Nonlinear Dynamics and Chaos

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Conservative Systems

Newton's law $F = ma$ is the source of many important second-order systems.

$$m\ddot{x} = F(x)$$

We are assuming that F is independent of both $\dot{x}$ and t. There is no damping or friction of any kind, and there is no time-dependent driving force.

Thus, energy is conserved!

Let's prove this by introducing the potential energy: $F(x) = -dV/dx.$

$$m\ddot{x} + \frac{dV}{dx} = 0.$$

Conservative Systems

Let's prove this by introducing the potential energy: $$m\ddot{x} + \frac{dV}{dx} = 0.$$

$$m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = 0 \quad \Longrightarrow \quad \frac{d}{dt}\left[\tfrac{1}{2}m\dot{x}^2 + V(x)\right] = 0$$

$$\frac{d}{dt}V(x(t)) = \frac{dV}{dx}\frac{dx}{dt}$$

Hence, for a given solution $x(t)$, the total energy is constant as a function of time.

$$E = \tfrac{1}{2}m\dot{x}^2 + V(x)$$

The energy is a conserved quantity, a constant of motion, or a first integral.

Systems for which a conserved quantity exists are called ***conservative systems.***

Conservative Systems

Given a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, a **conserved quantity** is a real-valued continuous function $E(\mathbf{x})$ that is constant on trajectories, i.e. $dE/dt = 0$.

We also require that E(**x**) be non-constant on every open set. Otherwise a constant function like $E(\mathbf{x}) = 0$ would qualify as a conserved quantity.

A conservative system cannot have any attracting fixed points. One generally finds saddles and centres in conservative systems.

Solution: Suppose $\mathbf{x}^*$ were an attracting fixed point. Then all points in its basin of attraction would have to be at the same energy $E(\mathbf{x}^*)$ (because energy is constant on trajectories and all trajectories in the basin flow to $\mathbf{x}^*$). Hence $E(\mathbf{x})$ must be a *constant function* for $\mathbf{x}$ in the basin. But this contradicts our definition of a conservative system, in which we required that $E(\mathbf{x})$ be nonconstant on all open sets. ■

Example: Conservative Systems

Consider a particle of mass $m = 1$ moving in a double-well potential:

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4.$$

1. Find and classify all the equilibrium points for the system.

2. Then plot the phase portrait and interpret the results physically.

Solution:

First, we compute the force: $-dV/dx = x - x^3$

And the equation of motion is: $\ddot{x} = x - x^3$

Example: Conservative Systems

And the equation of motion is: $\ddot{x} = x - x^3$

We can obtain the **vector field**:

$$\boxed{\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 \end{aligned}}$$

where y represents the particle's velocity.

Equilibrium points occur where $(\dot{x}, \dot{y}) = (0,0)$:

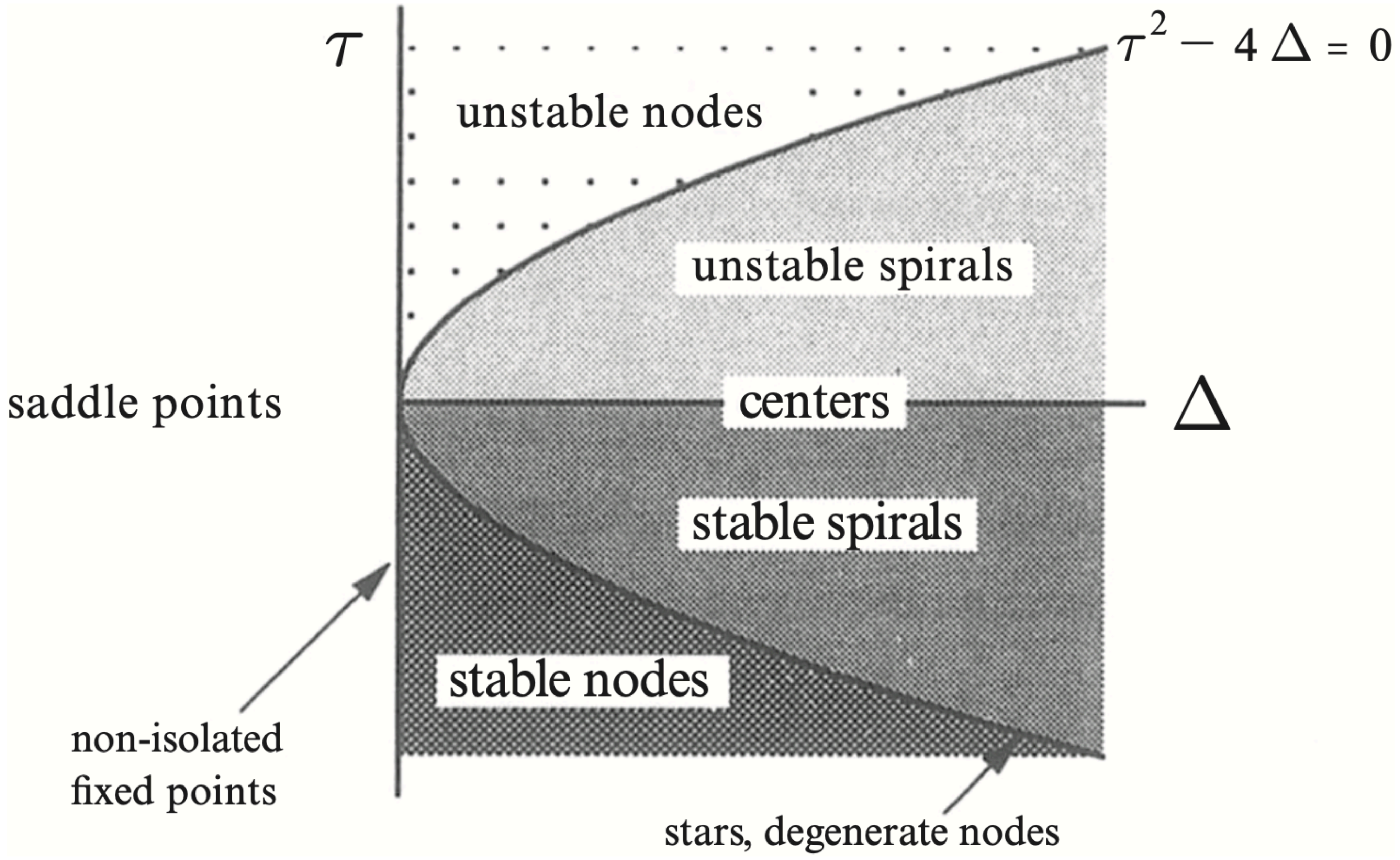
The equilibria are $(x^*, y^*) = (0,0)$ and $(x^*, y^*) = (\pm 1,0)$.

Example: Conservative Systems

To classify these fixed points we compute the Jacobian: $A = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$

At $(0,0)$, we have $\Delta = -1$, so the origin is a **saddle point**.

But when $(x^*, y^*) = (\pm 1,0)$, we find $\tau = 0$, $\Delta = 2$, so these equilibria are predicted to be **centres.**



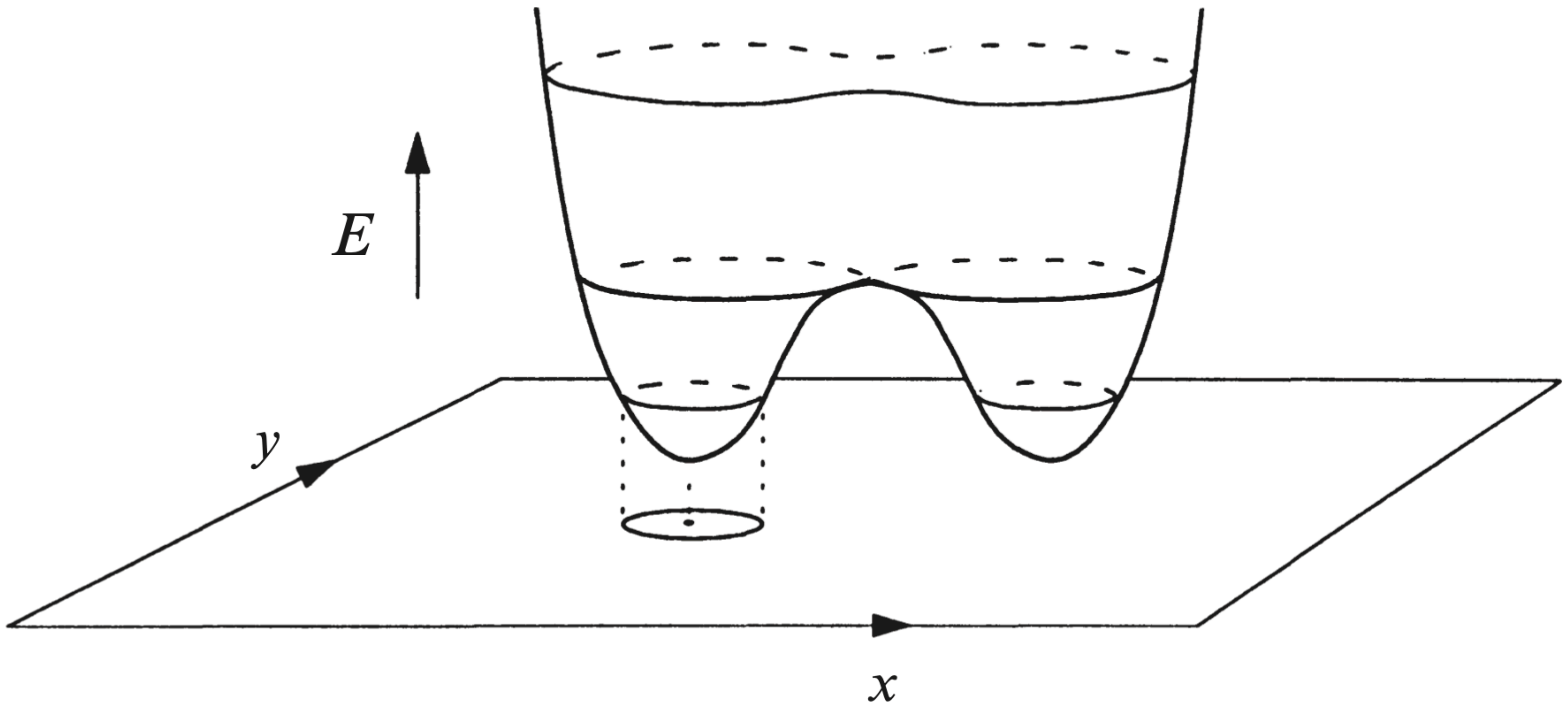
Energy conservation prevents small non-linear terms from destroying a centre, as was predicted by the linear approximation.

Example: Conservative Systems

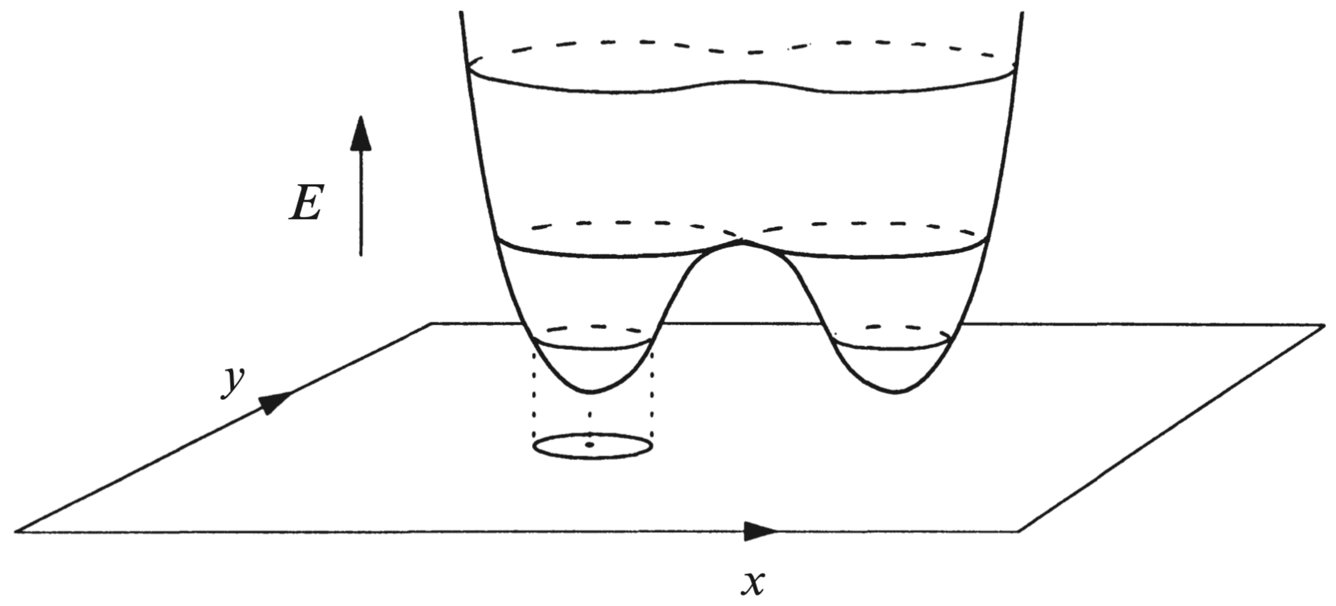
The trajectories are closed curves defined by the **contours** of constant energy:

$$E = \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 = \text{constant}.$$

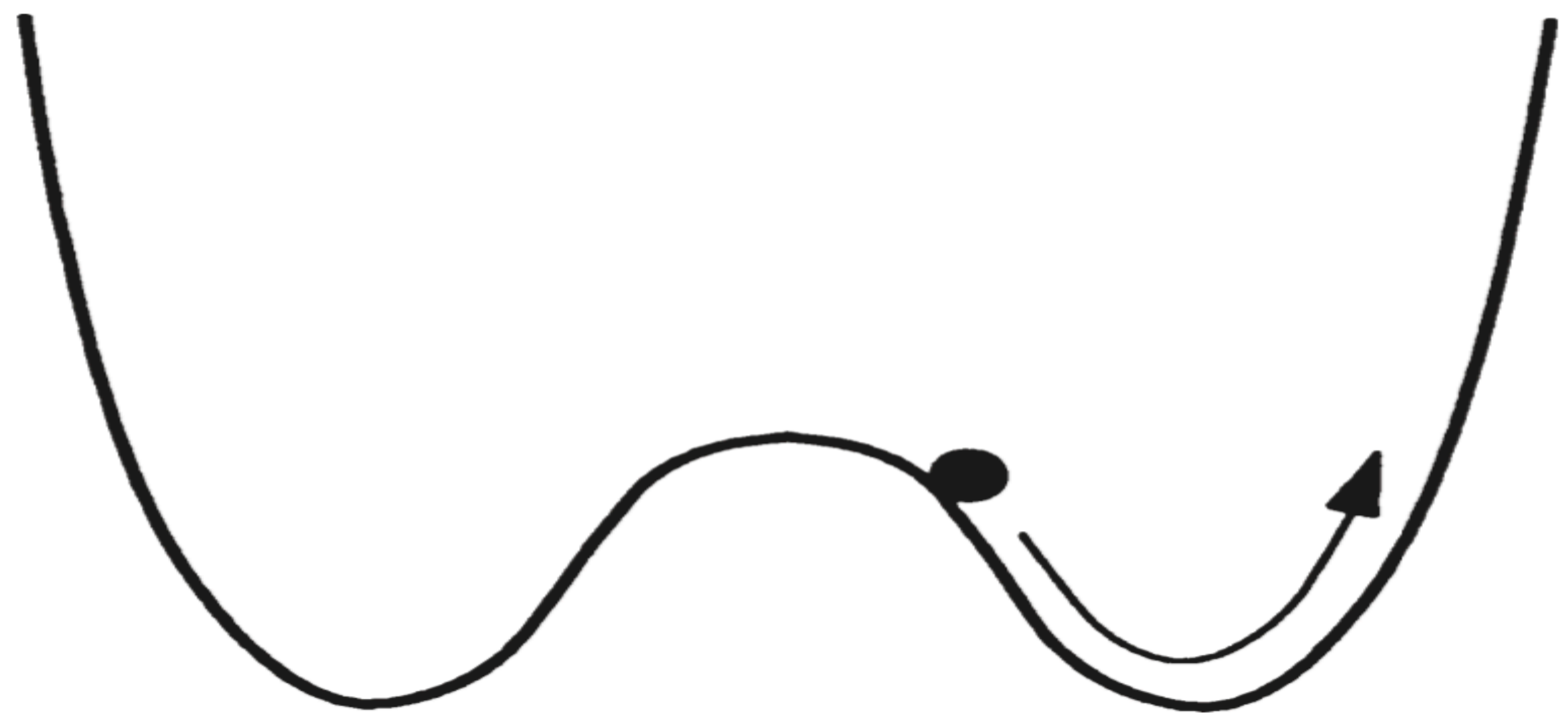
The resulting surface is called the **energy surface for the system.**



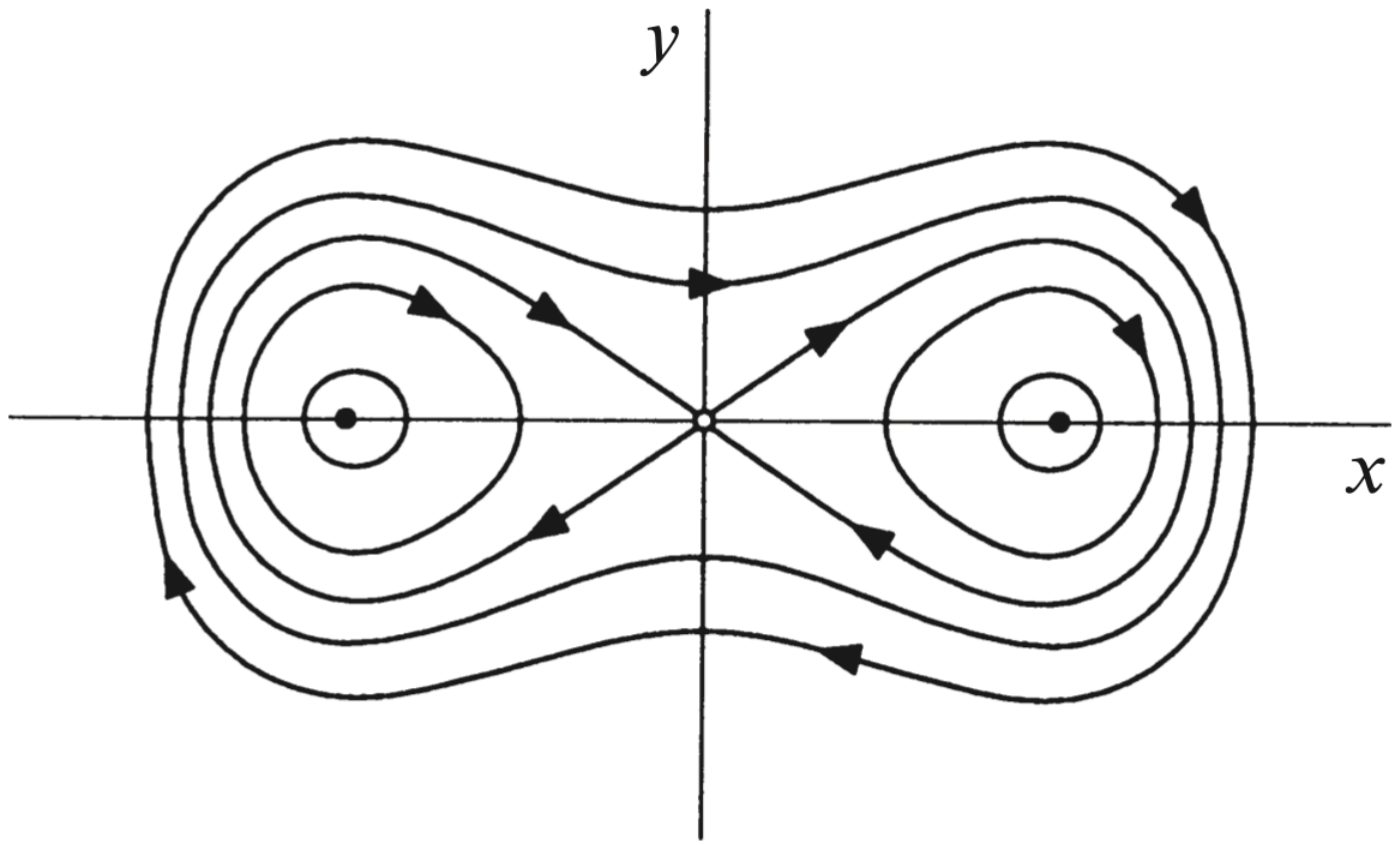
Example: Conservative Systems



Solutions of the system are typically periodic, except for the equilibrium solutions and two very special trajectories, **homoclinic orbits**: these are the trajectories that start and end at the same fixed point (in this case, the origin).



The neutrally stable equilibria correspond to **the particle at rest at the bottom of one of the wells**, and the small closed orbits represent small oscillations about these equilibria.



The saddle point represents the particle being **precariously balanced** at the top of the central barrier (the "hump") between the two wells.

Tutorial: Energy surface

Example:

$\dot{x} = y$

$\dot{y} = x - x^3$

Find the equilibrium points, the energy, and represent the energy surface.

Non-linear centres for conservative systems

Centres occur at the local minima of the energy function.

One expects neutrally stable equilibria and small oscillations to occur at the bottom of any potential well, no matter what its shape.

Theorem 6.5.1: (Nonlinear centers for conservative systems) Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbf{R}^2$, and $\mathbf{f}$ is continuously differentiable. Suppose there exists a conserved quantity $E(\mathbf{x})$ and suppose that $\mathbf{x}^*$ is an isolated fixed point (i.e., there are no other fixed points in a small neighborhood surrounding $\mathbf{x}^*$). If $\mathbf{x}^*$ is a local minimum of E, then all trajectories sufficiently close to $\mathbf{x}^*$ are closed.

Reversible Systems

Time-reversal symmetry: The dynamics of some on-linear systems look the same whether time runs forward or backward (e.g. undimmed pendulum).

Any mechanical system of the form $m\ddot{x} = F(x)$ is symmetric under time reversal.

If we make the change of variables $t \to -t$, the second derivative $\ddot{x}$ stays the same and so the equation is unchanged. The velocity $\dot{x}$ would be reversed.

The equivalent system is:

$$\dot{x} = y$$
$$\dot{y} = \frac{1}{m}F(x)$$

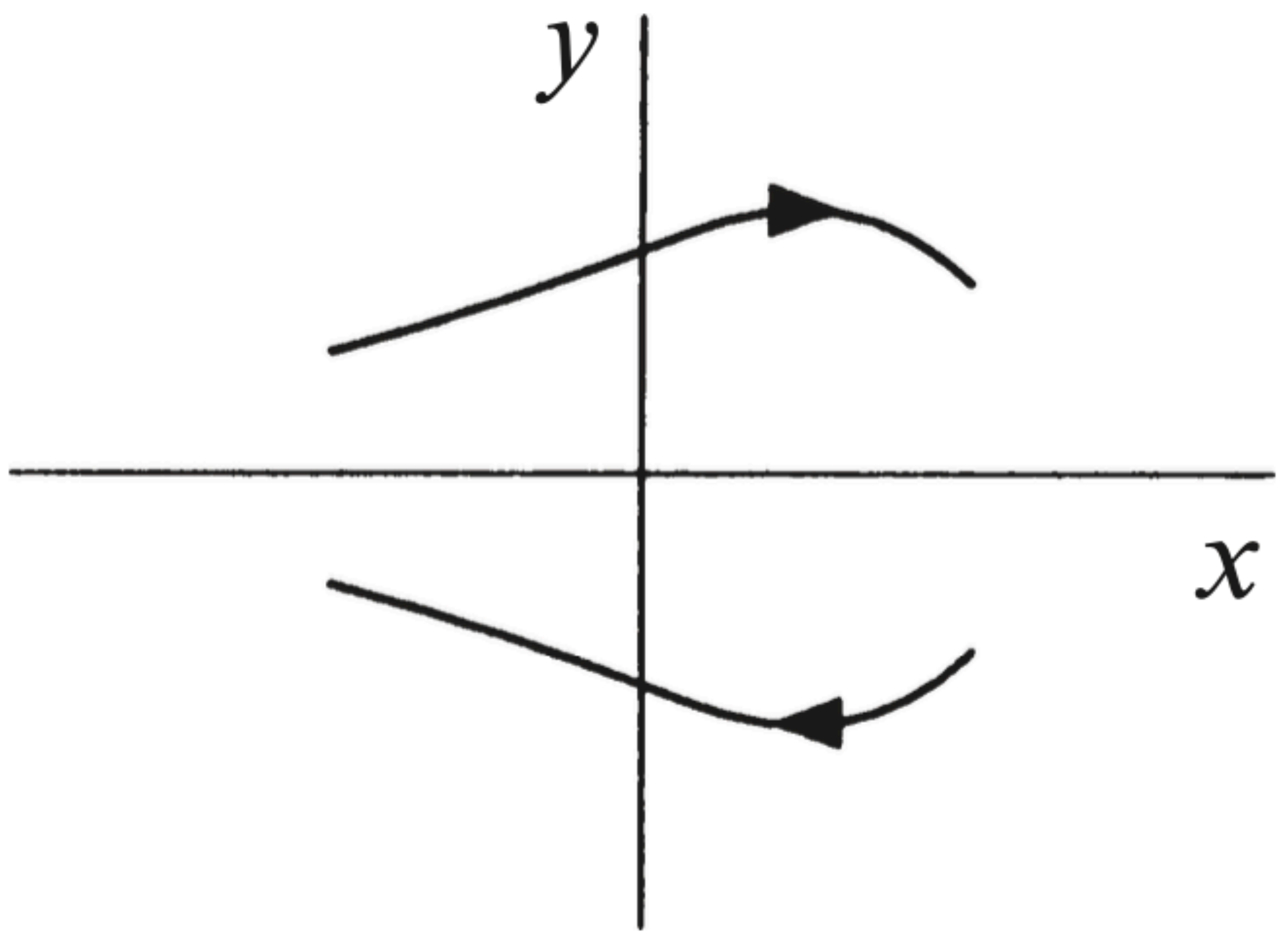
where y is the velocity.

Reversible Systems

$$\dot{x} = y$$
$$\dot{y} = \frac{1}{m} F(x)$$

If we make the change of variables $t \to -t$ and $y \to -y$, both equations stay the same.

If $(x(t), y(t))$ is a solution, then so is $(x(-t), -y(-t))$.



Therefore every trajectory has a twin:

they differ only by time-reversal and a reflection in the x-axis.

Reversible Systems: Definition

Reversible system: It is a second-order system that is invariant under $t \to -t$ and $y \to -y$. For example, any system of the form

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y),$$

where f is odd in y and g is even in y ,i.e.,

$f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$ is reversible.

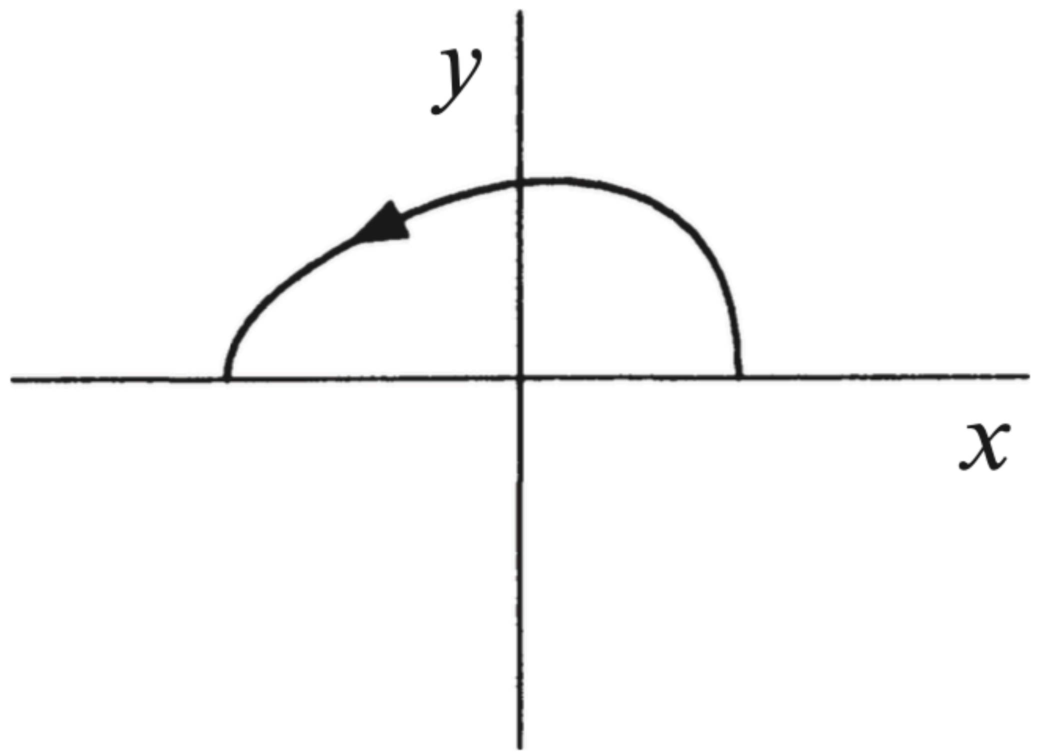
Reversible systems are different from conservative systems, but they have many of the same properties. For instance, **centres are robust in reversible systems as well.**

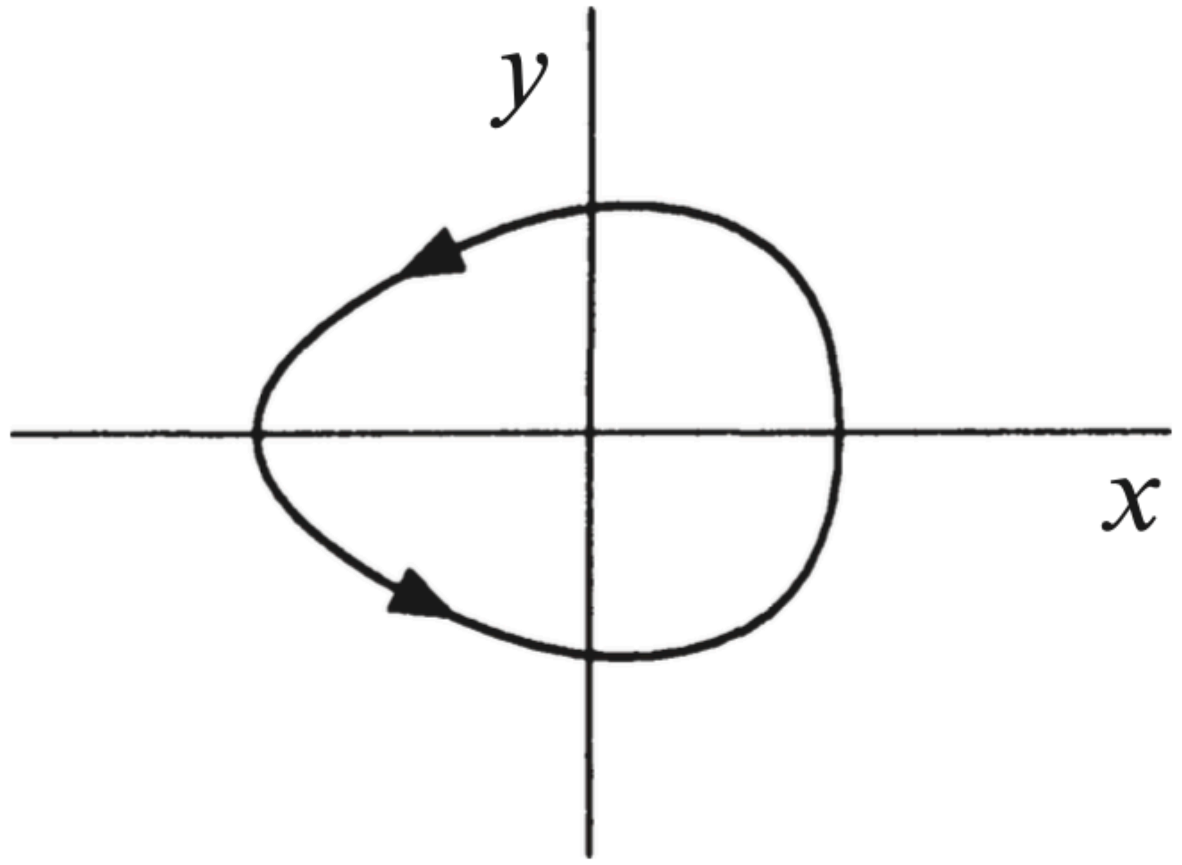
Non-linear centres for reversible systems

Theorem 6.6.1: (Nonlinear centers for reversible systems) Suppose the origin $\mathbf{x}^* = \mathbf{0}$ is a linear center for the continuously differentiable system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y), \end{aligned}$$

and suppose that the system is reversible. Then sufficiently close to the origin, all trajectories are closed curves.





The two trajectories form a closed orbit. **Hence all trajectories sufficiently close to the origin are closed.**

Example: centres for reversible systems

Show that the system:

$$\dot{x} = y - y^3$$

$$\dot{y} = -x - y^2$$

has a nonlinear centre at the origin, and plot the phase portrait.

The Jacobian at the origin is:

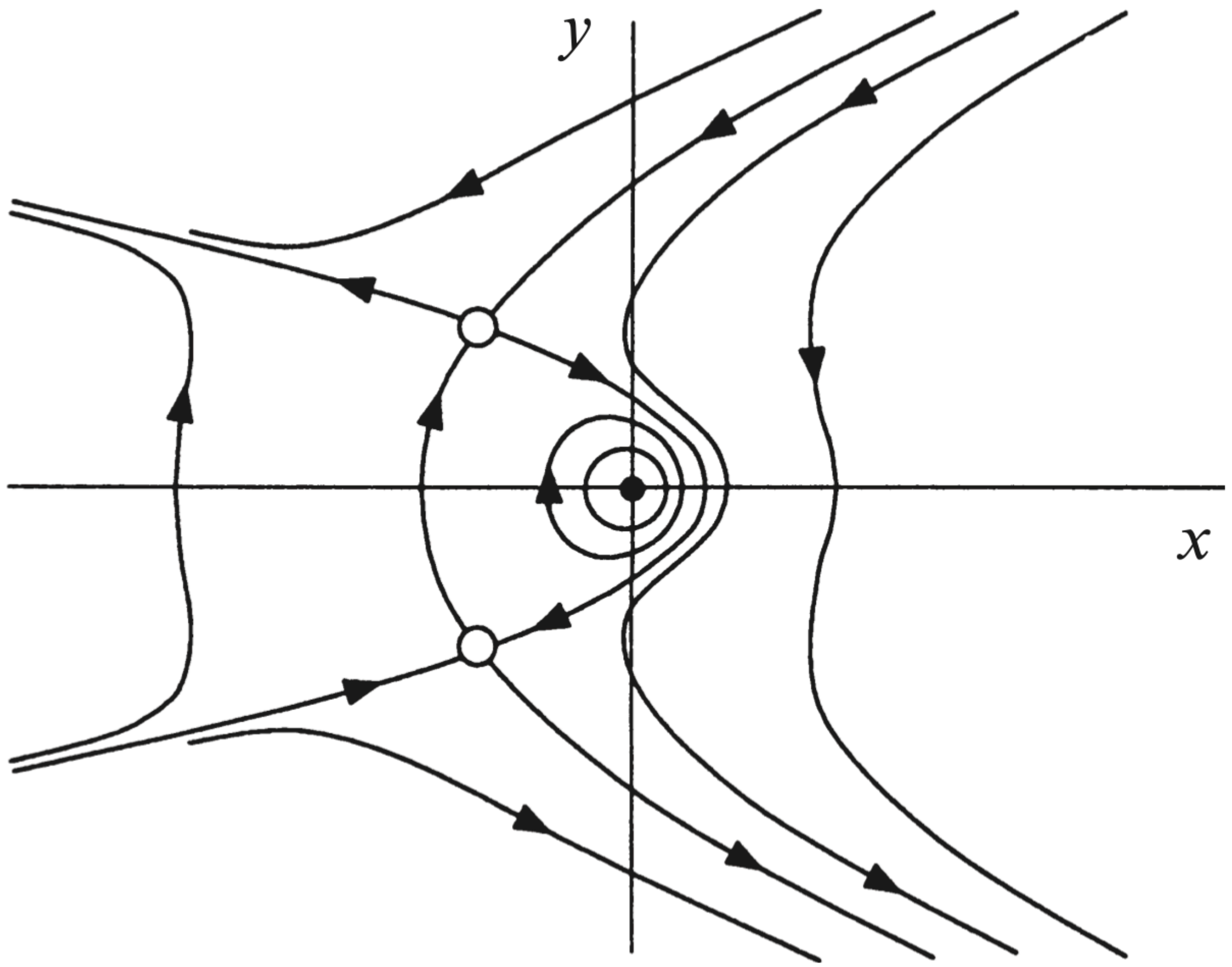
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This has $\tau = 0$, $\Delta > 0$, so the origin is a centre.

The system is reversible, since the equations are invariant under the transformation $t \to -t$, $y \to -y$. **The origin is a non-linear centre.**

Example: centres for reversible systems

Phase portrait:



The other **fixed (saddle) points** of the system are $(-1,1)$ and $(-1,-1)$.

The reversibility symmetry is apparent.

The twin saddle points are joined by a pair of trajectories, called **heteroclinic trajectories** or **saddle connections**.

Heteroclinic trajectories are much more common in reversible or conservative systems than in other types of systems.

Homoclinic Trajectory: is an orbit connects a single equilibrium point to itself.

Heteroclinic Trajectory: is an orbit that connects two different equilibrium points.

Tutorial:

Show that the system

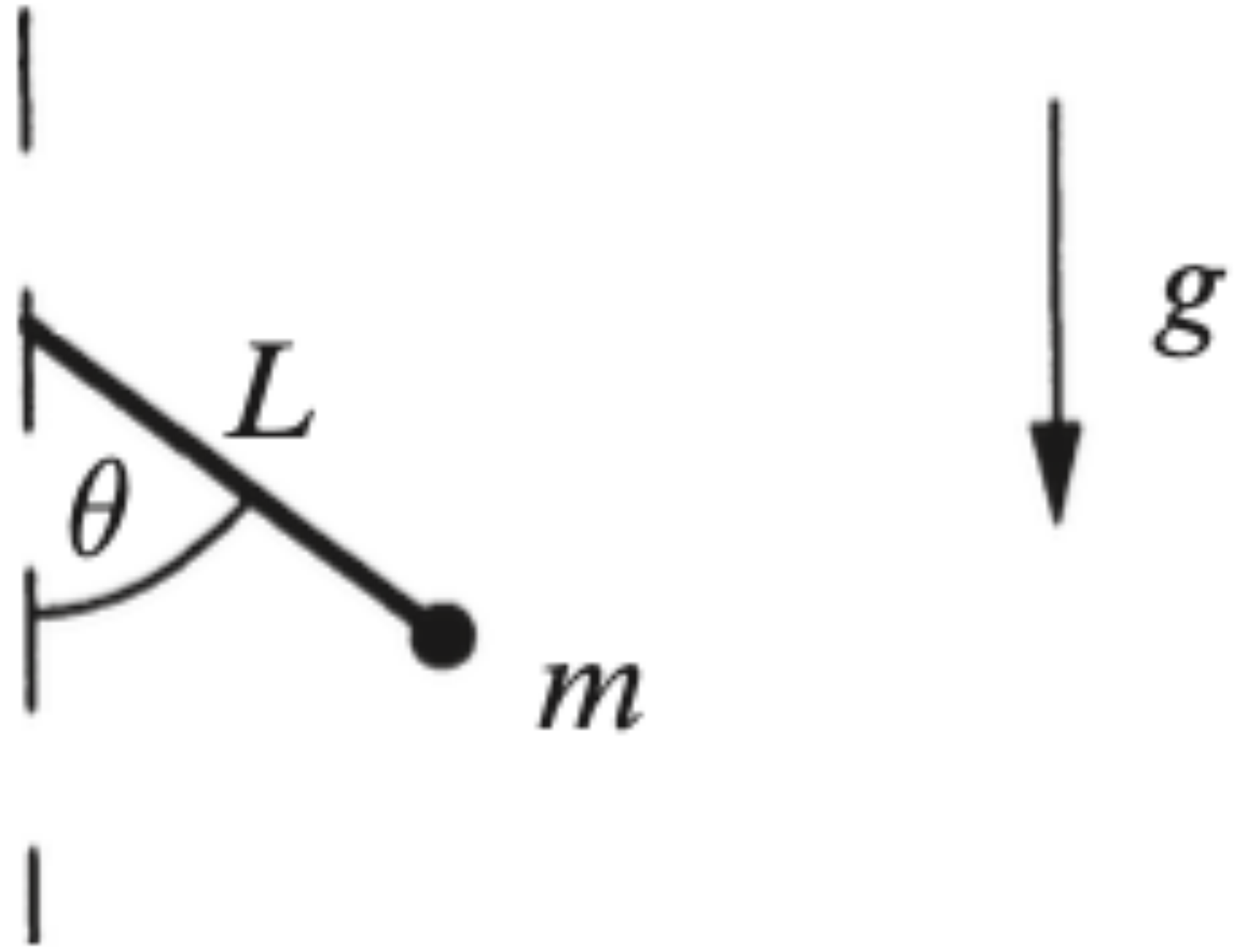
$$\dot{x} = -2\cos x - \cos y$$
$$\dot{y} = -2\cos y - \cos x$$

is reversible, but *not* conservative. Then plot the phase portrait.

Pendulum

The pendulum's essential non-linearity is usually sidestepped by the small-angle approximation $\sin(\theta) \approx \theta$.

Here we will use phase plane methods to analyse the pendulum, even in the large-angle regime where the pendulum whirls over the top.

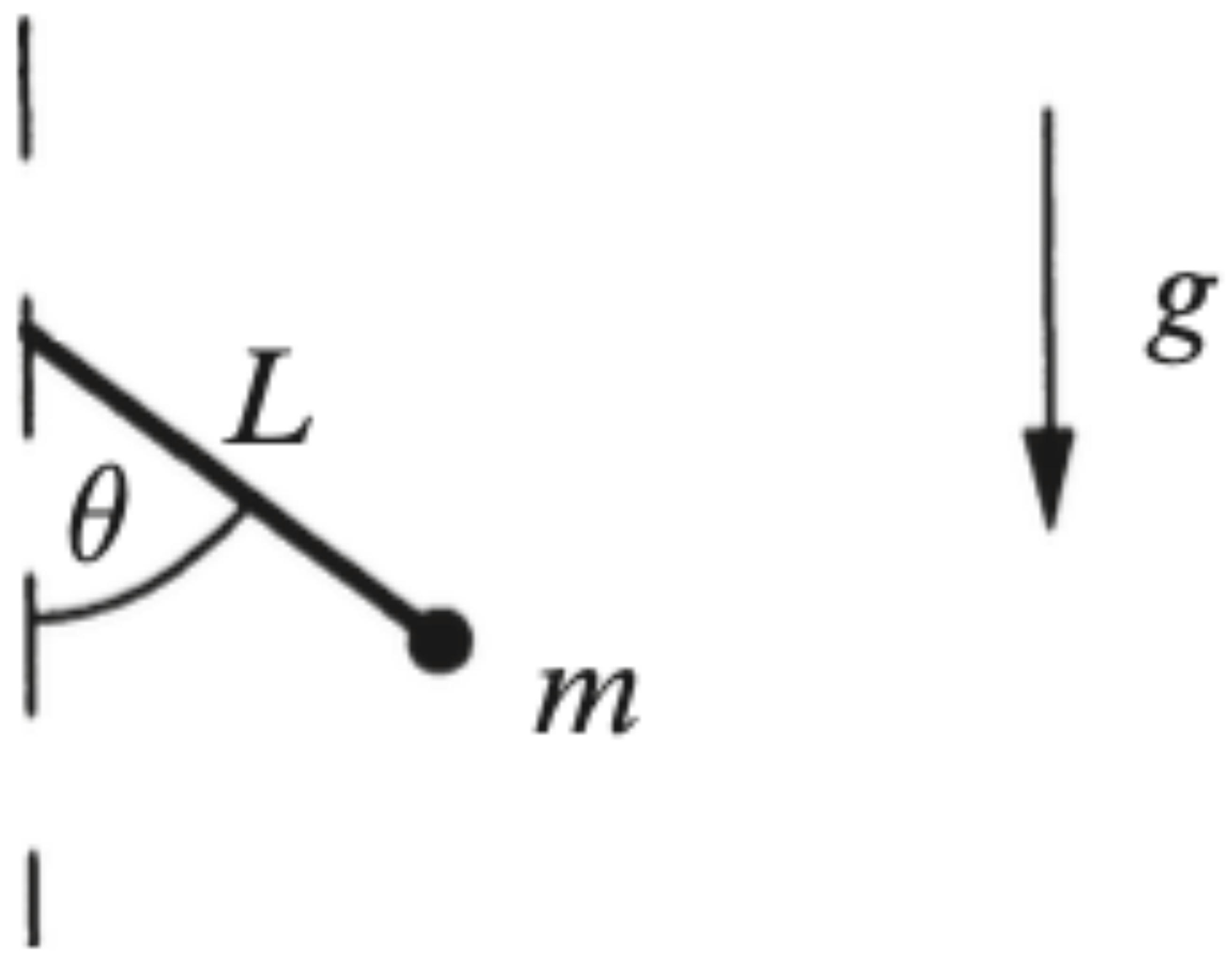


In the absence of damping and external driving, the motion of a pendulum is governed by:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where is the angle from the downward vertical, g is the acceleration due to gravity, and L is the length of the pendulum.

Pendulum



We non-dimensionalise the ODE by introducing a frequency $\omega = \sqrt{\dfrac{g}{L}}$ and a dimensionless time $\tau = \omega t$:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \sin\theta = 0$$

where we have differentiation with respect to τ.

In phase plane:

$$\dot{\theta} = v$$

$$\dot{v} = -\sin\theta$$

where v is the (dimensionless) angular velocity.

Pendulum: fixed points

The fixed points are $(\theta^*, v^*) = (k\pi, 0)$, where k is any integer. There's no physical difference between angles that differ by 2π.

We focus on 2 fixed points $(0,0)$ and $(\pi,0)$.

At $(0,0)$, the Jacobian is:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The origin is a **nonlinear centre**, for two reasons:

1. The system is reversible: the equations are invariant under the transformation $\tau \to -\tau$, $v \to -v$.

2. The system is also conservative.

Pendulum: fixed points -> origin

2. The system is also conservative.

$$\dot{\theta}(\ddot{\theta} + \sin\theta) = 0 \Rightarrow \frac{1}{2}\dot{\theta}^2 - \cos\theta = \textbf{constant}$$

The energy function has a local minimum at the origin: $E(\theta, v) = \frac{1}{2}v^2 - \cos\theta$

Taylor expansion: $\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots$

Thus: $E \approx \frac{1}{2}(v^2 + \theta^2) - 1$ for small (θ, v), so the origin is a **nonlinear centre** with closed orbits approximately circular:

$$\theta^2 + v^2 \approx 2(E + 1).)$$

Near $(0,0)$, the system behaves like a **simple harmonic oscillator.**

Pendulum: fixed points -> saddle point

At $(\pi, 0)$, the Jacobian is: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

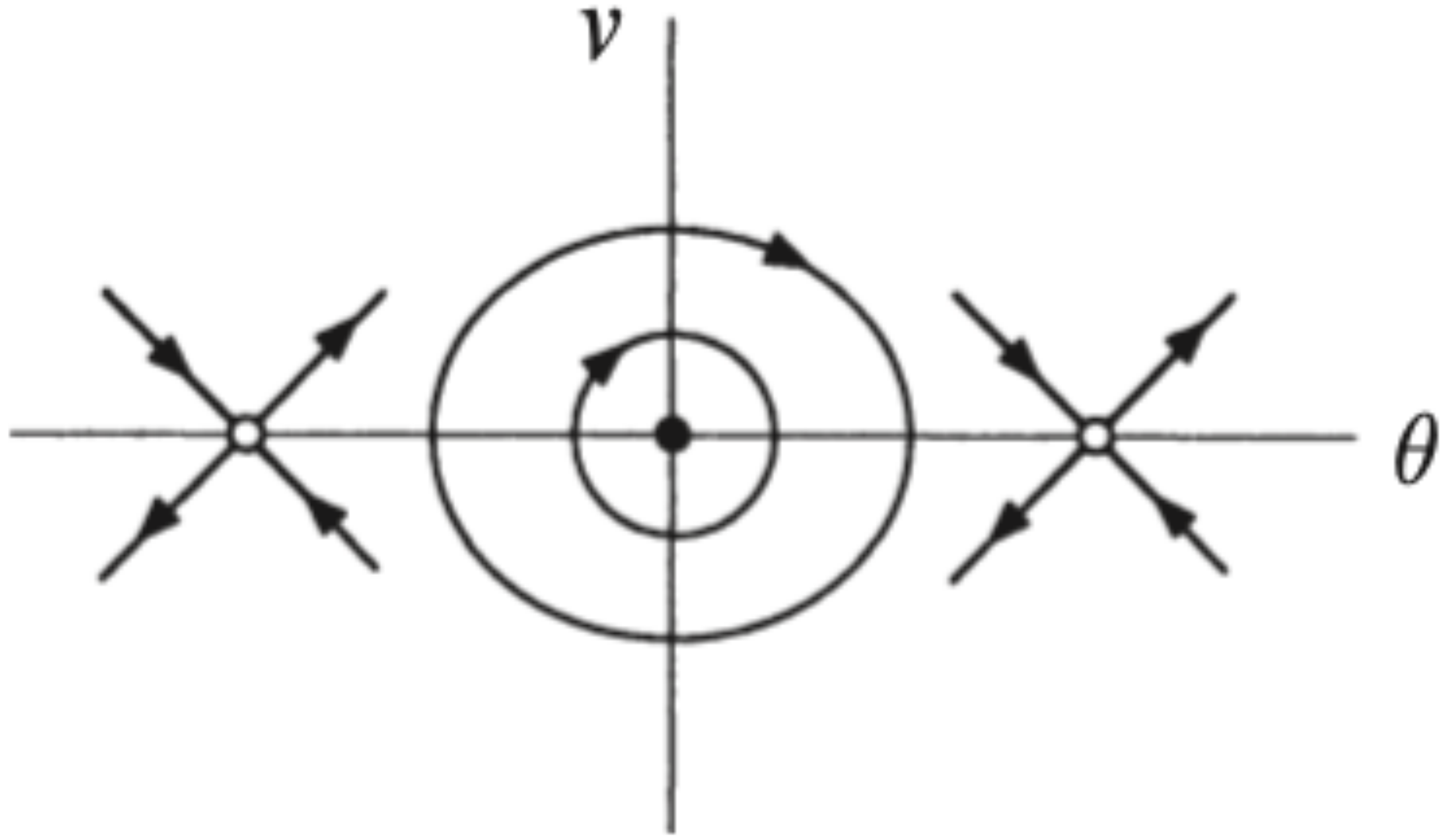
The characteristic equation is $\lambda^2 - 1 = 0$. Therefore, $\lambda_1 = -1$, $\lambda_2 = 1$, so the **fixed point is a saddle.**

The corresponding eigenvectors are $v_1 = (1, -1)$ and $v_2 = (1, 1)$.

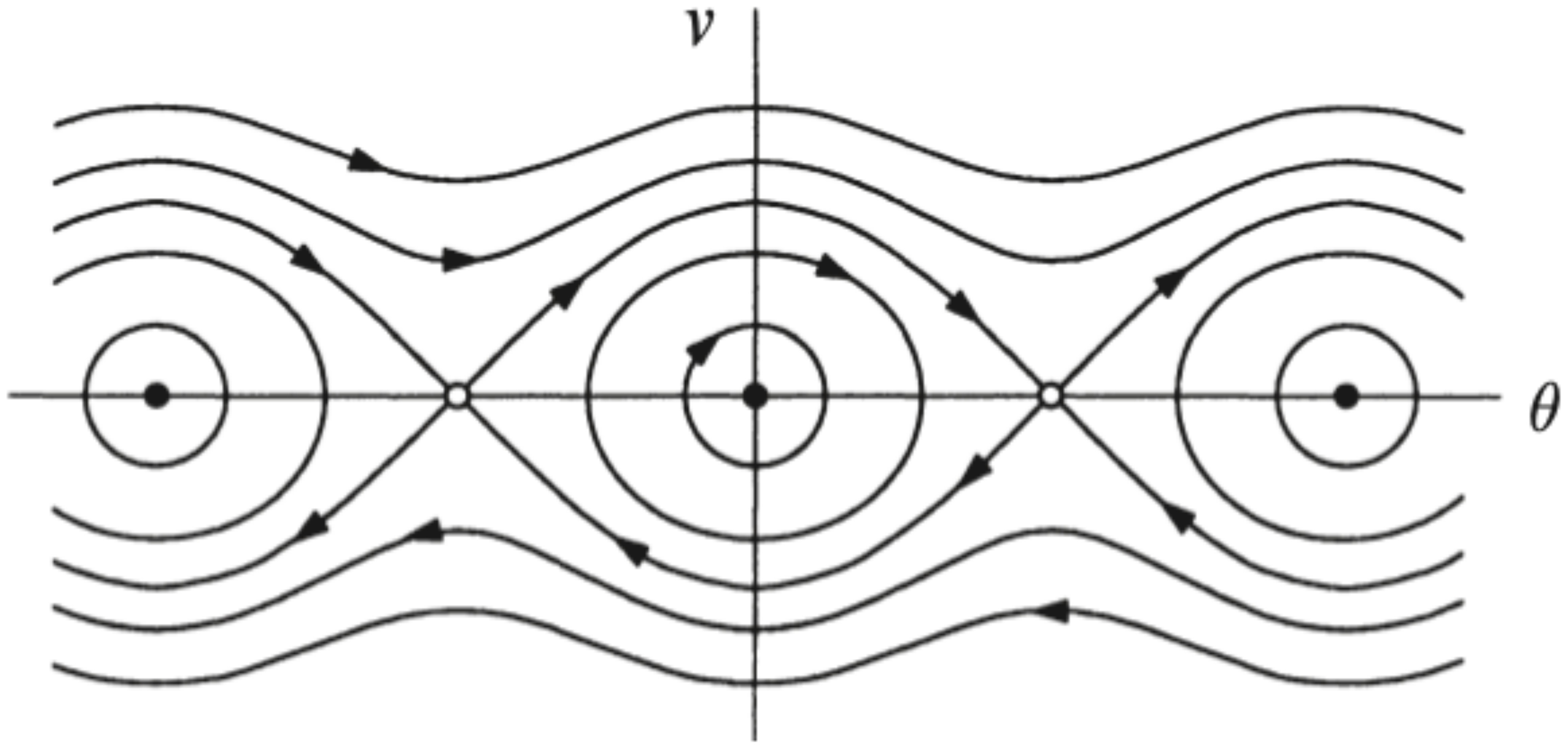
We can add the energy contours for different E:

$$E = \frac{1}{2}v^2 - \cos\theta$$

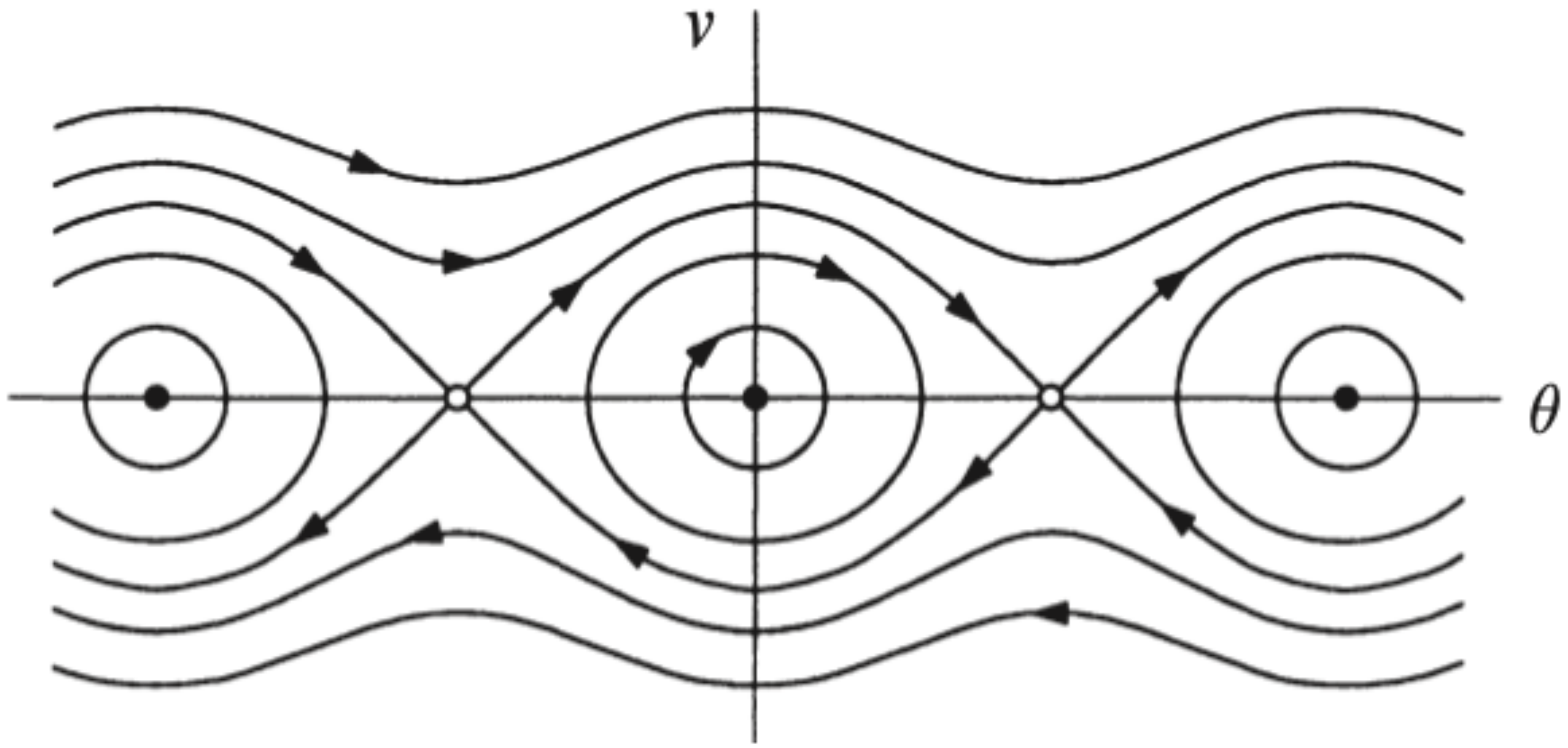
Phase portrait



Phase portrait with energy contours



Pendulum: phase portrait



The centre corresponds to a state of **neutrally stable equilibrium,** with the pendulum at rest and hanging straight down.

This is the **lowest possible energy state** ($E = -1$).

The small orbits surrounding the centre represent small oscillations about equilibrium, traditionally called **librations.**

As *E* increases, the orbits grow. The critical case is $E = 1$, corresponding to the **heteroclinic trajectories joining the saddles.**

The saddles represent an **inverted pendulum at rest.** The heteroclinic trajectories represent delicate motions in which the pendulum slows to a halt precisely as it approaches the inverted position.

Pendulum: phase portrait

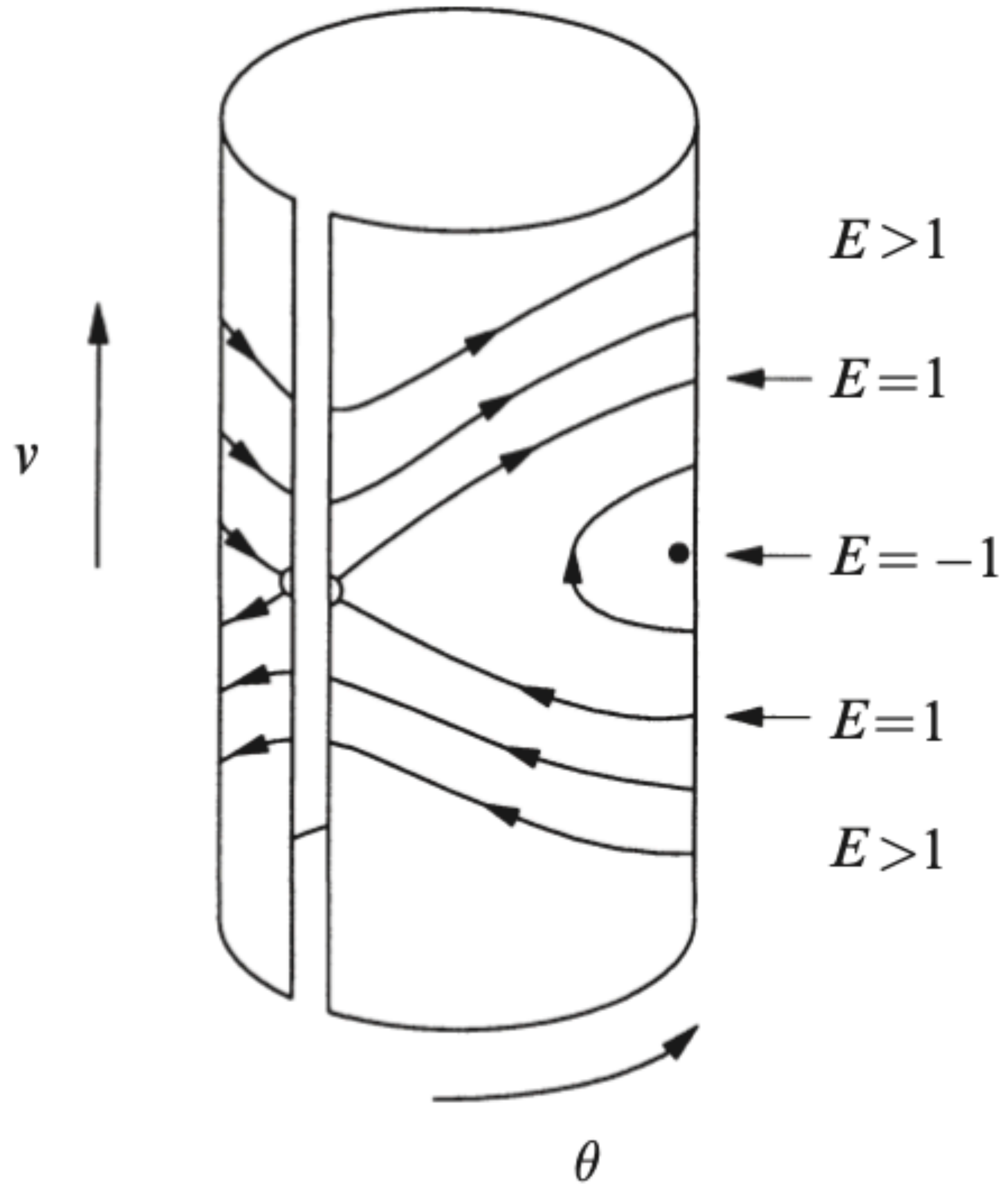
For $E > 1$, the pendulum whirls repeatedly over the top.

These **rotations** should also be regarded as periodic solutions, since and are the same physical position.

The natural phase space for the pendulum is the **cylindrical phase space.**

Fundamental geometric difference between v and θ: the angular velocity v is a real number, whereas θ is an angle

Cylindrical Phase Space



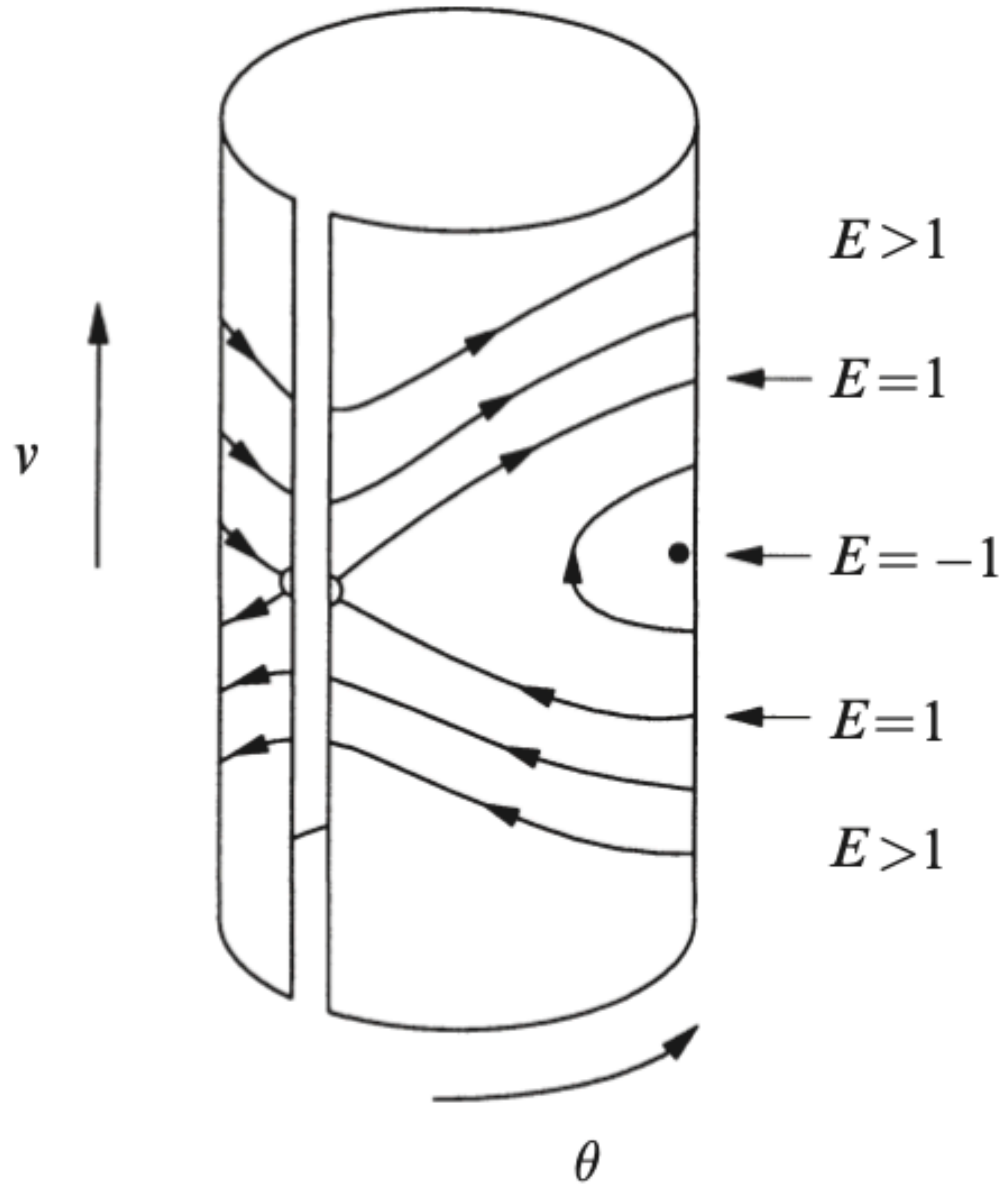
Pendulum: phase portrait

The periodic whirling motions look periodic —they are the closed orbits that encircle the cylinder for $E > 1$.

The heteroclinic trajectories become homoclinic orbits on the cylinder.

Both homoclinic orbits have the same energy and shape.

Cylindrical Phase Space



Pendulum:

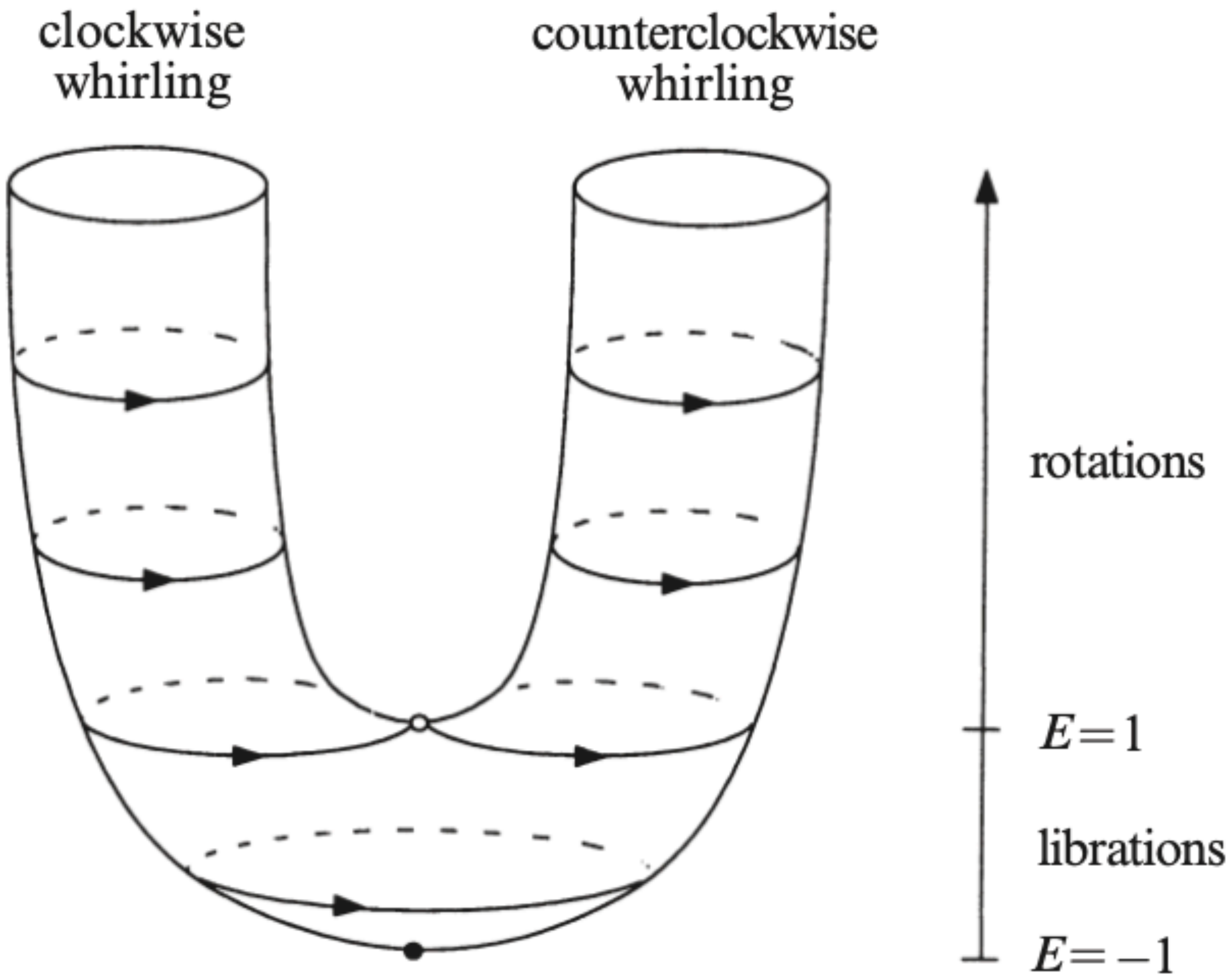
The orbits on the cylinder remain at constant height, while the cylinder gets bent into a **U-tube**.

The two arms of the tube are distinguished by the sense of rotation of the pendulum, either clockwise or counterclockwise.

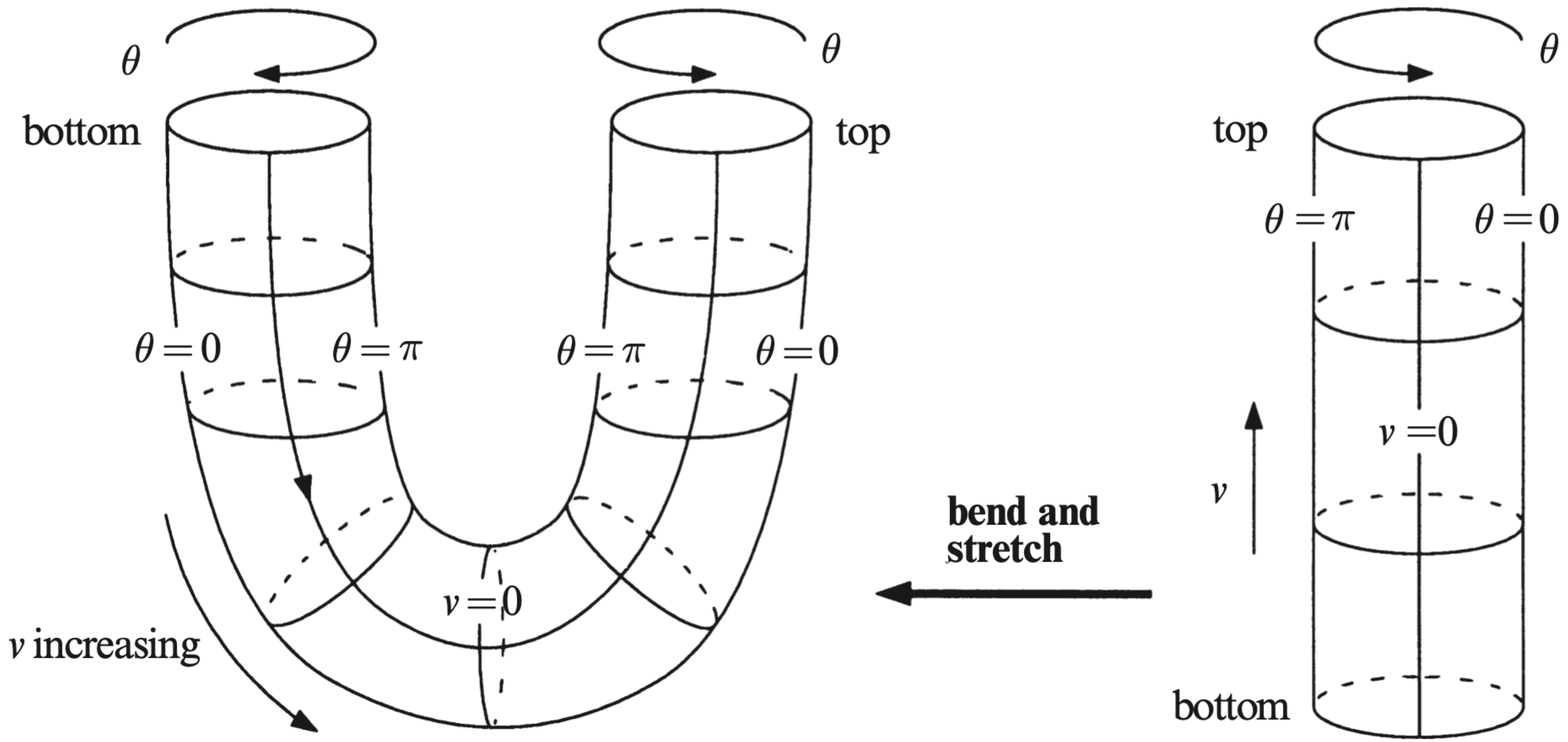
At low energies, this distinction no longer exists.

The **homoclinic orbits** lie at $E = 1$, the borderline between **rotations** and **librations**.

Energy U-tube



Pendulum:



Here we see the coordinate system. The direction of increasing θ has reversed when the bottom of the cylinder is bent around to form the U-tube.

Pendulum: damping

Suppose that we add a small amount of linear damping to the pendulum:

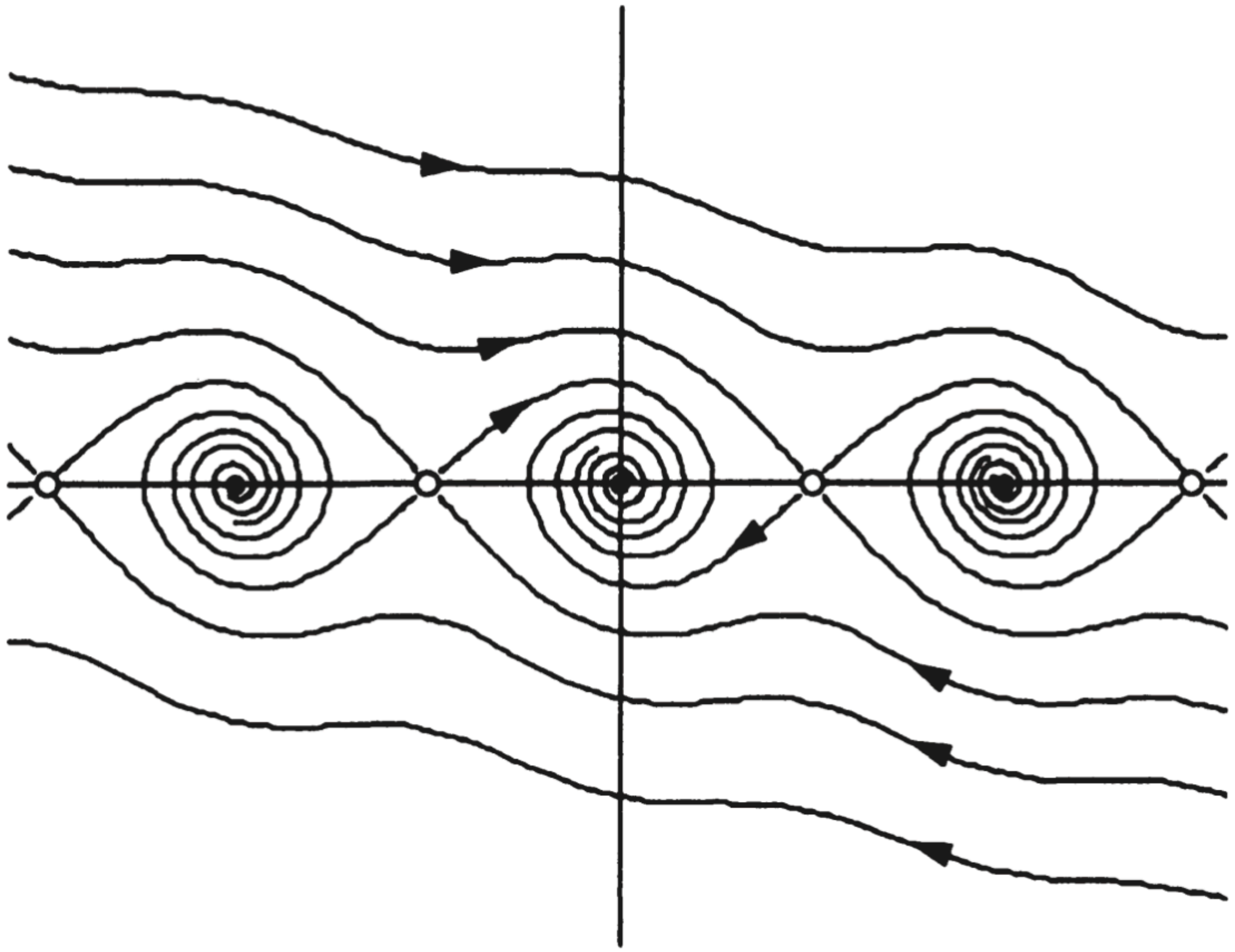
$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$$

where $b > 0$ is the damping strength.

The centres become stable spirals.

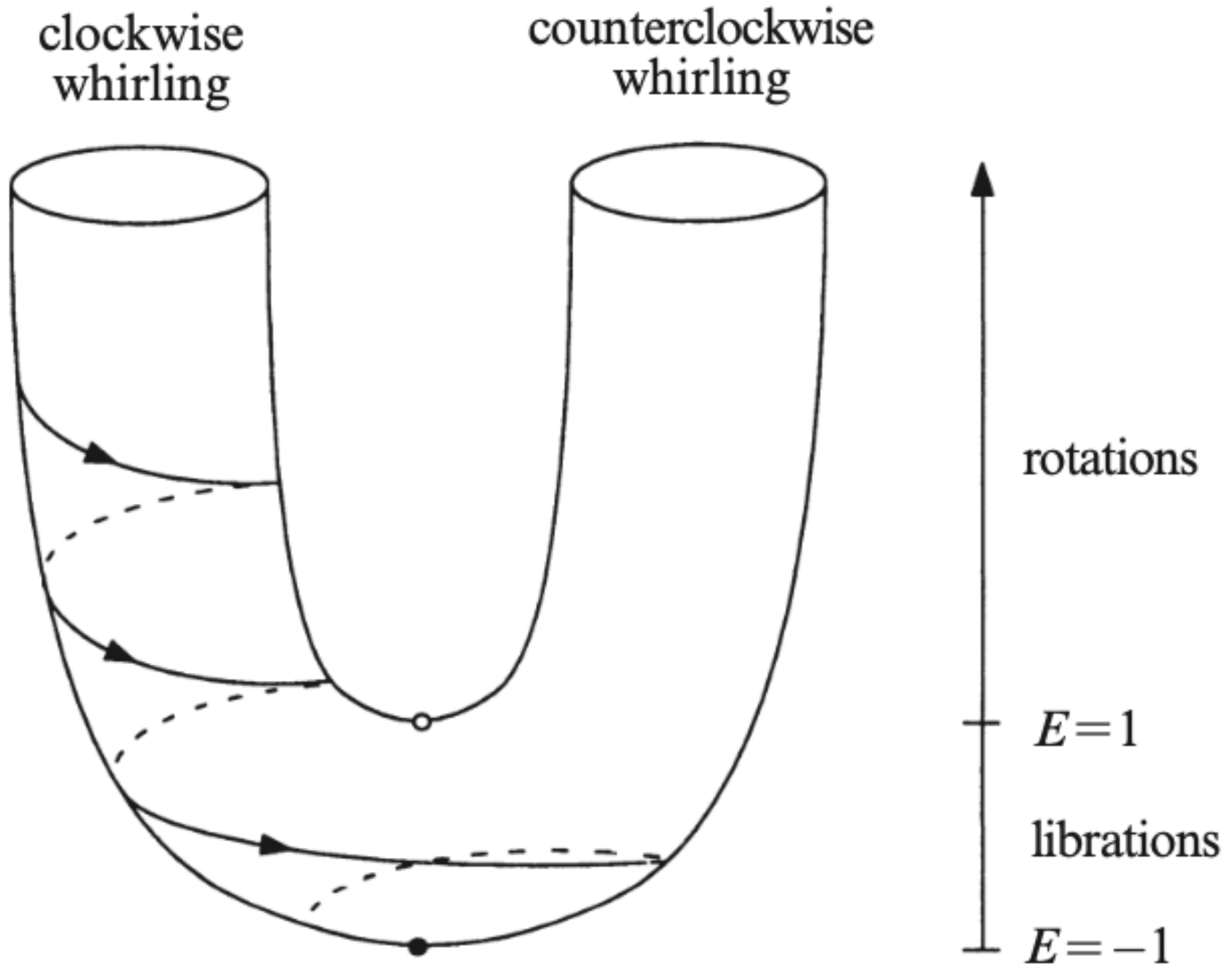
The saddles remain saddles.

Phase portrait



Pendulum: damping

U-tube



All trajectories continually lose altitude, except for the fixed points.

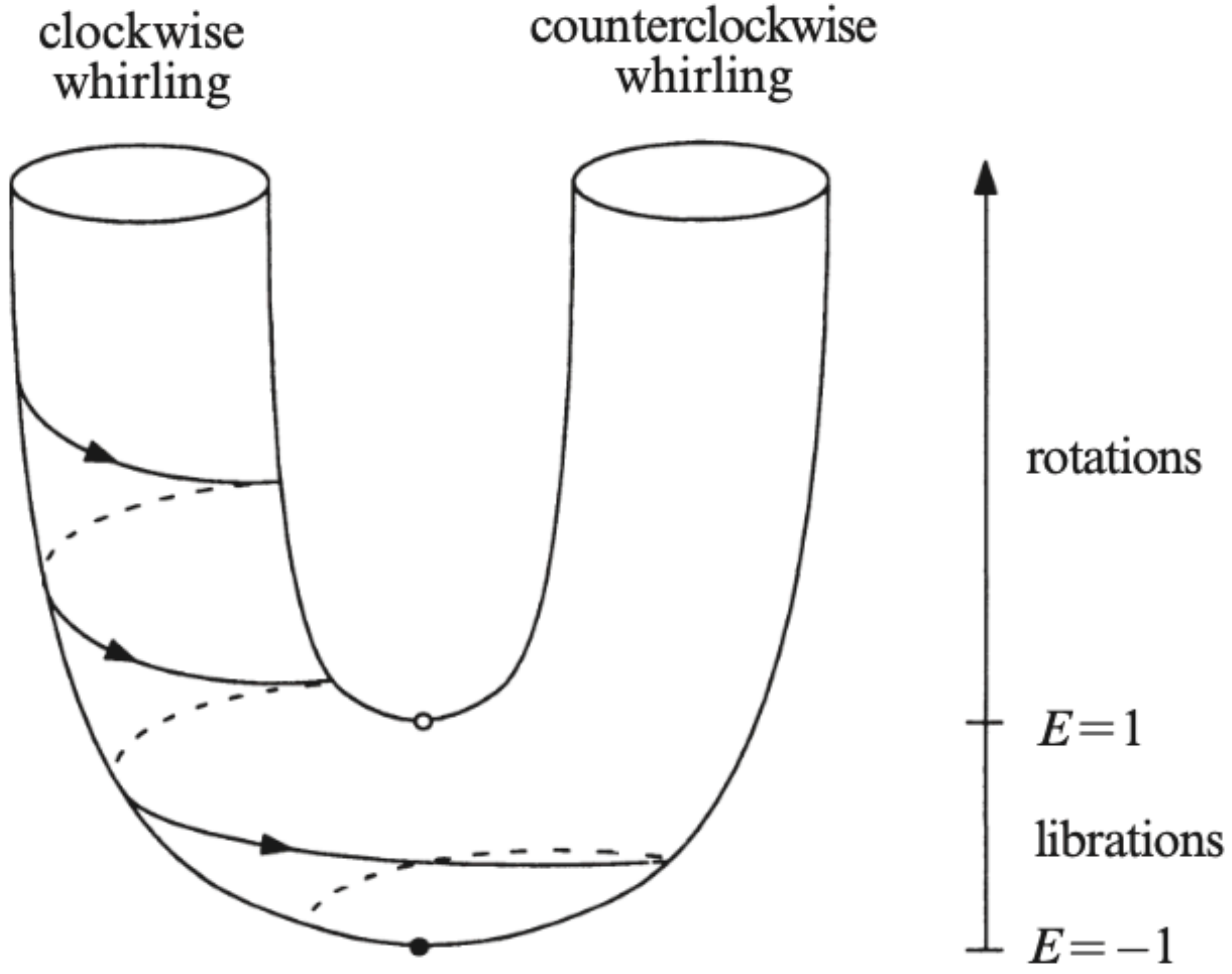
We can see this explicitly by computing the change in energy along a trajectory:

$$\frac{dE}{d\tau} = \frac{d}{d\tau}\left(\tfrac{1}{2}\dot{\theta}^2 - \cos\theta\right) = \dot{\theta}(\ddot{\theta} + \sin\theta) = -b\dot{\theta}^2 \leq 0$$

Hence E decreases monotonically along trajectories, except at fixed points where $\dot{\theta} = 0$.

Pendulum: damping

U-tube



The pendulum is initially whirling clockwise. As it loses energy, it has a harder time rotating over the top.

The corresponding trajectory spirals down the arm of the U-tube until $E < 1$.

The pendulum doesn’t have enough energy to whirl, and so it settles down into a small oscillation about the bottom.

Eventually the motion damps and the pendulum comes to rest at its stable equilibrium.