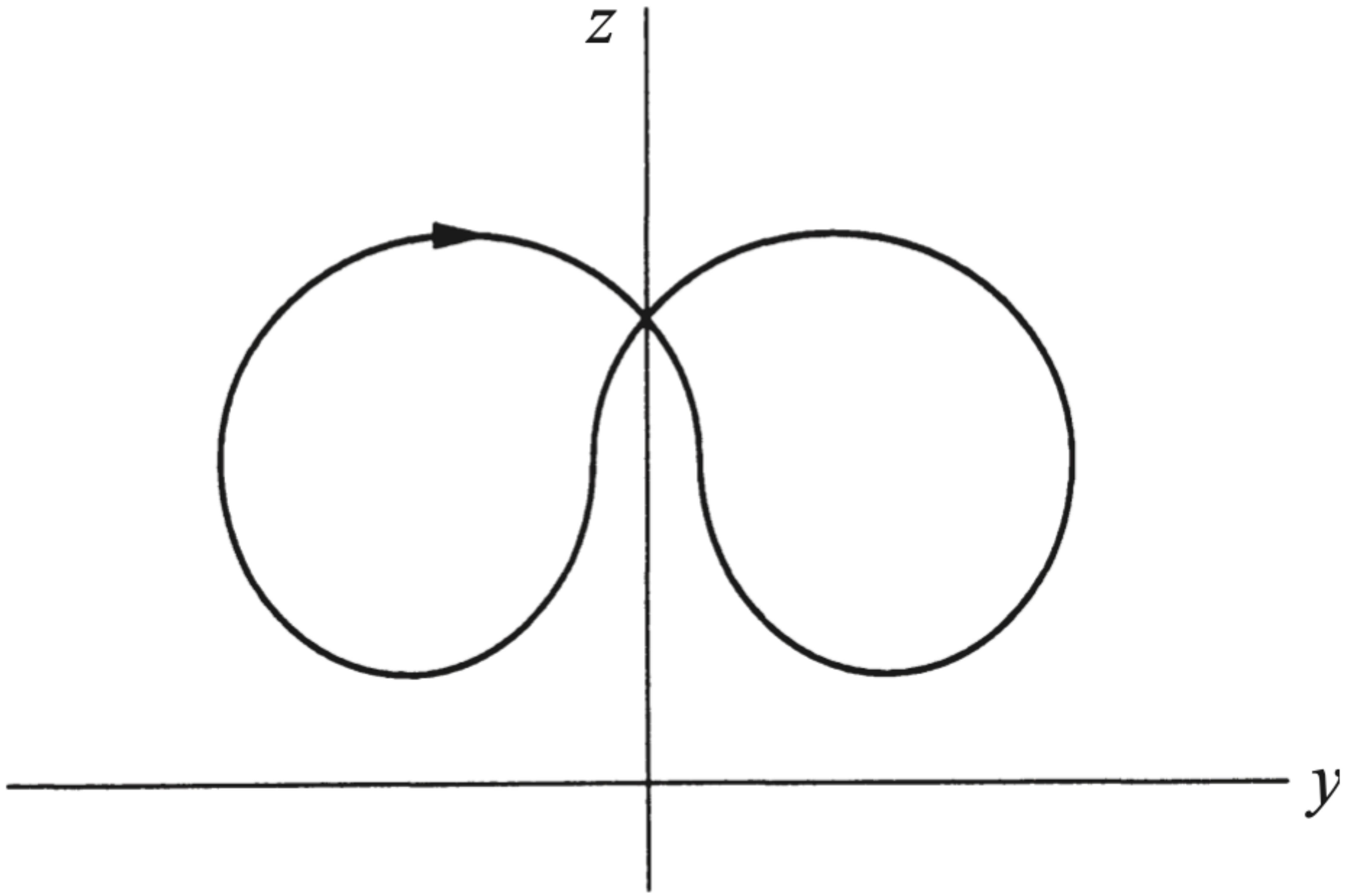
We need to analyse the fixed points on the map:

$$f(z^*) = z^*,$$

$$z_n = z_{n+1} = z_{n+2} = \ldots .$$

There is one fixed point, where the 45° diagonal intersects the graph. It represents a closed orbit.

Lorenz Map



To show that this closed orbit is unstable, consider a slightly perturbed trajectory that has:

$$z_n = z^* + \eta_n, \text{ where } \eta_n \text{ is small.}$$

After linearisation as usual, we find:

$$\eta_{n+1} \approx f'(z^*)\eta_n$$

$$|f'(z^*)| > 1 \quad \Rightarrow \quad |\eta_{n+1}| > |\eta_n|$$

The deviation grows in every iteration, so the original closed orbit is unstable.

All closed orbits are unstable.

Global behaviour