Nonlinear Dynamics and Chaos

PHYMSCFUN12

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Bifurcations

The dynamics of vector fields on the line is very limited: all solutions either settle down to equilibrium or head out to infinity.

What's interesting about 1D systems? ***Dependence on parameters.***

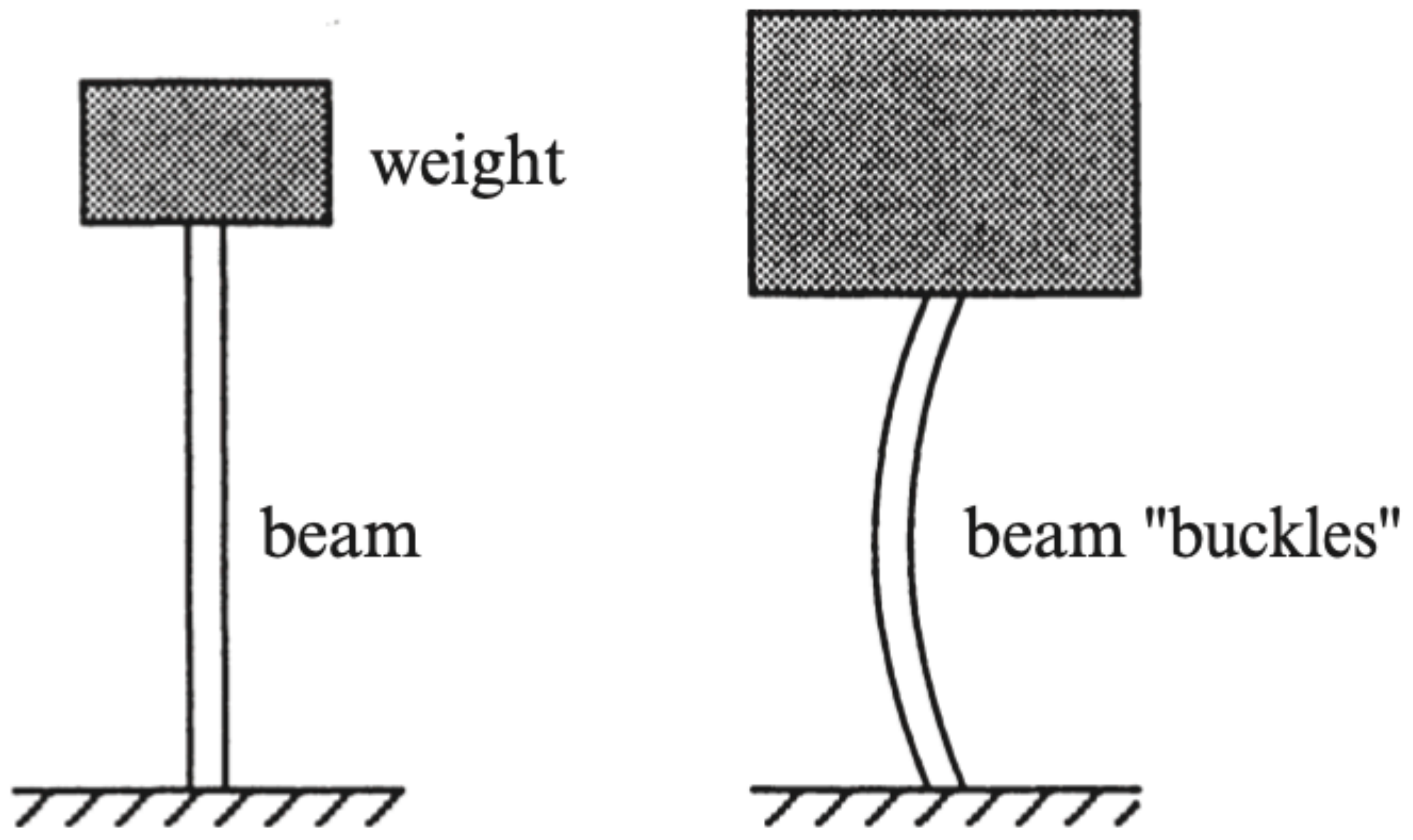
The qualitative structure of the flow can change as parameters are varied.

Fixed points can be created or destroyed, or their stability can change.

These qualitative changes in the dynamics are called ***bifurcations***, and the parameter values at which they occur are called ***bifurcation points.***

Bifurcations provide models of transitions and instabilities as some *control parameter* is varied.

Bifurcations



Weight is the control parameter and the deflection of the beam from vertical plays the role of the dynamical variable x.

We will study bifurcations of fixed points for flows on the line.

Examples:

1. Onset of coherent radiation in a laser
2. Outbreak of an insect population.

Saddle-node bifurcation

The saddle-node bifurcation is the basic mechanism by which fixed points are *created and destroyed*. Fixed points known as saddles and nodes can collide and annihilate in higher dimensions.

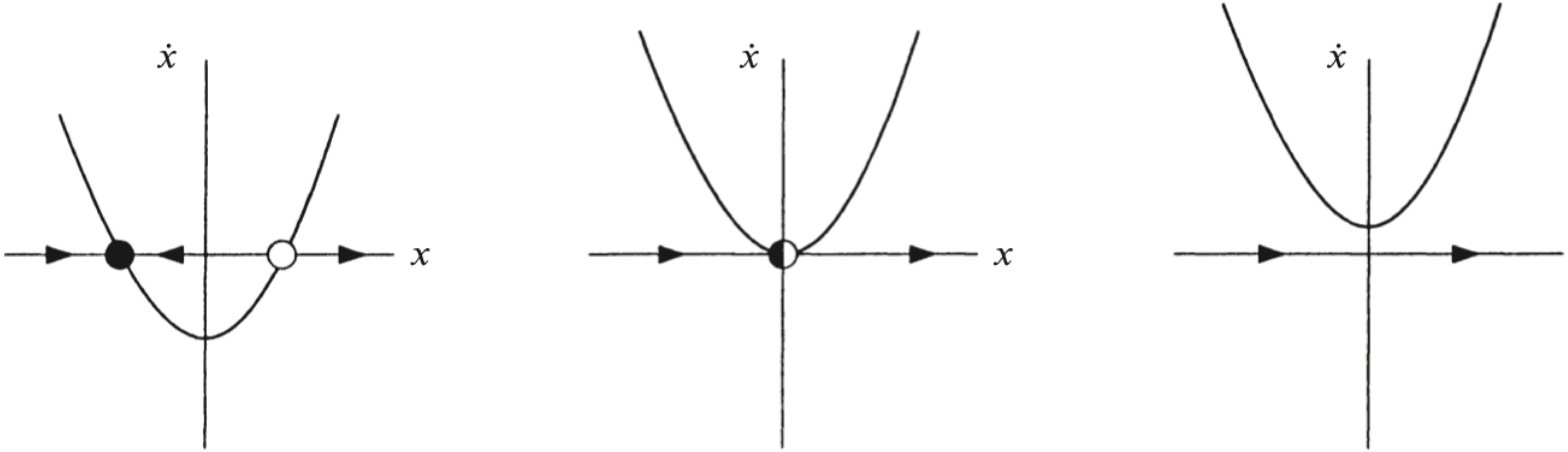
As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

The prototypical example of a saddle-node bifurcation is given by the first-order system:

$$\dot{x} = r + x^2$$

where *r* is a parameter, which may be positive, negative, or zero. When *r* is negative, there are two fixed points, one stable and one unstable.

Saddle-node bifurcation



(a) $r<0$ (b) $r=0$ (c) $r>0$

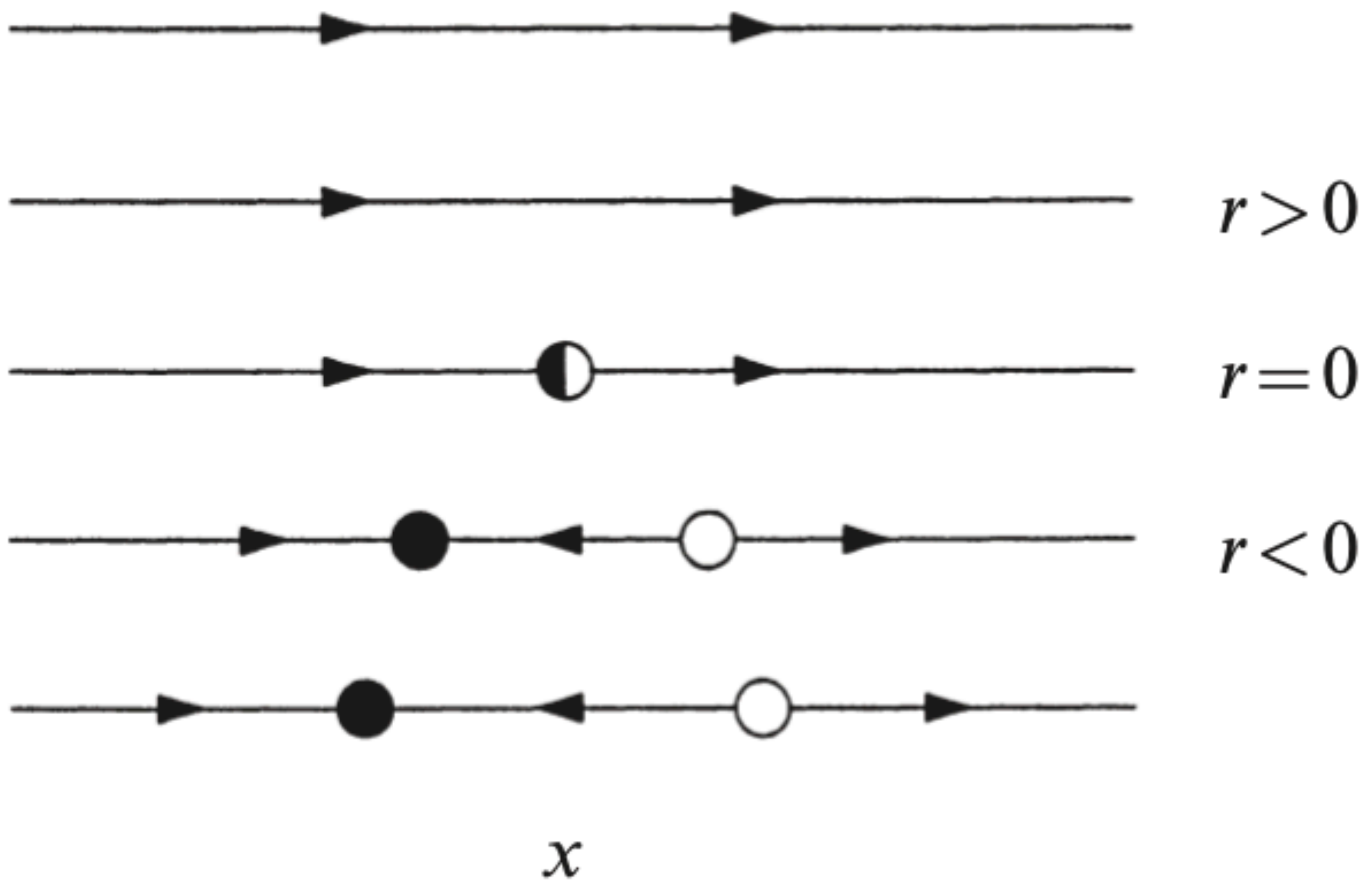
Half-stable fixed point

This type of fixed point is extremely delicate—it vanishes as soon as $r>0$, and now there are no fixed points at all.

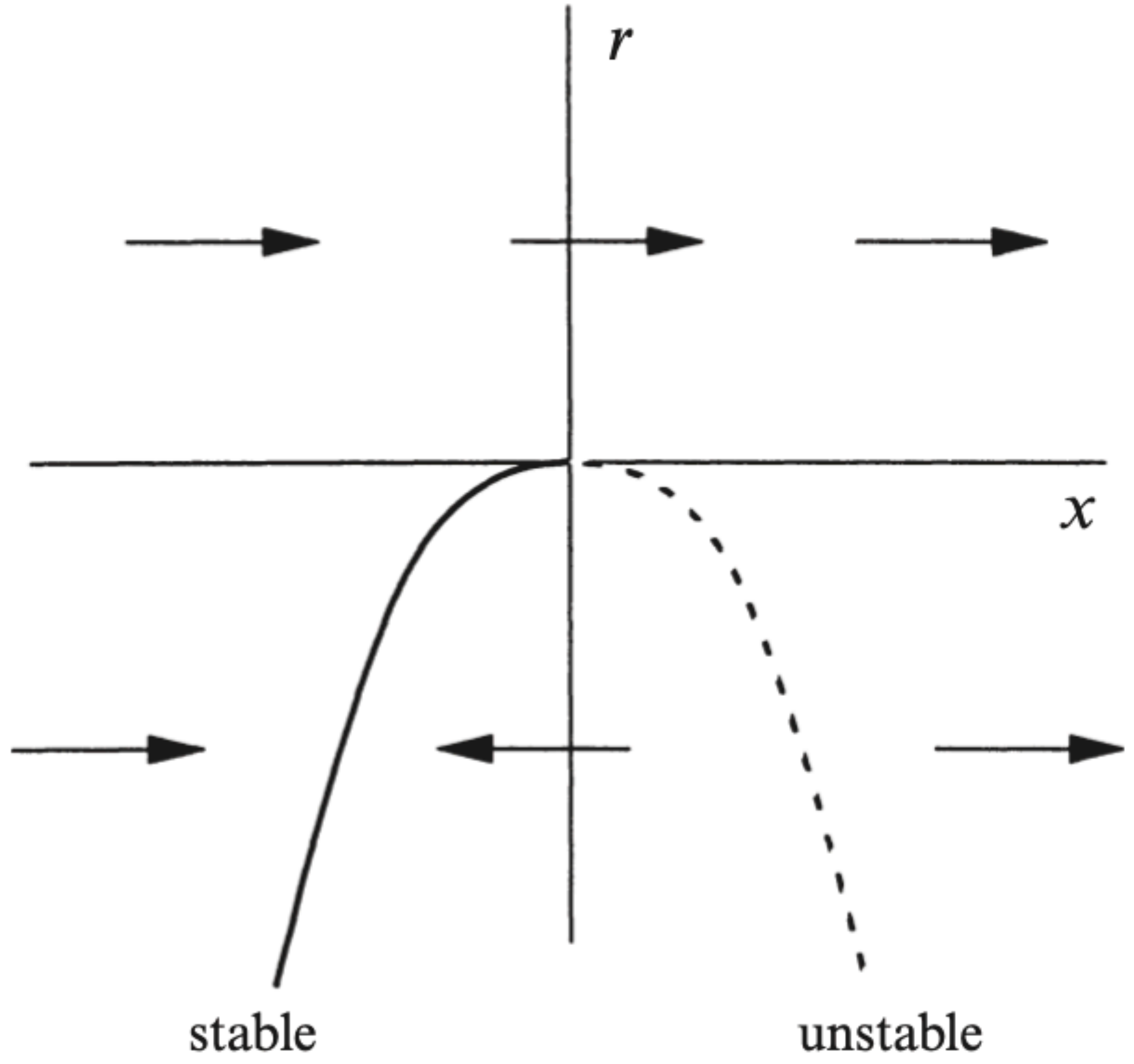
A *bifurcation* occurred at $r = 0$, since the vector fields for $r<0$ and $r>0$ are qualitatively different. Bifurcation means splitting into two branches!

Saddle-node bifurcation: Graphical Conventions

A stack of vector fields for discrete values of *r.*



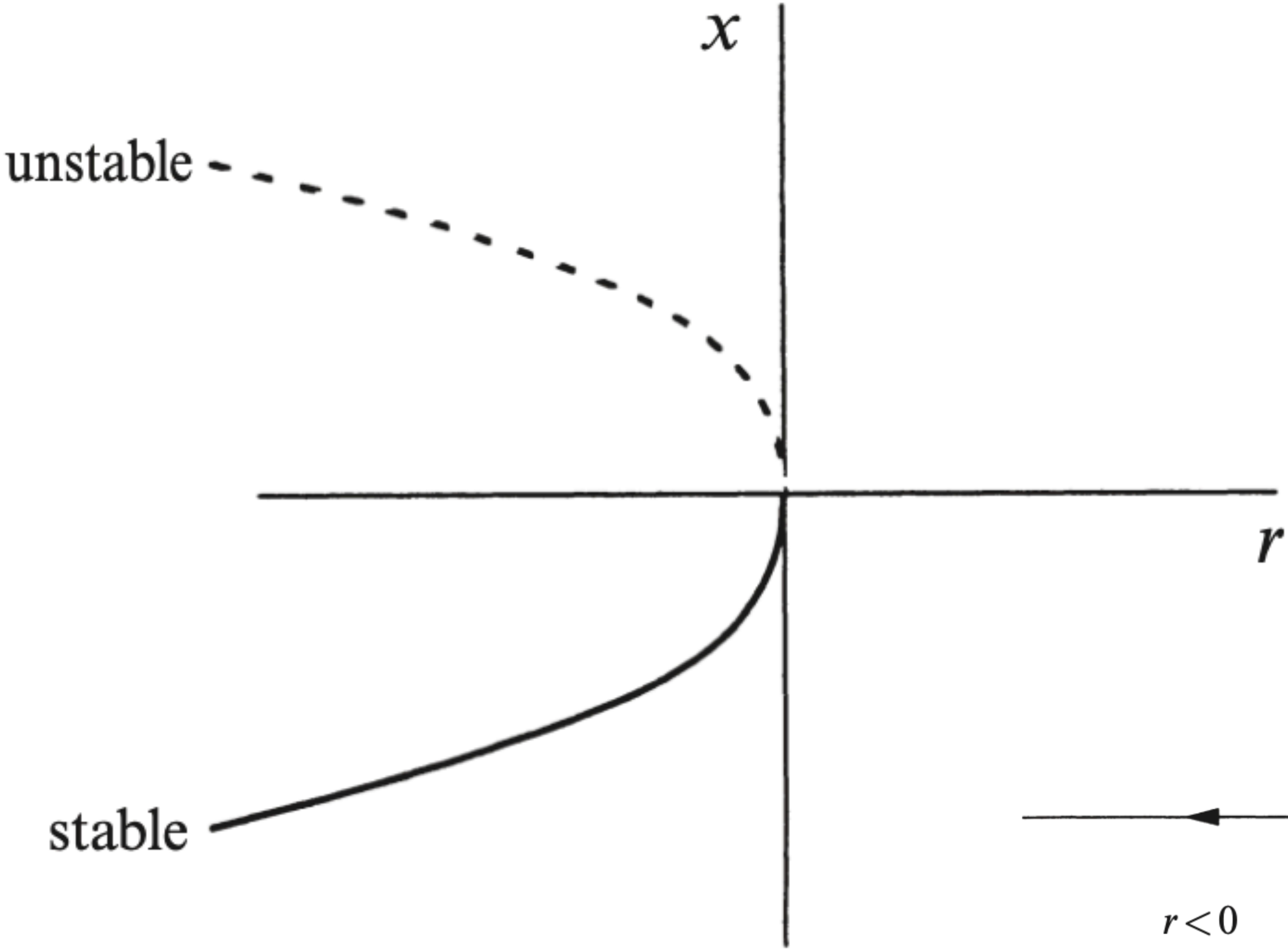
Dependence of the fixed points on *r*



Continuous stack of vector fields, a solid line for stable points and a broken line for unstable ones.

Saddle-node bifurcation: Bifurcation diagram

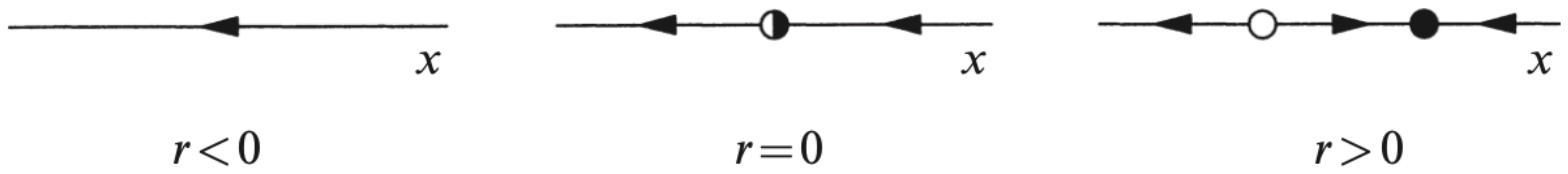
Axes are inverted.



Other names:

- fold bifurcation
- turning-point bifurcation
- blue sky bifurcation

$$\dot{x} = r - x^2$$



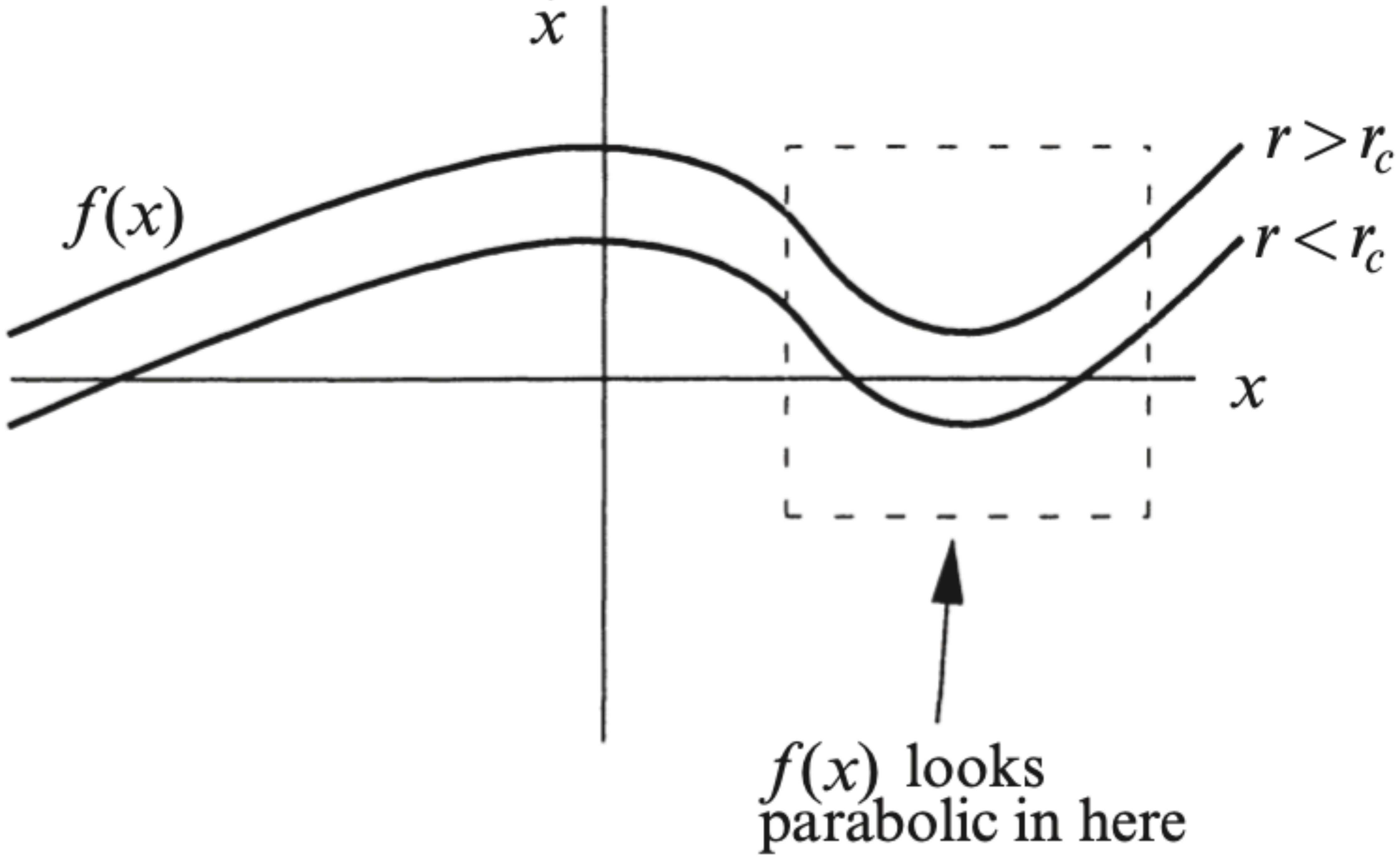
Saddle-node bifurcation: Normal forms

Normal forms for the saddle-node bifurcation:

$$\dot{x} = r + x^2$$

$$\dot{x} = r - x^2$$

These ODEs are representative of *all* saddle-node bifurcations.



Close to a saddle-node bifurcation, the dynamics typically look like them.

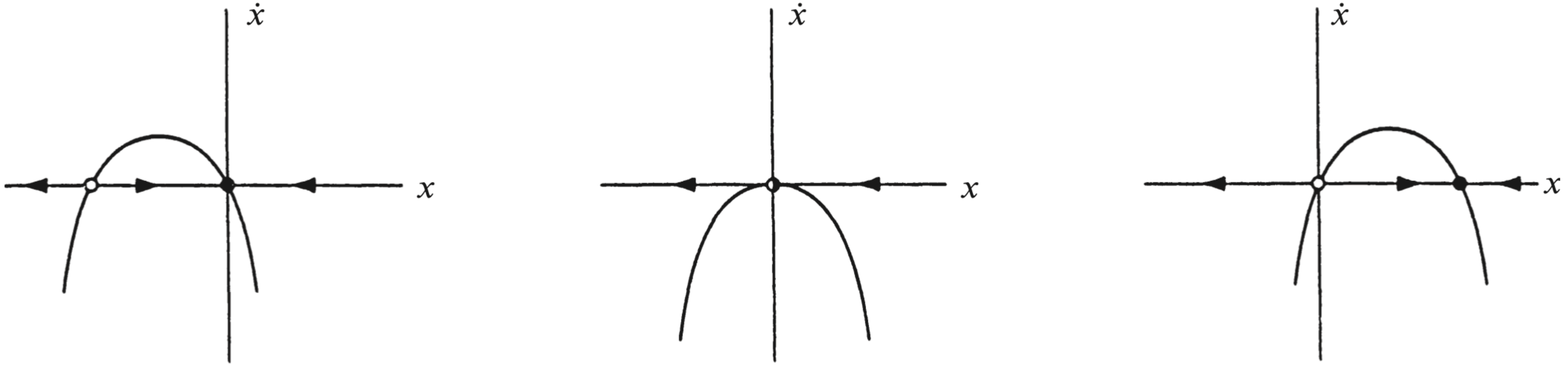
Transcritical Bifurcation

The transcritical bifurcation is the standard mechanism for changes in the stability of fixed points.

A fixed point must exist for all values of a parameter and can never be destroyed.

The normal form for a transcritical bifurcation is: $\dot{x} = rx - x^2$

where *x* and *r* to be either positive or negative

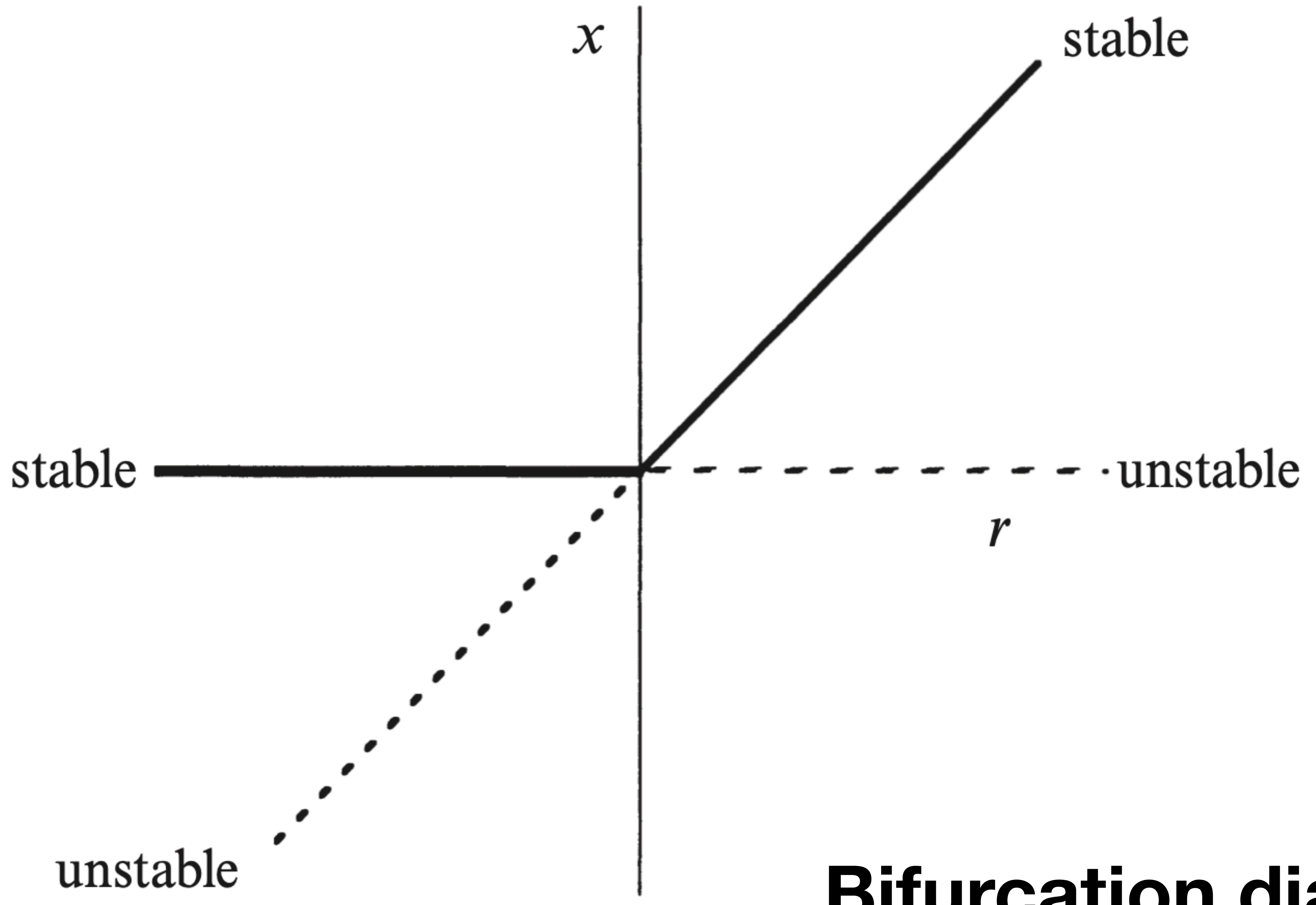


(a) $r<0$ (b) $r=0$ (c) $r>0$

Transcritical Bifurcation

An exchange of stabilities has taken place between the two fixed points:

In the transcritical case, the two fixed points don’t disappear after the bifurcation.



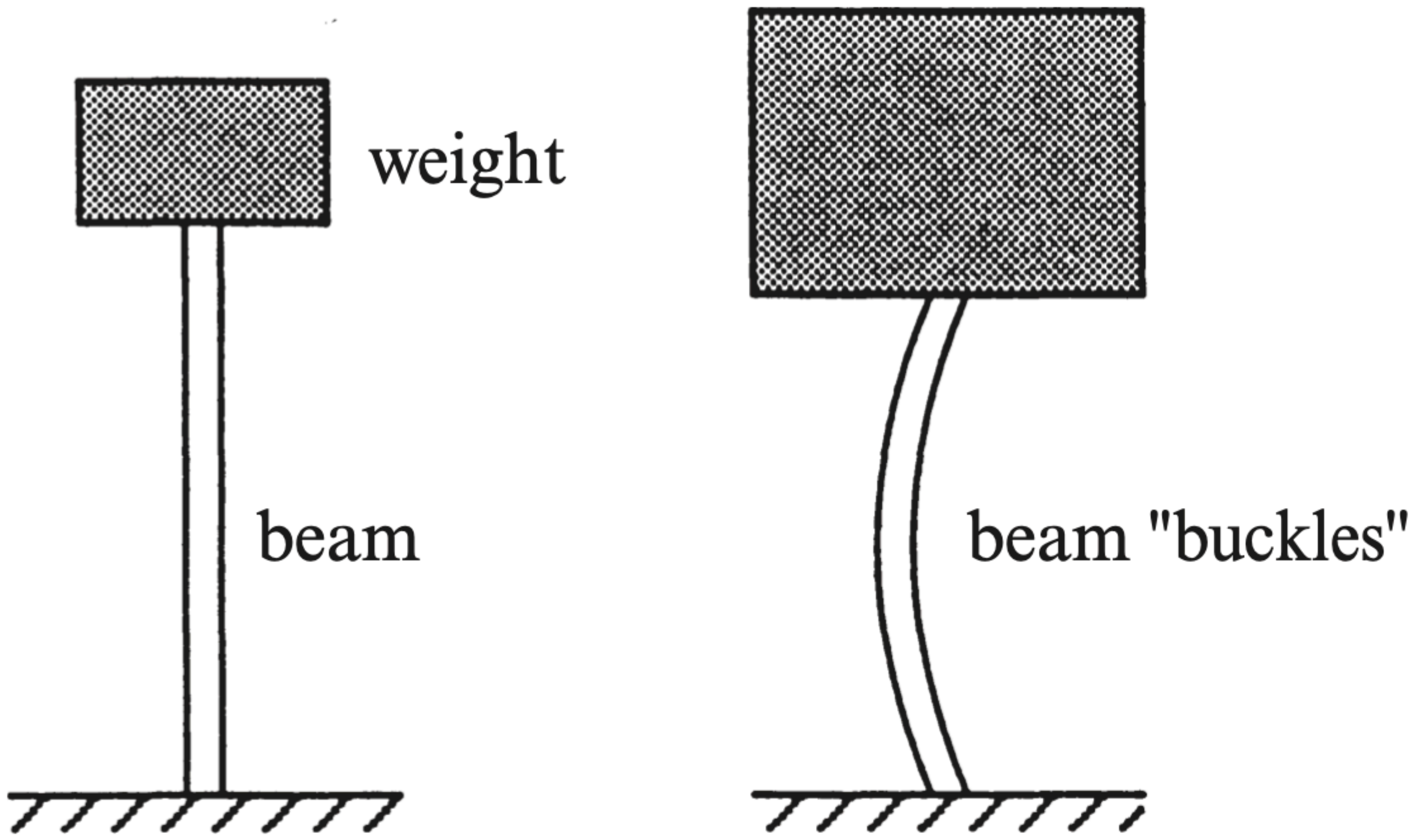
Parameter r is the independent variable, fixed points $x^* = 0$ and $x^* = r$ are dependent variables.

Bifurcation diagram for the transcritical bifurcation

Pitchfork Bifurcation

This bifurcation is common in physical problems that have a symmetry, e.g. a spatial symmetry between left and right.

Fixed points appear and disappear in symmetrical pairs.



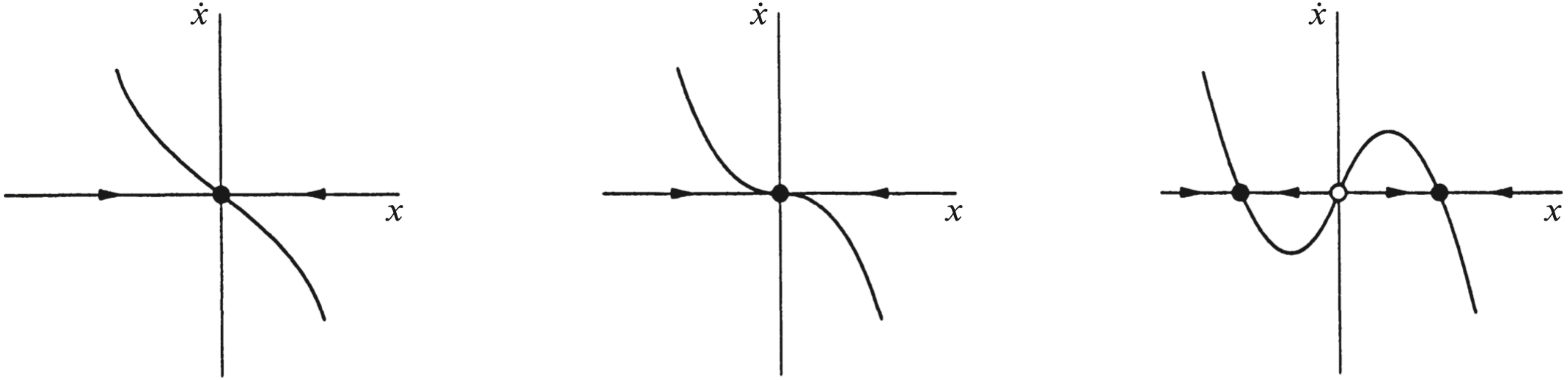
The beam is stable in the vertical position if the load is small. There is a stable fixed point corresponding to zero deflection.

If the load exceeds the buckling threshold, the beam may buckle to either the left or the right. The vertical position is unstable, and two new symmetrical fixed points.

Types of pitchfork bifurcation

Supercritical Pitchfork Bifurcation, with normal form: $$\dot{x} = rx - x^3$$

ODE is **invariant** under the change of variables x and -x. Invariance is the mathematical expression of the left-right symmetry.



(a) $r<0$

(b) $r=0$

(c) $r>0$

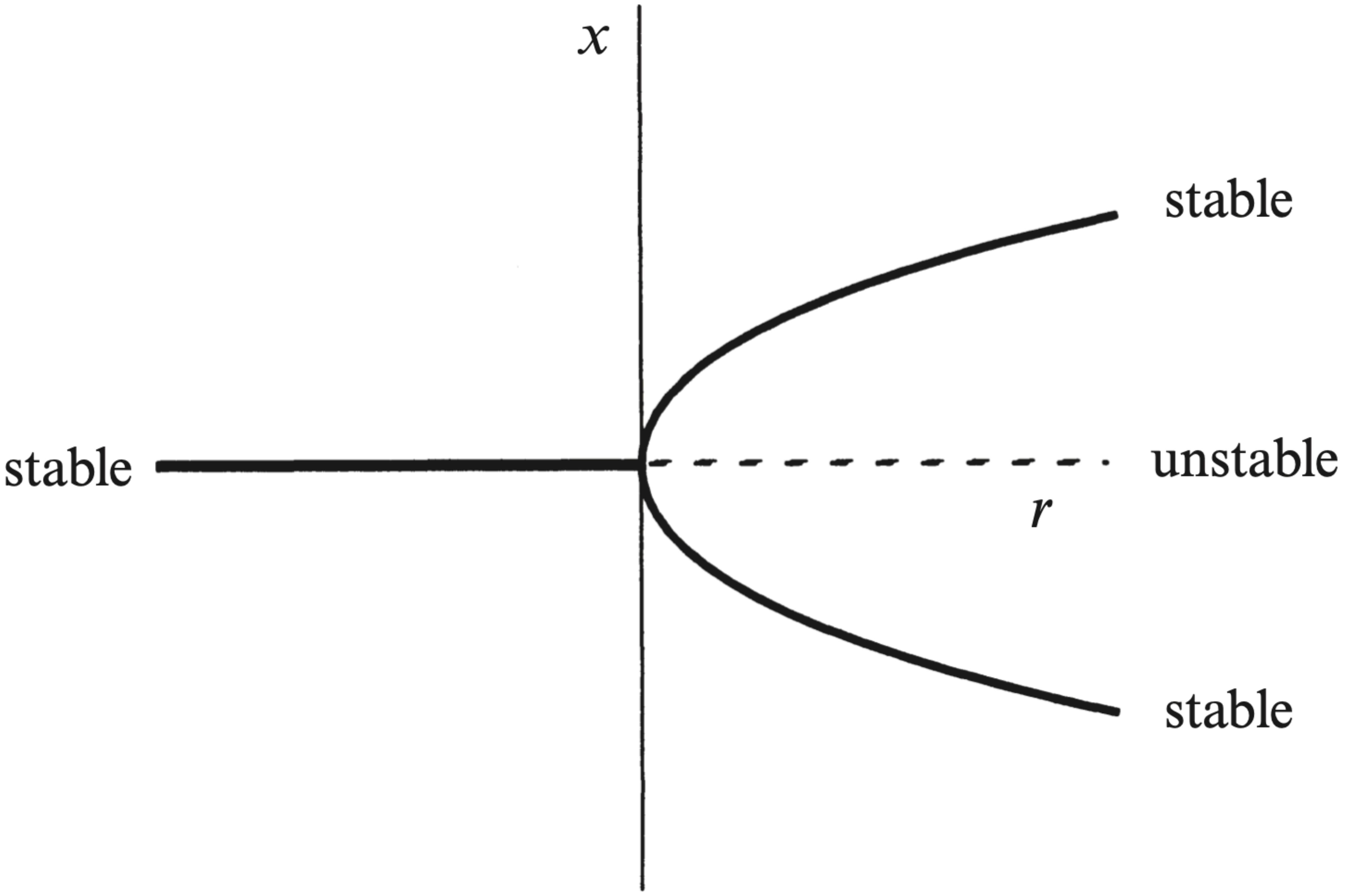
The origin is the only fixed point, and it is stable.

Critical slowing down

Origin has become unstable. Two new stable fixed points.

Bifurcation diagram

Supercritical Pitchfork Bifurcation, with normal form: $\dot{x} = rx - x^3$

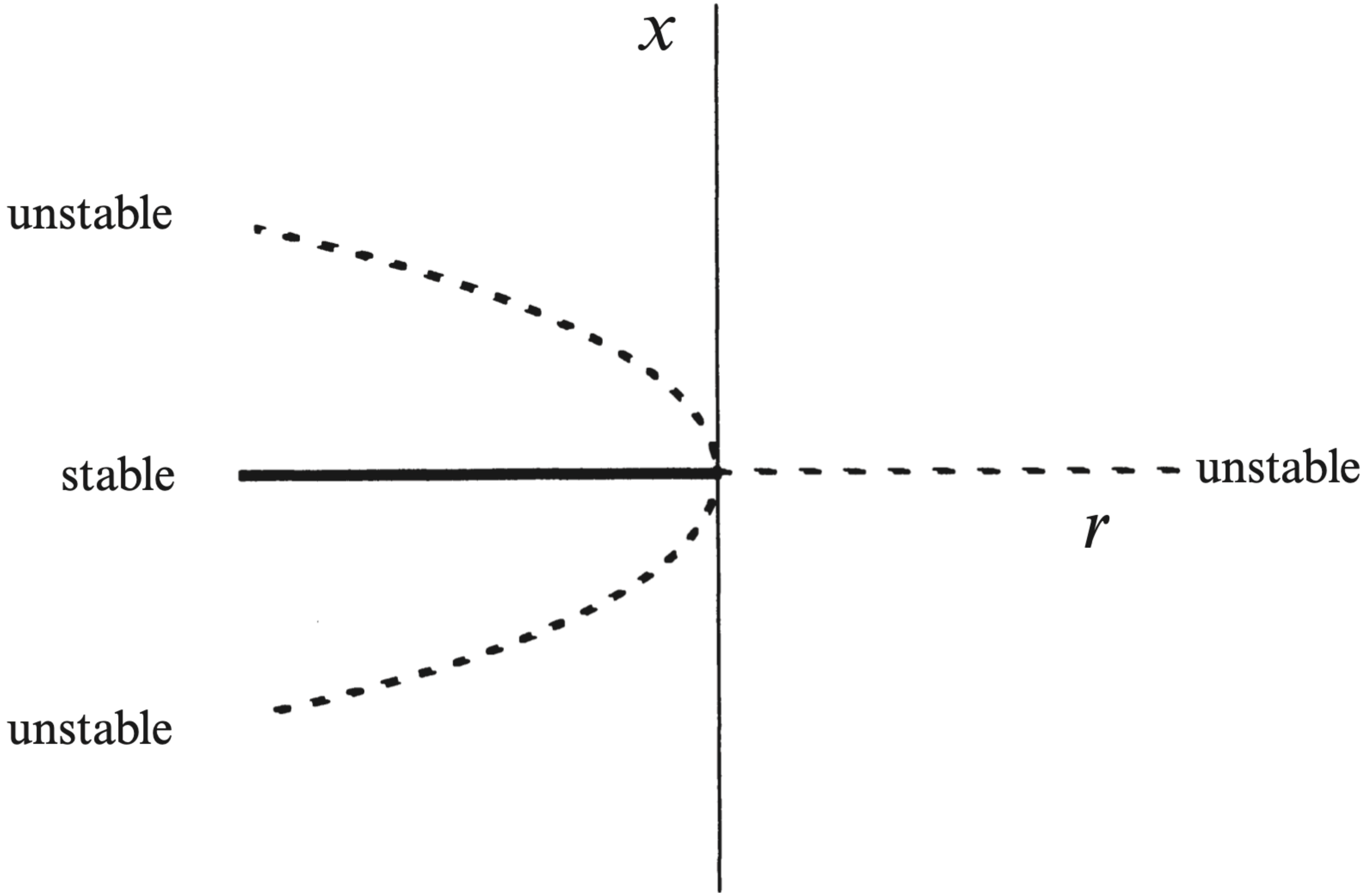


The cubic term is stabilising: it acts as a restoring force that pulls $x(t)$ back toward $x = 0$.

Types of pitchfork bifurcation

Subcritical Pitchfork Bifurcation, with normal form: $\dot{x} = rx + x^3$

The cubic term is destabilising.



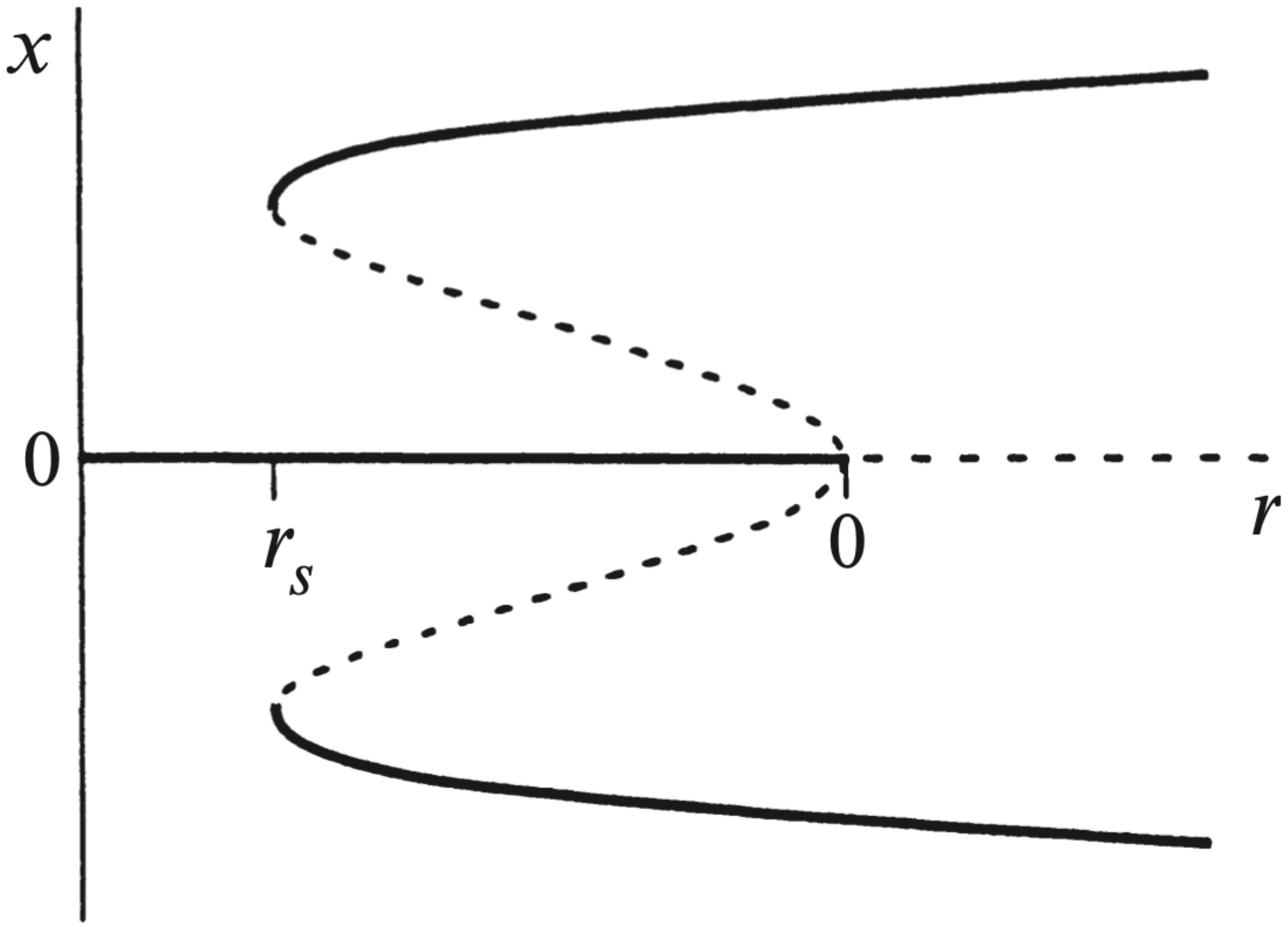
Nonzero fixed points are unstable, and exist only below the bifurcation for $r < 0$ **(subcritical).**

The cubic term help in driving the trajectories out to infinity (blow up).

In real physical systems, such an explosive instability is usually opposed by the stabilising influence of **higher-order terms**.

Types of pitchfork bifurcation

Subcritical Pitchfork Bifurcation, with normal form: $\dot{x} = rx + x^3$

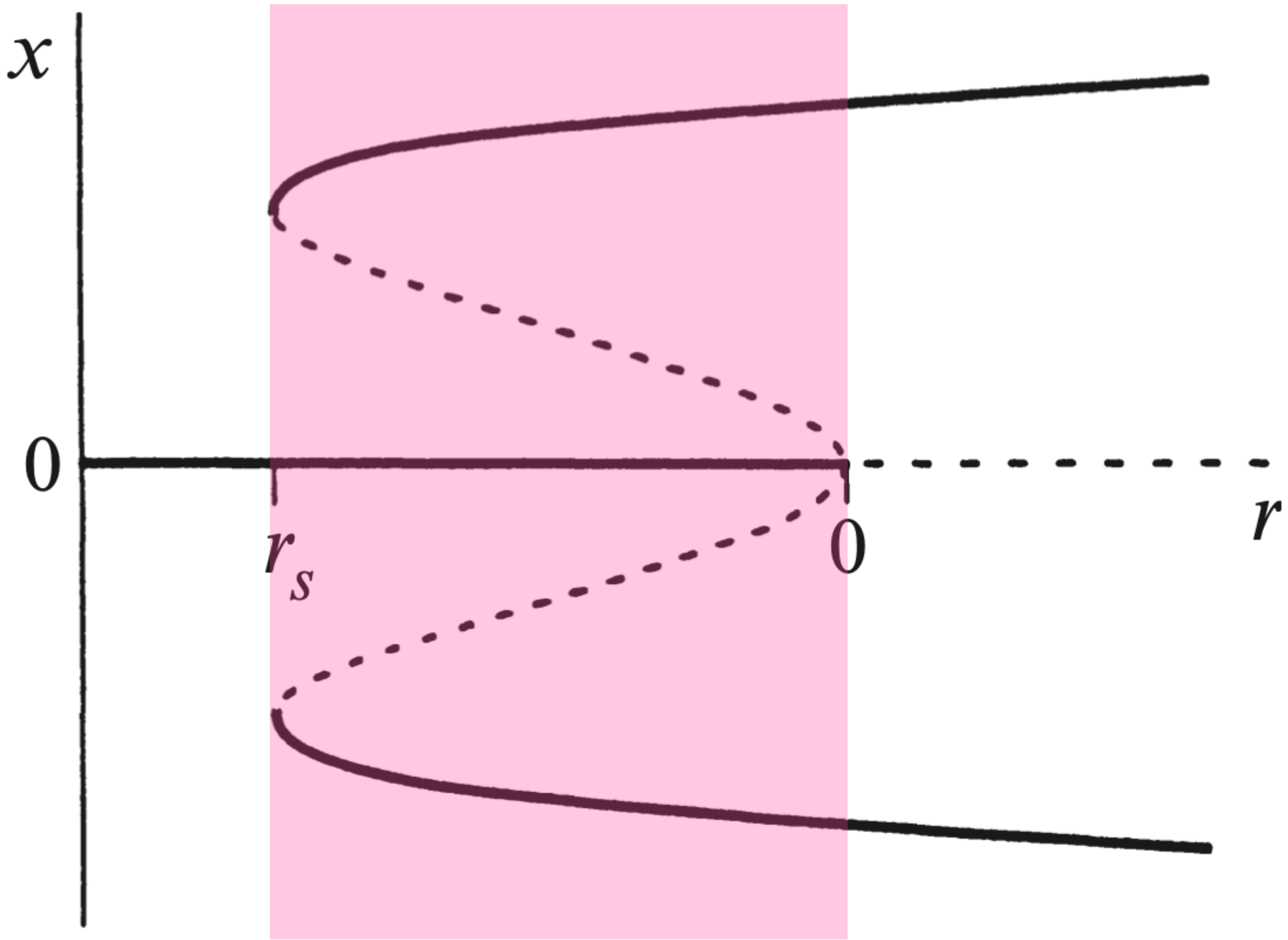


In real physical systems, such an explosive instability is usually opposed by the stabilising influence of higher-order terms.

$$\dot{x} = rx + x^3 - x^5$$

Stable large-amplitude branches exist for all $r > r_s$.

Subcritical Pitchfork Bifurcation

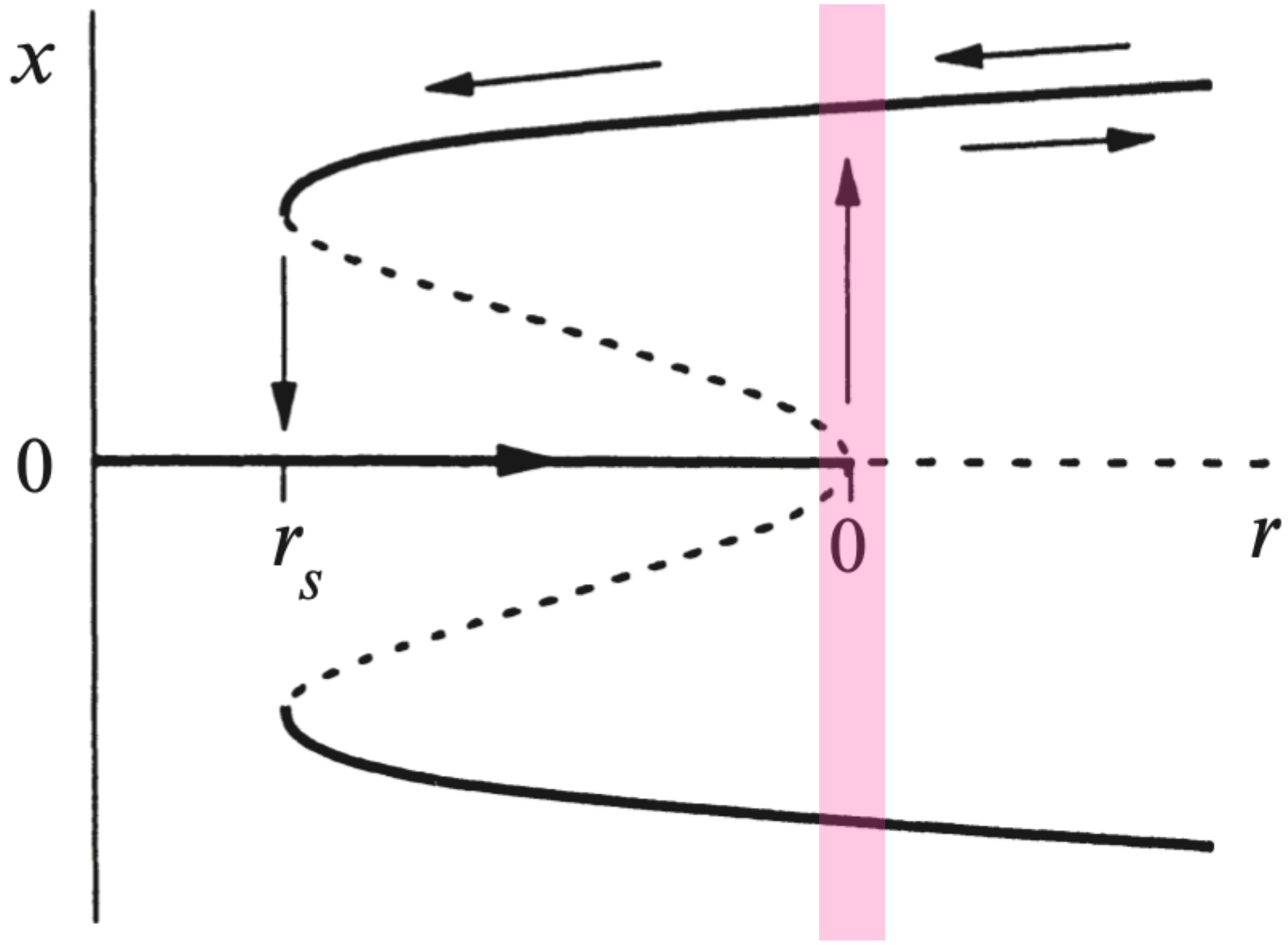


In the range $r_s < r < 0$, two qualitatively different stable states coexist (origin and the large-amplitude fixed points).

The initial condition x_0 determines which fixed point is approached as $t \to \infty$.

The origin is stable to small perturbations, but not to large ones. The origin is locally stable, but not globally stable.

Subcritical Pitchfork Bifurcation: jumps and hysteresis



The existence of different stable states allows for the possibility of ***jumps*** and ***hysteresis*** as r is varied.

If r is now decreased, the state remains on the large-amplitude branch, even when r is decreased below 0! **This lack of reversibility as a parameter is varied is called *hysteresis.***

We have to lower r even further (down past r_s) to get the state to jump back to the origin.

The bifurcation at r_s is a **saddle-node bifurcation.**

Imperfect Bifurcations and Catastrophes

In many real-world circumstances, the symmetry is only approximate—an imperfection leads to a slight difference between left and right.

Consider the system: $$\dot{x} = h + rx - x^3$$

If $h = 0$, we have the normal form for a **supercritical pitchfork bifurcation**, and there's a perfect symmetry between x and $-x$. But this symmetry is broken when $h \neq 0$; for this reason we refer to h as an ***imperfection parameter***.

We have two independent parameters: h and r. We'll think of r as fixed, and then examine the effects of varying h.

Imperfect Bifurcations and Catastrophes

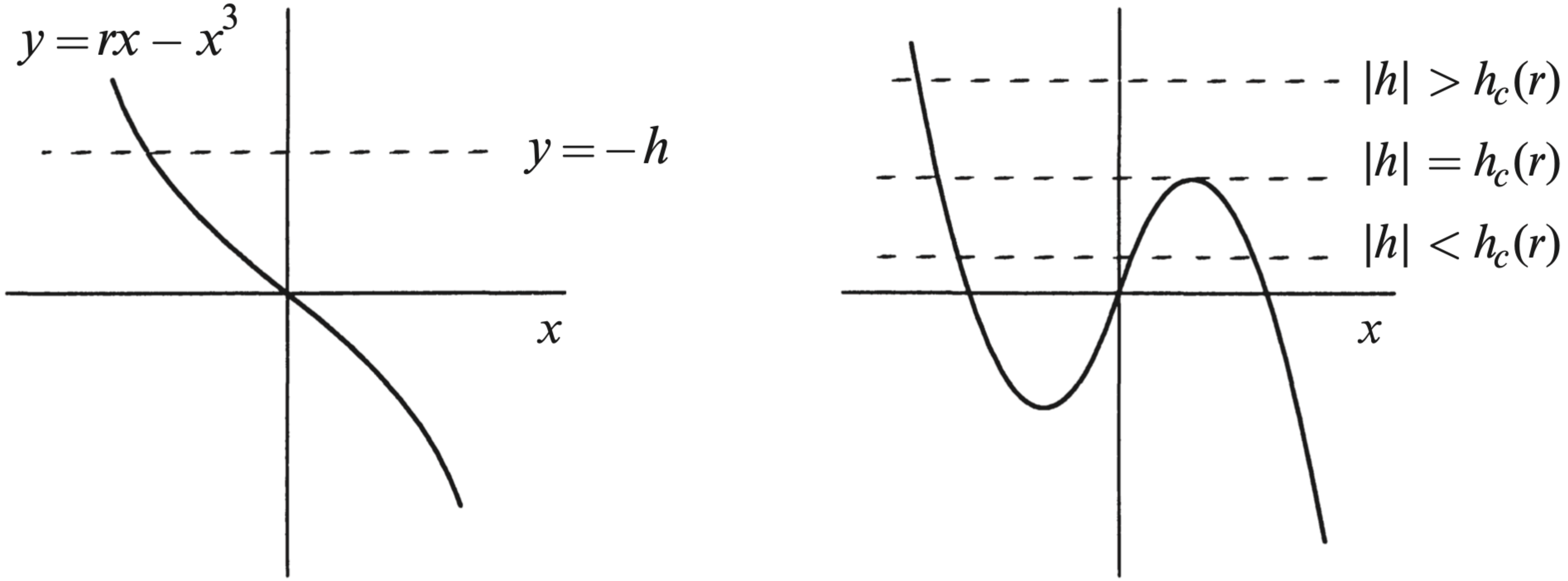
We need to find the roots: $\dot{x} = h + rx - x^3$

$$\tfrac{d}{dx}(rx - x^3) = r - 3x^2 = 0$$

$$x_{\max} = \sqrt{\frac{r}{3}}$$

$$rx_{\max} - (x_{\max})^3 = \frac{2r}{3}\sqrt{\frac{r}{3}}$$

$$h_c(r) = \frac{2r}{2}\sqrt{\frac{r}{3}}$$

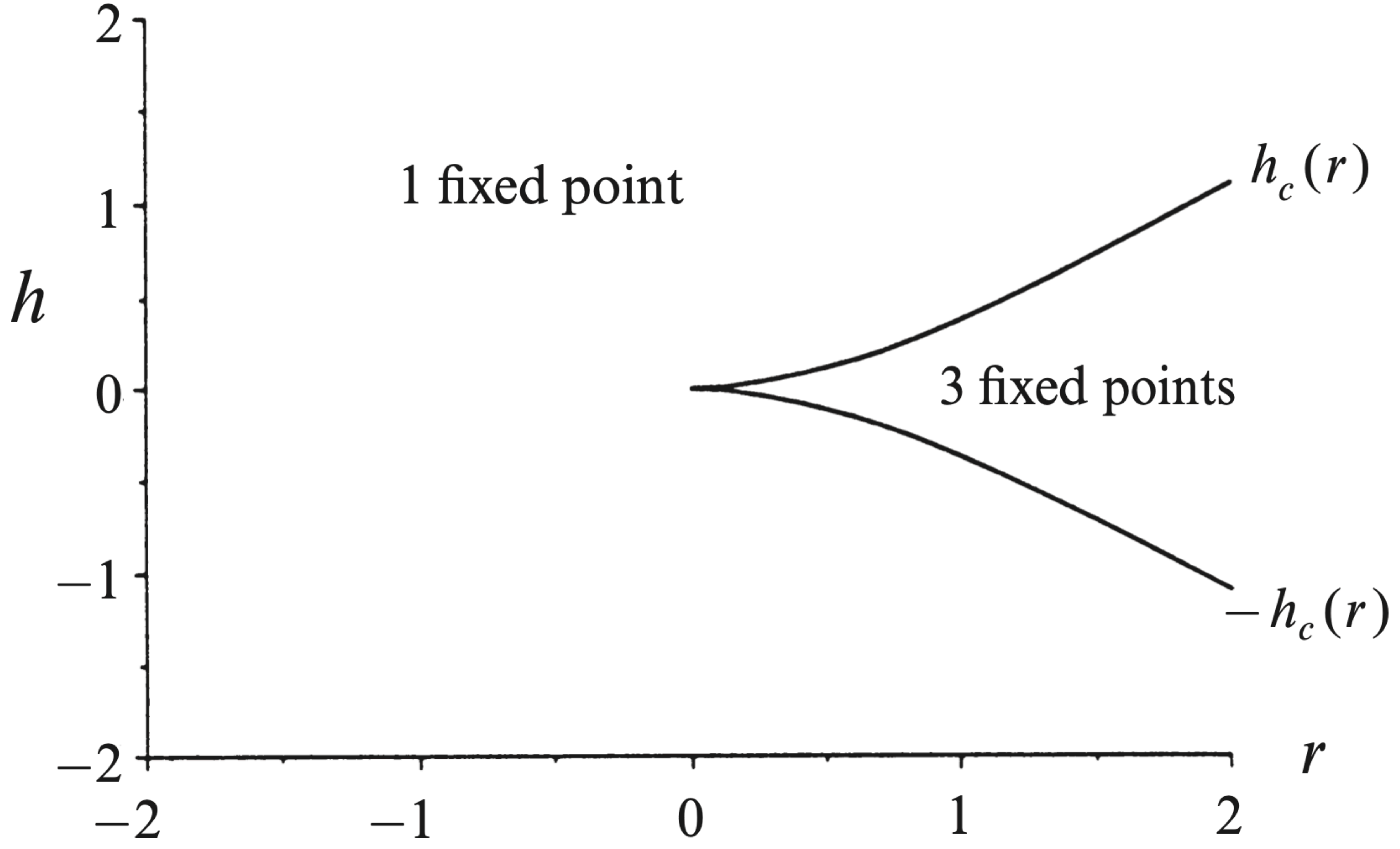


(a) $r \leq 0$ (b) $r > 0$

Saddle-node bifurcations occur when: $h = h_c(r)$

Parameter (r,h) space

Stability diagram



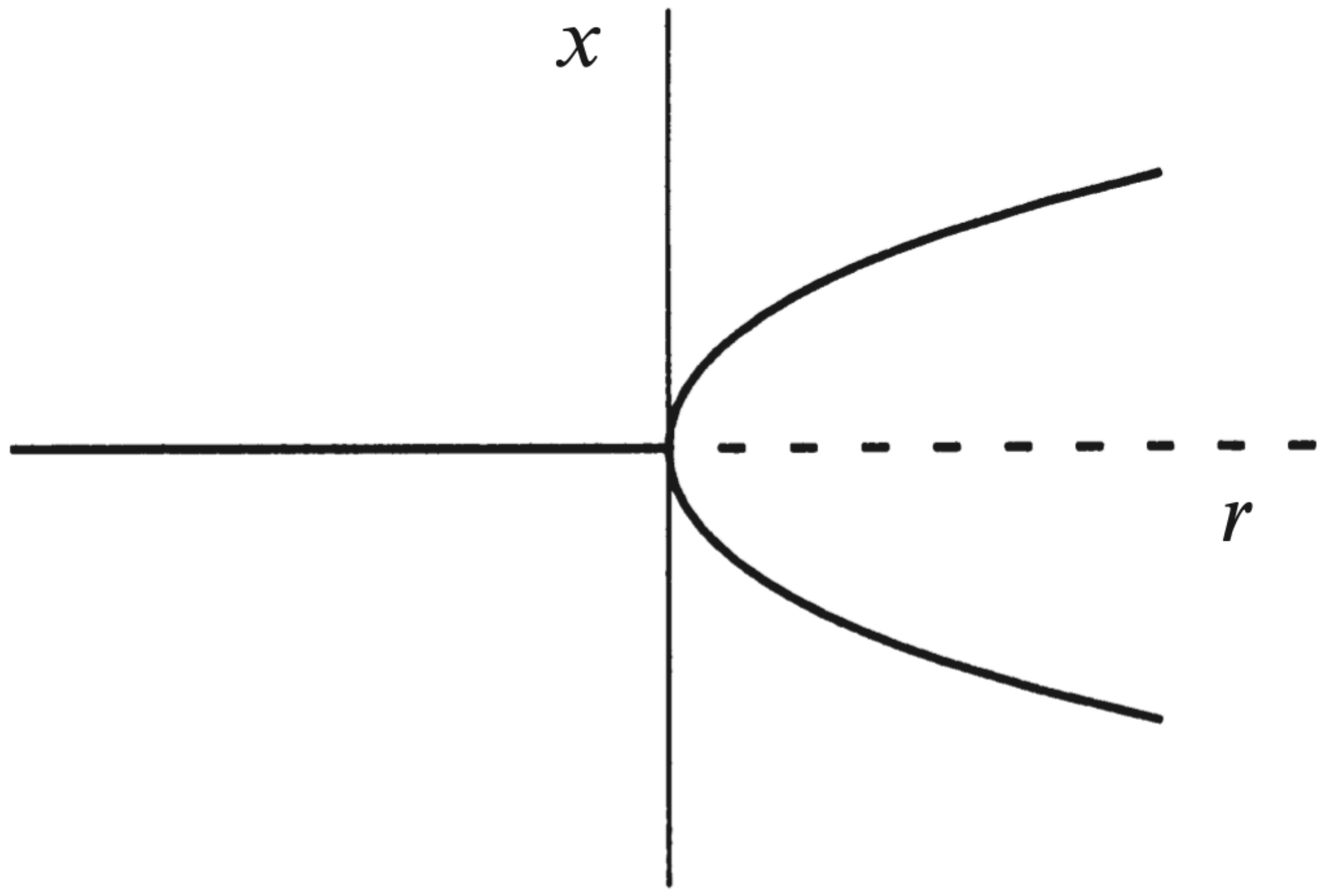
Bifurcation curves $h = \pm\, h_c(r)$

Saddle-node bifurcations occur all along the boundary of the regions, except at the cusp point

$(r, h) = (0{,}0)$ is called a cusp point (codimension-2 bifurcation.

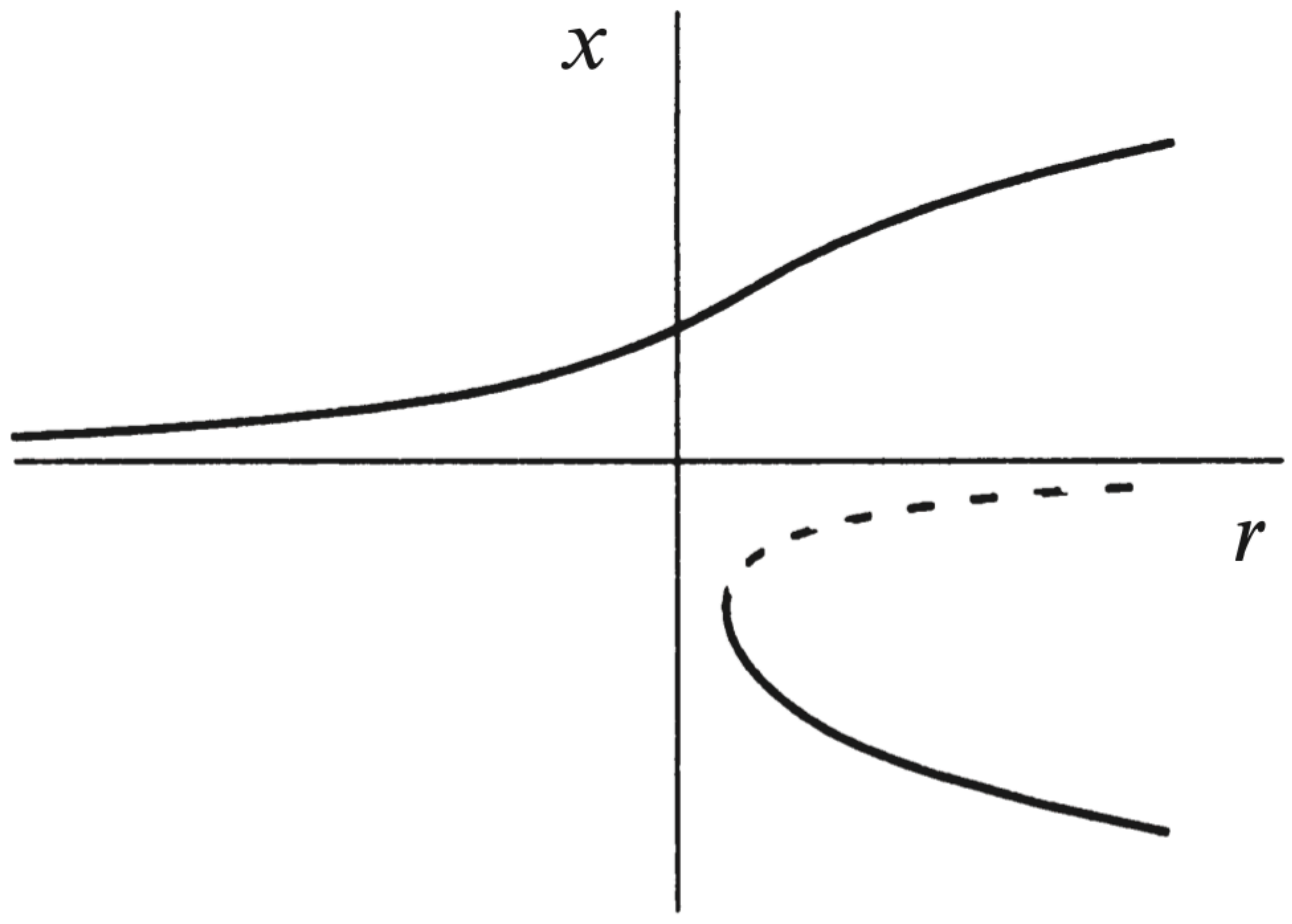
Bifurcation diagrams

Bifurcation diagram for fixed h



(a) $h = 0$

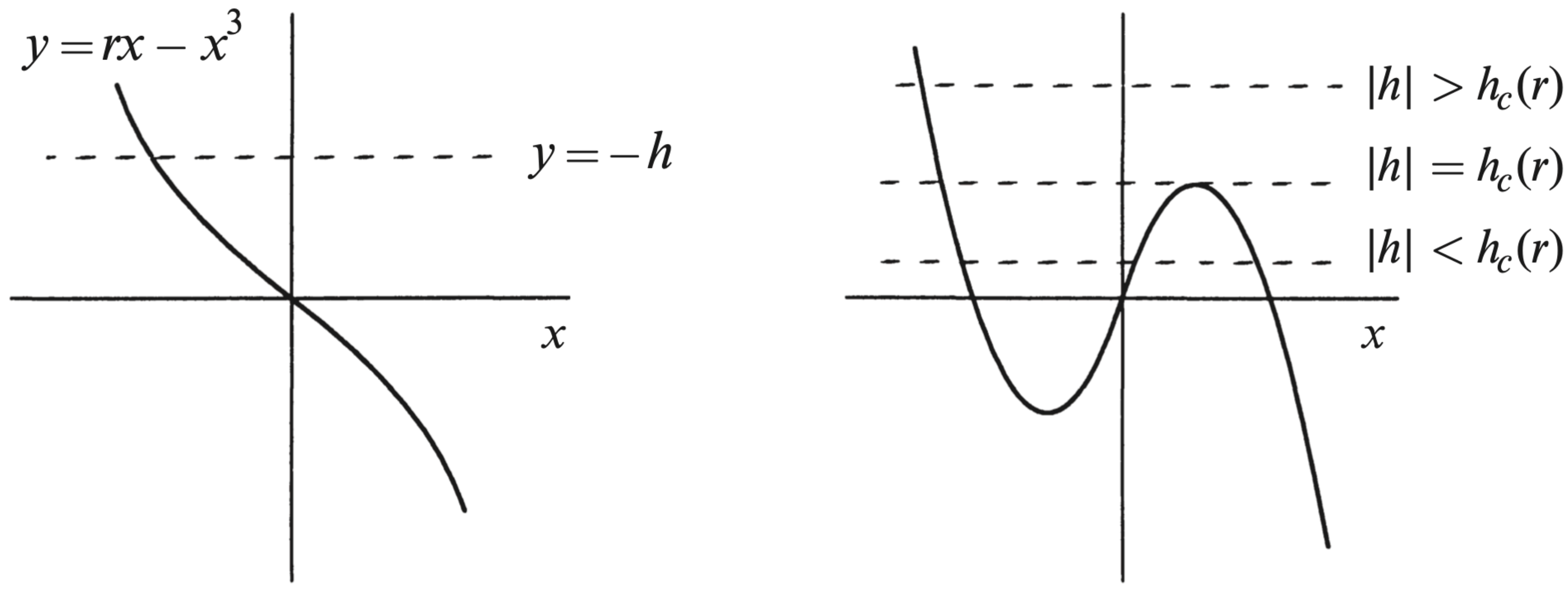
Pitchfork diagram



(b) $h \neq 0$

Pitchfork disconnects into two pieces

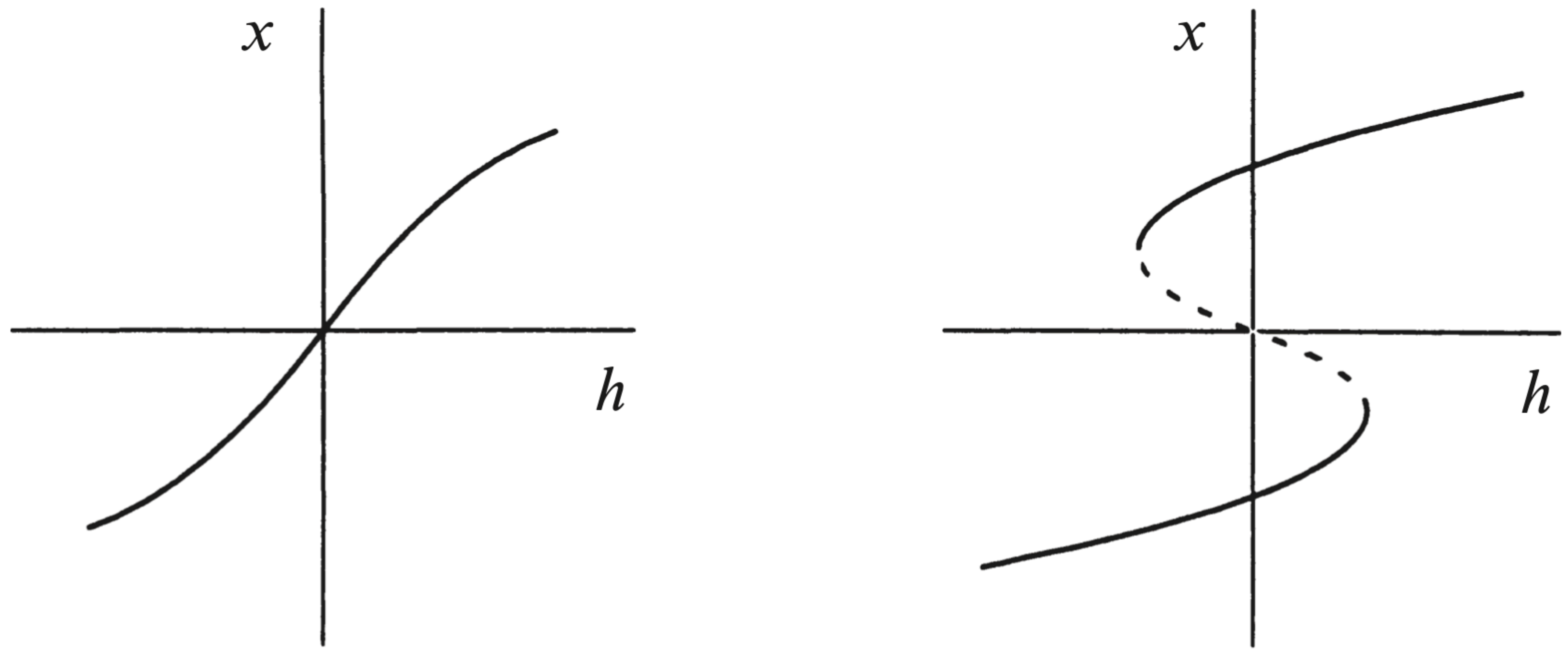
Bifurcation diagrams



(a) $r \leq 0$ (b) $r > 0$

Rotated by 90deg.

Bifurcation diagram for fixed r



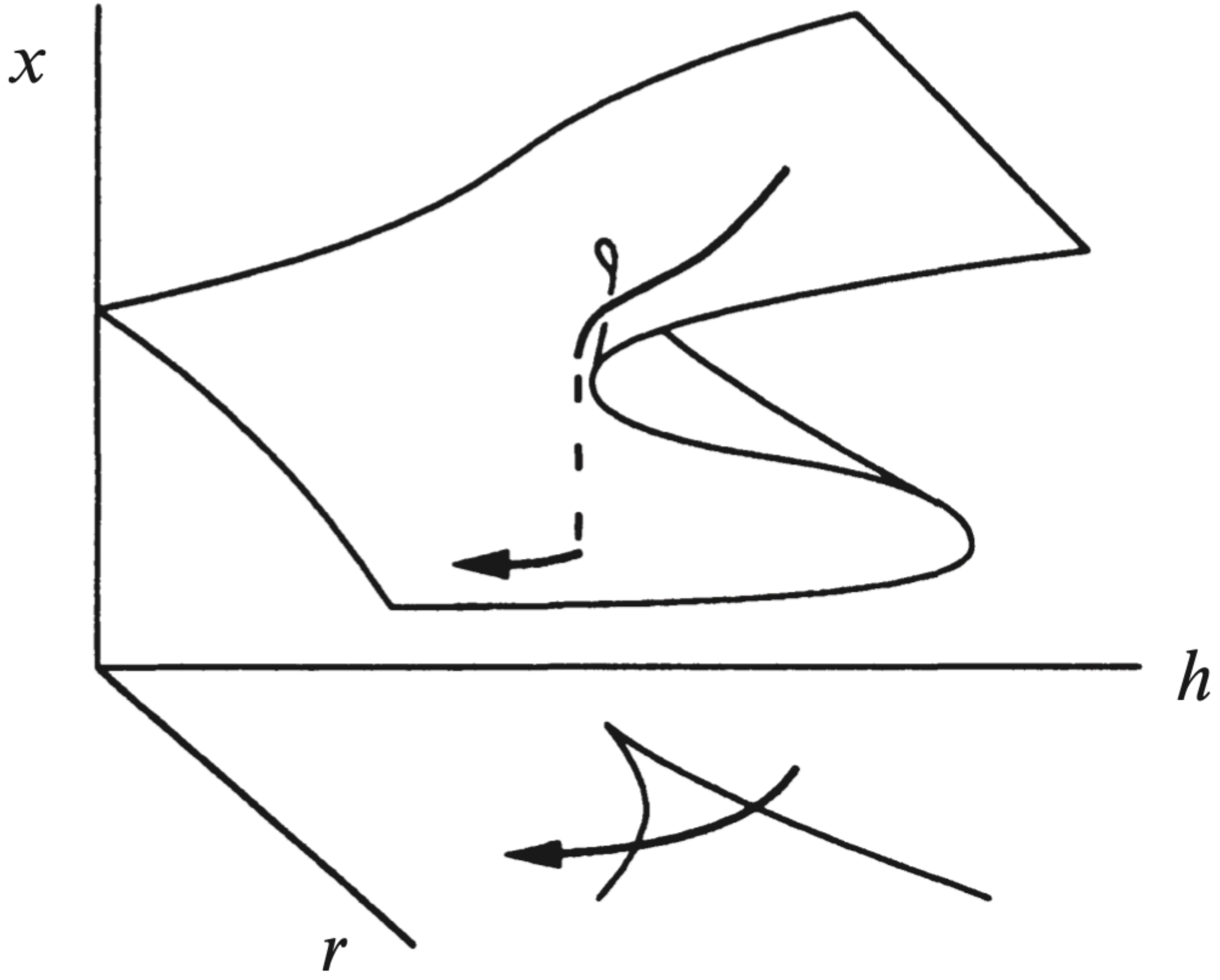
(a) $r \leq 0$

(b) $r > 0$

One stable fixed point for each h.

Pitchfork disconnects into two pieces

Cusp catastrophe surface

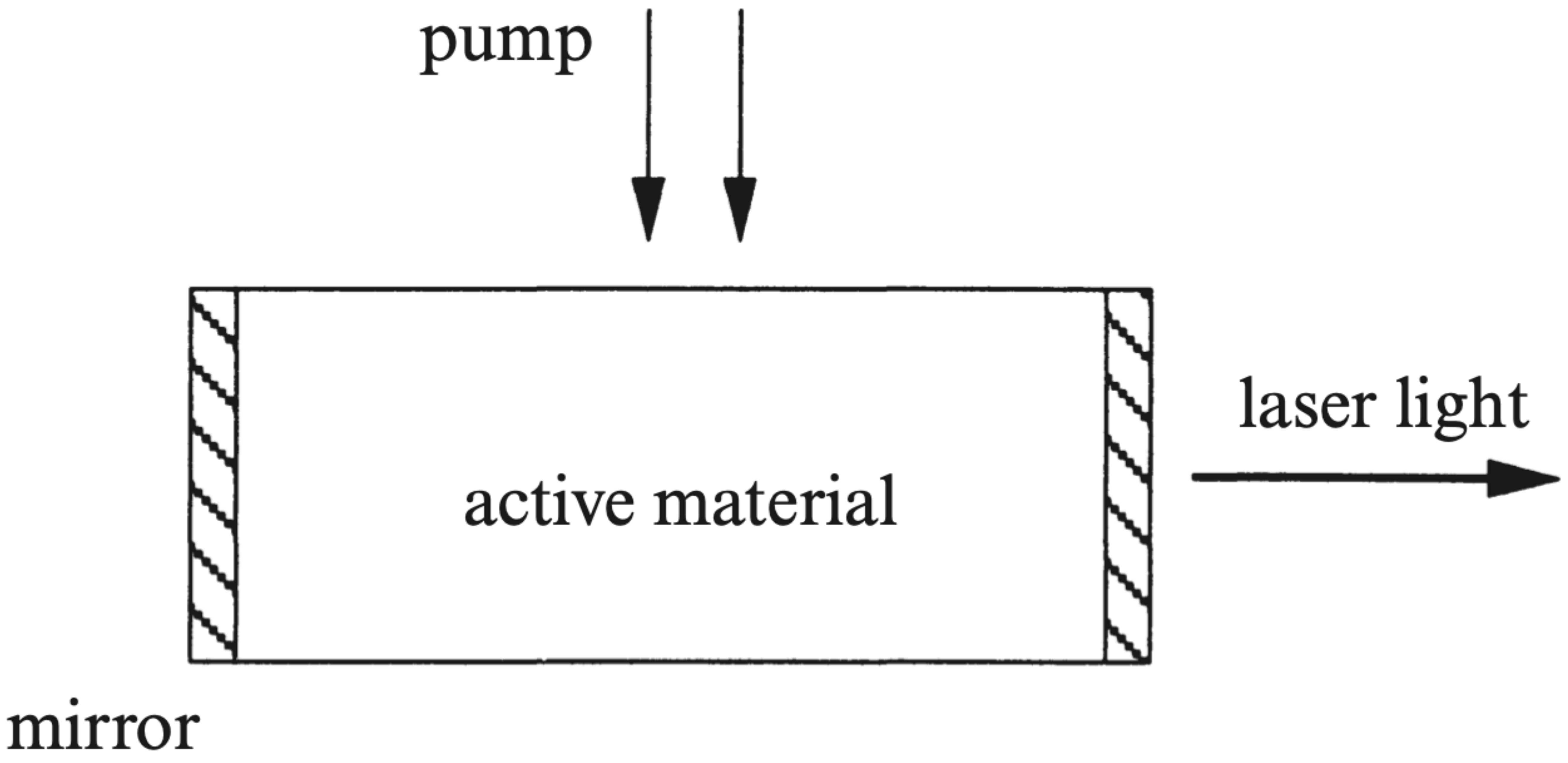


Catastrophe:

As parameters change, the state of the system can be carried over the edge of the **upper surface**, after which it drops discontinuously to the **lower surface.**

Applications: Laser Threshold

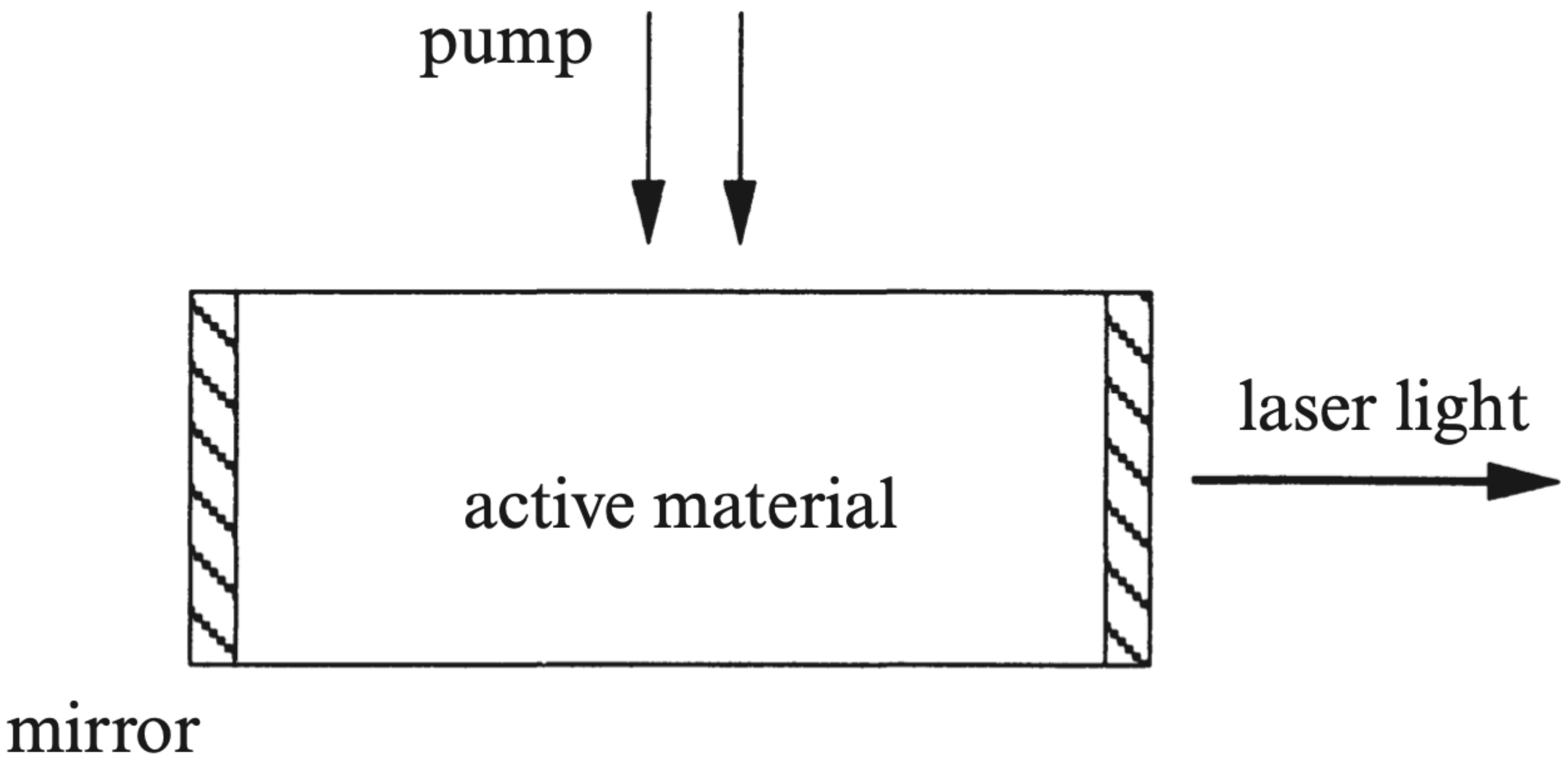
We'll consider a particular type of laser known as a solid-state laser, which consists of a collection of special "laser-active" atoms embedded in a solid-state matrix, bounded by partially reflecting mirrors at either end.



An external energy source is used to excite or "pump" the atoms out of their ground states.

Each atom can be thought of as a little antenna radiating energy.

Applications: Laser Threshold



Under relatively weak pumping, the laser acts like a **lamp**: the excited atoms oscillate independently of one another and emit randomly phased light waves.

When the pump strength exceeds a certain threshold, the atoms begin to oscillate in phase—the lamp has turned into a **laser**.

Trillions of little antennas act like one giant antenna and produce a **beam of radiation**, much more coherent and intense than that produced below the **laser threshold.**

Applications: Laser Threshold

This sudden onset of coherence is amazing, considering that the atoms are being excited completely at random by the pump!

Hence the process is *self-organising*: the coherence develops because of a cooperative interaction among the atoms themselves.

Simplified (non-quantum) Model:

$$\begin{aligned}\dot{n} &= \text{gain} - \text{loss} \\ &= GnN - kn.\end{aligned}$$

The dynamical variable is the number of photons $n(t)$ in the laser field.

Applications: Laser Threshold

Simplified (non-quantum) Model:

$$\begin{aligned}\dot{n} &= \text{gain} - \text{loss} \\ &= GnN - kn.\end{aligned}$$

Gain: process of stimulated emission, in which photons stimulate excited atoms to emit additional photons. Because this process occurs via random encounters between photons and excited atoms, it occurs at a rate proportional to *n* and to the number of excited atoms, denoted by $N(t)$.

The parameter $G > 0$ is known as the gain coefficient.

Loss: escape of photons through the end faces of the laser. The parameter $k > 0$ is a rate constant; its reciprocal $= 1/k$ represents the typical lifetime of a photon in the laser.

Applications: Laser Threshold

Simplified (non-quantum) Model:

$$\begin{aligned}\dot{n} &= \text{gain} - \text{loss} \\ &= GnN - kn.\end{aligned}$$

Key physical idea: after an excited atom emits a photon, it drops down to a lower energy level and is no longer excited. N decreases by the emission of photons.

To capture this effect, $N(n)$. Suppose that in the absence of laser action, the pump keeps the number of excited atoms fixed at N_0. Then the actual number of excited atoms will be reduced by the laser process. Specifically, we assume:

$$N(t) = N_0 - \alpha n$$

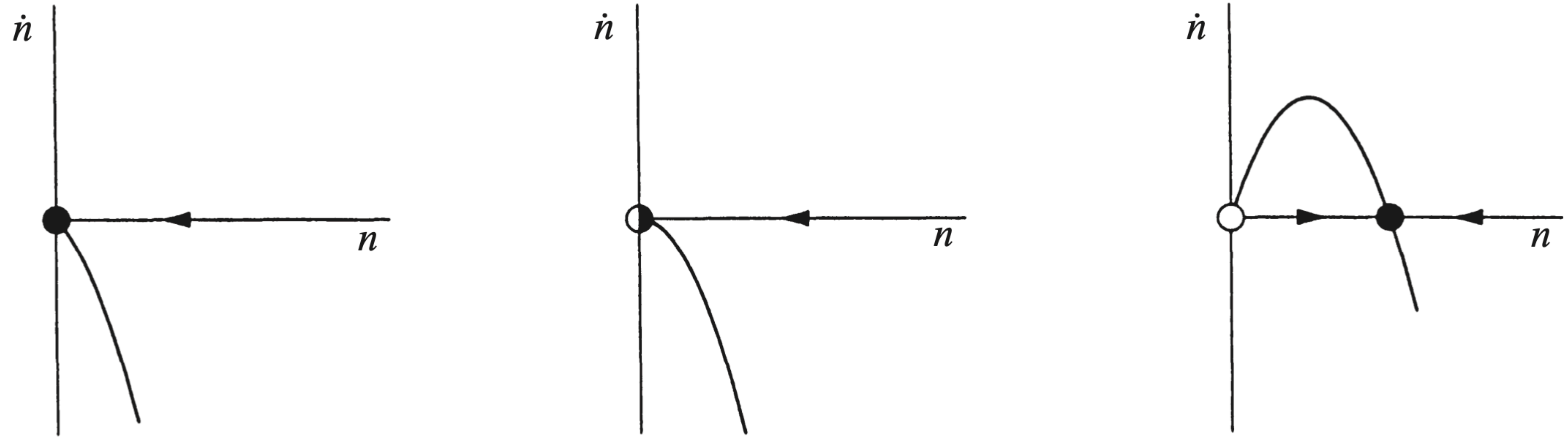
where $\alpha > 0$ is the rate at which atoms drop back to their ground states.

Applications: Laser Threshold

First-order system for $n(t)$:

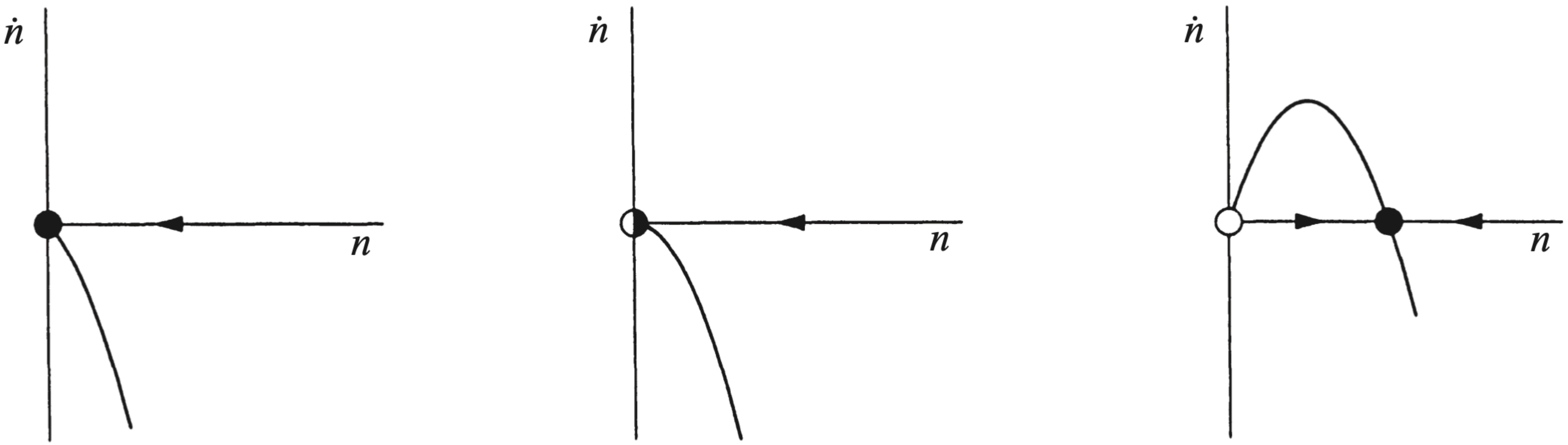
$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2$$

Vector field for different values of the pump strength N_0



$N_0 < k/G$ $N_0 = k/G$ $N_0 > k/G$

Applications: Laser Threshold



$N_0 < k/G$ $\quad$ $N_0 = k/G$ $\quad$ $N_0 > k/G$

$n* = 0$ is stable

No stimulated emission and the laser acts like a lamp

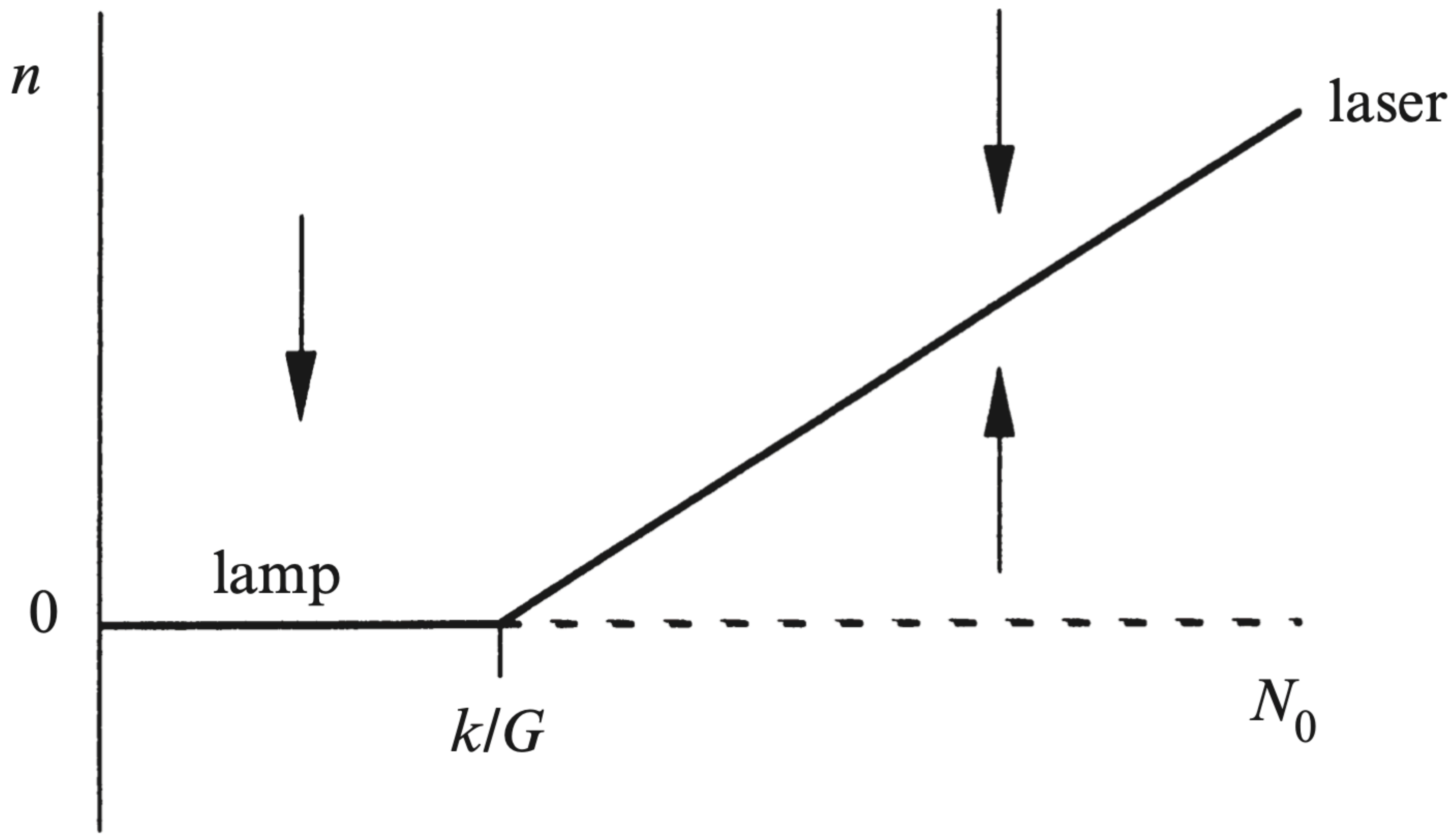
Transcritical bifurcation when $N_0 = k/G$ (**laser threshold**)

$n* = 0$ loses stability

Spontaneous laser action:

$$n^* = (GN_0 - k)/\alpha G > 0$$

Applications: Laser Threshold



Spontaneous laser action:

$$n^* = (GN_0 - k)/\alpha G > 0$$

Caveats:

Dynamics of the excited atoms

Existence of spontaneous emission

Applications: Insect Outbreak

Model for the sudden outbreak of an insect called the spruce budworm that attacks the leaves of the balsam fir tree.

https://en.wikipedia.org/wiki/Abies_balsamea

https://en.wikipedia.org/wiki/Choristoneura

When an outbreak occurs, the budworms can defoliate and kill most of the fir trees in the forest in about four years (**bifurcation and catastrophe**).

Applications: Insect Outbreak

Ludwig et al. (1978) model:

Interaction between budworms and the forest, exploiting a **separation of time scales.**

The **budworm population evolves on a fast time scale** (they can increase their density fivefold in a year, so they have a characteristic time scale of months).

The t**rees grow and die on a slow time scale** (they can completely replace their foliage in about 7–10 years, and their life span in the absence of budworms is 100–150 years.)

As far as the budworm dynamics are concerned, **the forest variables may be treated as constants**. At the end of the analysis, we will allow the forest variables to drift very slowly—this drift ultimately triggers an outbreak.

Applications: Insect Outbreak

Ludwig et al. (1978) model:

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N)$$

In the absence of predators, the budworm population $N(t)$ is assumed to grow logistically with growth rate R and carrying capacity K.

The carrying capacity depends on the amount of foliage left on the trees, and so it is a slowly drifting parameter (fixed).

The term $p(N)$ represents the death rate due to *predation:*

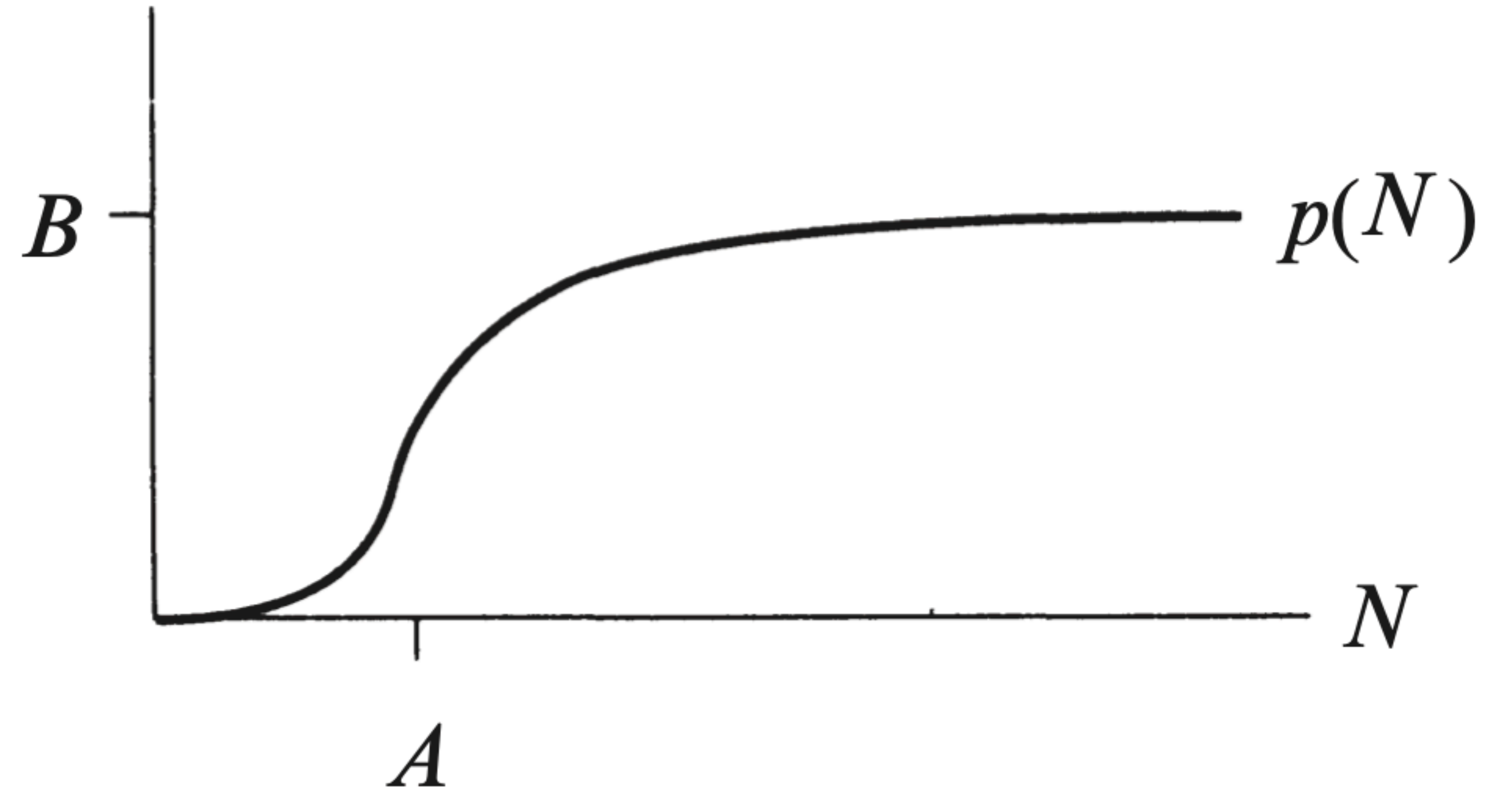
$$p(N) = \frac{BN^2}{A^2 + N^2}$$

$$A, B > 0.$$

Applications: Insect Outbreak

Ludwig et al. (1978) model:

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}$$



The model has four parameters: *R*, *K*, *A*, and *B*.

What do we mean by an "outbreak" in the context of this model?

The budworm population suddenly jumps from a low to a high level

Are there solutions with this character?

It is convenient to recast the model into a dimensionless form.

Applications: Insect Outbreak

Dimensionless Formulation:

We will scale the equation so that all the dimensionless groups are pushed into the logistic part of the dynamics, with none in the predation part.

We divide by B and let: $x = N/A$

$$\dot{N} = RN\left(1-\frac{N}{K}\right) - \frac{BN^2}{A^2+N^2} \quad \Rightarrow \quad \frac{A}{B}\frac{dx}{dt} = \frac{R}{B}Ax\left(1-\frac{Ax}{K}\right) - \frac{x^2}{1+x^2}$$

We should introduce a dimensionless time and dimensionless groups *r* and *k*:

$$\tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}$$

Applications: Insect Outbreak

Dimensionless Formulation & Analysis of Fixed Points

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

Here r and k are the **dimensionless growth rate and carrying capacity,** respectively.

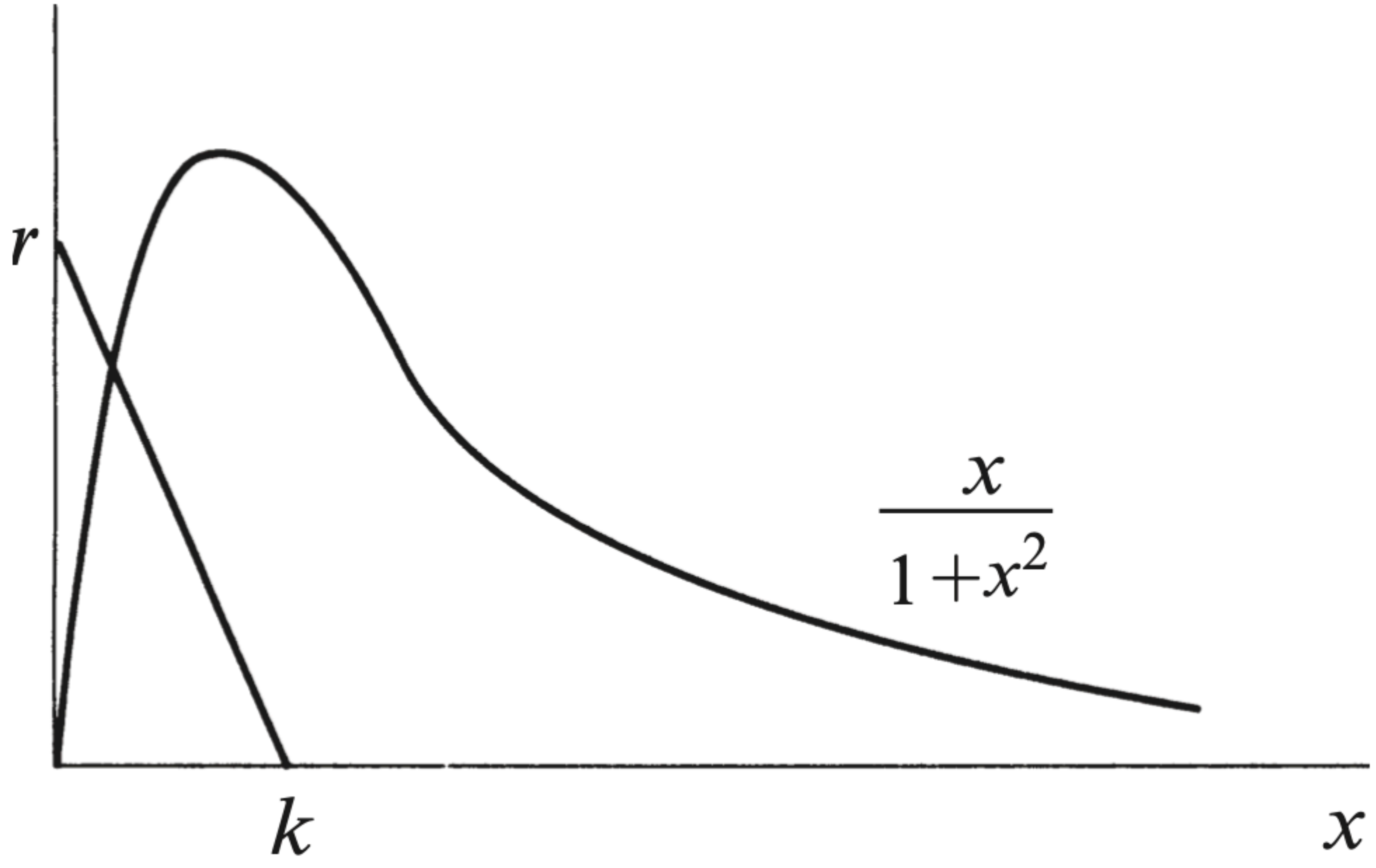
We have a fixed (always unstable) point at $x^* = 0$. The predation is extremely weak for small x, and so the budworm population grows exponentially for x near zero.

Other fixed points:

$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$$

Applications: Insect Outbreak

Analysis of Fixed Points



$$r\left(1-\frac{x}{k}\right)=\frac{x}{1+x^2}$$

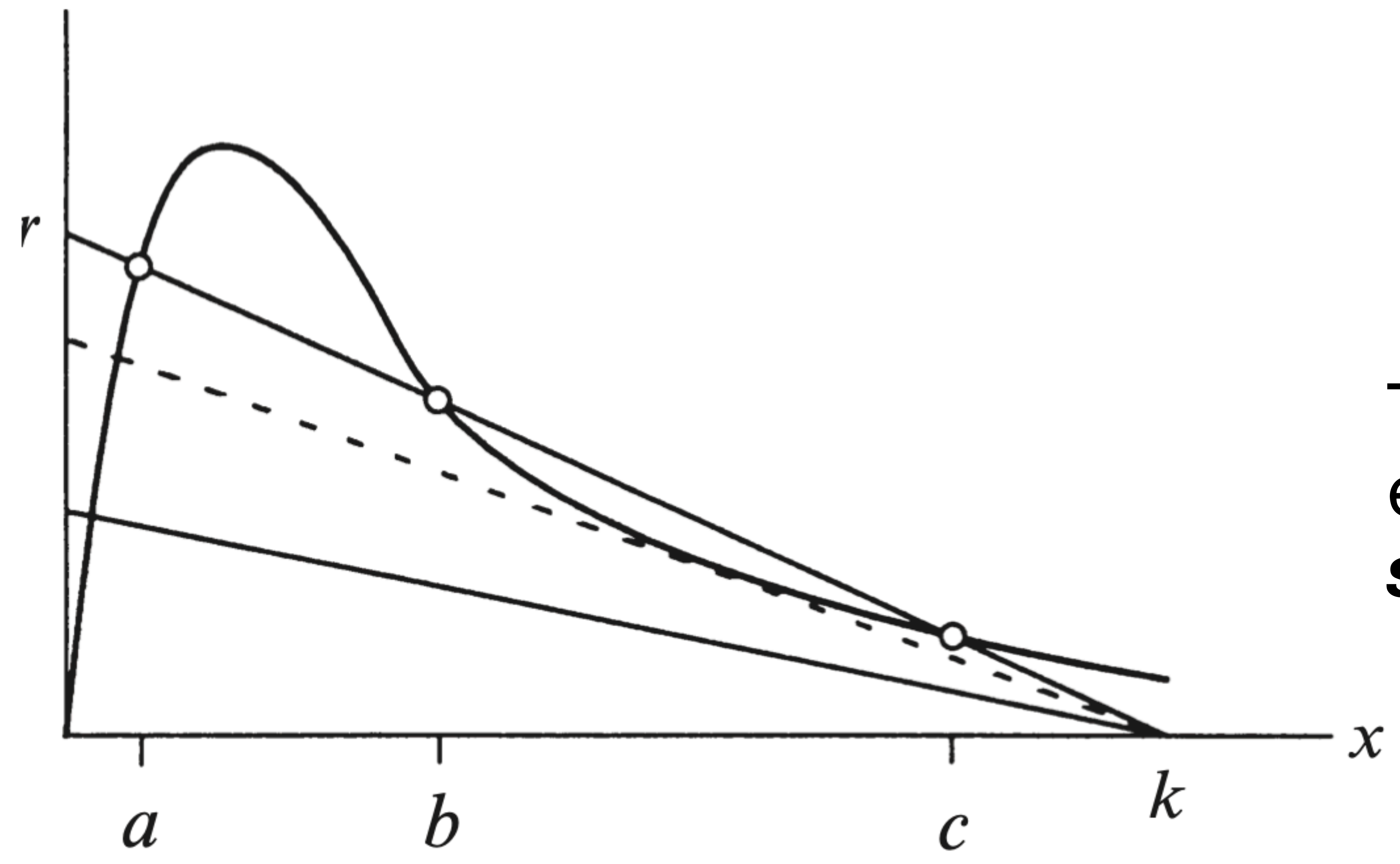
The r-k line moves out, but not the curve (non-dimensionalisation).

If *k* is sufficiently small, there is exactly one intersection for any *r >0*.

For large *k*, we can have one, two, or three intersections, depending on the value of *r,*

Applications: Insect Outbreak

Analysis of Fixed Points



$$r\left(1-\frac{x}{k}\right)=\frac{x}{1+x^2}$$

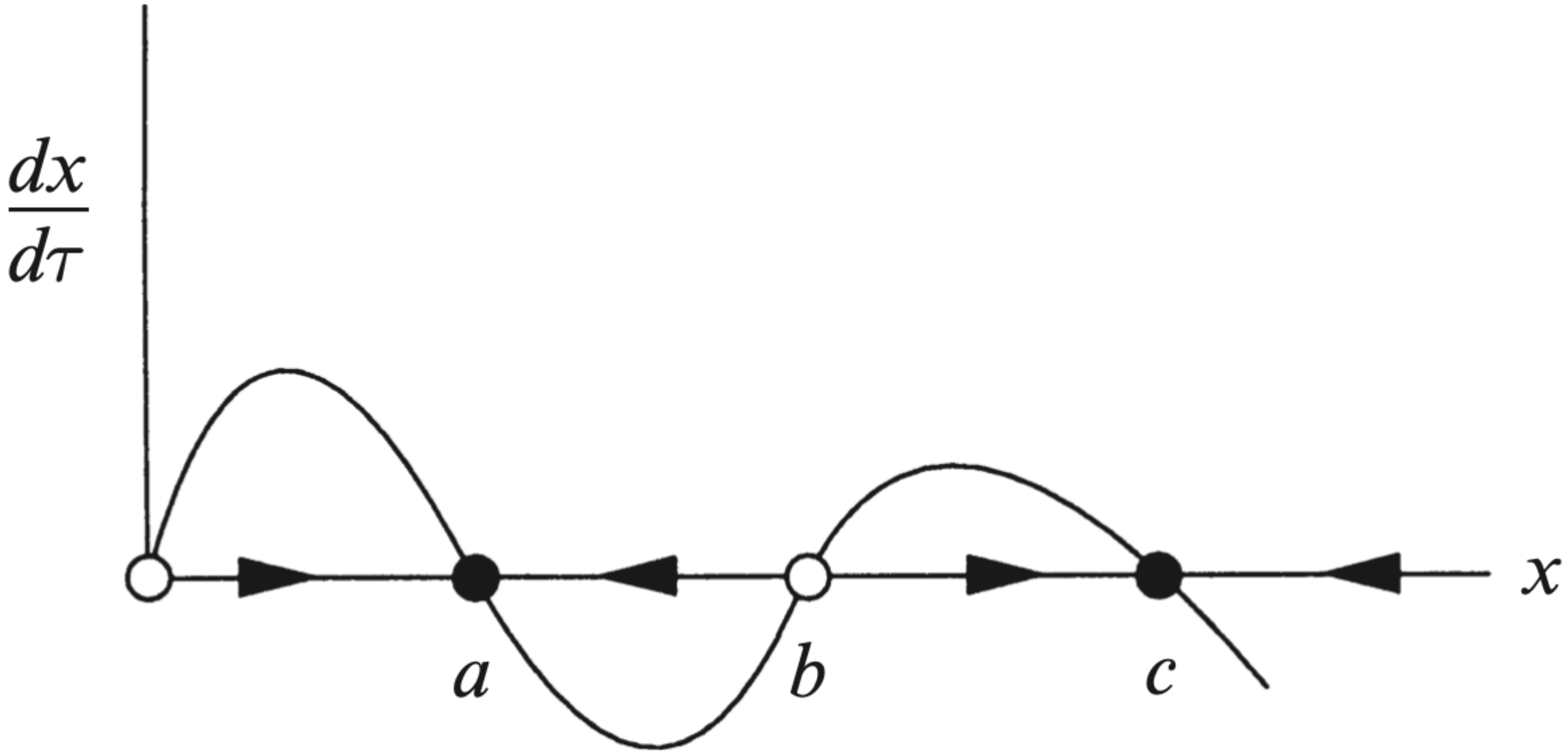
As we decrease *r* with *k* fixed, the line rotates counterclockwise about *k*.

Then the fixed points *b* and *c* approach each other and eventually coalesce in a ***saddle-node bifurcation.***

Applications: Insect Outbreak

Stability Analysis:

$x^* = 0$ is unstable, and also observe that the stability type must alternate as we move along the x-axis.



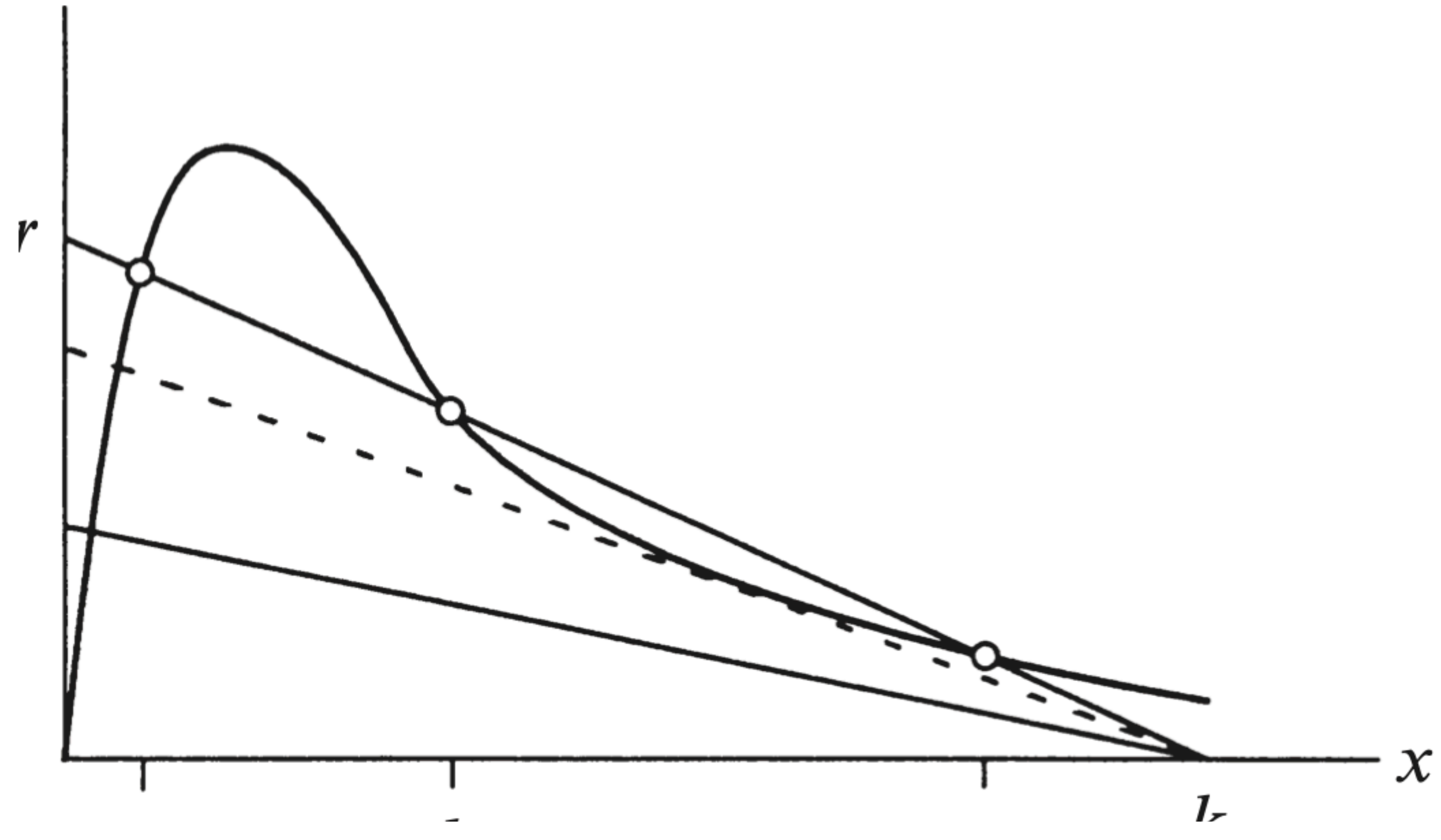
The smaller stable fixed point a is called the refuge level of the budworm population, while the larger stable point c is the outbreak level.

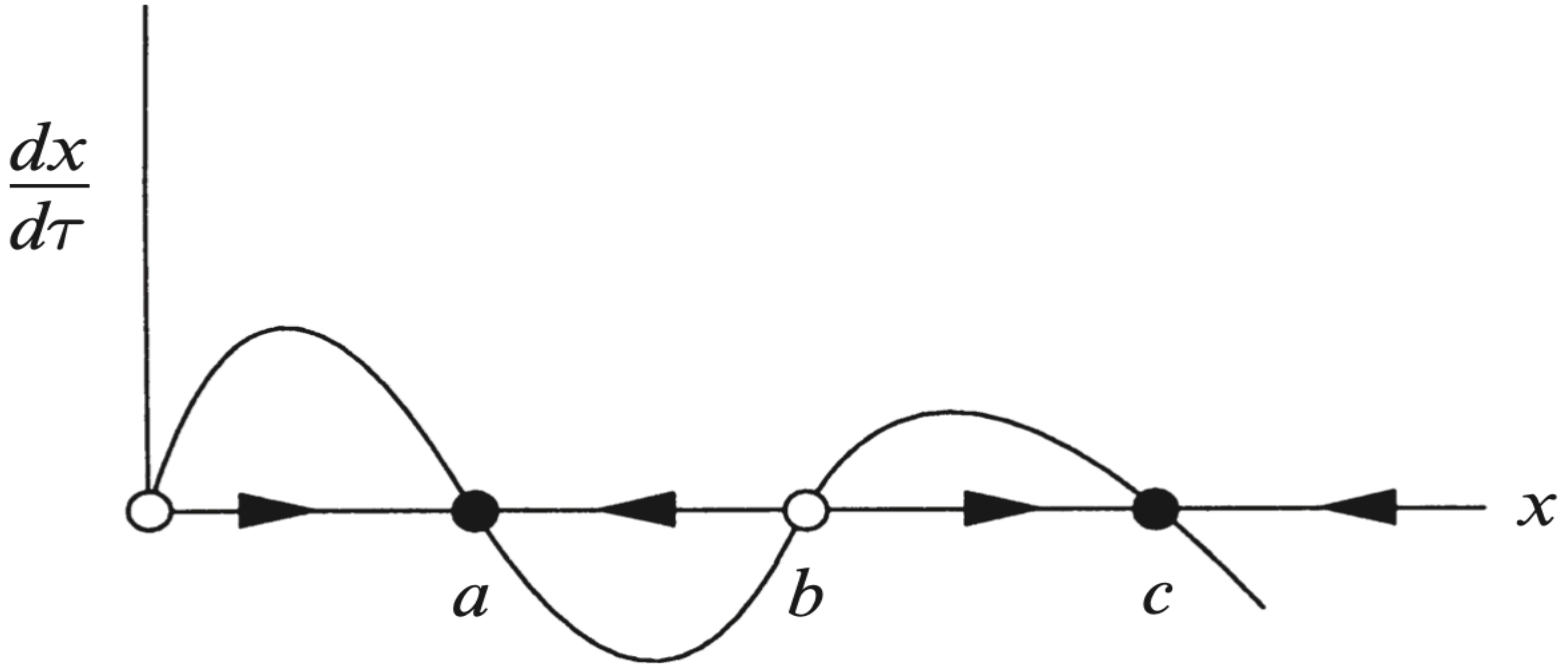
From the point of view of pest control, one would like to keep the population at a and away from c.

a is stable, b is unstable, and c is stable.

Applications: Insect Outbreak

Stability Analysis:





An outbreak occurs if and only if $x_0 > b$, so b is the threshold.

An outbreak can also be triggered by a saddle-node bifurcation. If the parameters *r* and *k* drift in such a way that the fixed point *a* disappears, then the population will jump suddenly to the outbreak level *c.*

The situation is made worse by the hysteresis effect—even if the parameters are restored to their values before the outbreak, the population will not drop back to the refuge level.

Applications: Insect Outbreak

Calculating the Bifurcation Curves:

We compute the curves in (k, r) space where the system undergoes saddle-node bifurcations.

The bifurcation curves can only be written in the ***parametric form:*** $(k(x), r(x))$

For a saddle-node bifurcation is that the line $r(1 - x/k)$ intersects the curve $x/(1 + x^2)$ tangentially.

$$r\left(1-\frac{x}{k}\right)=\frac{x}{1+x^2} \qquad \frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right]=\frac{d}{dx}\left[\frac{x}{1+x^2}\right] \Rightarrow -\frac{r}{k}=\frac{1-x^2}{(1+x^2)^2}.$$

Applications: Insect Outbreak

Calculating the Bifurcation Curves:

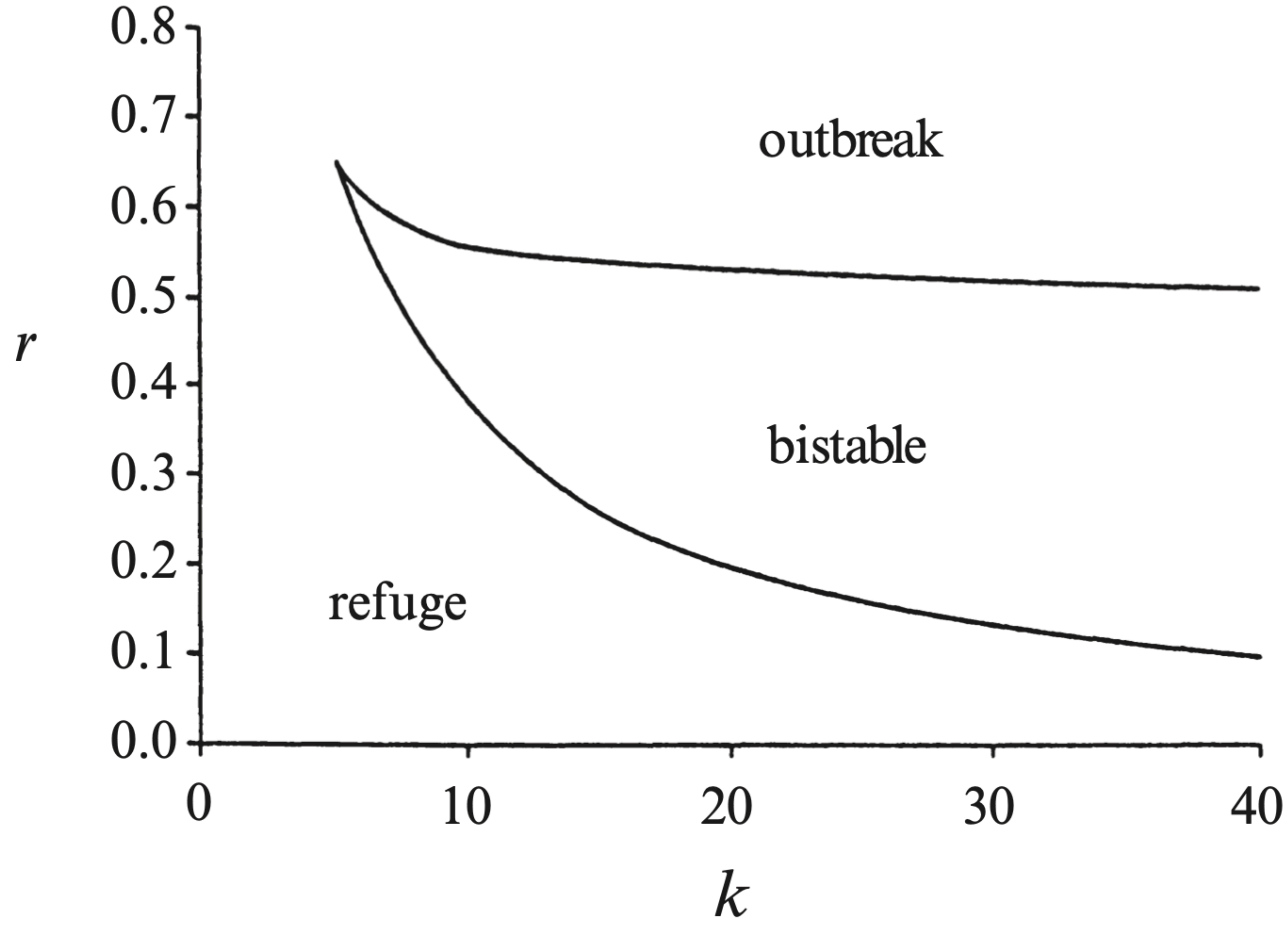
After substitutions:

$$r = \frac{2x^3}{(1+x^2)^2}$$

$$k = \frac{2x^3}{x^2-1}$$

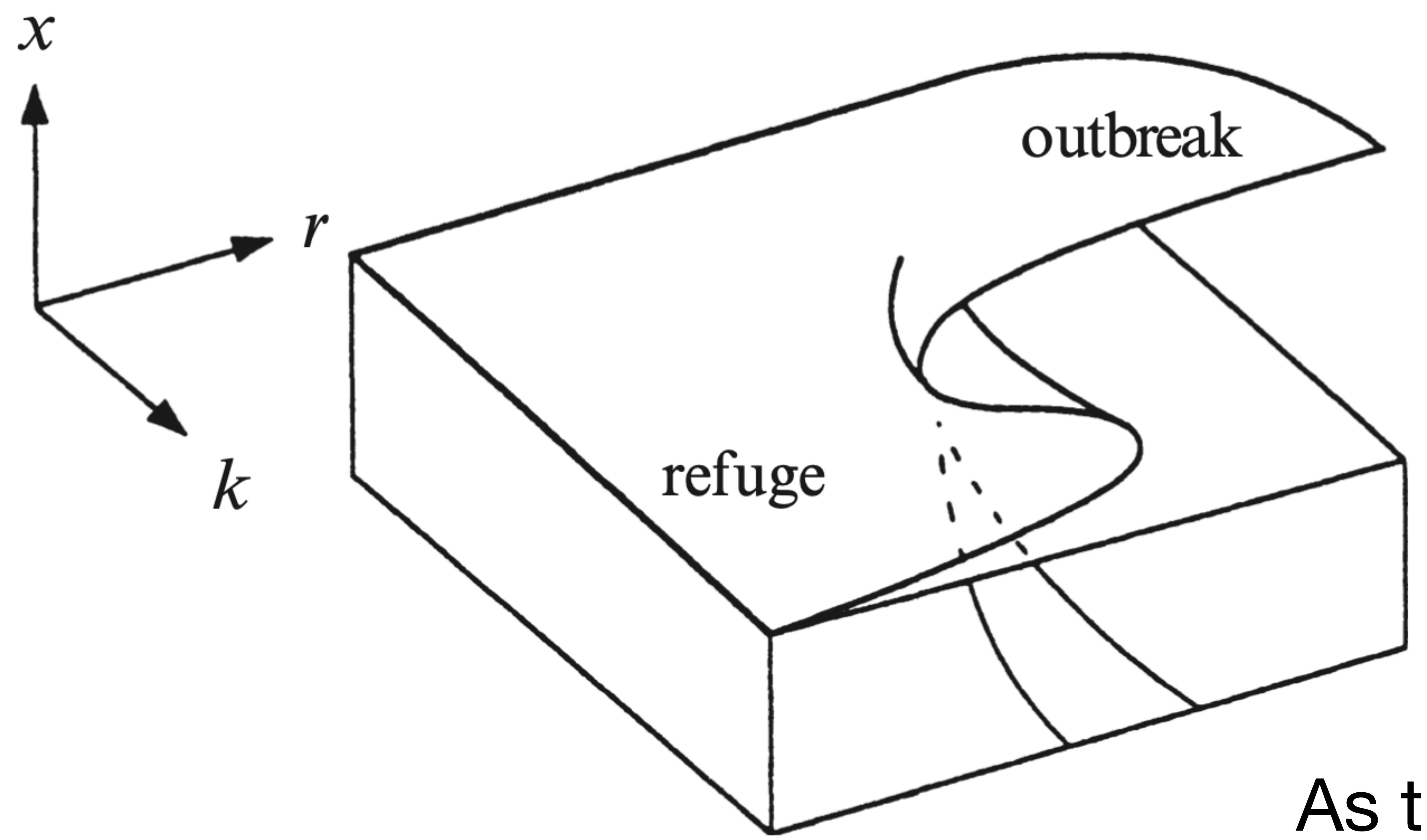
The condition $k > 0$ implies that x must be restricted to the range $x > 1$.

Stability diagram



Applications: Insect Outbreak

Cusp catastrophe surface



Ludwig et al. (1978), *r* increases as the forest grows, while *k* remains fixed.

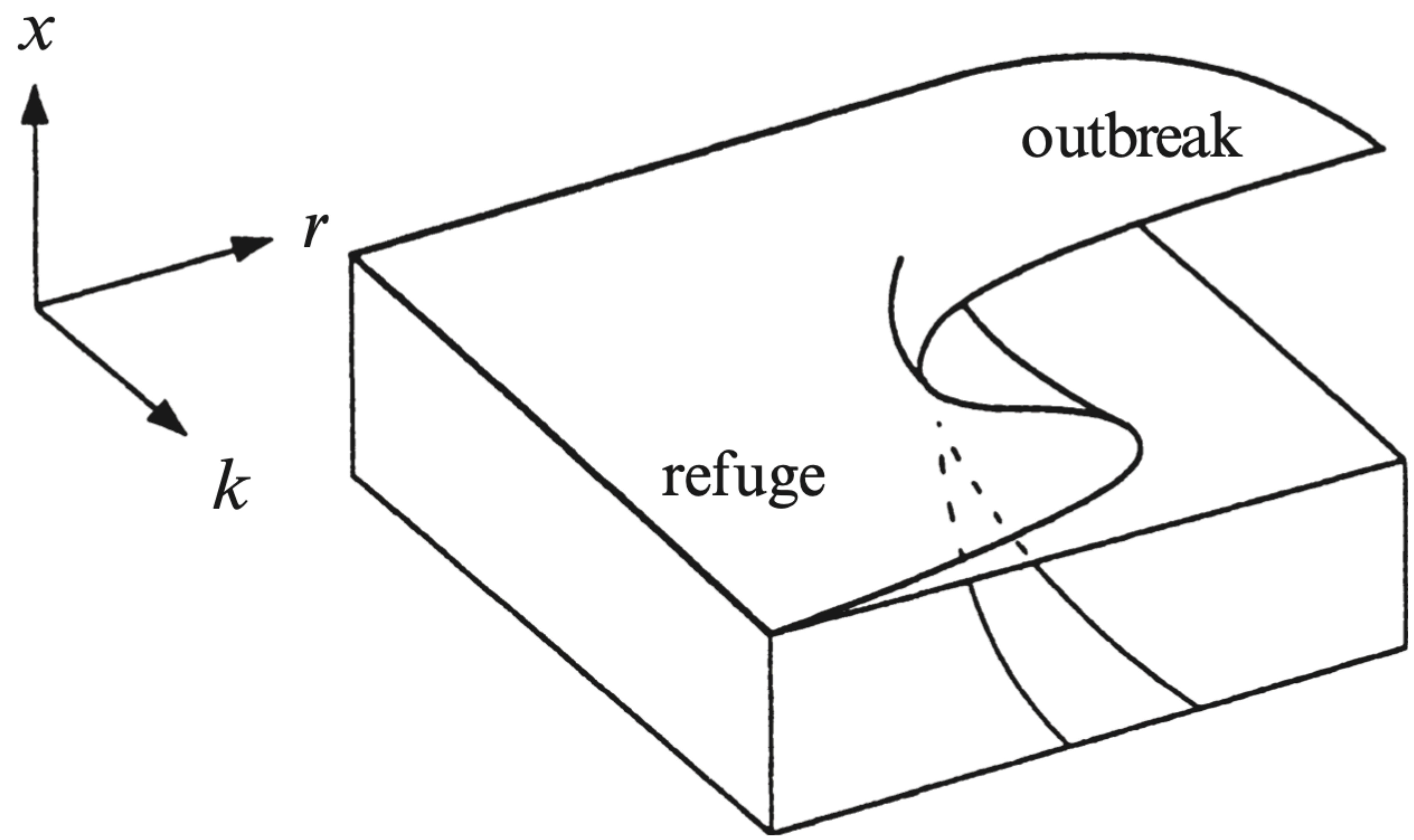
For a young forest, typically $k \approx 300$ and $r < 1/2$ so the parameters lie in the bistable region.

The budworm population is kept down by birds, which find it easy to search the small number of branches per acre.

As the forest grows, the size of the trees increases and therefore the point (*k*,*r*) drifts upward in parameter space toward the outbreak region.

Applications: Insect Outbreak

Cusp catastrophe surface



$r \approx 1$ for a fully mature forest, which lies dangerously in the outbreak region.

After an outbreak occurs, the fir trees die and the forest is taken over by birch trees, which are less efficient at using nutrients and eventually the fir trees come back—this recovery takes about 50–100 years.