Nonlinear Dynamics and Chaos

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Linear Systems in 2D

A **two-dimensional linear system** is a system of the form:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

$$\Rightarrow \quad \dot{\mathbf{x}} = A\mathbf{x}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where a, b, c, d are parameters.

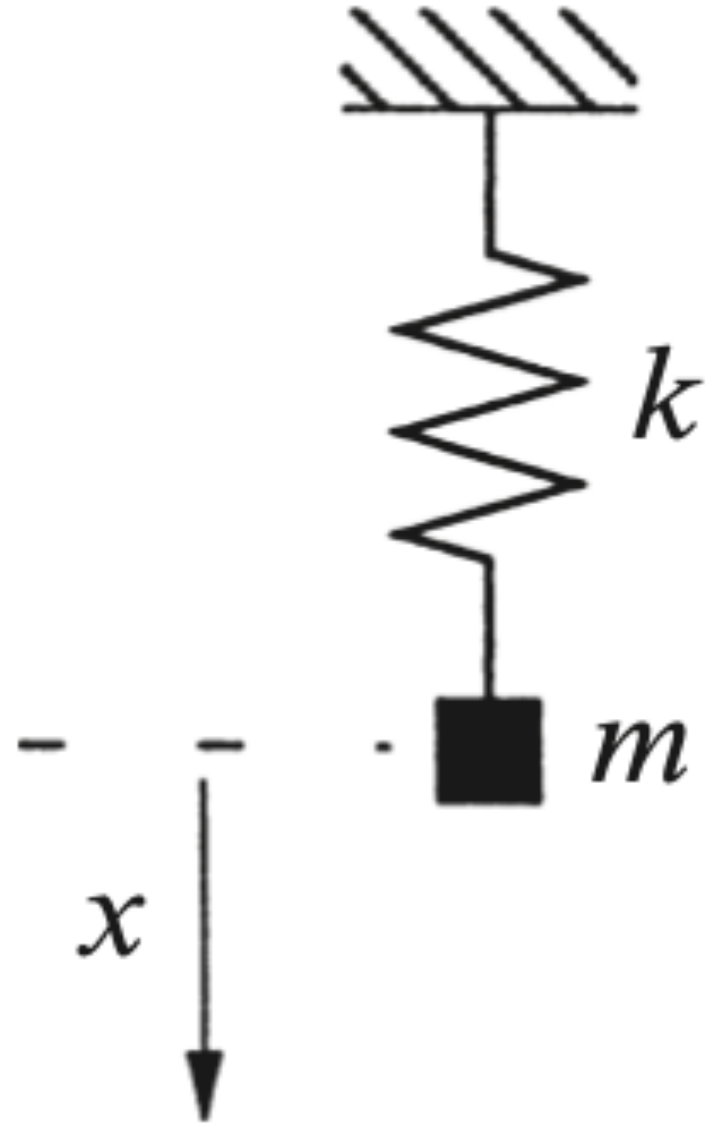
Linearity:

If x_1 and x_2 are solutions, any linear combination $c_1 x_1 + c_2 x_2$ is also a solution.

Fixed point at $x^* = 0$:

Notice that $\dot{x} = 0$ when $x = 0$, so $x^* = 0$ is always a fixed point for any choice of A.

Linear Systems in 2D



Example 1: Simple Harmonic Oscillator

Linear ODE: $m\ddot{x} + kx = 0$

The motion in the phase plane is determined by a vector field that comes from this ODE. We need to find a **vector field** that characterises the state of the system:

The ODE determines the future states of the system given **position** x **and velocity** v

ODE system:

$$\dot{x} = v$$
$$\dot{v} = -\frac{k}{m}x$$

$\omega^2 = k/m$

$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x$$

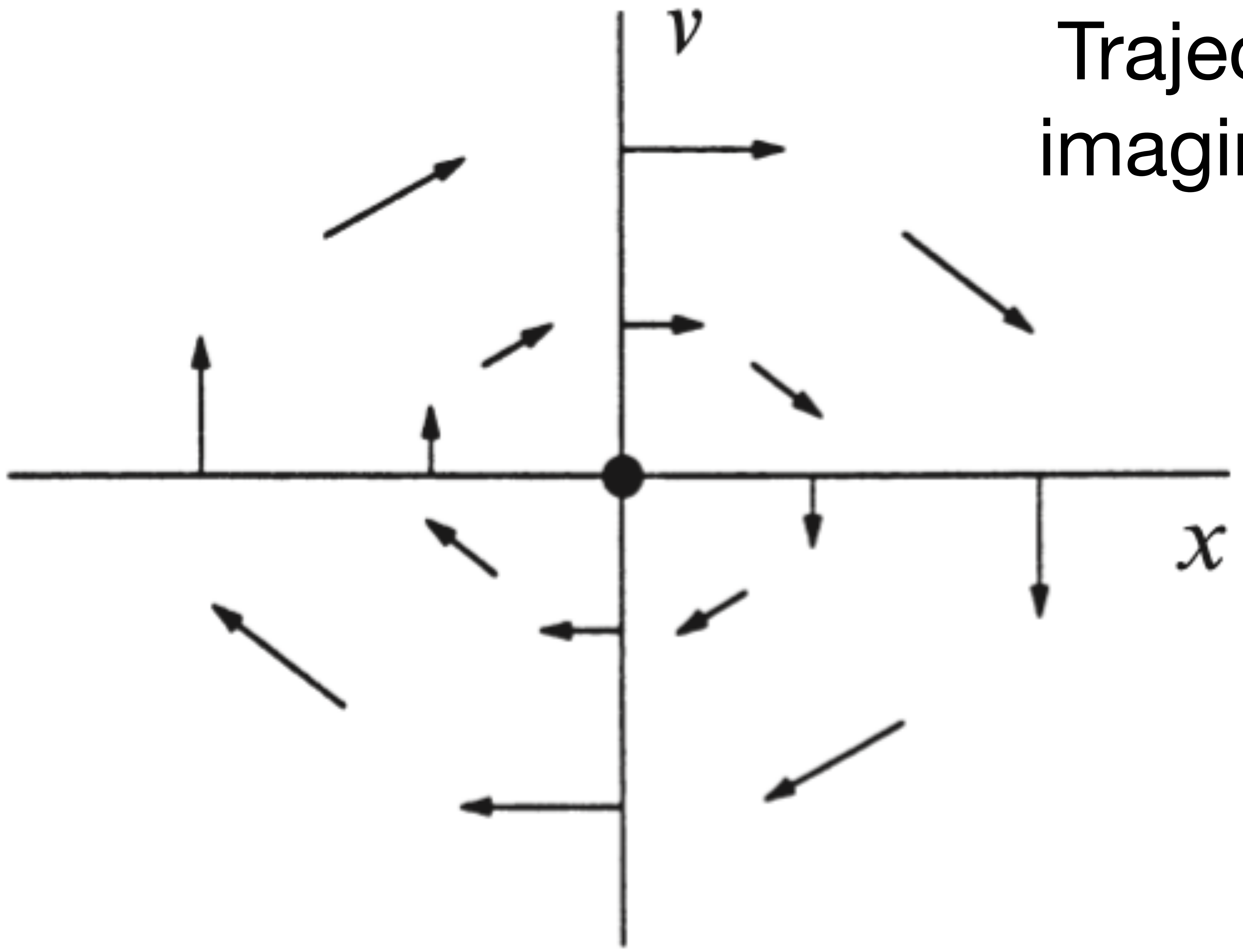
Linear Systems in 2D

ODE system:

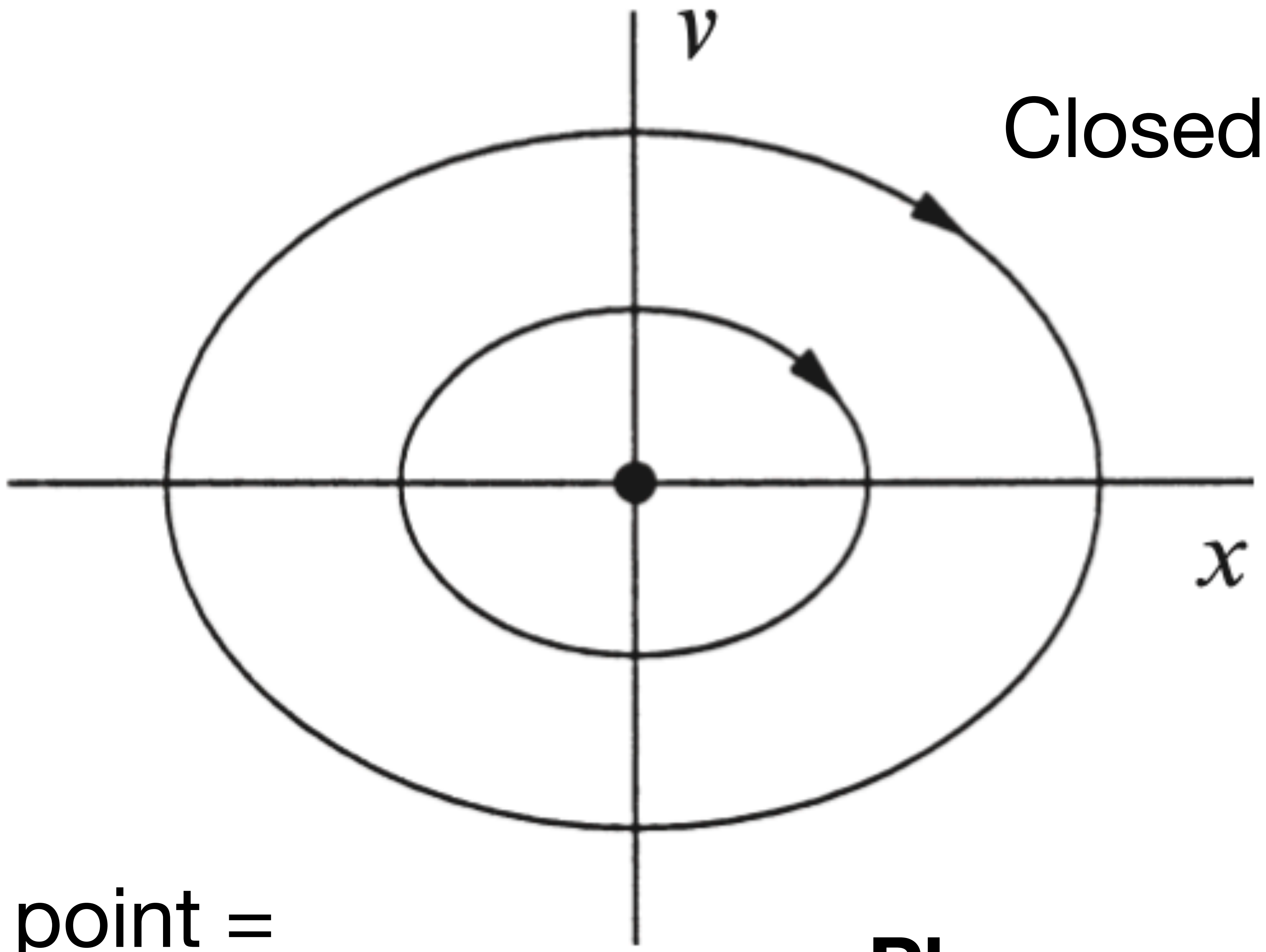
$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x$$

Example 1: Simple Harmonic Oscillator

The ODE system assigns a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at each point (x, v), and therefore represents a **vector field** on the phase plane.



Trajectories = imaginary fluid



Closed orbits

Phase point = imaginary particle

Phase portrait

Linear Systems in 2D

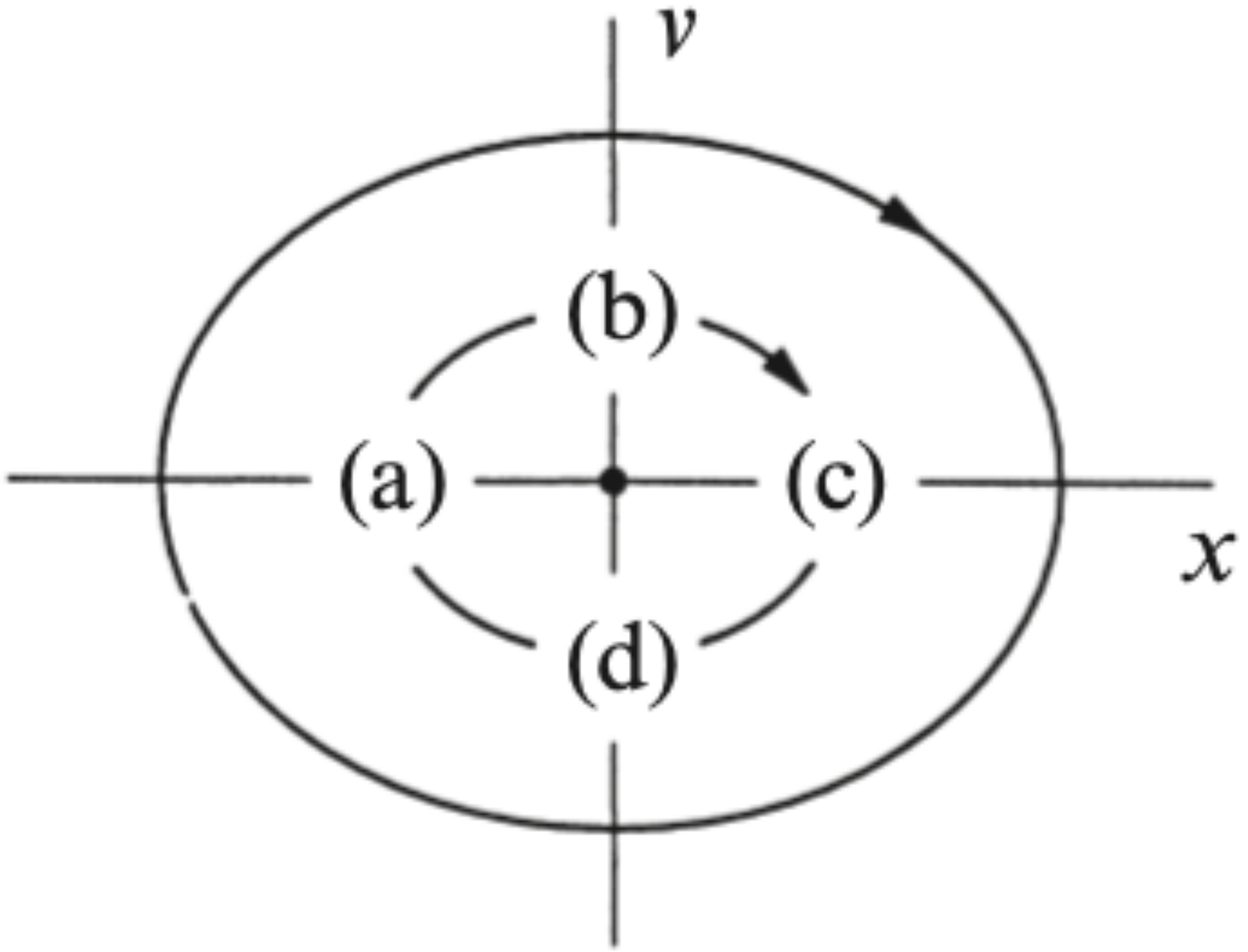
ODE system:

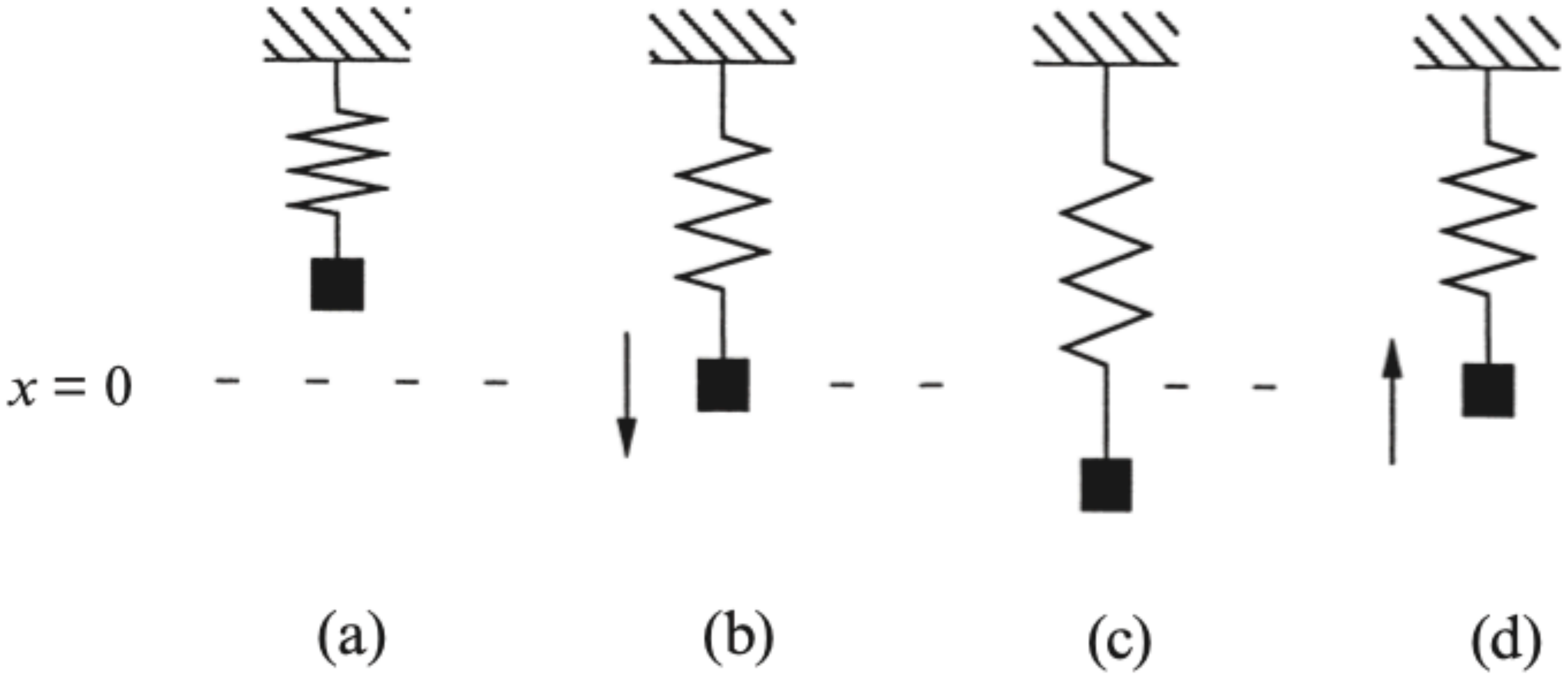
$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x$$

Example 1: Simple Harmonic Oscillator

The fixed point $(x, v) = (0,0)$ corresponds to static equilibrium of the system.

The closed orbits are elipses, $\omega^2 x^2 + v^2 = C$, where $C > 0$ is a constant.





Linear Systems in 2D

Uncoupled ODE system:

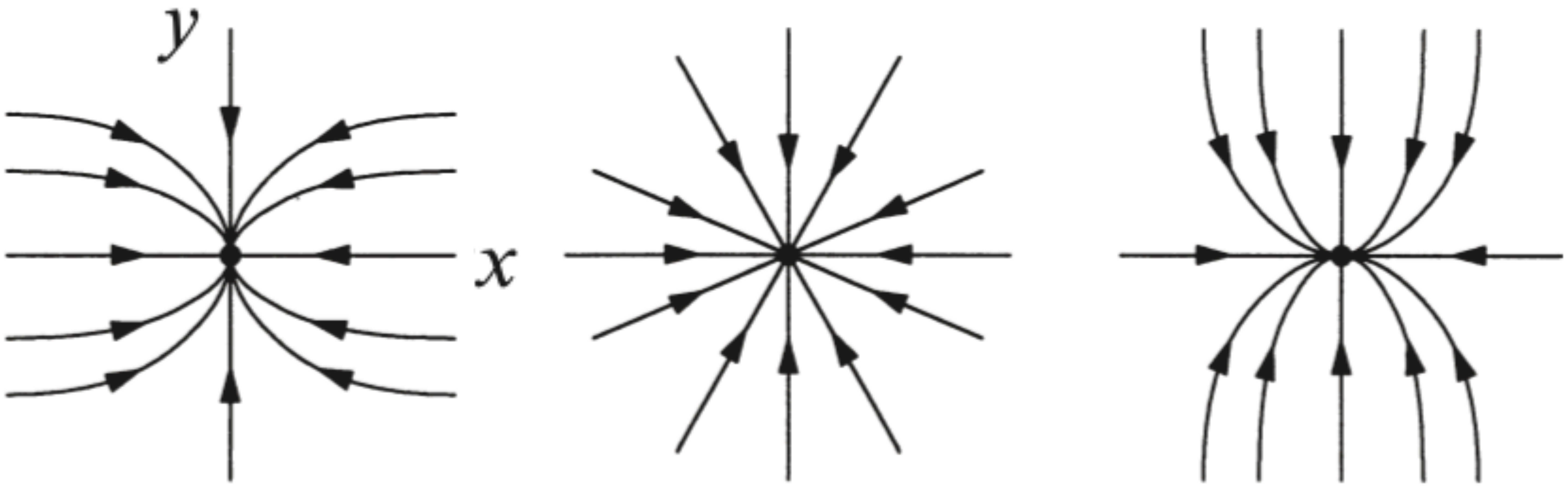
Example 2:

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$$

$$\dot{x} = ax \qquad \dot{y} = -y$$

$$x(t) = x_0 e^{at} \qquad y(t) = y_0 e^{-t}$$

a varies from $-\infty$ to $+\infty$.

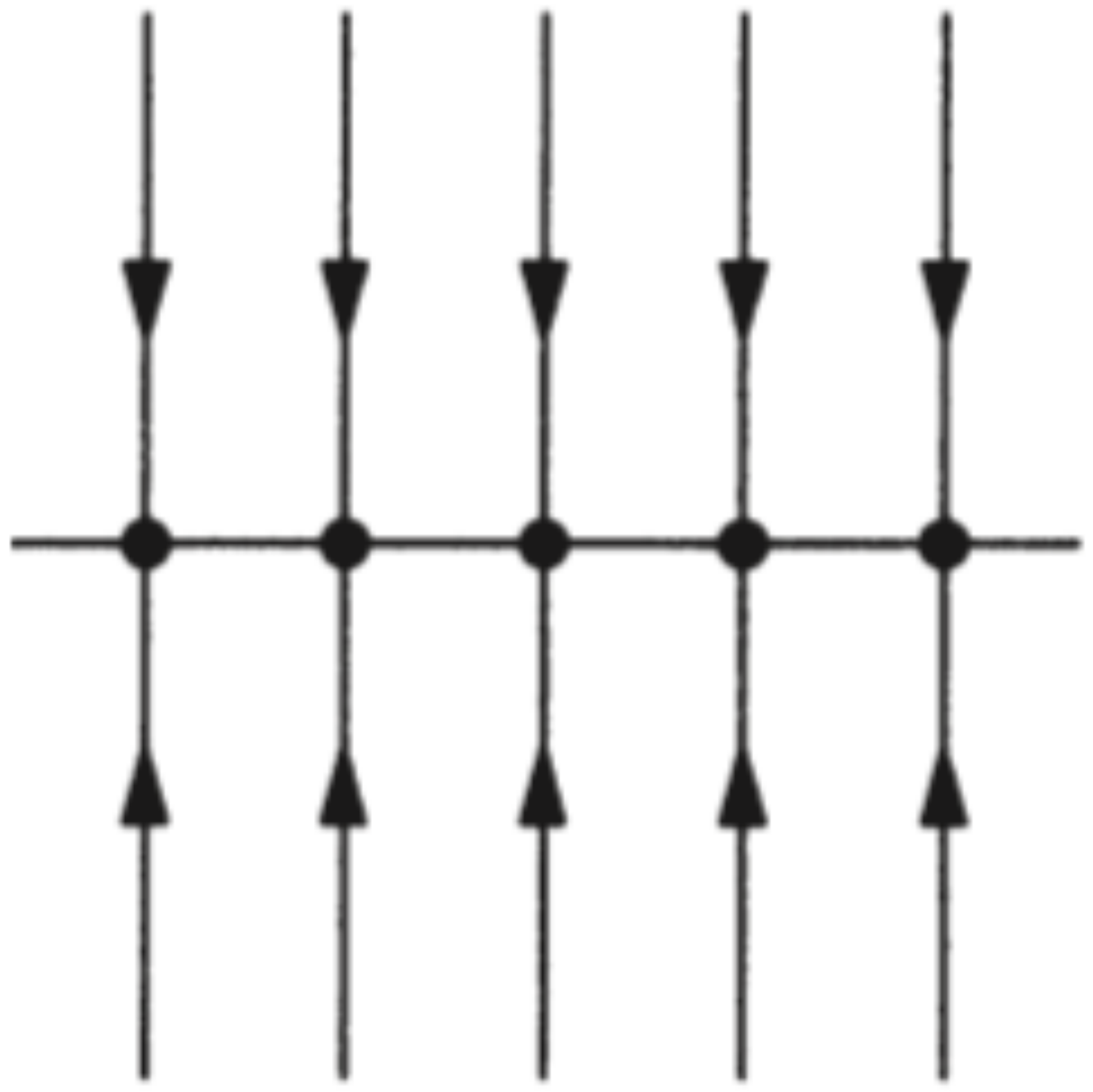


(a) $a < -1$ (b) $a = -1$ (c) $-1 < a < 0$

$x^* = 0$ **(stable node)** $x^* = 0$ **(star node)**

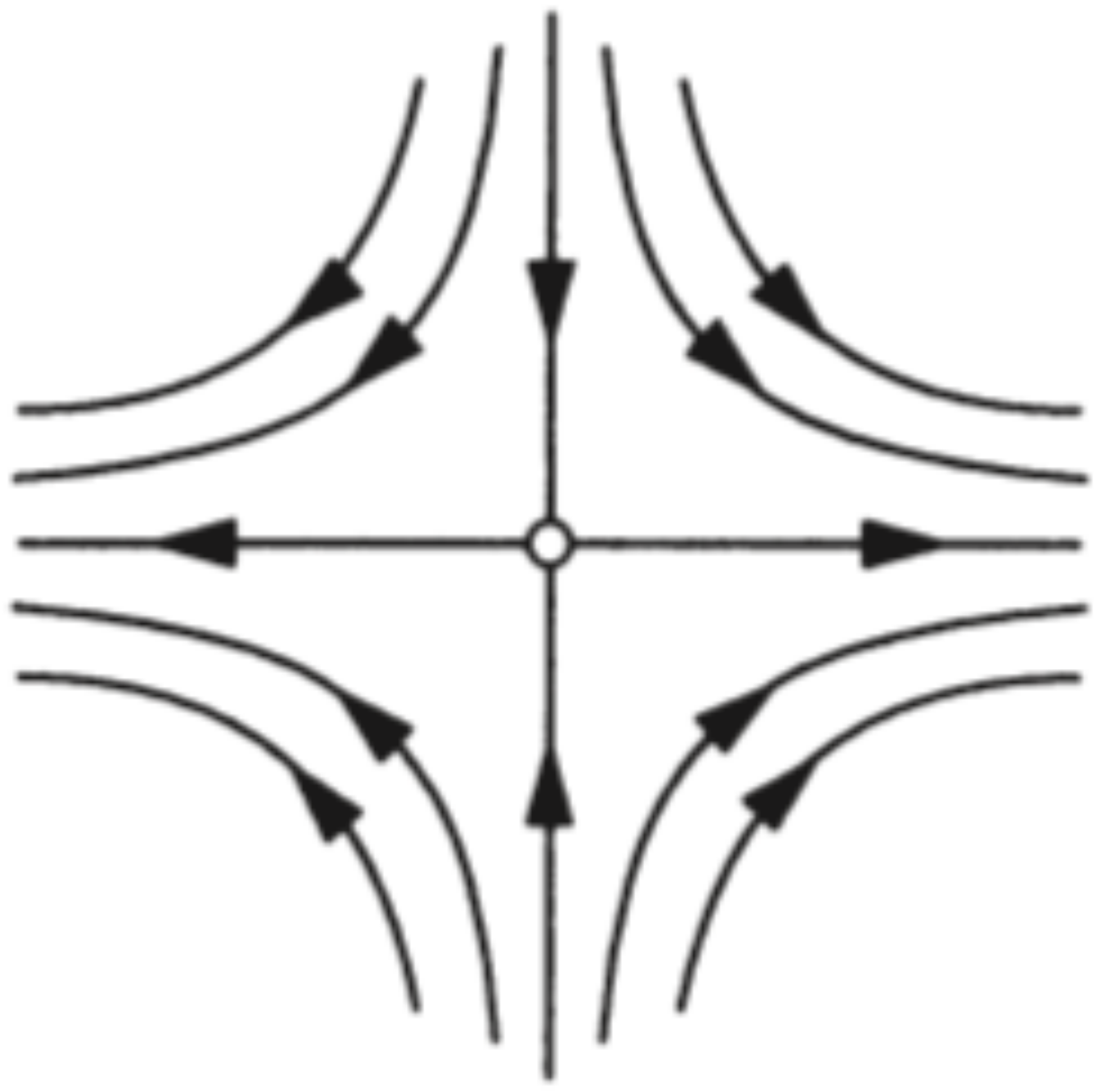
Linear Systems in 2D

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t} \end{aligned}$$



(d) $a=0$

(line of fixed points)



(e) $a>0$

$x^* = 0$ **(saddle node)**

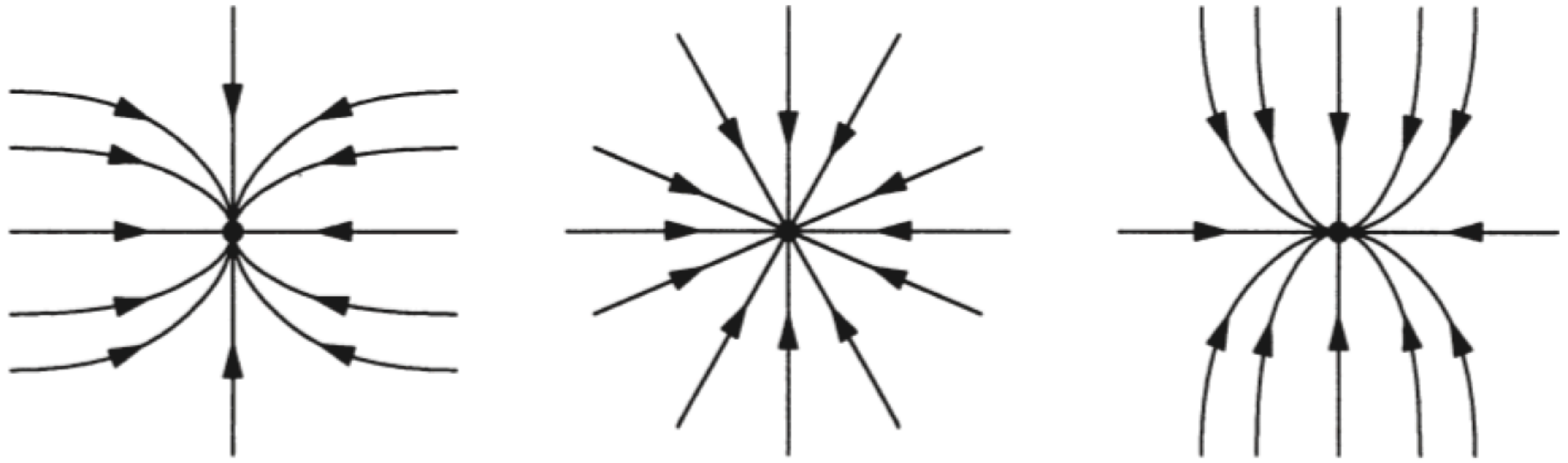
The y-axis is called the **stable manifold** of the saddle point x^* defined as the set of initial conditions $\mathbf{x_0}$ such that $\mathbf{x(t)} \to \mathbf{x}^*$ as $t \to \infty$.

The x-axis is called the **unstable manifold** of the saddle point x^* defined as the set of initial conditions $\mathbf{x_0}$ such that $\mathbf{x(t)} \to \mathbf{x}^*$ as $t \to -\infty$.

Concepts and Stability

$\mathbf{x}^* = \mathbf{0}$ is an **attracting fixed point:**

$\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$

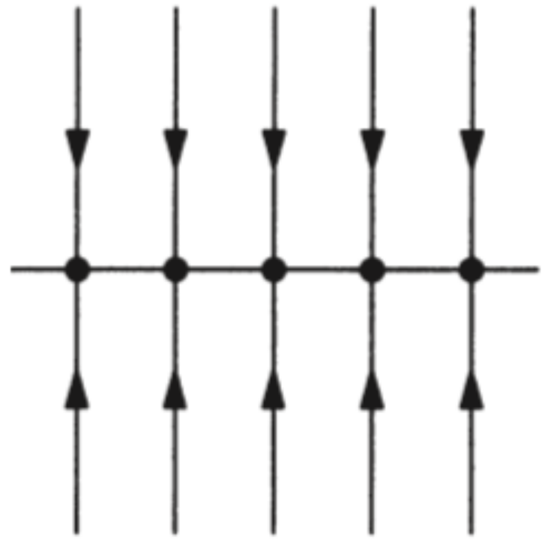


(a) $a < -1$ (b) $a = -1$ (c) $=1 < a < 0$

Stability: behaviour of trajectories for all time.

$\mathbf{x}^* = \mathbf{0}$ is **Liapunov stable** if all trajectories that start sufficiently close to $\mathbf{x}^*$ remain close to it for all time.

$\mathbf{x}^* = \mathbf{0}$ is **neutrally stable** when a fixed point is Liapunov stable but not attracting.



(d) $a=0$

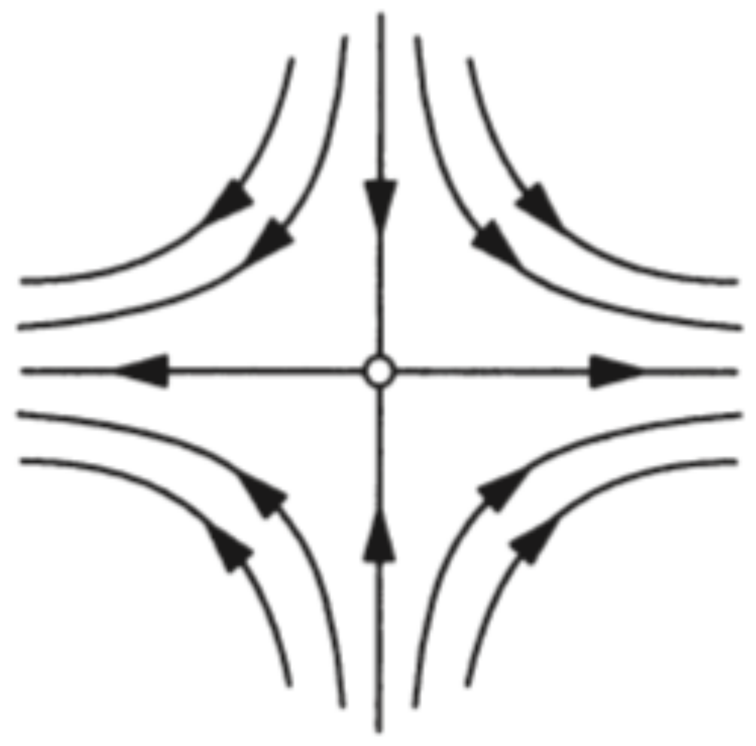
Neutral stability is commonly encountered in mechanical systems in the absence of friction (e.g. SAO equilibrium point).

Concepts and Stability

Stability

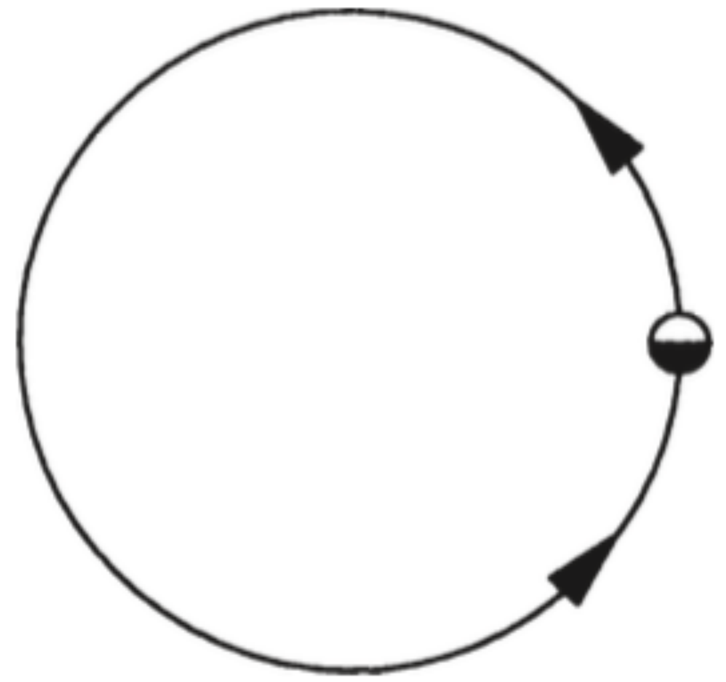
If a fixed point is both Liapunov stable and attracting, we call it **stable**, or sometimes **asymptotically stable.**

x* is ***unstable*** when it is neither attracting nor Liapunov stable.



(e) $a>0$

Example: Vector field on the circle



$$\dot{\theta} = 1 - \cos\theta$$

Here $\theta^* = 0$ attracts all trajectories as $t \to \infty$, but it is not Liapunov stable.

Classification of Linear Systems

Analysing Fixed Points in 2D Systems

Linear Equation: $\dot{\mathbf{x}} = A\mathbf{x}$

For the general case, we seek trajectories of the form: $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$

If such solutions exist, they correspond to exponential motion along the line spanned by the vector **v**.

The desired straight-line solutions exist if **v** is an ***eigenvector*** of *A* with corresponding ***eigenvalue*** λ**:**

$$\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}. \quad \Rightarrow \quad A\mathbf{v} = \lambda\mathbf{v}$$

The stability and dynamics of the fixed point at $x^* = 0$ are determined by the **eigenvalues** (λ_1, λ_2) of the matrix A.

Classification of Linear Systems

Eigenvalues:

2x2 matrix: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - \tau\lambda + \Delta = 0$

Where: $$\begin{aligned} \tau &= \mathrm{trace}(A) = a + d, \\ \Delta &= \det(A) = ad - bc. \end{aligned}$$

The eigenvalues depend only on the trace and determinant of the matrix *A:*

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \Rightarrow \lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

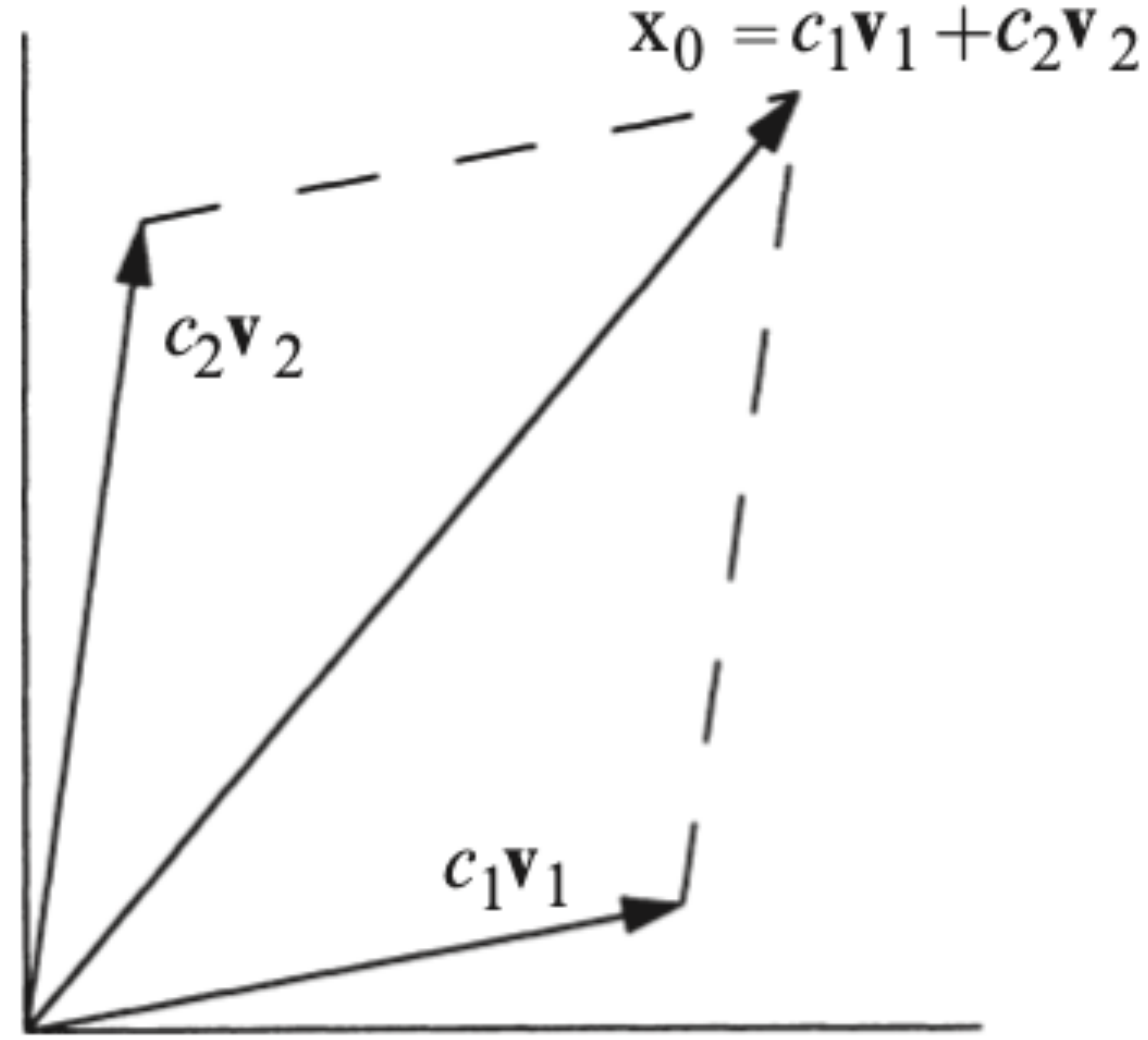
Classification of Linear Systems

General solution:

Linear Equation: $\dot{\mathbf{x}} = A\mathbf{x}$

Any initial condition x_0 can be written as a linear combination of eigenvectors:

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$



The general solution for **x**(t) is:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

Classification of Linear Systems

Stable & Unstable Nodes:

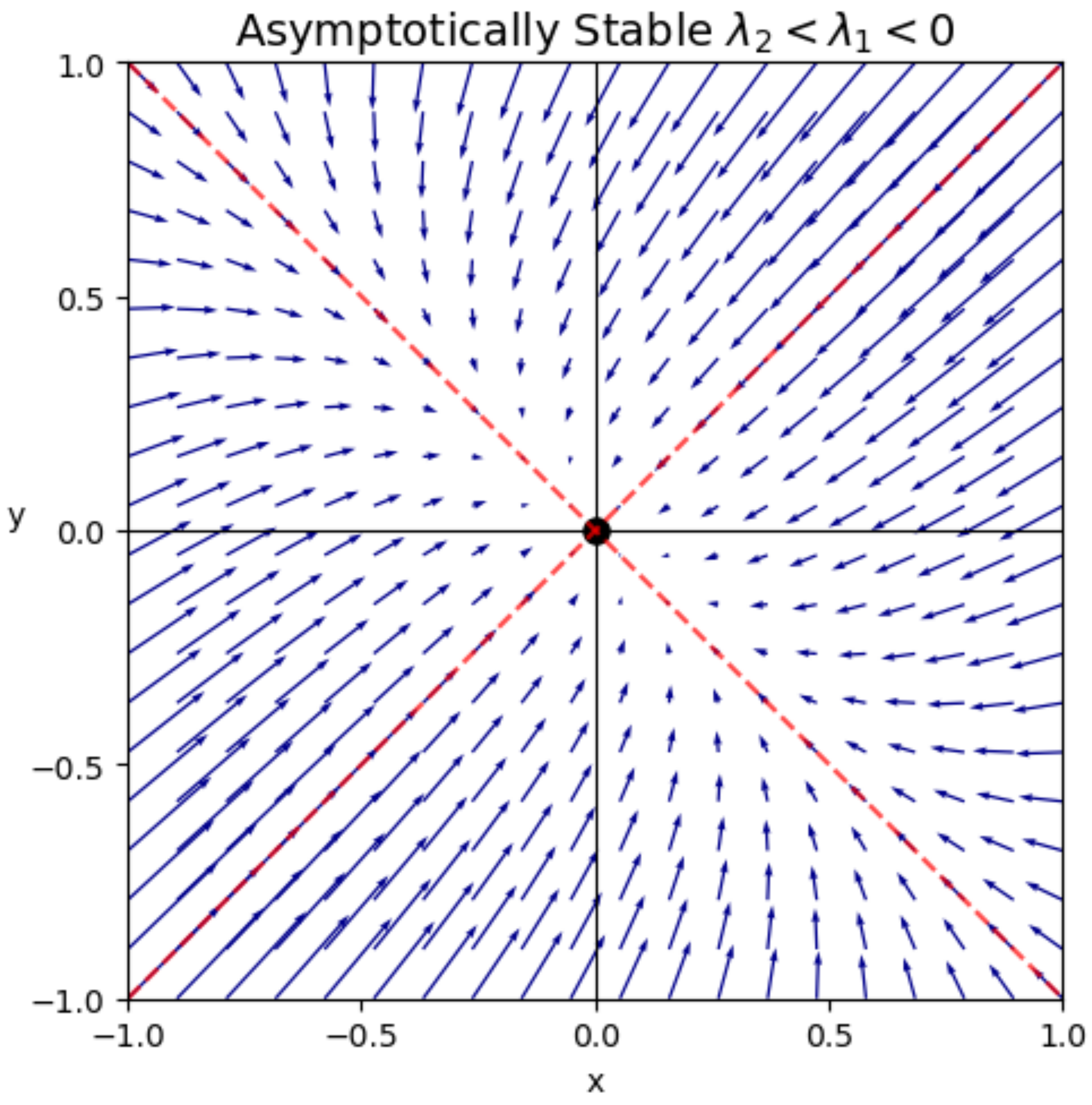
Type	Description	Condition	Phase Portrait
Stable Node	λ_1 , λ_2 are real, distinct, and negative	$\lambda_2 < \lambda_1 <0$	Trajectories flow inward (**stable**). Approach the origin tangent to the eigenvector of λ_1 (the smaller absolute value. **Asymptotically stable.**
Unstable Node	λ_1 , λ_2 are real, distinct, and positive	$0< \lambda_1 < \lambda_2$	Trajectories flow outward (**unstable**). Depart tangent to the eigenvector of λ2 (the larger value).

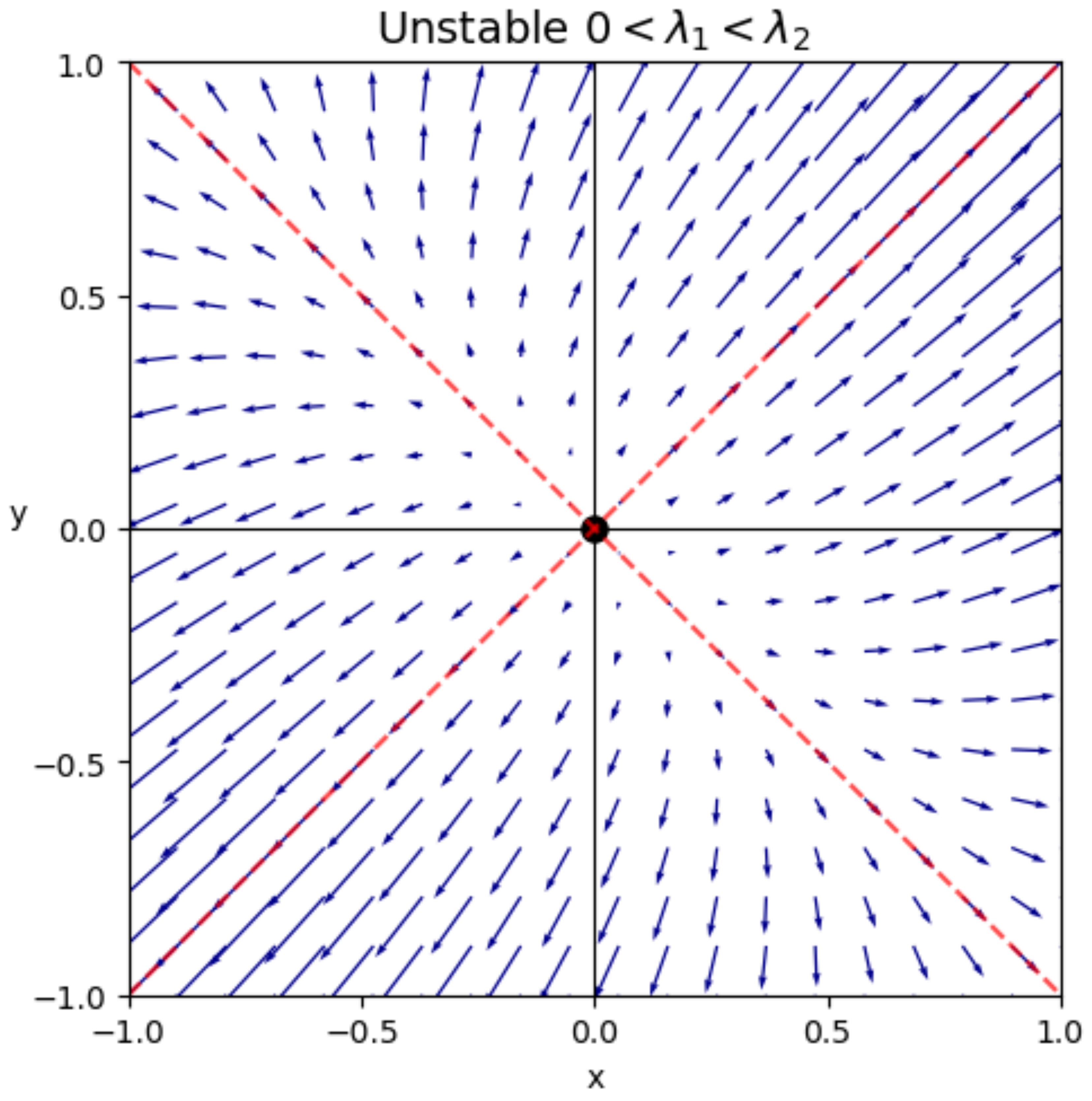
Classification of Linear Systems

Stable & Unstable Nodes:

$$\mathbf{A}_{\text{stable}} = \begin{pmatrix} -2.0 & -1.0 \\ -1.0 & -2.0 \end{pmatrix} \qquad \mathbf{A}_{\text{unstable}} = \begin{pmatrix} 2.0 & 1.0 \\ 1.0 & 2.0 \end{pmatrix}$$

Eigendirection	Eigenvalue	Behaviour
Fast	λ2 (Larger absolute value)	Trajectories quickly align toward this direction first.
Slow	λ1 (Smaller absolute value)	Trajectories become tangent to this direction near the origin.





Classification of Linear Systems

Saddle Point:

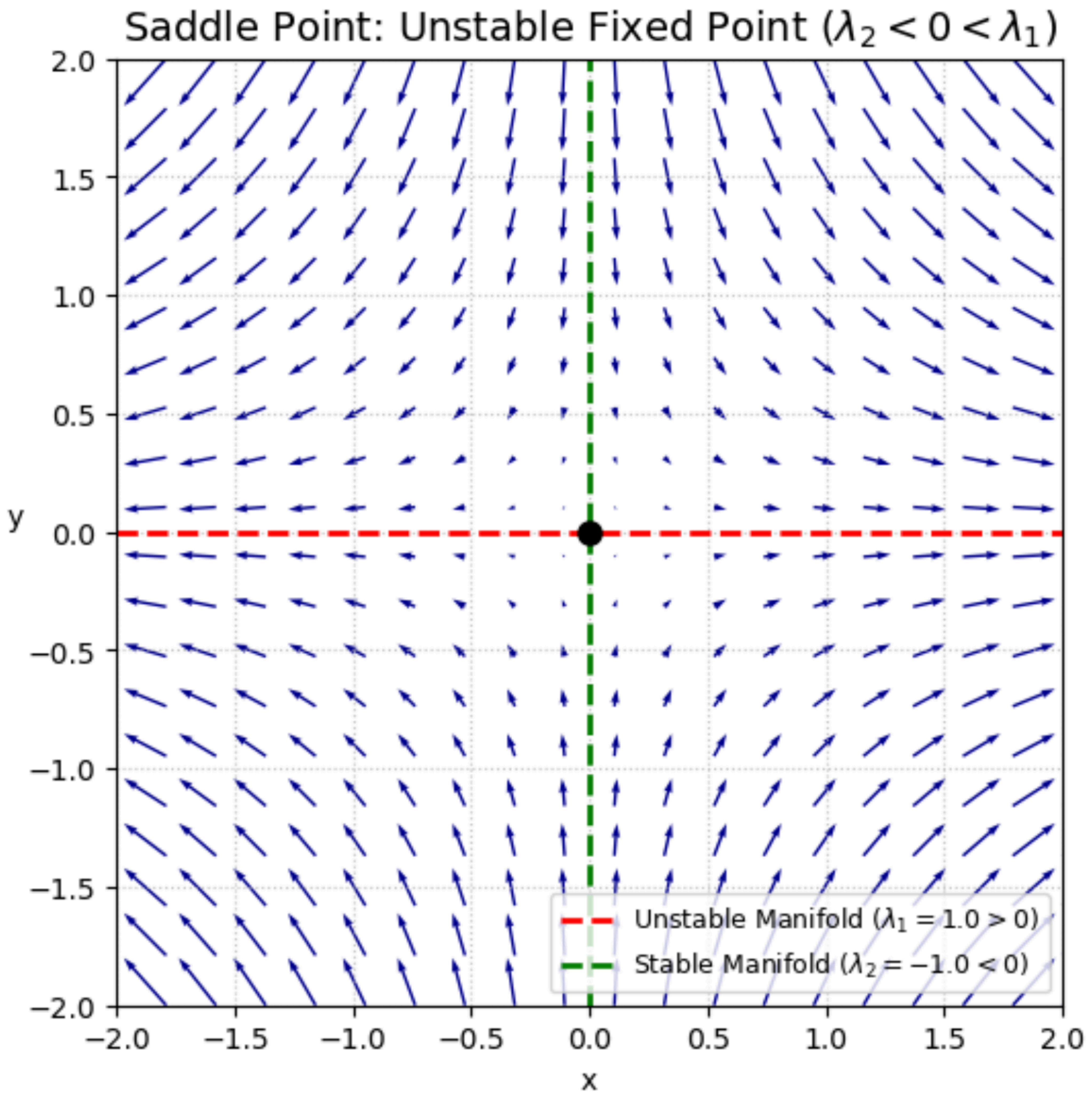
Type	Description	Condition	Phase Portrait
Saddle Point	λ_1 , λ_2 are real, distinct, and have opposite signs	$\lambda_1 < 0 < \lambda_2$	**Unstable.** Defined by two key manifolds: **Stable Manifold** (along v_1) and **Unstable Manifold** (along v_2). Only trajectories starting on the stable manifold approach the origin.

Classification of Linear Systems

Saddle Point:

$$\mathbf{A}_{\text{saddle}} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & -1.0 \end{pmatrix}$$

Eigenvalue (λ)	Eigenvector (v)	Axis	Flow Type	Manifold Type
$\lambda 1 = 1.0$	$v1 = (1\ 0)$	x-axis (y=0)	Repelling (Outward)	Unstable Manifold (W^u)
$\lambda 2 = -1.0$	$v2 = (0\ 1)$	y-axis (x=0)	Attracting (Inward)	Stable Manifold (W^s)



Classification of Linear Systems

Spirals and Centre: The case of complex eigenvalues

Example: Harmonic oscillator slightly damped.

The eigenvalues are: $$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

Complex eigenvalues occur when: $\tau^2 - 4\Delta < 0$

Eigenvalues: $\lambda_{1,2} = \alpha \pm i\omega$ with: $\alpha = \tau/2, \; \omega = \tfrac{1}{2}\sqrt{4\Delta - \tau^2}$

General solution:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

We have exponentially decaying oscillations if $\alpha = Re(\lambda) < 0$ (**stable spiral**) and growing oscillations if $\alpha = Re(\lambda) > 0$ (**unstable spiral**).

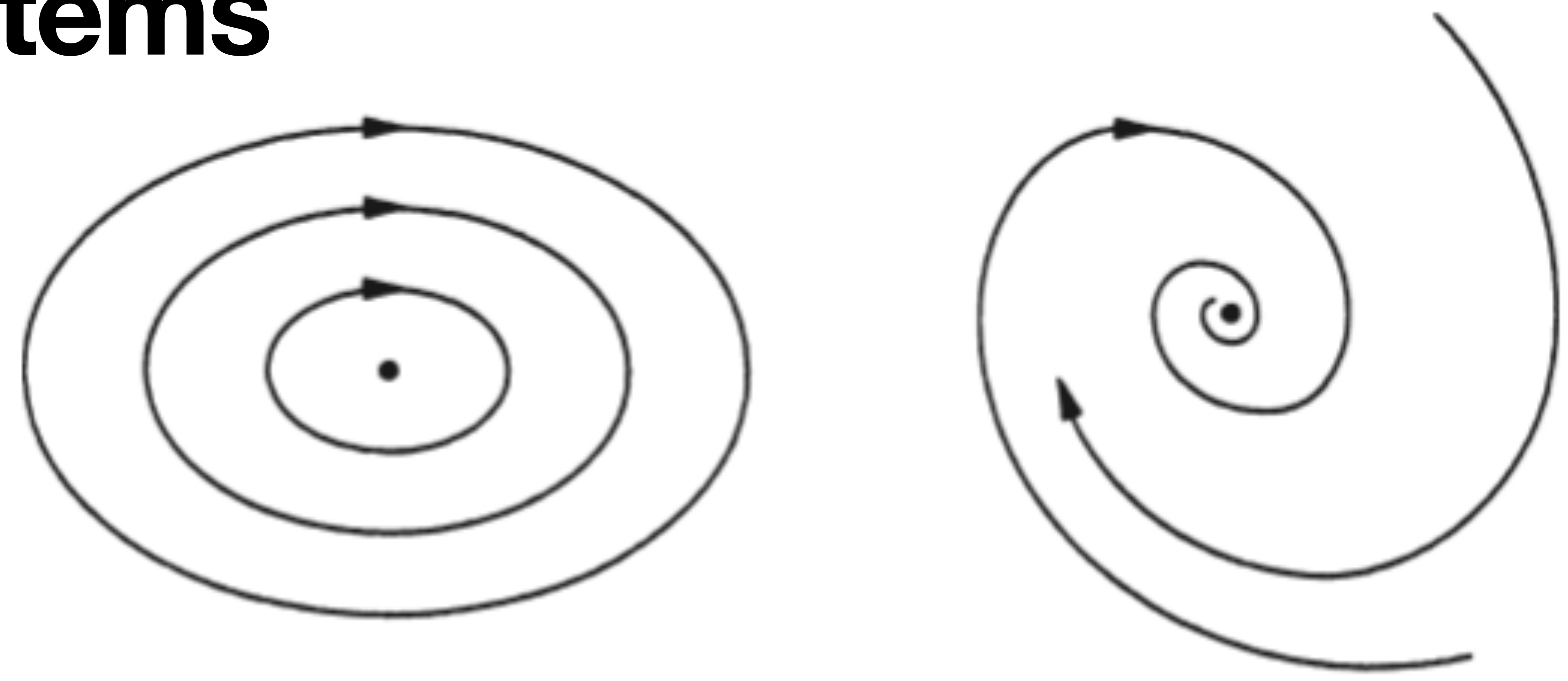
Classification of Linear Systems

Spirals and Centre: The case of complex eigenvalues

Type	Description	Condition	Phase Portrait
Stable Spiral	Complex with negative real part.	$\alpha<0$	Trajectories spiral inward toward the origin (Stable). The fixed point is an attracting focus. Asymptotically Stable.
Unstable Spiral	Complex with positive real part	$\alpha>0$	Trajectories spiral outward away from the origin (Unstable). The fixed point is a repelling focus.

Classification of Linear Systems

Spirals and Centre



(a) center (b) spiral

Type	Description	Condition	Phase Portrait
Centre	Purely imaginary	$\alpha=0$	Trajectories are closed orbits (ellipses). Stable, but not asymptotically stable (they stay close, but don't converge).

Classification of Linear Systems

Degenerate Cases (Repeated Eigenvalues)

Type	Description	Condition	Phase Portrait
Star/Proper Node	Real, equal, negative, with two eigenvectors	λ1 = λ2 $v_1 \neq v_2$	Trajectories are straight lines flowing directly inward (Stable). Highly symmetric convergence. Asymptotically Stable.
Degenerate/ Improper Node	Real, equal, negative, with one eigenvector.	λ1 = λ2 $v_1 = v_2$	Trajectories flow inward but are tangent to a single direction (the single eigenvector). Less symmetric convergence than a Star Node. Asymptotically Stable.

Classification of Linear Systems

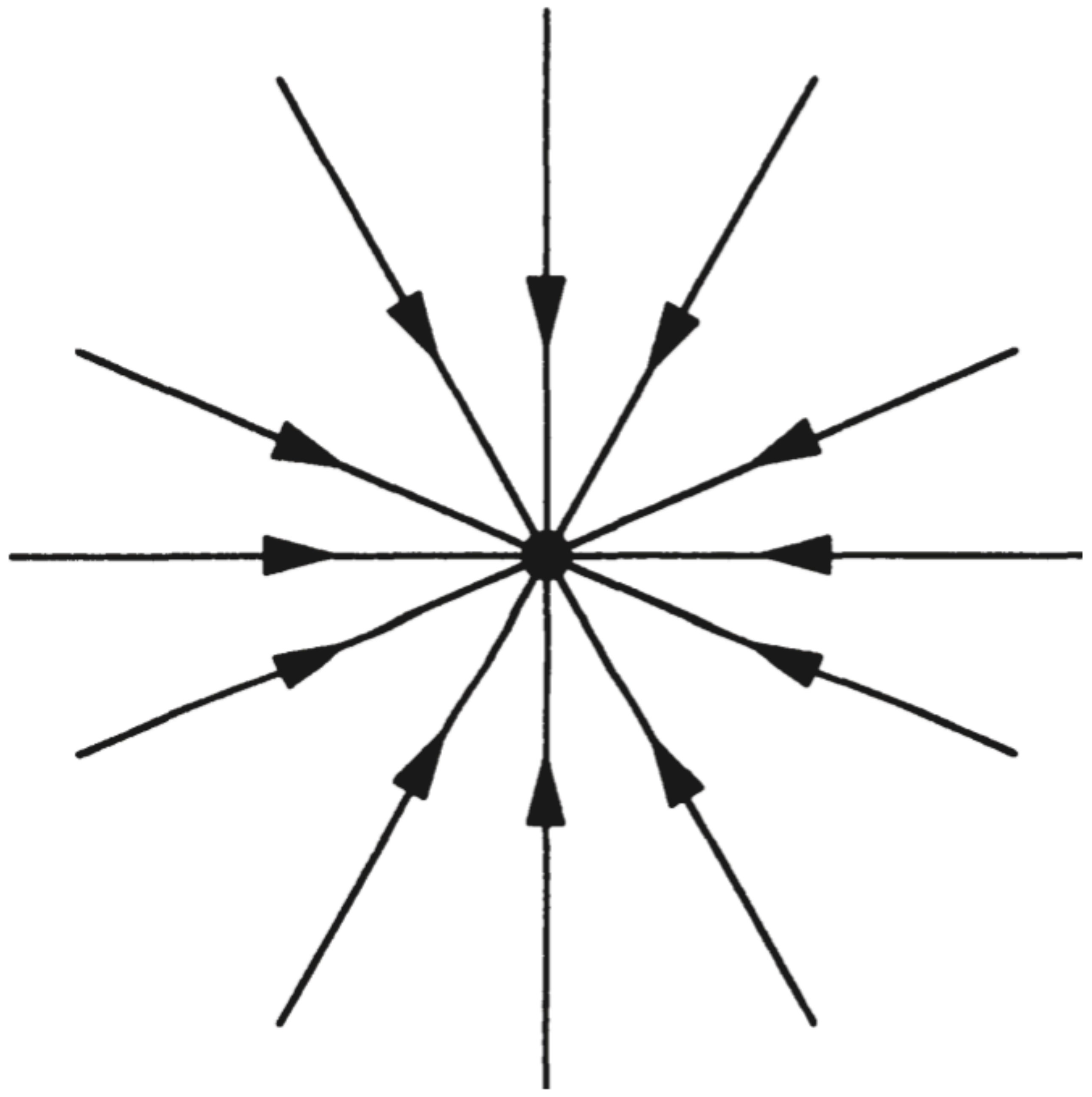
Repeated Eigenvalues, different eigenvectors:

If there are two independent eigenvectors, then they span the plane and so every vector is an eigenvector with this same eigenvalue λ.

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \quad \Rightarrow \quad A\mathbf{x}_0 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda\mathbf{v}_1 + c_2\lambda\mathbf{v}_2 = \lambda\mathbf{x}_0$$

$\mathbf{x_0}$ is also an eigenvector with eigenvalue λ. Multiplication by A simply stretches every vector by a factor λ.

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

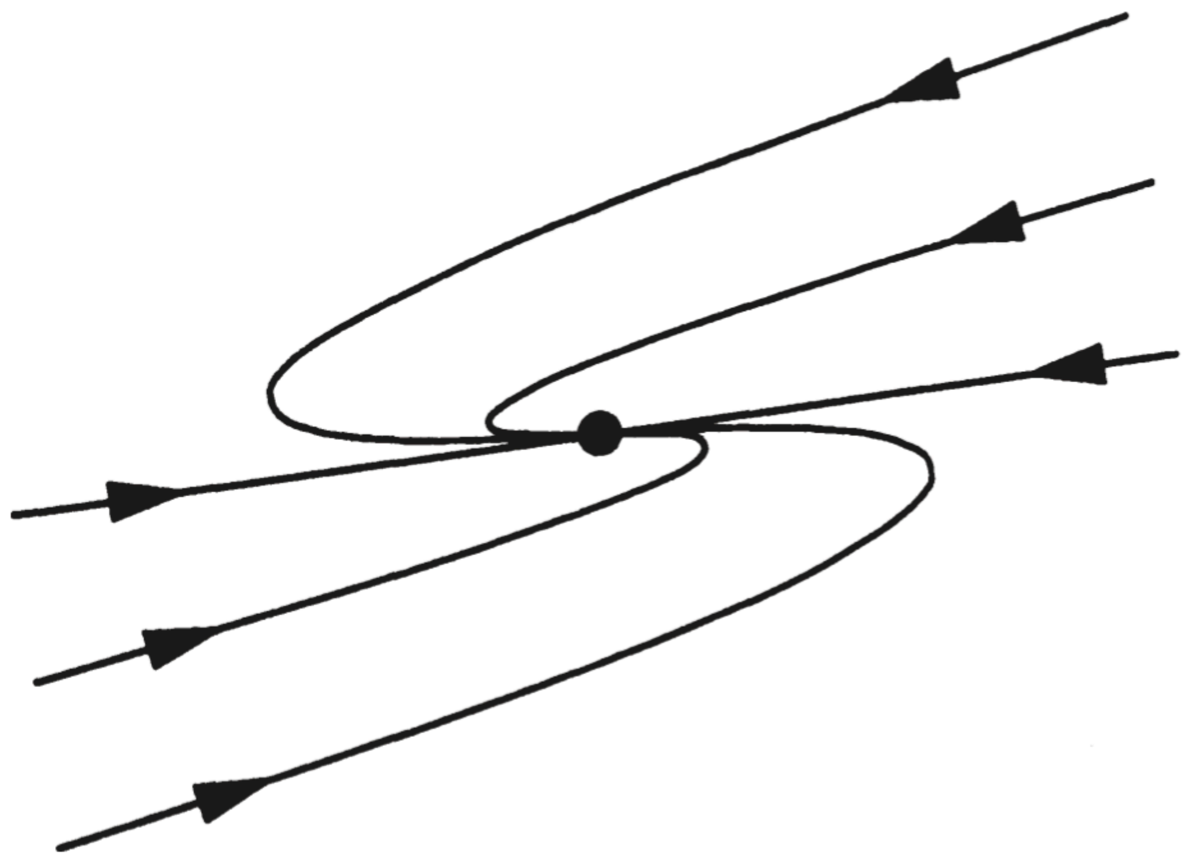


If $\lambda \neq 0$, all trajectories are straight lines through the origin ($\mathbf{x}(\mathbf{t}) = e^{\lambda t}\mathbf{x_0}$) and the fixed point is a **star node.**

Classification of Linear Systems

Repeated Eigenvalues, same eigenvector:

The other possibility is that there's only one eigenvector.
The eigenspace corresponding to is one-dimensional.

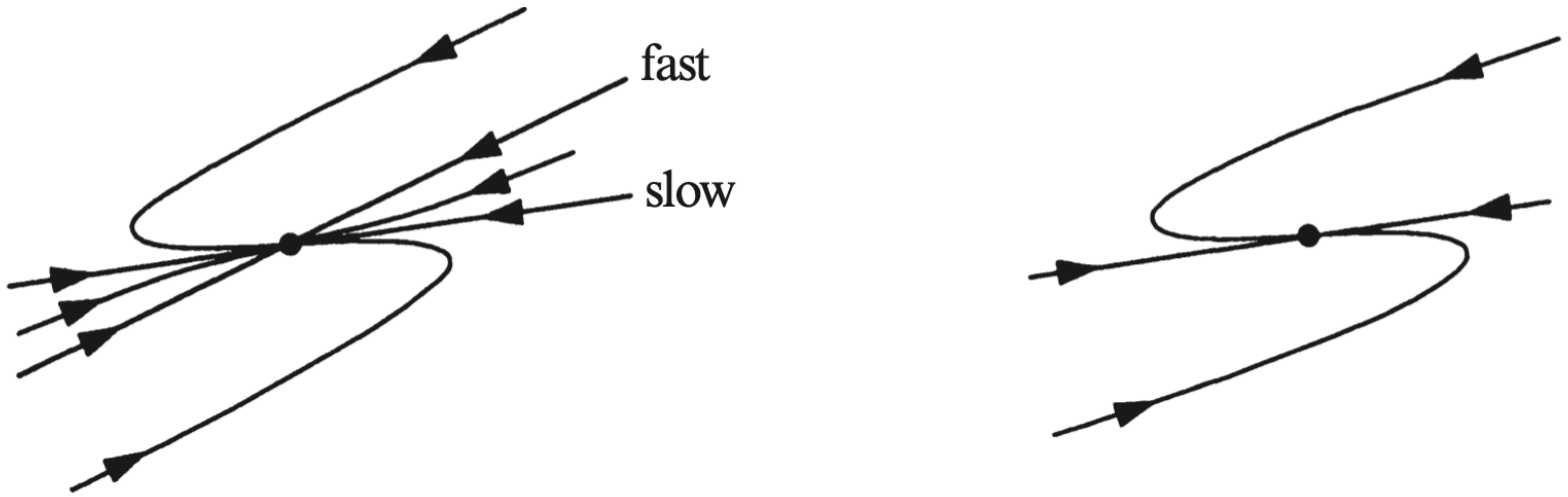


$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$$

with $b \neq 0$ has only a one-dimensional eigenspace.

When there's only one eigendirection, the fixed point is a **degenerate node.**

A degenerate node is created by deforming an ordinary node.

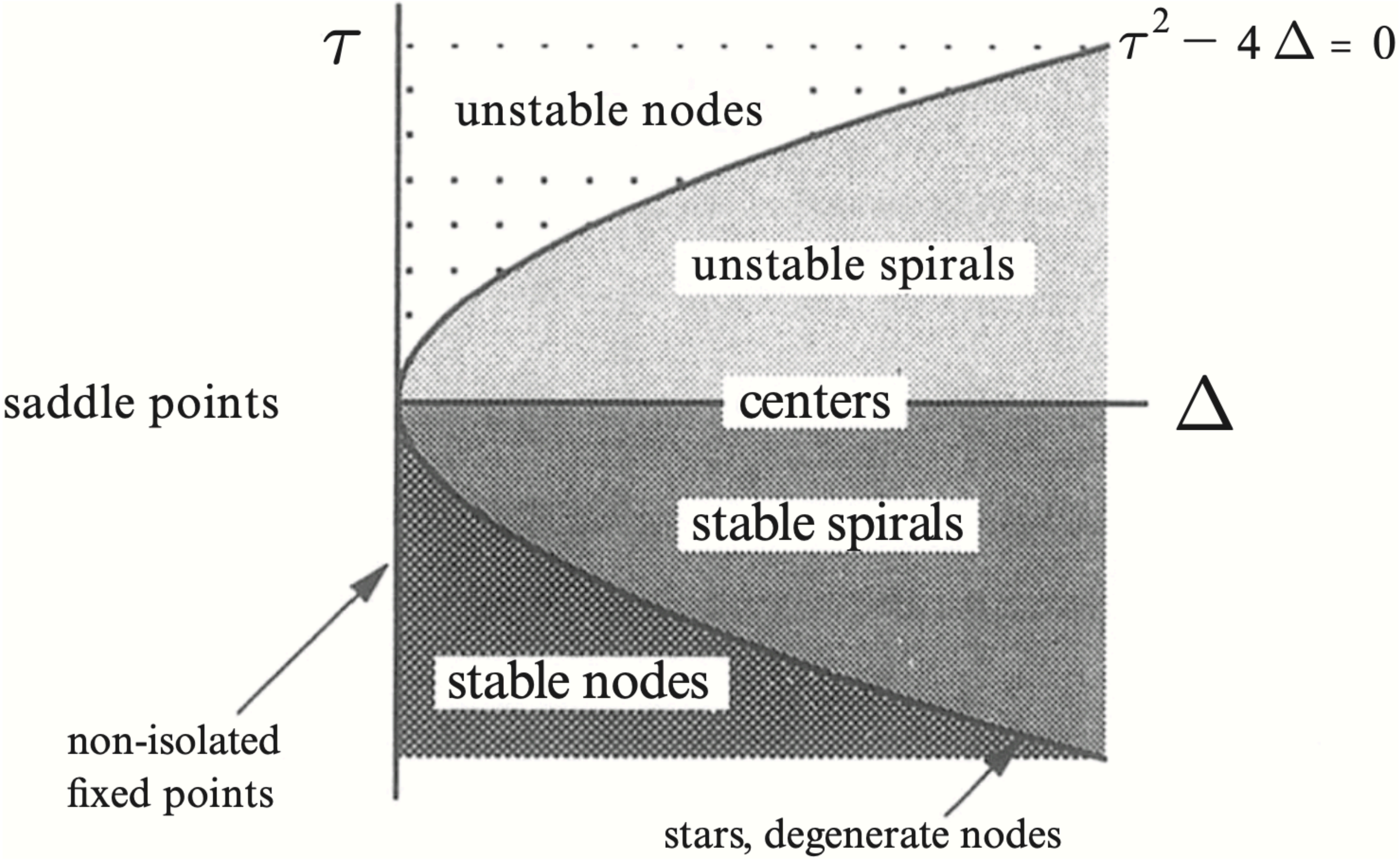


(a) node

(b) degenerate node

Classification of Fixed Points

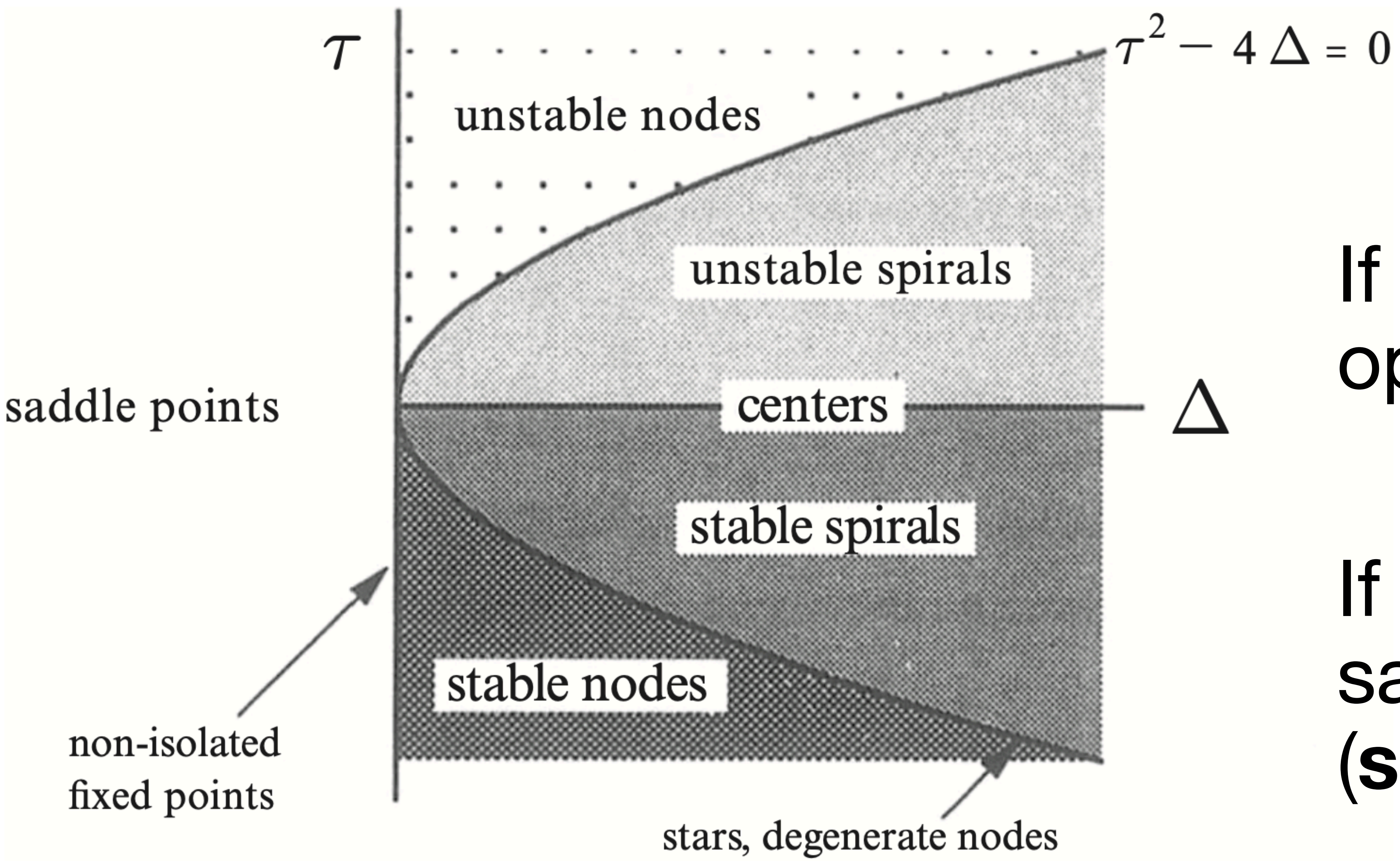
We can show the type and stability of all the different fixed points on a single diagram.



Classification of Fixed Points

The axes of the diagram are the **trace** τ and the **determinant** Δ of the matrix *A.* All of the information in the diagram is implied by the following formulas:

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \qquad \tau = \lambda_1 + \lambda_2 .$$

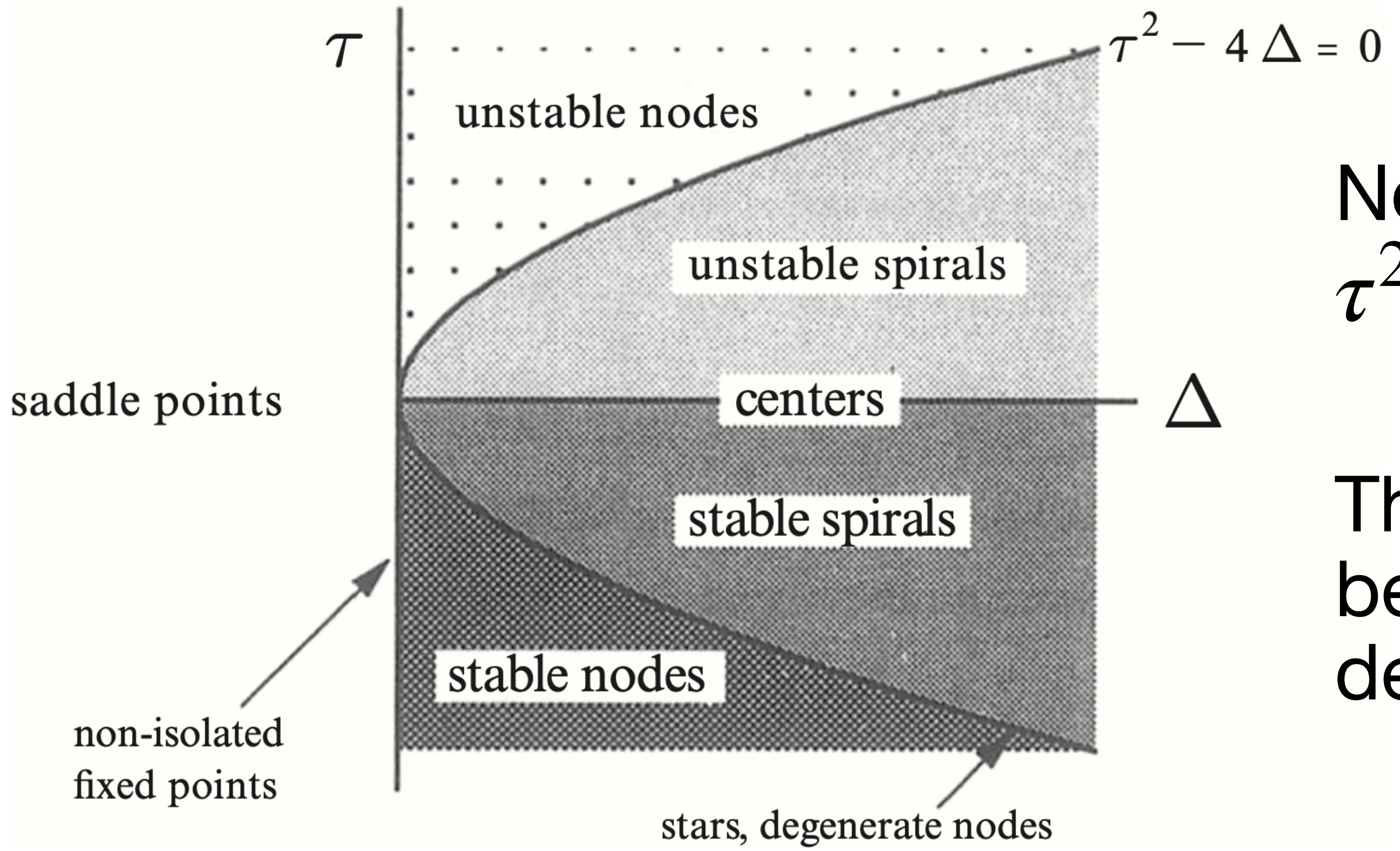


$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \tau\lambda + \Delta = 0.$$

If $\Delta < 0$, the eigenvalues are real and have opposite signs. The fixed point is a **saddle point.**

If $\Delta > 0$, the eigenvalues are either real with the same sign (**nodes**), or complex conjugate (**spirals and centres**).

Classification of Fixed Points



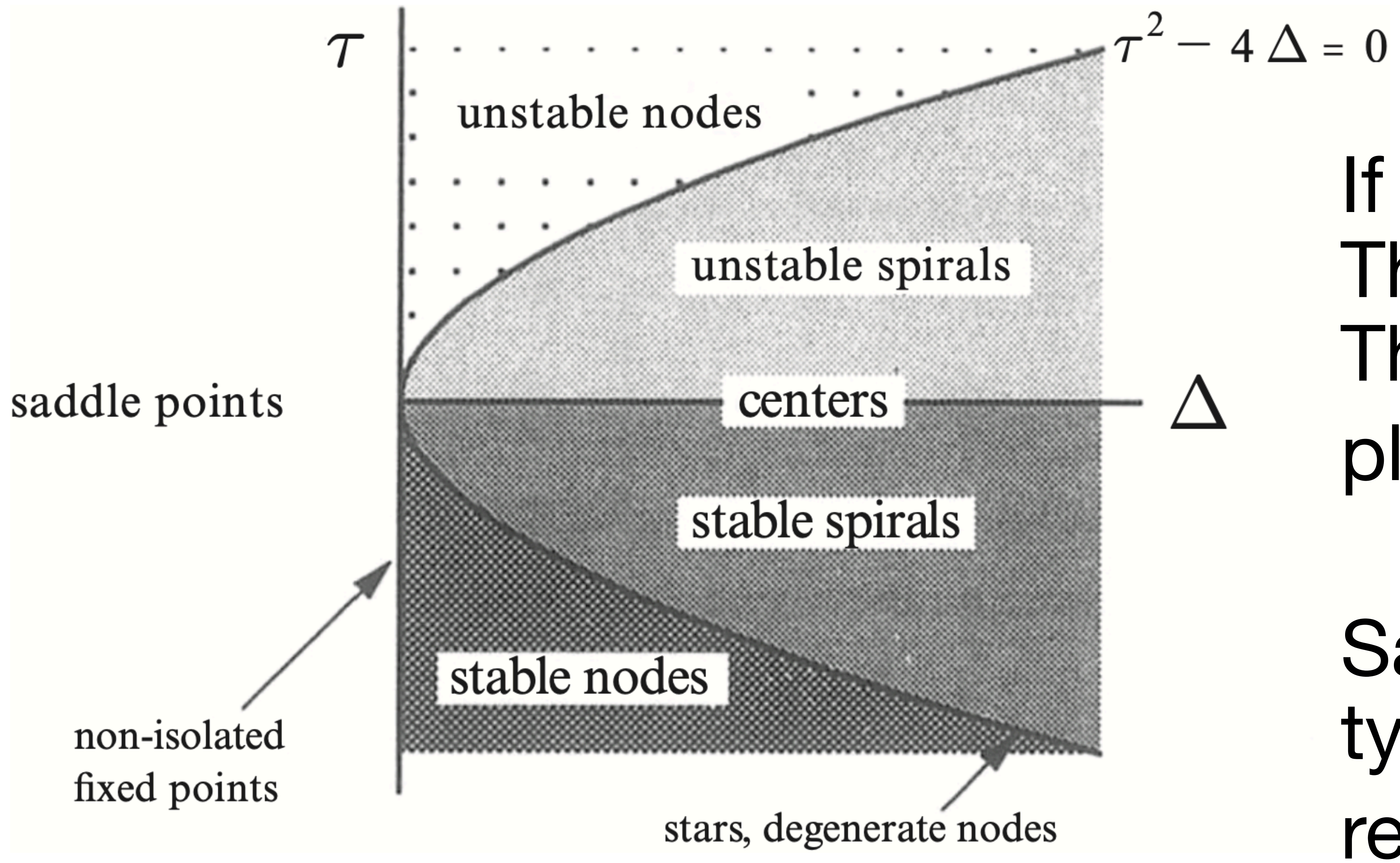
Nodes satisfy $\tau^2 - 4\Delta > 0$ and spirals satisfy $\tau^2 - 4\Delta < 0$.

The parabola $\tau^2 - 4\Delta = 0$ is the borderline between nodes and spirals; star nodes and degenerate nodes live on this parabola.

The stability of the nodes and spirals is determined by τ. When $\tau < 0$, both eigenvalues have negative real parts, so the **fixed point is stable.**

Unstable spirals and nodes have $\tau > 0$. Neutrally stable centers live on the borderline $\tau = 0$, where the eigenvalues are purely imaginary.

Classification of Fixed Points



If $\Delta = 0$, at least one of the eigenvalues is zero. Then the origin is not an isolated fixed point. There is either a whole line of fixed points, or a plane of fixed points, if $A = 0$.

Saddle points, nodes, and spirals are the major types of fixed points; they occur in large open regions of the (Δ, τ) plane.

Centers, stars, degenerate nodes, and non-isolated fixed points are ***borderline cases*** that occur along curves in the (Δ, τ) plane.

Of these borderline cases, **centres are the most important**. They occur very commonly in frictionless mechanical systems where energy is conserved.