Announcements: MSc

28-30 November: In person classes on YT campus

7-9 January: Theoretical physics symposium on YT campus

Nonlinear Dynamics and Chaos

PHYMSCFUN12

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MSc in Fundamental Physics

Yachay Tech University - 2025

Bifurcations in 2D

In 2D we still find that fixed points can be created or destroyed or destabilised as parameters are varied. The same is true of closed orbits.

We will describe the ways in which oscillations can be turned on or off.

If the phase portrait changes its topological structure as a parameter is varied, we say that a ***bifurcation*** has occurred ("topological equivalence").

Examples include changes in the number or stability of fixed points, closed orbits, or saddle connections as a parameter is varied.

The Saddle-Node, Transcritical, and Pitchfork Bifurcations of fixed points discussed earlier have analogs in 2D (and N-dimensions), but the action is confined to a 1D subspace along which the bifurcations occur.

Saddle-Node Bifurcation

The saddle-node bifurcation is the basic mechanism for the creation and destruction of fixed points.

Prototypical ODE system:

$$\dot{x} = \mu - x^2$$

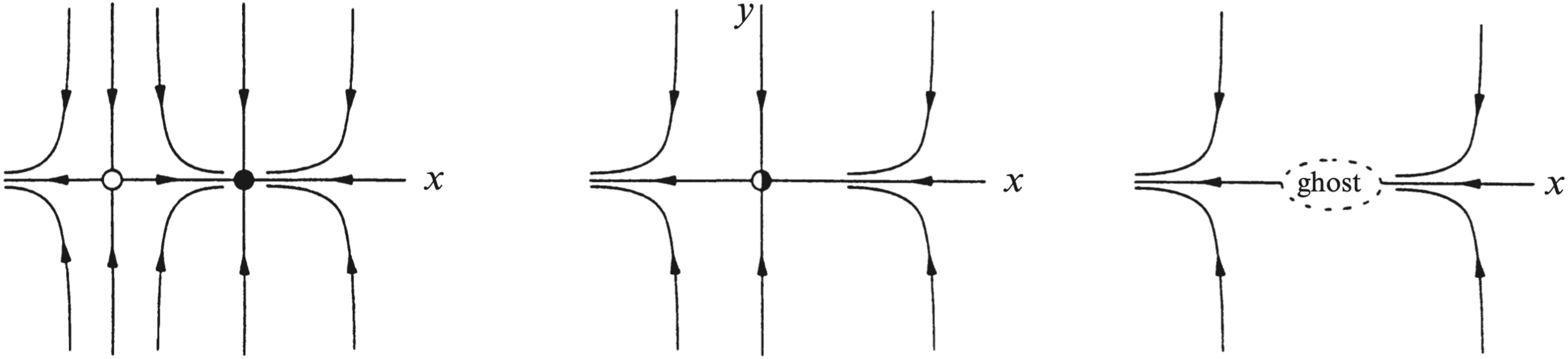
$$\dot{y} = -y.$$

In the *x*-direction we see a saddle-node bifurcation behaviour, while in the *y*-direction the motion is exponentially damped.

Saddle-Node Bifurcation

$$\dot{x} = \mu - x^2$$

$$\dot{y} = -y.$$

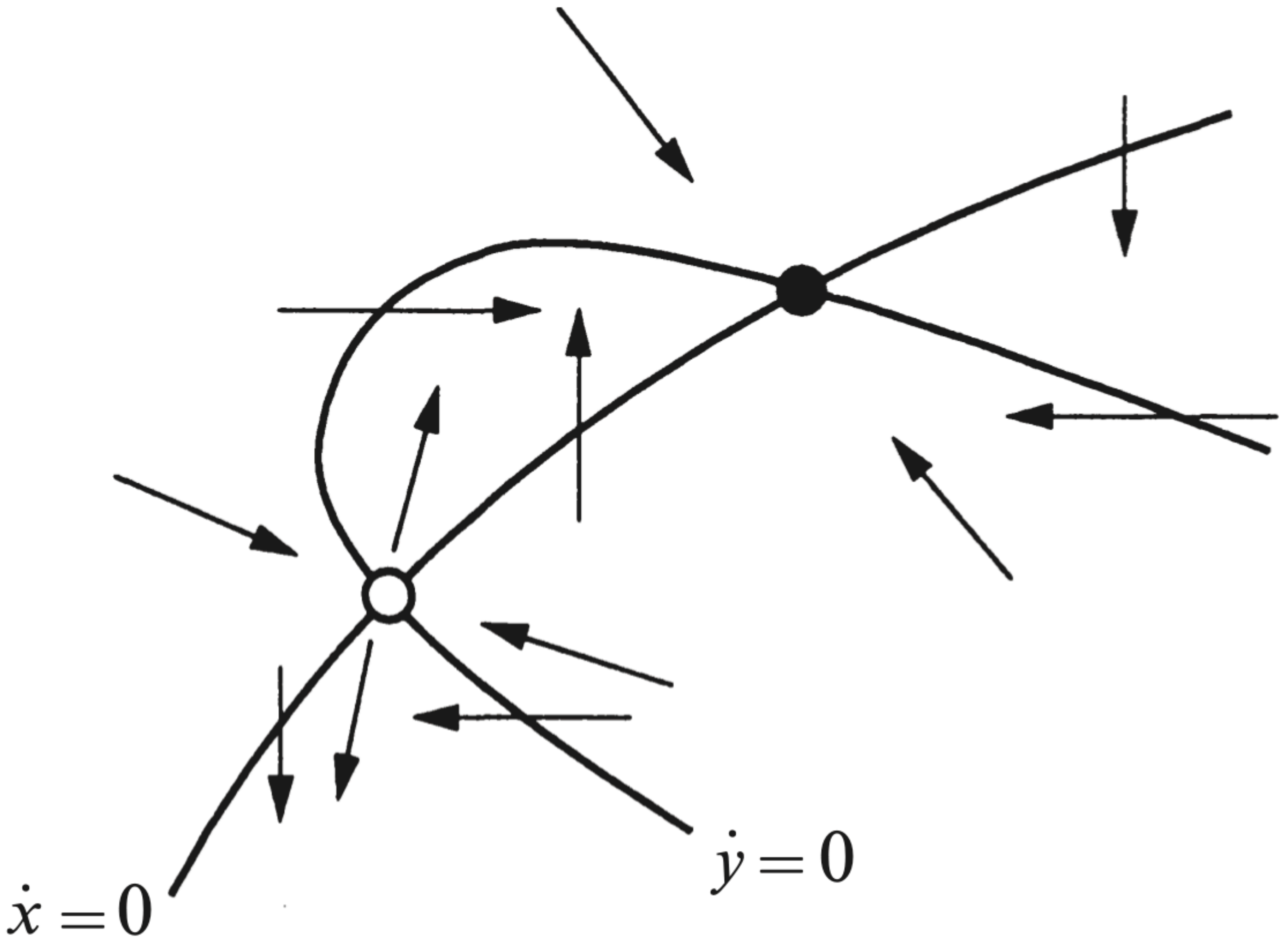


$\mu > 0$ $\mu = 0$ $\mu < 0$

Even after the fixed points have annihilated each other, they continue to influence the flow. They leave a **ghost,** a bottleneck region that sucks trajectories in and delays them before allowing passage out the other side.

The time spent in the bottleneck generically increases as $(\mu - \mu_c)^{-1/2}$, where μ_c is the value at which the saddle-node bifurcation occurs.

Saddle-Node Bifurcation



Consider a 2D system:

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

that depends on μ. For some value of μ the nullclines intersect.

Each intersection corresponds to a fixed point since $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously.

To see how the fixed points move as changes, we just have to watch the intersections.

After the nullclines pull apart, there are no intersections and the fixed points disappear. *All* saddle-node bifurcations have this character locally.

Example: Genetic Control System

Consider a 2D system:

$$\dot{x} = -ax + y$$

$$\dot{y} = \frac{x^2}{1+x^2} - by$$

where x and y are proportional to the concentrations of the protein and the messenger RNA from which it is translated, respectively, and $a, b > 0$ are parameters that govern the rate of degradation of x and y.

The activity of a certain gene is assumed to be directly induced by two copies of the protein for which it codes. The gene is stimulated by its own product, potentially leading to an autocatalytic feedback process.

Example: Genetic Control System

Consider a 2D system:

$$\dot{x} = -ax + y$$

y activates x,

x gets degraded normally,

x increases if y is big enough.

$$\dot{y} = \frac{x^2}{1+x^2} - by$$

x activates y, but not indefinitely as it saturates,

y decays if nothing sustains it.

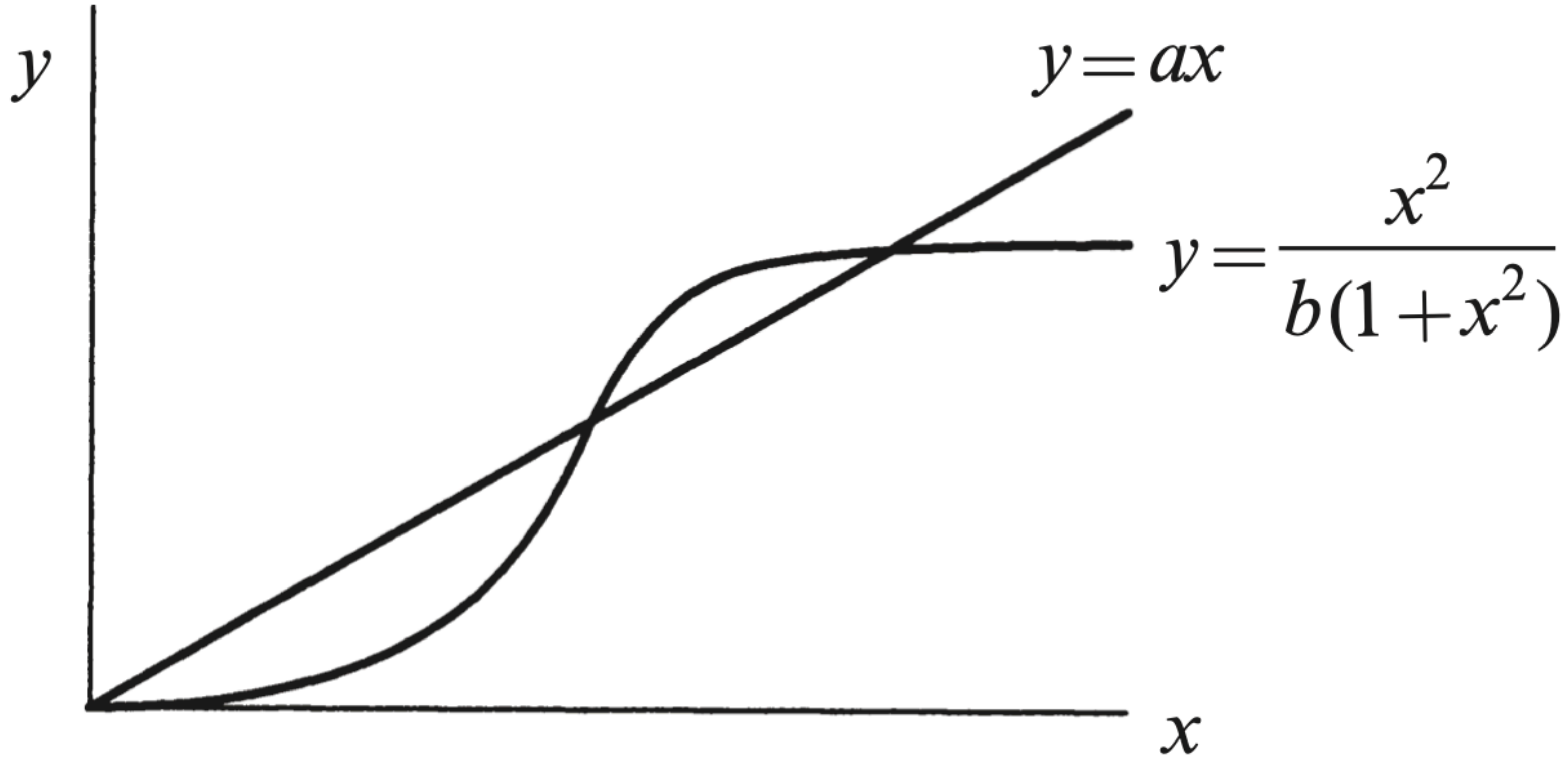
Example: Genetic Control System

$$\dot{x} = -ax + y$$
$$\dot{y} = \frac{x^2}{1+x^2} - by$$

The nullclines are given by a line and a sigmoidal curve:

$$y = ax$$

$$y = \frac{x^2}{b(1+x^2)}$$



Suppose we vary a, for a small $a < a_c$ there are 3 fixed points.

As a increases, the top 2 intersections approach each other and collide when the line intersects the curve tangentially.

For larger values of a, those fixed points disappear, leaving the origin as the only fixed point.

Example: Genetic Control System

$$\dot{x} = -ax + y$$
$$\dot{y} = \frac{x^2}{1+x^2} - by$$

To find a_c, we compute the fixed points directly and find where they coalesce. The nullclines intersect when:

$$ax = \frac{x^2}{b(1+x^2)}$$

One solution is: $(x^*, y^*) = (0,0)$

The other intersections satisfy the quadratic equation: $ab(1 + x^2) = x$

which has two solutions: $x^* = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}$ $\quad 1 - 4a^2 b^2 > 0 \Rightarrow 2ab < 1$

These solutions coalesce when $2ab = 1$. Hence: $a_c = 1/2b$

The fixed point at the bifurcation is $x^* = 1$.

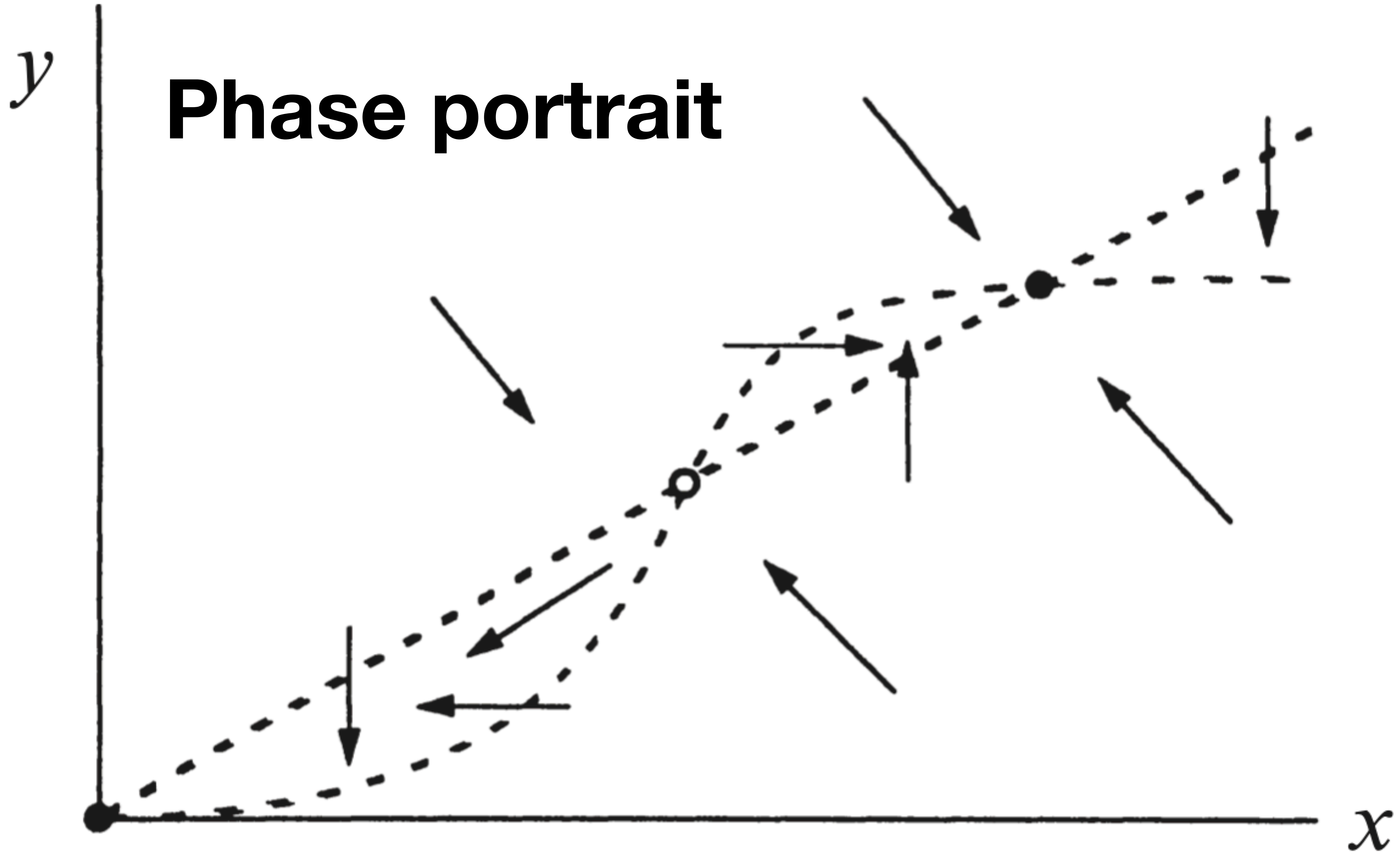
Tutorial: Numerical integration

$$\dot{x} = -ax + y$$

$$\dot{y} = \frac{x^2}{1+x^2} - by$$

Example: Genetic Control System

$$y = ax$$
$$y = \frac{x^2}{b(1+x^2)}$$



The vector field is vertical on the line $y = ax$ and horizontal on the sigmoidal curve.

The middle fixed point is a saddle and the other two are sinks. To confirm this, we turn now to the classification of the fixed points.

The Jacobian matrix at (x, y) is:

$$A = \begin{pmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{pmatrix}$$

A has trace $\tau = -(a + b) < 0$, so all the fixed points are either sinks or saddles, depending on the value of the determinant Δ.

Example: Genetic Control System

At (0,0), $\Delta = ab > 0$, so the origin is always a stable fixed point (*stable node).*

$$\tau^2 - 4\Delta > (a-b)^2 > 0.$$

At the other two fixed points:

$$\Delta = ab - \frac{2x^*}{\left(1+(x^*)^2\right)^2} = ab\left[1 - \frac{2}{1+(x^*)^2}\right] = ab\left[\frac{(x^*)^2 - 1}{1+(x^*)^2}\right]$$

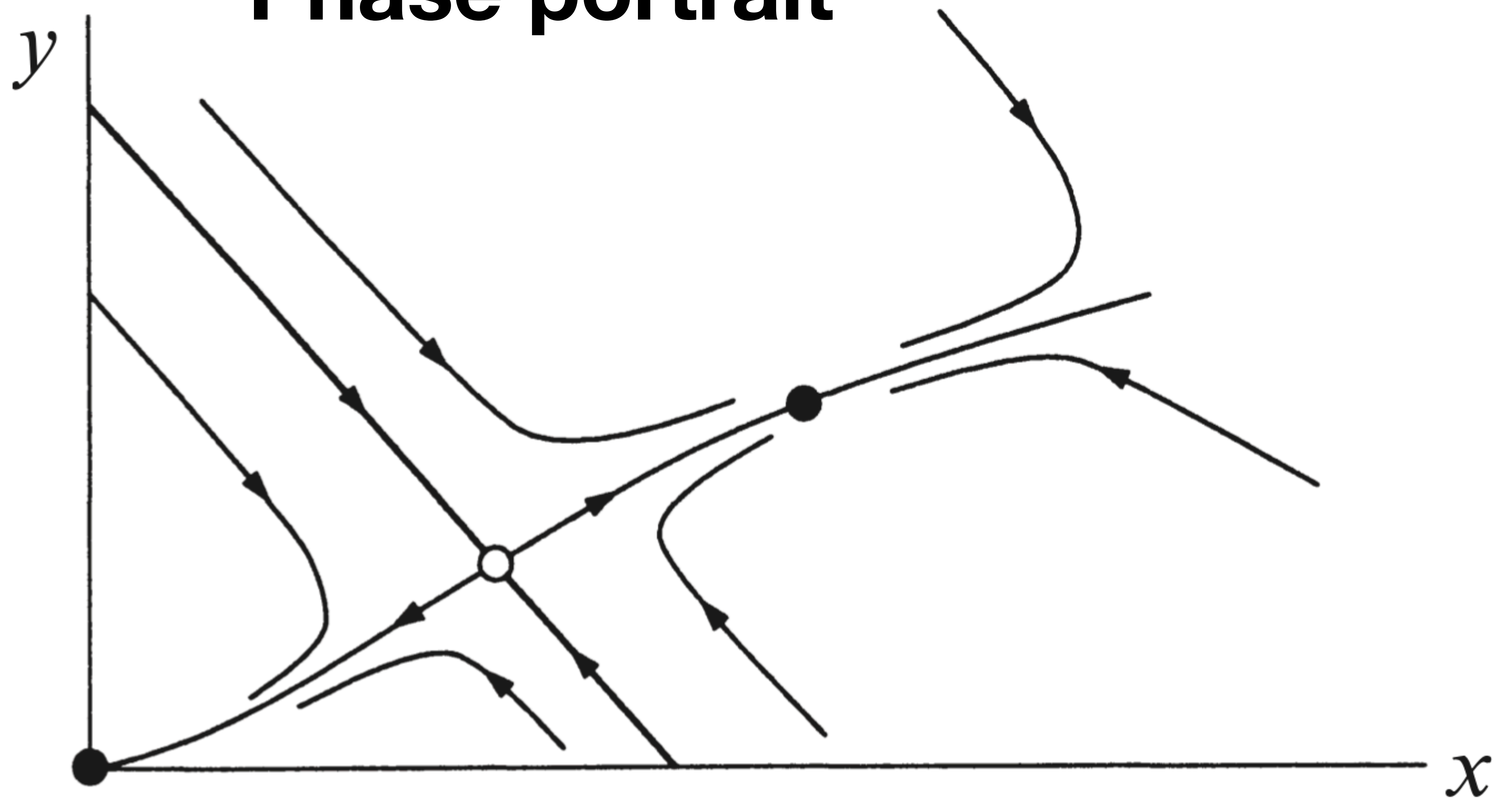
So $\Delta < 0$ for the "middle" fixed point, which has $0 < x^* < 1$ (*saddle point).*

The fixed point with $x^* > 1$ is always a *stable node*, since $\Delta < ab$ and therefore:

$$\tau^2 - 4\Delta > (a-b)^2 > 0.$$

Example: Genetic Control System

Phase portrait

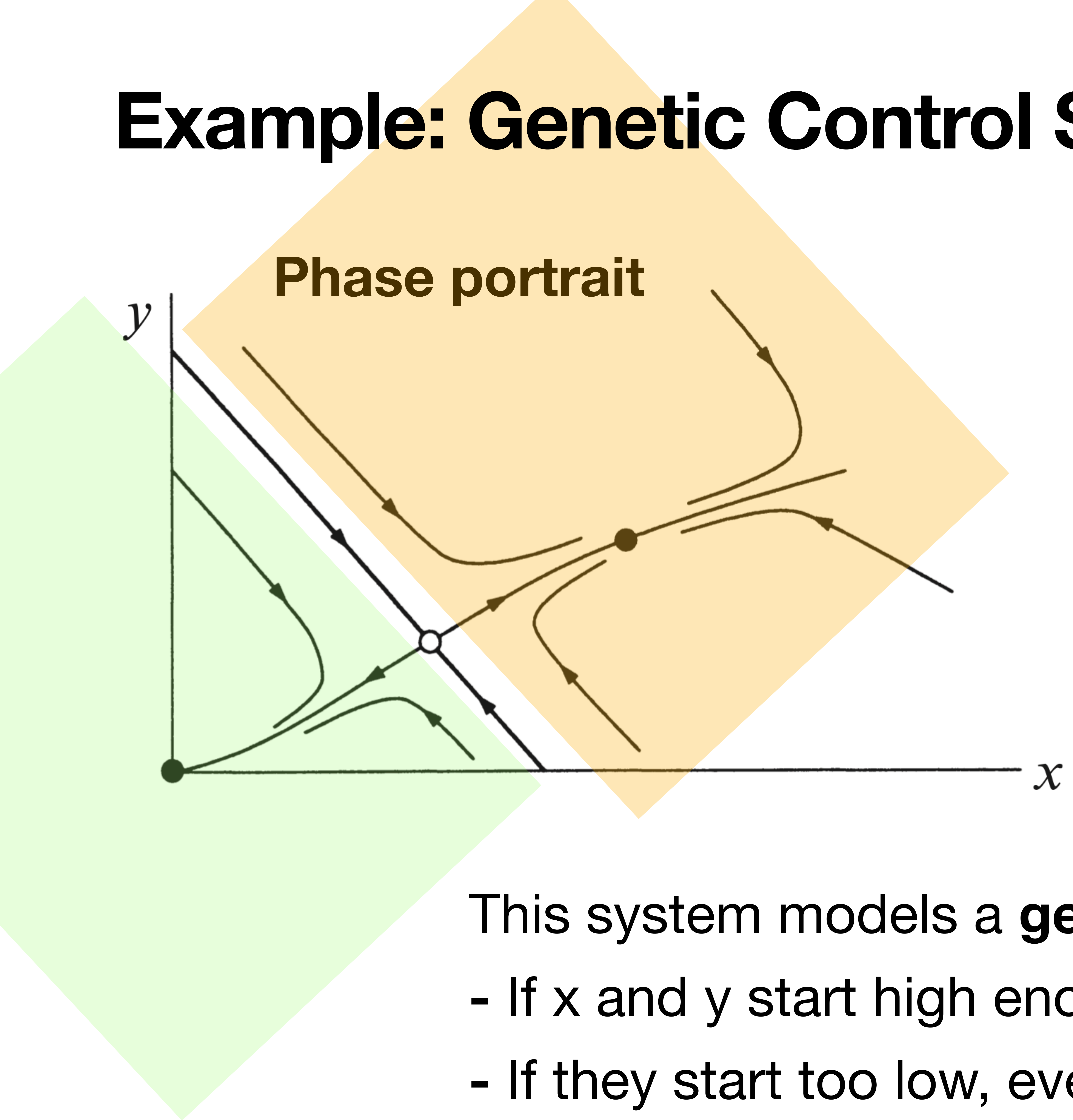


The unstable manifold of the saddle is necessarily trapped in the narrow channel between the two nullclines.

The *stable* manifold separates the plane into two regions, each a basin of attraction for a sink.

Biological interpretation: the system can act like a *biochemical switch*, but only if the mRNA and protein degrade slowly (their decay rates must satisfy $ab < 1/2$).

Example: Genetic Control System



Both x and y **decay** naturally.

The system represents a **positive feedback genetic circuit** where two molecules help each other get produced.

This system models a **genetic switch:**

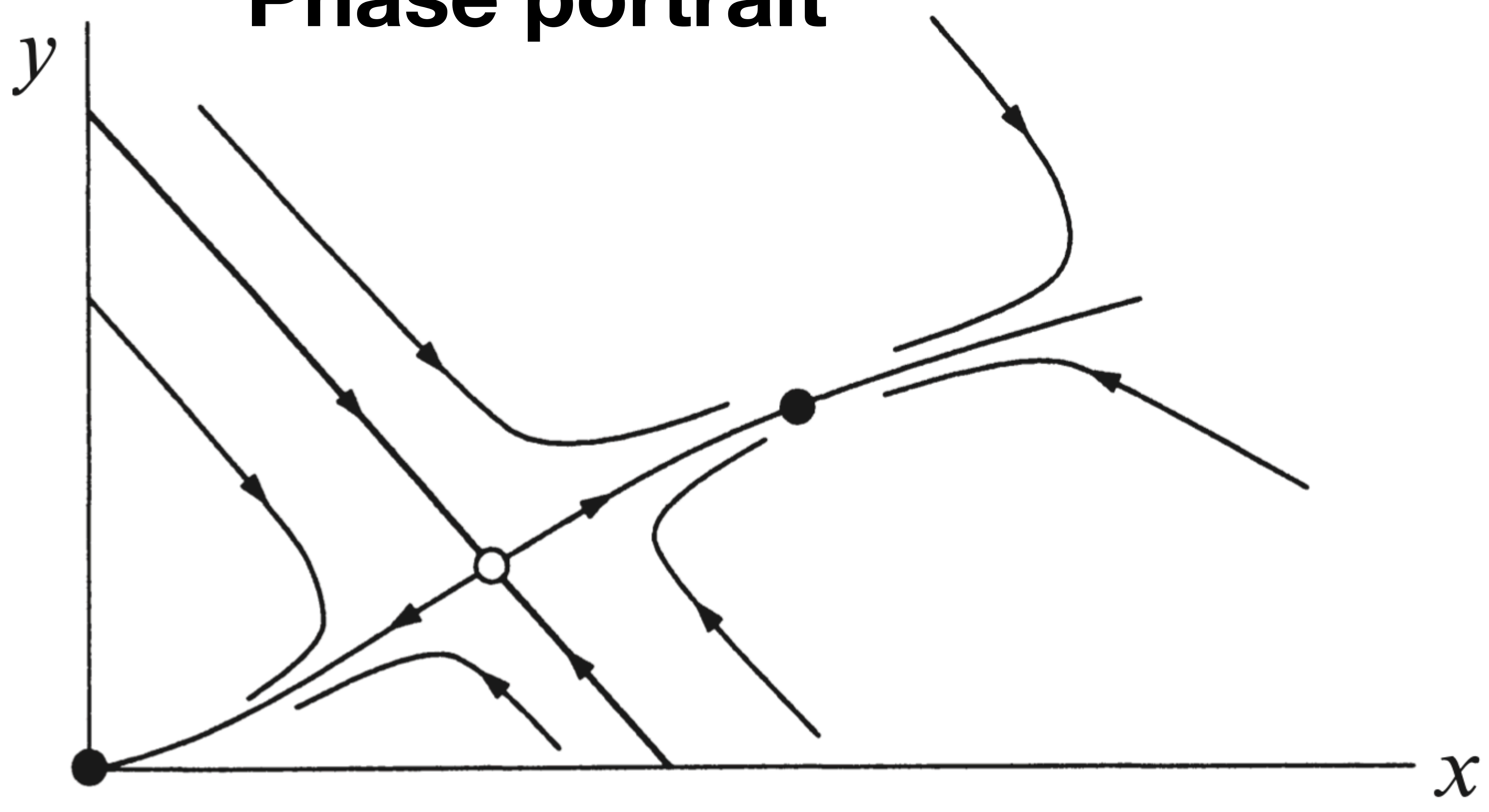
- If x and y start high enough, they keep each other turned **on.**

- If they start too low, everything decays → stable "off" state.

- The unstable point is the **decision threshold**.

Example: Genetic Control System

Phase portrait



There are 2 stable steady states:

At (0,0) the gene is silent and there is no protein around to turn it on.

When *x* and *y* are large, the gene is active and sustained by the high level of protein.

The stable manifold of the saddle acts like a threshold.

The bifurcation is a fundamentally 1D event, with the fixed points sliding toward each other along the unstable manifold.

1D bifurcations are the building blocks of analogous bifurcations in higher dimensions.

Example: Transcritical and Pitchfork Bifurcations

We can also construct prototypical examples of transcritical and pitchfork bifurcations at a stable fixed point.

In the *x*-direction the dynamics are given by normal forms, and in the *y*-direction the motion is exponentially damped.

$$\dot{x} = \mu x - x^2, \qquad \dot{y} = -y \qquad \text{(transcritical)}$$

$$\dot{x} = \mu x - x^3, \qquad \dot{y} = -y \qquad \text{(supercritical pitchfork)}$$

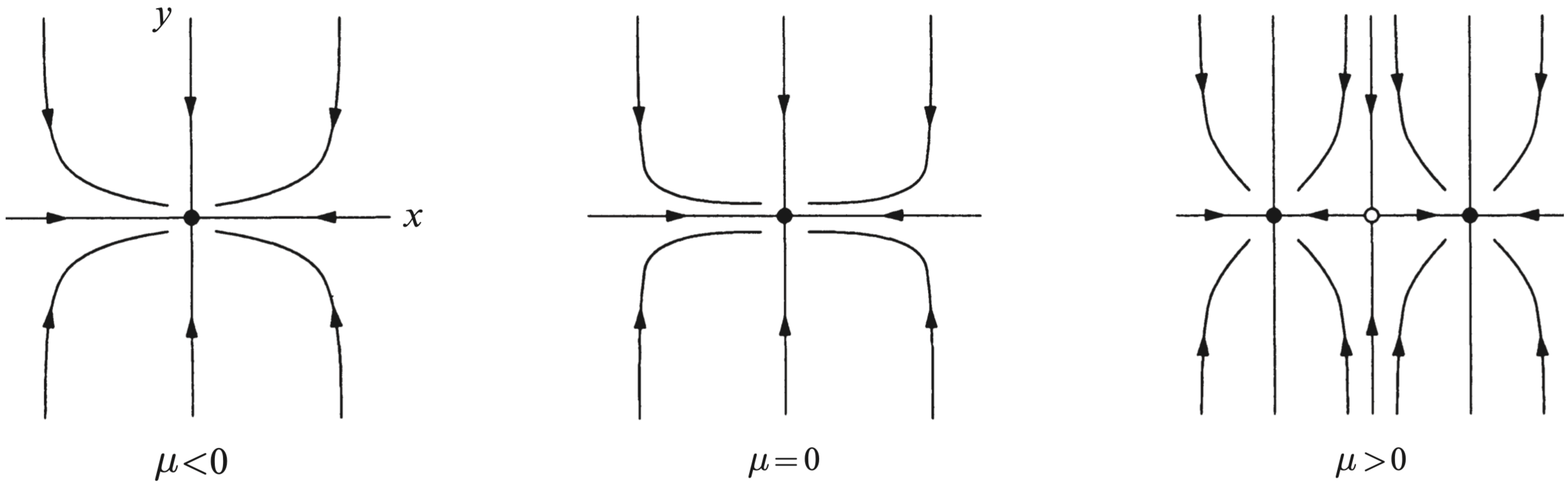
$$\dot{x} = \mu x + x^3, \qquad \dot{y} = -y \qquad \text{(subcritical pitchfork)}$$

The analysis in each case follows the same pattern, so we will discuss only the supercritical pitchfork bifurcation, which are common in systems that have a symmetry.

Example: Supercritical Pitchfork Bifurcation

Let's plot the phase portraits for the supercritical pitchfork system:

$$\dot{x} = \mu x - x^3, \qquad \dot{y} = -y$$



For $\mu < 0$, the only fixed point is a stable node at the origin.

For $\mu = 0$, we have critical slowing down along x.

For $\mu > 0$, (0,0)loses stability and 2 new stable fixed points are located at

$$(x^*, y^*) = (\pm\sqrt{\mu}, 0)$$

Example: Supercritical Pitchfork Bifurcation

Show that a supercritical pitchfork bifurcation occurs at the origin in the system:

$$\dot{x} = \mu x + y + \sin x$$
$$\dot{y} = x - y$$

and determine the bifurcation value μ_c. Plot the phase portrait near the origin for μ slightly greater than μ_c.

The system is invariant under the change of variables $x \rightarrow -x$, $y \rightarrow -y$, so the phase portrait must be symmetric under reflection through the origin. The origin is a fixed point for all μ, and its Jacobian is:

$$A = \begin{pmatrix} \mu + 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \tau = \mu, \Delta = -(\mu + 2)$$

Example: Supercritical Pitchfork Bifurcation

$$A = \begin{pmatrix} \mu+1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \tau = \mu,\ \Delta = -(\mu+2)$$

The origin is a stable fixed point if $\mu < -2$ and a saddle if $\mu > -2$. This suggests that a **pitchfork bifurcation** occurs at $\mu_c = -2$. To confirm this, we seek a symmetric pair of fixed points close to the origin for μ close to μ_c.

The fixed points satisfy: $y = x$ ➡ $(\mu+1)x + \sin x = 0$

Now suppose *x* is small and nonzero, and expand the sine as a power series:

$$(\mu+1)x + x - \frac{x^3}{3!} + O(x^5) = 0$$

Pair of fixed points at: $x^* \approx \pm\sqrt{6(\mu+2)}$

Example: Supercritical Pitchfork Bifurcation

A supercritical pitchfork bifurcation occurs at $\mu_c = -2$. If the bifurcation had been subcritical, the pair of fixed points would exist when the origin was stable, not after it has become a saddle.

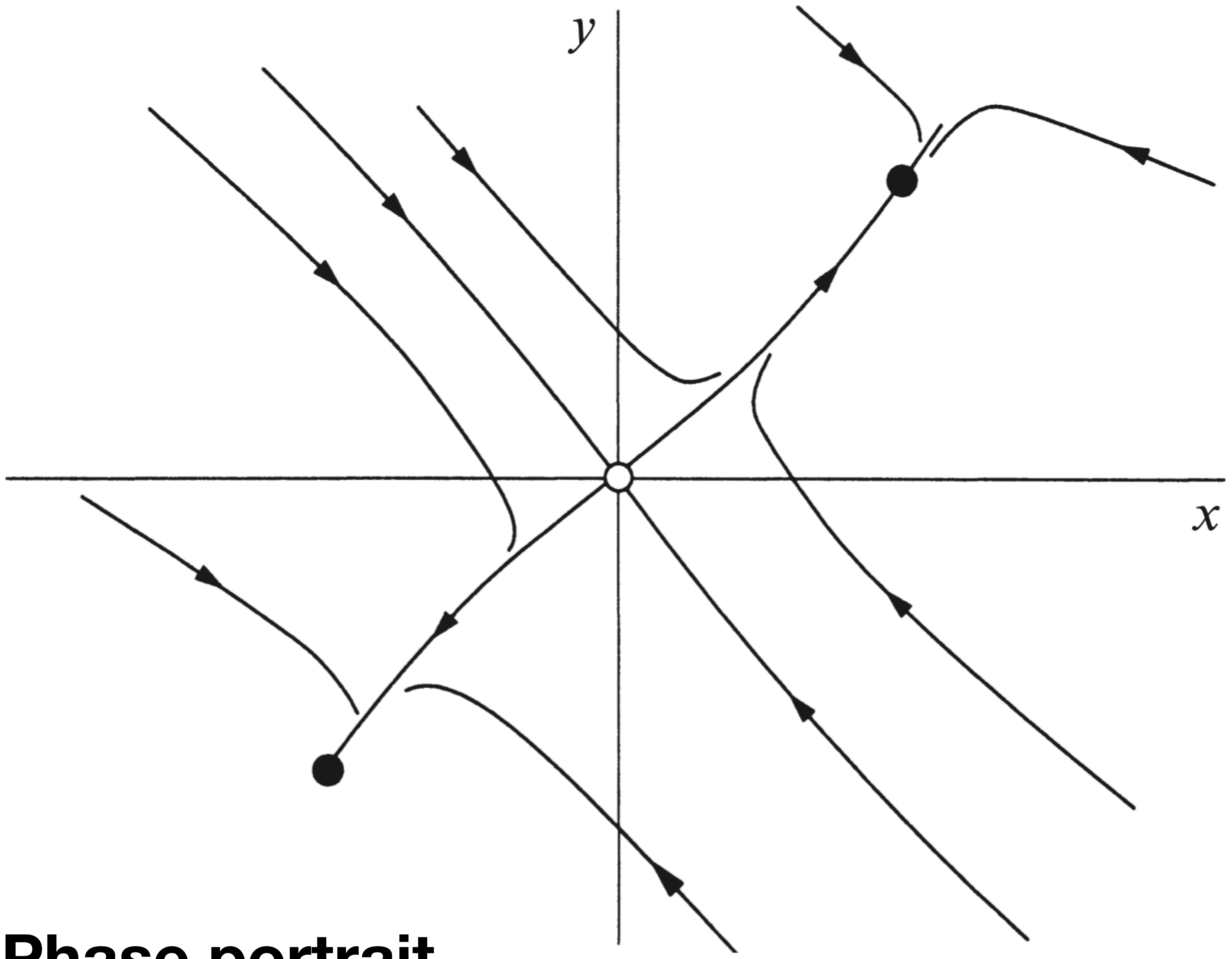
To draw the phase portrait near (0,0), we find the eigenvectors of the Jacobian at the origin. A simple approximation is that the Jacobian is close to that *at* the bifurcation:

$$A \approx \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

which has eigenvectors (1,1) and (1,-1), with eigenvalues 0 and 2, respectively.

For slightly greater than -2, the origin becomes a saddle and so the zero eigenvalue becomes slightly positive.

Example: Supercritical Pitchfork Bifurcation



Phase portrait

The bifurcation occurs when $\Delta = 0$, or equivalently, when one of the eigenvalues equals zero.

The saddle-node, transcritical, and pitchfork bifurcations are all examples of **zero-eigenvalue bifurcations**.

Such bifurcations always involve the collision of two or more fixed points.

Hopf Bifurcations

This is a new kind of bifurcation, one that has no counterpart in 1D systems. It provides a way for a fixed point to lose stability without colliding with any other fixed points.

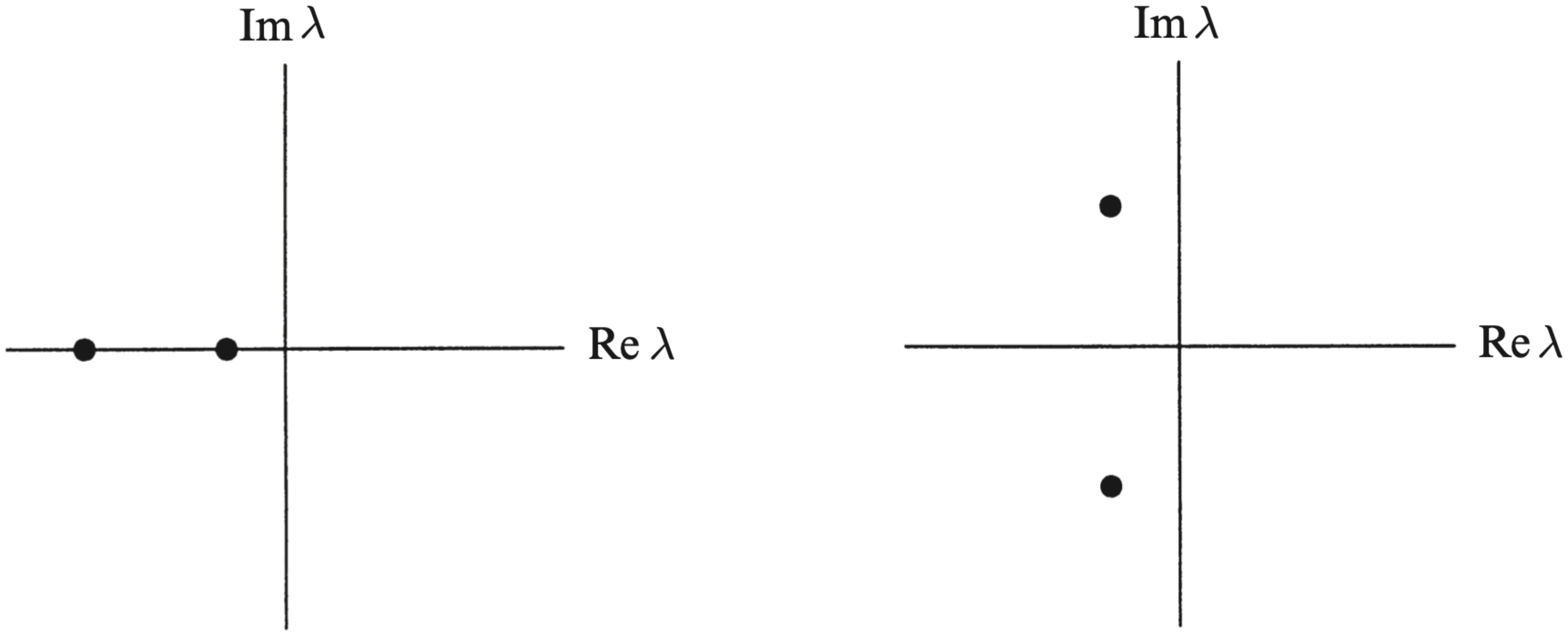
Suppose a 2D system has a stable fixed point. What are all the possible ways it could lose stability as a parameter varies?

The eigenvalues of the Jacobian are the key. If the fixed point is stable, the eigenvalues λ_1 and λ_2 must both lie in the left half-plane $Re(\lambda) < 0$.

Hopf Bifurcations

There are 2 possible pictures:

The eigenvalues are both real and negative or they are complex conjugates.

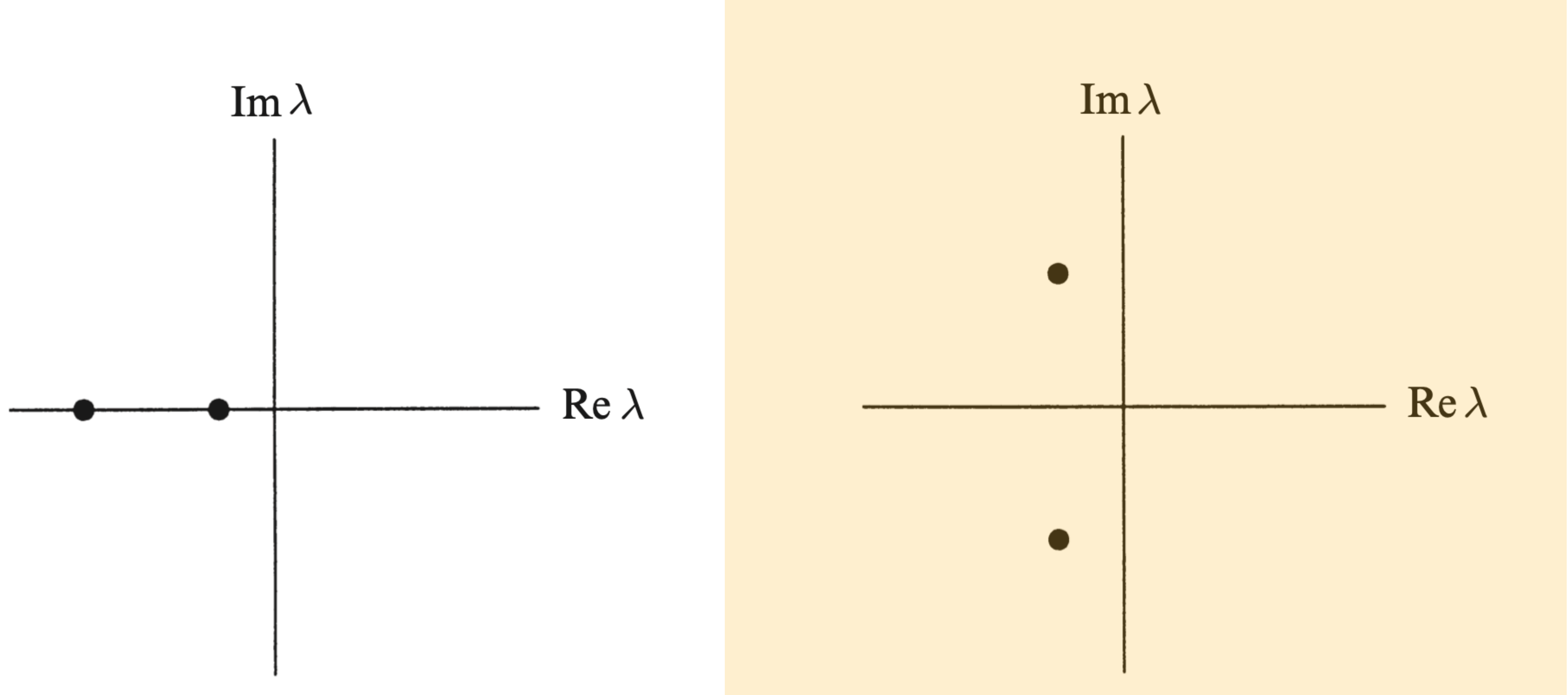


(a) (b)

To destabilise the fixed point, we need one or both of the eigenvalues to cross into the right half-plane as μ varies.

Hopf Bifurcations

At the $\lambda = 0$ crossing, we get saddle-node, transcritical, and pitchfork bifurcations.



(a) (b)

We consider the other possible scenario, in which **two complex conjugate eigenvalues simultaneously cross the imaginary axis into the right half-plane.**

Supercritical Hopf Bifurcation

We have a physical system that settles down to equilibrium through exponentially damped oscillations. Small disturbances decay after “ringing” for a while (a).

The decay rate depends on a control parameter μ. If the decay becomes slower and slower and finally changes to *growth* at a critical value μ_c, the equilibrium state will lose stability.

(a) $\mu < \mu_c$

(b) $\mu > \mu_c$

The resulting motion is a small-amplitude, sinusoidal, limit cycle oscillation about the former steady state (b).

The system has undergone a ***supercritical Hopf bifurcation.***

Supercritical Hopf Bifurcation

In terms of the flow in phase space, a supercritical Hopf bifurcation occurs when a stable spiral changes into an unstable spiral surrounded by a small, nearly elliptical limit cycle. Hopf bifurcations can occur in phase spaces of any dimension $n \geq 2$.

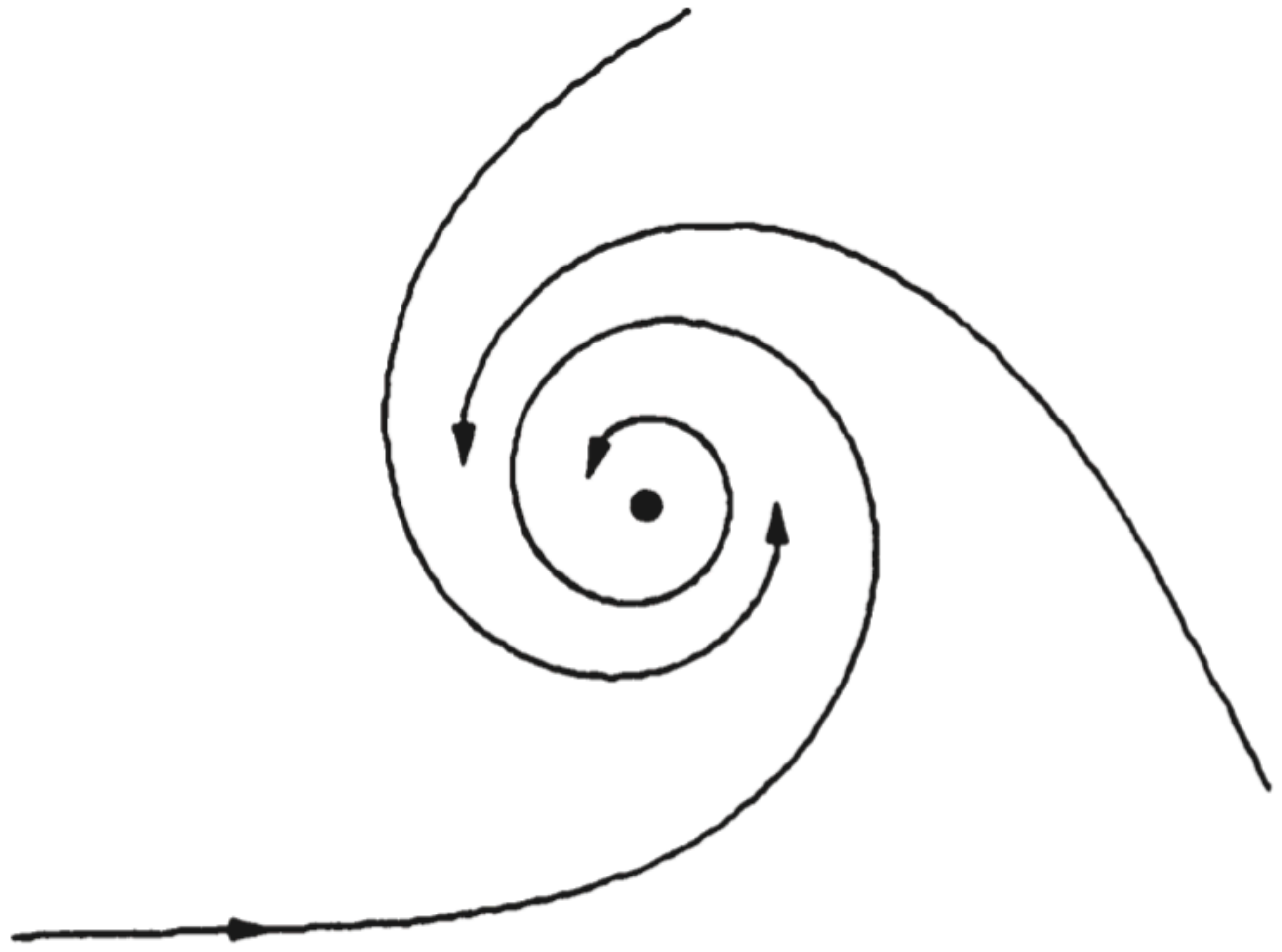
Example:

$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + br^2$$

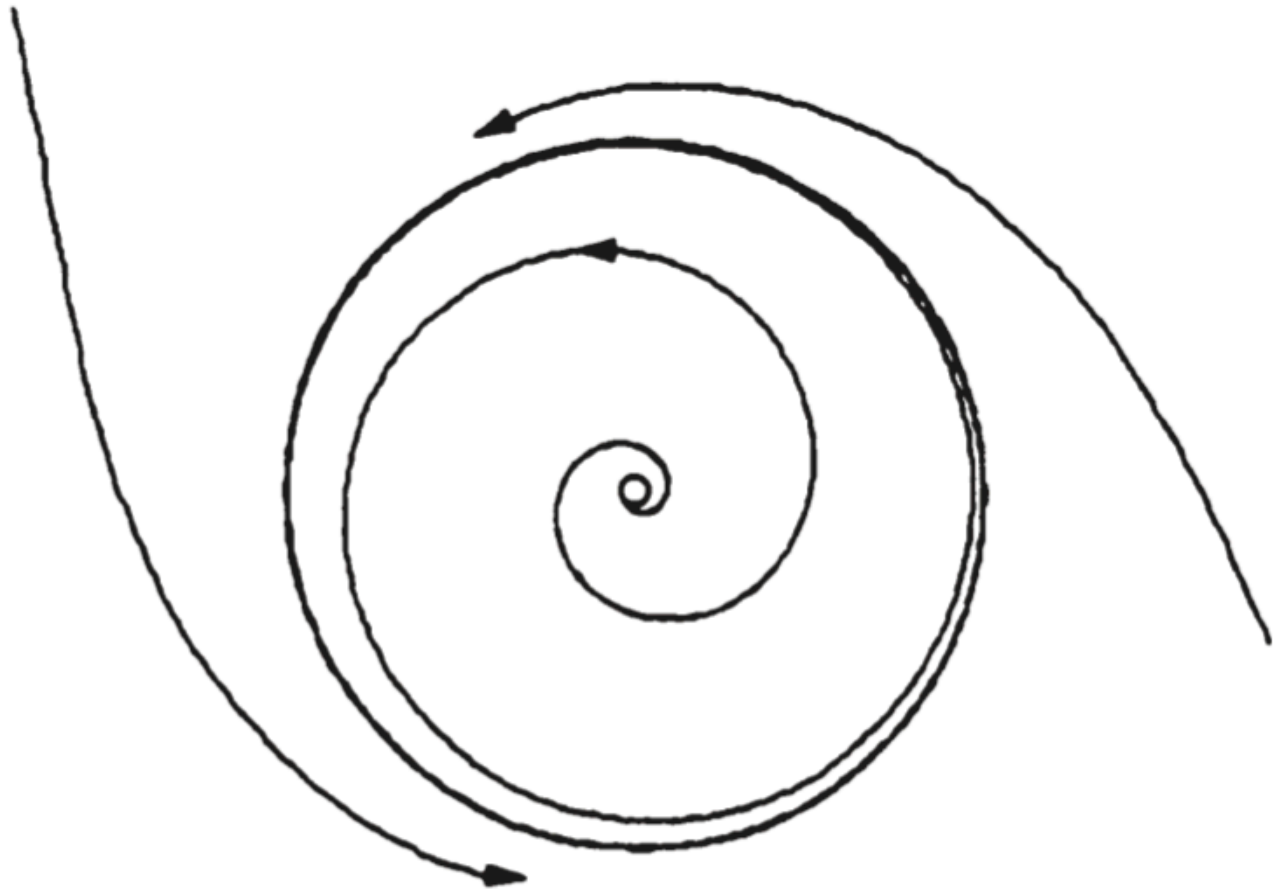
There are 3 parameters:

μ controls the stability of the fixed point at the origin,
ω gives the frequency of infinitesimal oscillations, and
b determines how frequency depends on amplitude for larger amplitude oscillations.

Supercritical Hopf Bifurcation



$\mu < 0$



$\mu > 0$

When $\mu < 0$ the origin $r = 0$ is a stable spiral whose sense of rotation depends on the sign of ω.

Finally, for $\mu > 0$ there is an unstable spiral at the origin and a stable circular limit cycle at $r = \sqrt{\mu}$.

For $\mu = 0$ the origin is still a stable spiral, though a very weak one: the decay is only algebraically fast.

Supercritical Hopf Bifurcation

To see how the eigenvalues behave during the bifurcation, we rewrite the system in Cartesian coordinates ($x = r\cos(\theta)$, $y = r\sin(\theta)$).

$$
\begin{aligned}
\dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\
&= (\mu r - r^3)\cos\theta - r(\omega + br^2)\sin\theta \\
&= \left(\mu - [x^2 + y^2]\right)x - \left(\omega + b[x^2 + y^2]\right)y \\
&= \mu x - \omega y + \text{cubic terms}
\end{aligned}
$$

$$\dot{y} = \omega x + \mu y + \text{cubic terms}$$

The Jacobian at the origin is: $$A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

which has eigenvalues: $$\lambda = \mu \pm i\omega$$

Supercritical Hopf Bifurcation

$$A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

The eigenvalues are: $\lambda = \mu \pm i\omega$

The eigenvalues cross the imaginary axis from left to right asincreases from negative to positive values.

Rules of thumb:

1. The size of the limit cycle grows continuously from zero, and increases proportional to $\sqrt{\mu - \mu_c}$, for μ close to μ_c.

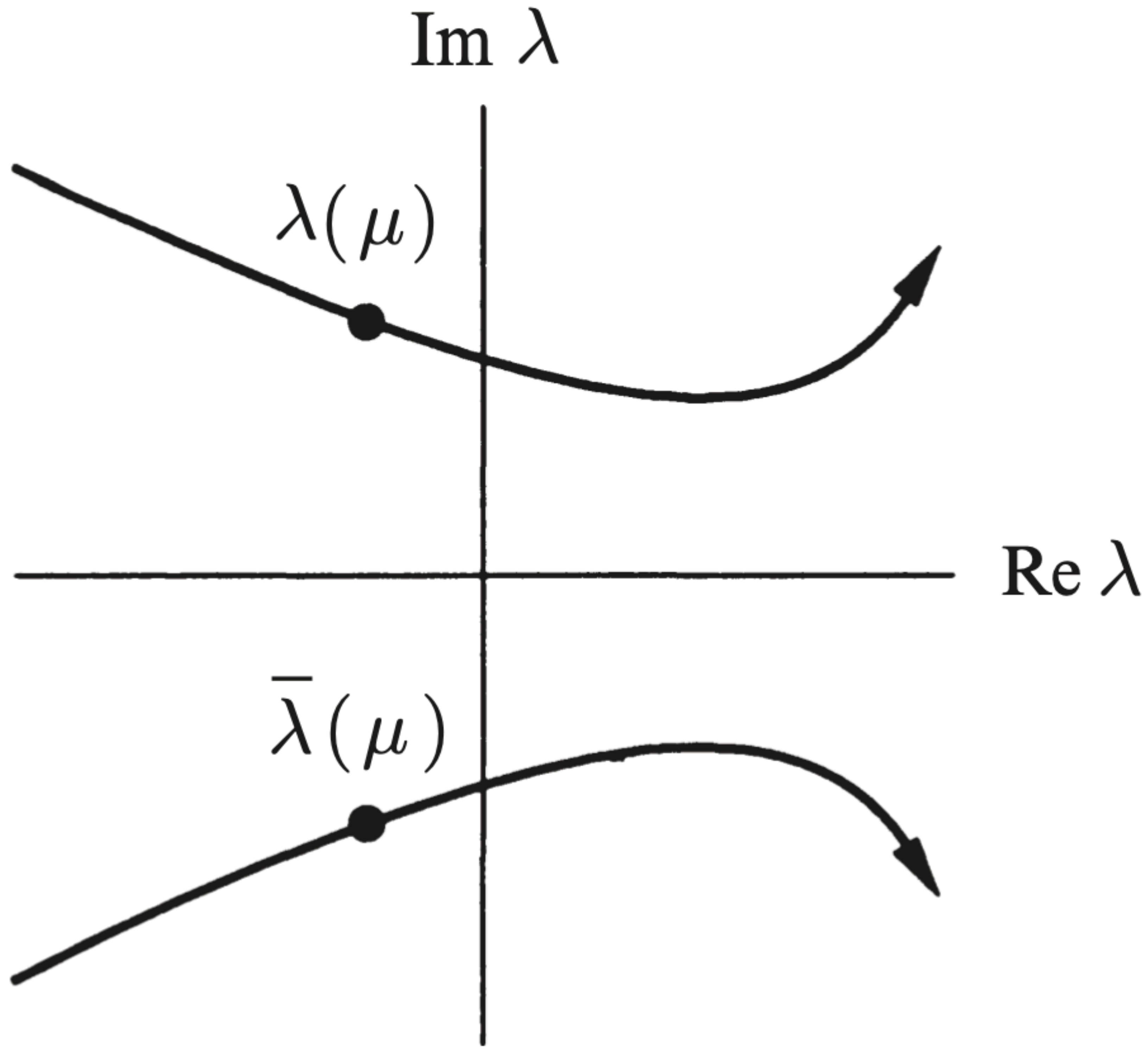
2. The frequency of the limit cycle is given approximately by $\omega = \text{Im}\,\lambda$, evaluated at $\mu = \mu_c$. This formula is exact at the birth of the limit cycle, and correct within $O(\mu - \mu_c)$ for μ close to μ_c. The period is therefore $T = (2\pi/\text{Im}\,\lambda) + O(\mu - \mu_c)$.

Supercritical Hopf Bifurcation

Some notes:

In Hopf bifurcations encountered in practice, the limit cycle is elliptical, not circular, and its shape becomes distorted as moves away from the bifurcation point. **Our example is only typical topologically, not geometrically**.

Second, in our idealised case the eigenvalues move on horizontal lines as varies, i.e., $Im(\lambda)$ is strictly independent of μ. Normally, **the eigenvalues would follow a curvy path and cross the imaginary axis with nonzero slope.**



Subcritical Hopf Bifurcation

Like pitchfork bifurcations, Hopf bifurcations come in both super- and subcritical varieties. The subcritical case is always much more dramatic, and potentially dangerous in engineering applications.

Why is that dangerous in engineering?

Many engineering systems rely on stable, predictable behaviour:
- Aircraft wings
- Bridges
- Rotating shafts
- Power grids
- Chemical reactors
- Control systems

Small changes in parameters (speed, load, flow rate, temperature)
- can cause sudden onset of large vibrations
- which can cause structural failure or loss of control.

Subcritical Hopf Bifurcation

Like pitchfork bifurcations, Hopf bifurcations come in both super- and subcritical varieties. The subcritical case is always much more dramatic, and potentially dangerous in engineering applications.

After the bifurcation, **the trajectories must jump to a distant attractor,** which may be a fixed point, another limit cycle, infinity, or (in 3D) a chaotic attractor.

Example:

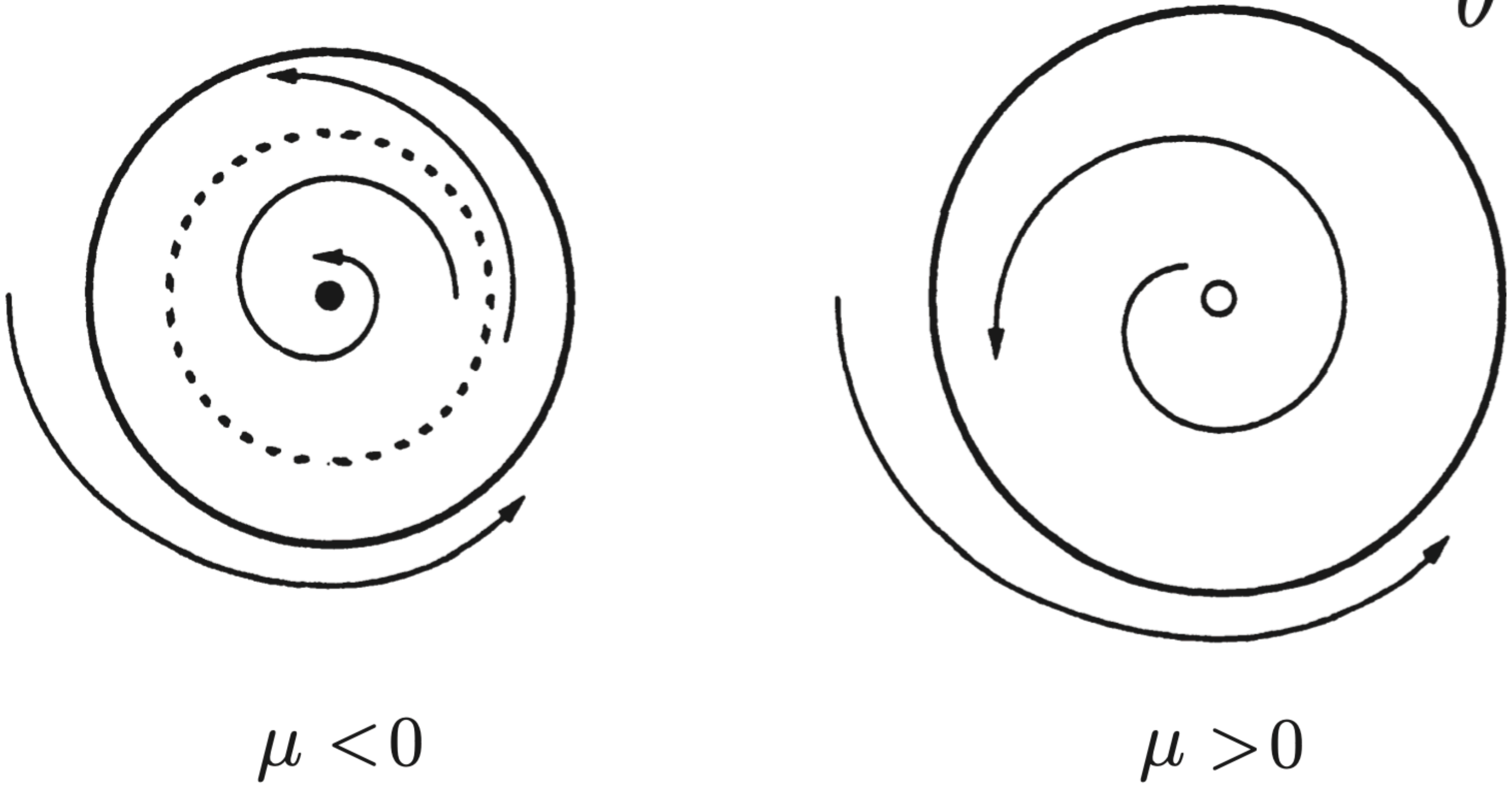
$$\dot{r} = \mu r + r^3 - r^5$$

$$\dot{\theta} = \omega + b r^2 .$$

The important difference from the earlier supercritical case is that the cubic term r^3 is now **destabilising;** it helps to drive trajectories away from the origin.

Subcritical Hopf Bifurcation

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + br^2.$$



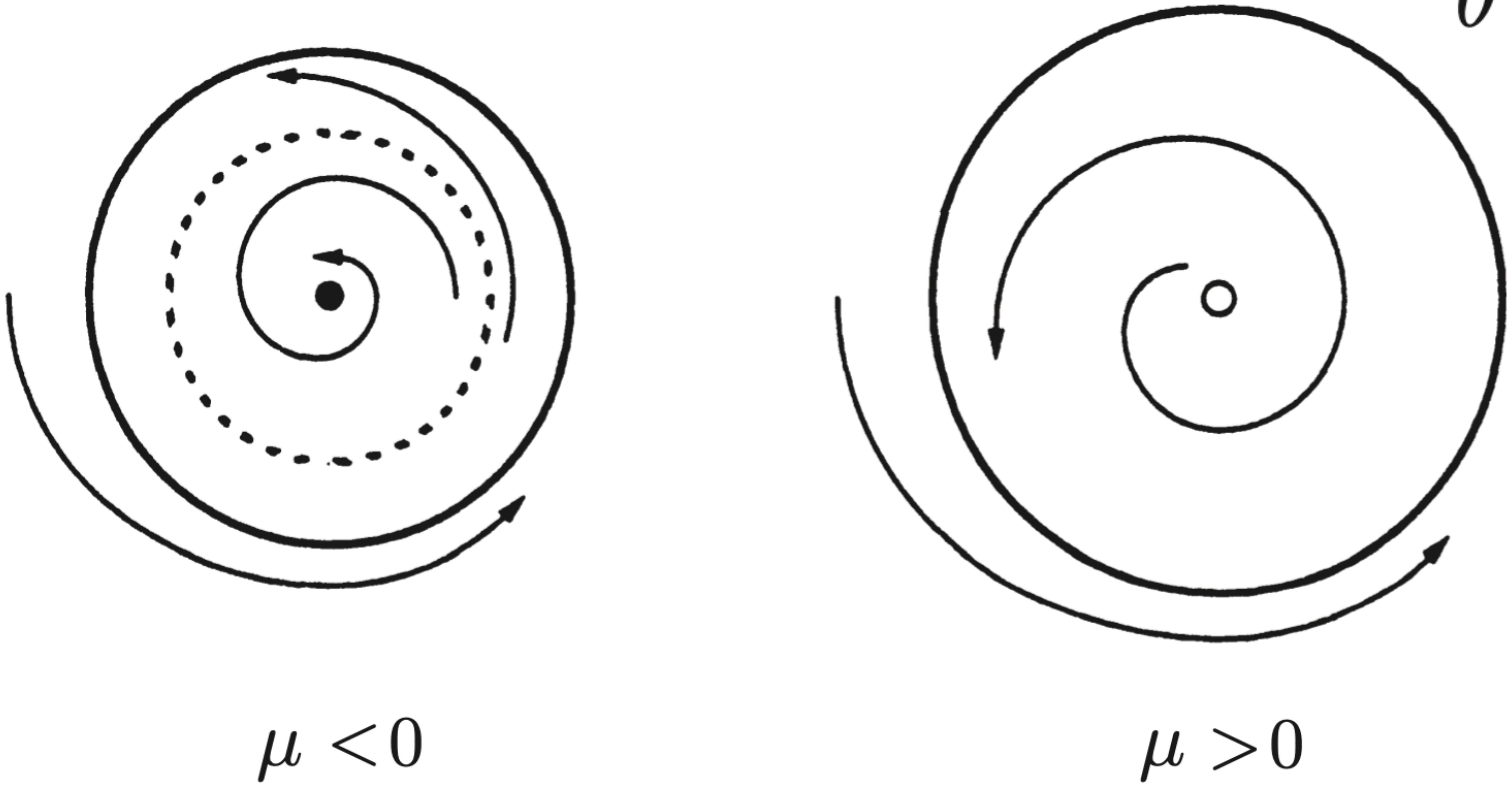
For $\mu < 0$ there are two attractors, a stable limit cycle and a stable fixed point at the origin. Between them lies an unstable cycle.

As μ increases, the unstable cycle tightens like a noose around the fixed point.

A **subcritical Hopf bifurcation** occurs at $\mu = 0$, where the unstable cycle shrinks to zero amplitude and engulfs the origin, rendering it unstable.

Subcritical Hopf Bifurcation

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + b r^2.$$

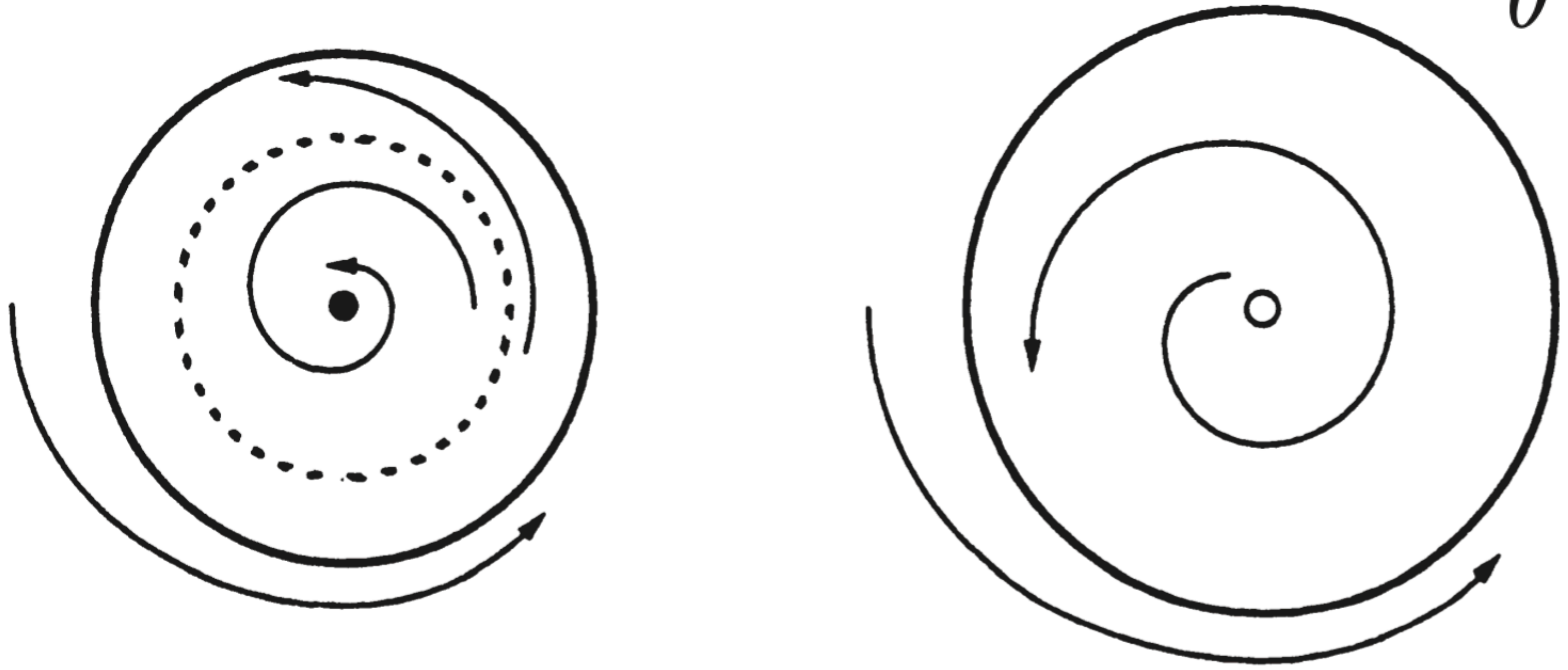


For $\mu > 0$, the large-amplitude limit cycle is suddenly the only attractor. Solutions that used to remain near the origin are now forced to grow into large-amplitude oscillations.

Note that the system exhibits **hysteresis:** once large-amplitude oscillations have begun, they cannot be turned off by bringing μ back to zero.

Subcritical Hopf Bifurcation

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + br^2.$$



$\mu < 0$ $\mu > 0$

The large oscillations will persist until $\mu = -1/4$ where the stable and unstable cycles collide and annihilate. This destruction of the large-amplitude cycle occurs via another type of bifurcation to be discussed in the next section.

Subcritical, Supercritical, or Degenerate Bifurcation?

Given that a Hopf bifurcation occurs, how can we tell if it is sub- or supercritical?

The linearisation doesn't provide a distinction: in both cases, a pair of eigenvalues moves from the left to the right half-plane.

We can use the computer:

If a small, attracting limit cycle appears immediately after the fixed point goes unstable, and if its amplitude shrinks back to zero as the parameter is reversed, the bifurcation is supercritical.

Otherwise, it is probably subcritical, in which case the nearest attractor might be far from the fixed point, and the system may exhibit hysteresis as the parameter is reversed. Of course, computer experiments are not proofs and you should check the numerics carefully before making any firm conclusions.

Degenerate Bifurcation

An example of this type is given by the damped pendulum:

$$\ddot{x} + \mu \dot{x} + \sin x = 0$$

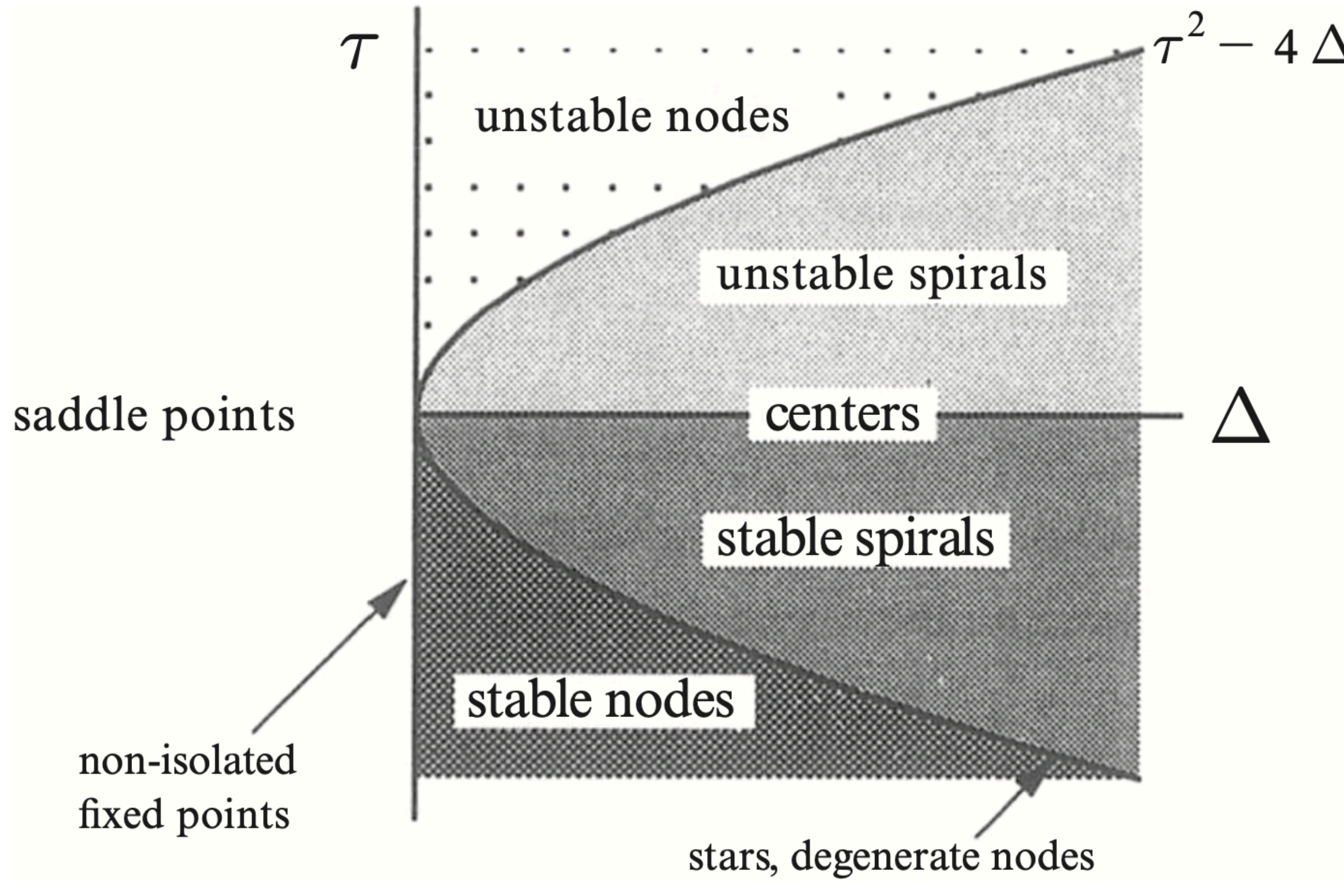
As we change the damping μ from positive to negative, the fixed point at the origin changes from a stable to an unstable spiral.

However at $\mu = 0$ we do *not* have a true Hopf bifurcation because there are no limit cycles on either side of the bifurcation.

Instead, at $\mu = 0$ we have a continuous band of closed orbits surrounding the origin. These are not limit cycles! (a limit cycle is an **isolated closed orbit**.)

This degenerate case typically arises when a nonconservative system suddenly becomes conservative at the bifurcation point. Then **the fixed point becomes a nonlinear center, rather than the weak spiral required by a Hopf bifurcation.**

Example: Hopf Bifurcation



Consider the system:

$$\dot{x} = \mu x - y + xy^2$$
$$\dot{y} = x + \mu y + y^3$$

Show that a Hopf bifurcation occurs at the origin as μ varies.

Is the bifurcation subcritical, supercritical, or degenerate?

The Jacobian at the origin is:

$$A = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

$\tau = 2\mu$, $\Delta = \mu^2 + 1 > 0$, and $\lambda = \mu \pm i$.

Hence, as μ increases through zero, the origin changes from a stable spiral to an unstable spiral. This suggests that a Hopf bifurcation takes place at $\mu = 0$.

Example: Hopf Bifurcation

Consider the system:

$$\dot{x} = \mu x - y + xy^2$$
$$\dot{y} = x + \mu y + y^3$$

To decide whether the bifurcation is subcritical, supercritical, or degenerate, we use simple reasoning and numerical integration.

If we transform the system to polar coordinates, we find that: $\dot{r} = \mu r + ry^2$

Hence $\dot{r} \geq \mu r$, which implies that for $\mu > 0$, $r(t)$ grows *at least* as fast as $r_0 \, e^{\mu t}$.

All trajectories are repelled out to infinity!

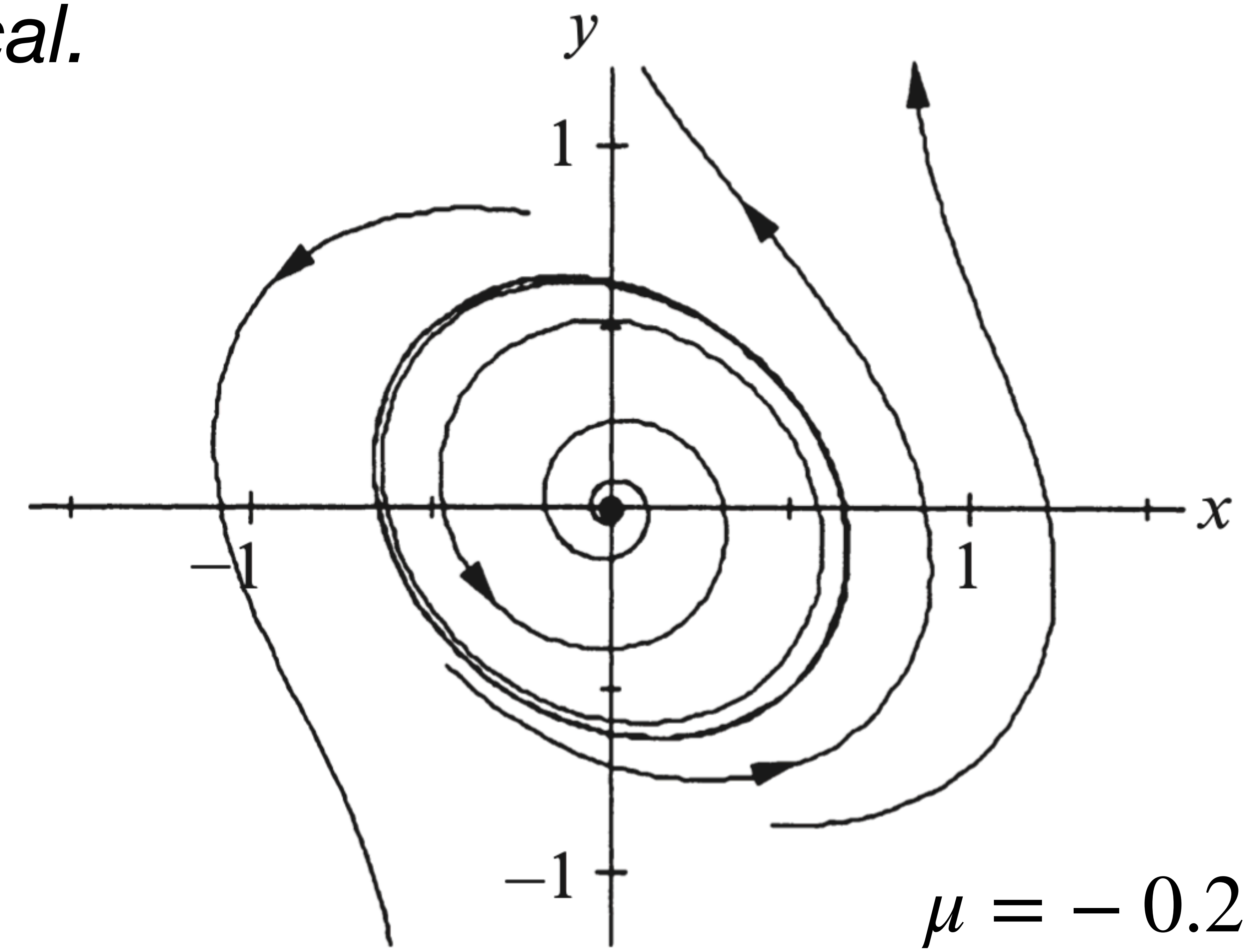
The unstable spiral is not surrounded by a stable limit cycle; hence the bifurcation cannot be supercritical.

Example: Hopf Bifurcation

Could the bifurcation be degenerate?

That would require that the origin be a nonlinear center when $\mu = 0$. But $\dot{r}$ is strictly positive away from the *x*-axis, so closed orbits are still impossible.

Thus, we expect that the bifurcation is *subcritical.*



$\mu = -0.2$

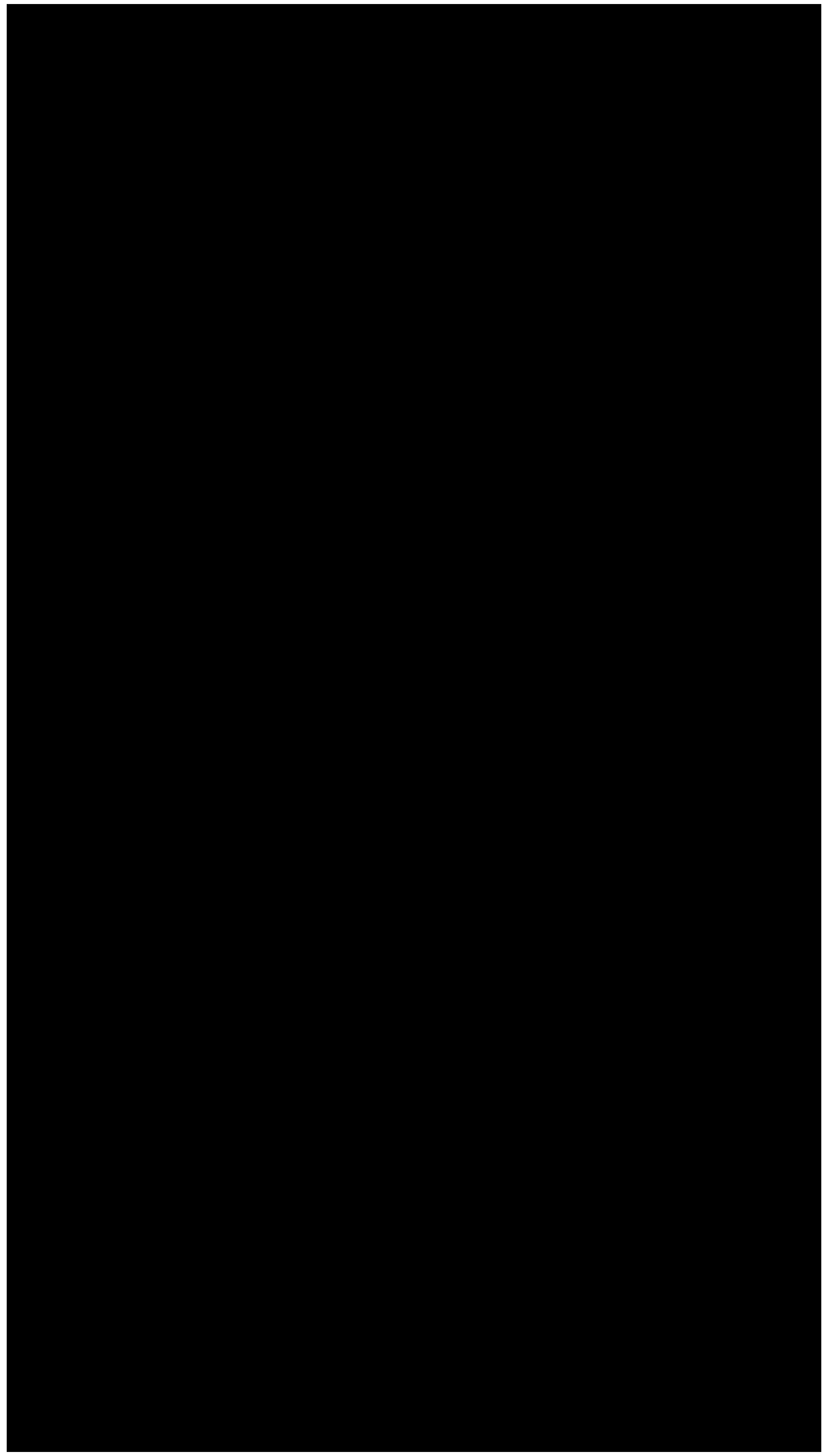
An unstable limit cycle surrounds the stable fixed point, just as we expect in a **subcritical bifurcation.**

Tutorial: Numerical integration

$$\dot{x} = \mu x - y + xy^2$$
$$\dot{y} = x + \mu y + y^3$$

Hopf Bifurcation: Oscillating Chemical Reactions

Chemical oscillators:



https://www.youtube.com/shorts/ieh9qIkkMJQ

It was believed that all solutions of chemical reagents must go monotonically to equilibrium.

Belousov found that chemical reactions can oscillate spontaneously.

Zhabotinsky confirmed that Belousov was right all along.

Belousov-Zhabotinsky (BZ) Reaction

Hopf Bifurcation: Oscillating Chemical Reactions

Analogy to biology:

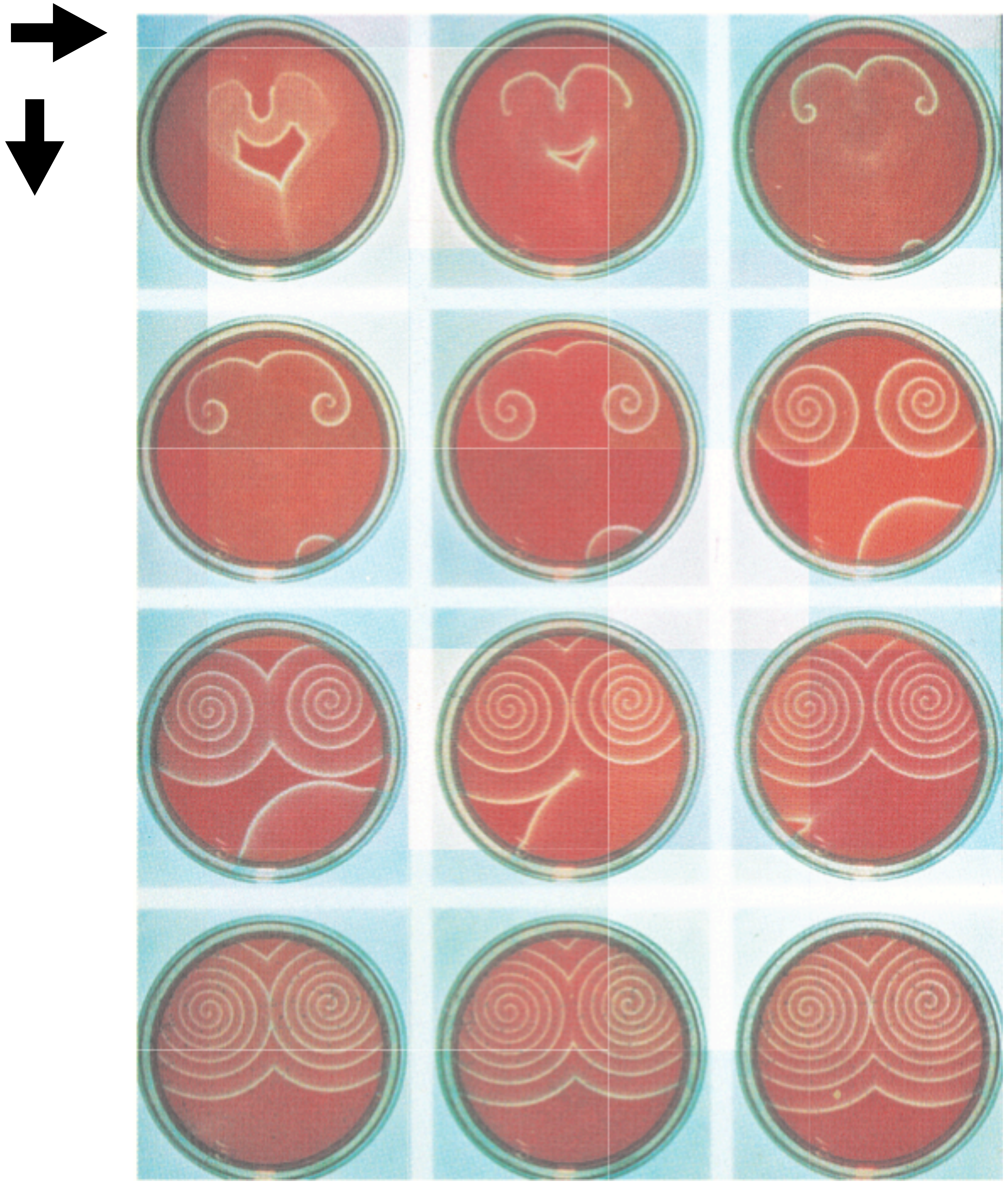
Propagating waves of oxidation were observed in thin unstirred layers of BZ reagent.

These waves annihilate upon collision, just like waves of excitation in neural or cardiac tissue.

The waves always take the shape of expanding concentric rings or spirals.

Spiral waves and their 3D analogs, “scroll waves”, appear to be implicated in certain cardiac arrhythmias.

Hopf Bifurcation: Oscillating Chemical Reactions



Spiral waves of chemical activity in a shallow dish of the BZ reaction.

The IC shown in the upper left was created by touching the liquid with a hot wire, thereby inducing an expanding circular wave of oxidation.

As time evolves, the waves propagate by diffusion through the motionless red liquid.

When 2 waves collide, they annihilate each other.

Ultimately the system organises itself into a pair of counterrotating spirals.

Hopf Bifurcation: Oscillating Chemical Reactions

Chlorine Dioxide–Iodine–Malonic Acid Reaction

Lengyel et al. (1990) proposed and analysed a model of another oscillating reaction, the chlorine dioxide-iodine-malonic acid ($\mathrm{ClO_2} - \mathrm{I_2} - \mathrm{MA}$) reaction:

$$\mathrm{MA} + \mathrm{I_2} \rightarrow \mathrm{IMA} + \mathrm{I^-} + \mathrm{H^+}; \quad \frac{d[\mathrm{I_2}]}{dt} = -\frac{k_{1a}[\mathrm{MA}][\mathrm{I_2}]}{k_{1b} + [\mathrm{I_2}]} \qquad (1)$$

$$\mathrm{ClO_2} + \mathrm{I^-} \rightarrow \mathrm{ClO_2^-} + \tfrac{1}{2}\mathrm{I_2}; \quad \frac{d[\mathrm{ClO_2}]}{dt} = -k_2[\mathrm{ClO_2}][\mathrm{I^-}] \qquad (2)$$

$$\mathrm{ClO_2^-} + 4\mathrm{I^-} + 4\mathrm{H^+} \rightarrow \mathrm{Cl^-} + 2\mathrm{I_2} + 2\mathrm{H_2O};$$

$$\frac{d[\mathrm{ClO_2^-}]}{dt} = -k_{3a}[\mathrm{ClO_2^-}][\mathrm{I^-}][\mathrm{H^+}] - k_{3b}[\mathrm{ClO_2^-}][\mathrm{I_2}]\frac{[\mathrm{I^-}]}{u + [\mathrm{I^-}]^2} \qquad (3)$$

Hopf Bifurcation: Oscillating Chemical Reactions

Chlorine Dioxide–Iodine–Malonic Acid Reaction

By approximating the concentrations of the slow reactants as *constants* and making other reasonable simplifications, Lengyel et al. (1990) reduce the system to a 2-variable, non-dimensional model:

$$\dot{x} = a - x - \frac{4xy}{1+x^2}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

where *x* and *y* are the dimensionless concentrations of I^- and $\mathrm{ClO_2}^-$.

$a, b > 0$ depend on the empirical rate constants and on the concentrations assumed for the slow reactants.

Hopf Bifurcation: Oscillating Chemical Reactions

$$\dot{x} = a - x - \frac{4xy}{1+x^2}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

Prove that the system has a closed orbit in the positive quadrant $x, y > 0$ if *a* and *b* satisfy certain constraints, to be determined.

The nullclines help us to construct a trapping region,

e.g. $\dot{x} = 0$:

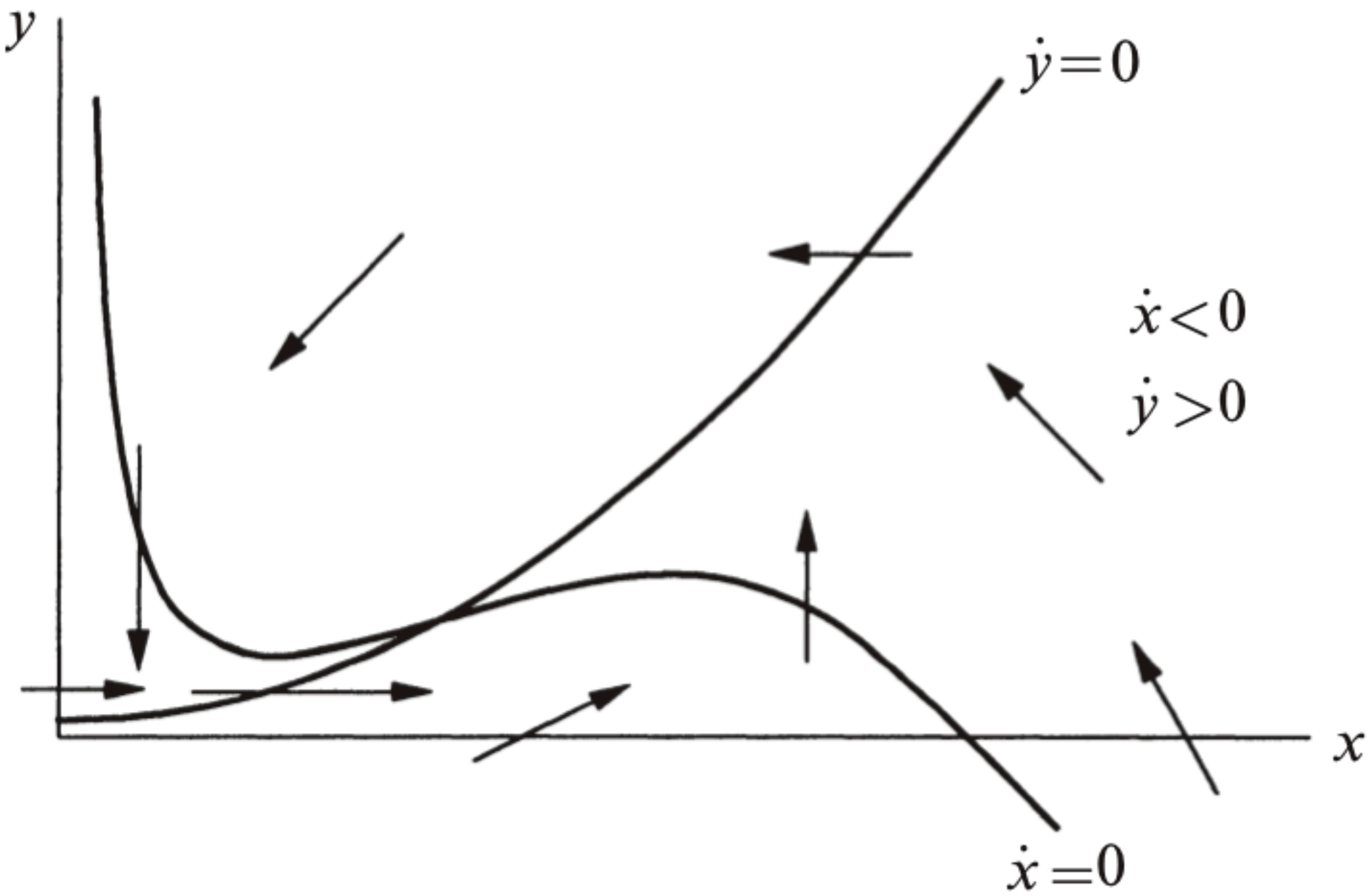
$$y = \frac{(a-x)(1+x^2)}{4x}$$

e.g. $\dot{y} = 0$:

$$y = 1 + x^2$$

Hopf Bifurcation: Oscillating Chemical Reactions

Phase portrait



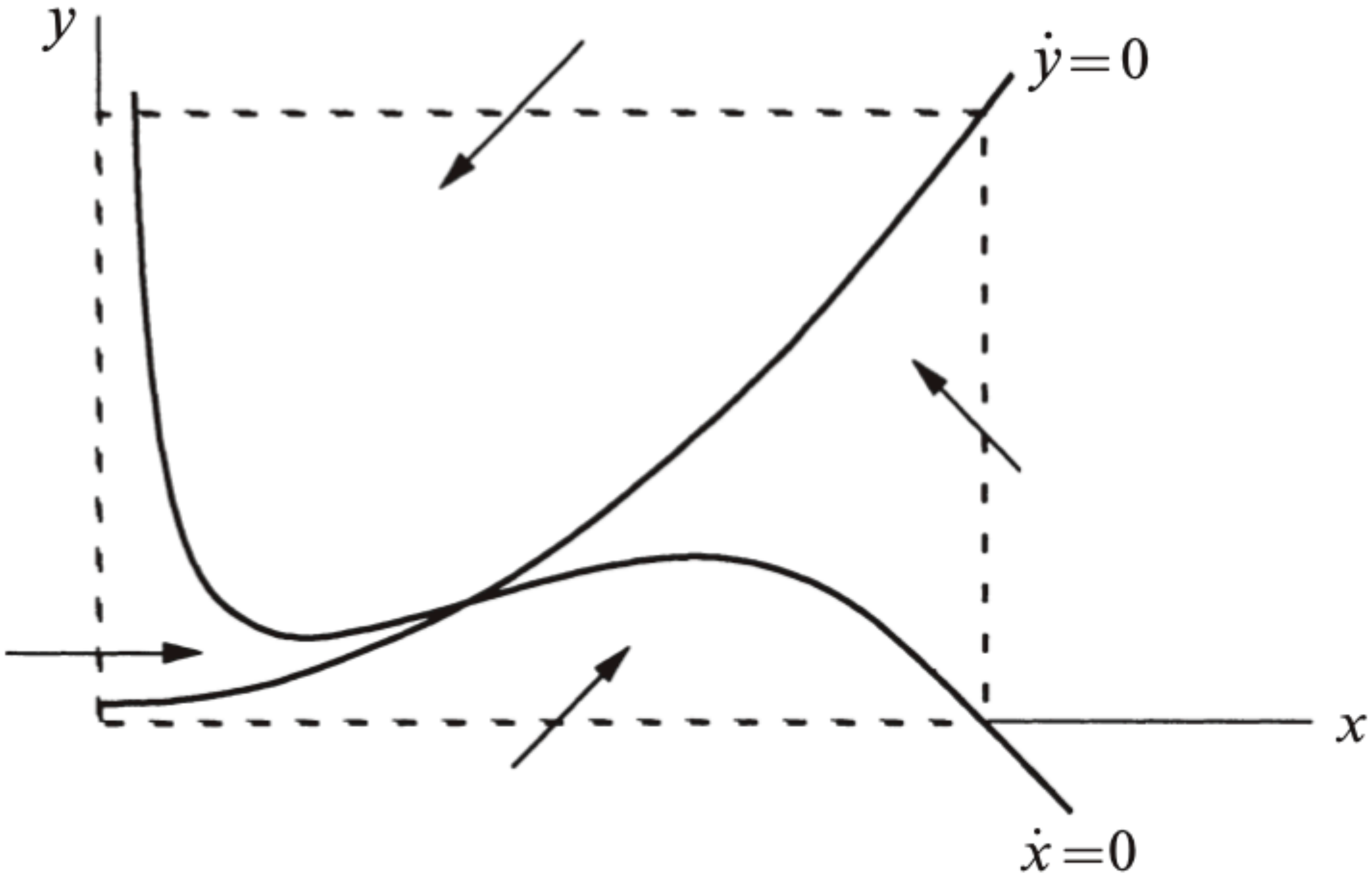
Nullclines:

$$y = \frac{(a-x)(1+x^2)}{4x}$$

$$y = 1 + x^2$$

Hopf Bifurcation: Oscillating Chemical Reactions

Phase portrait: "trapping region"



Trapping Region

All the vectors on the boundary point into the box.

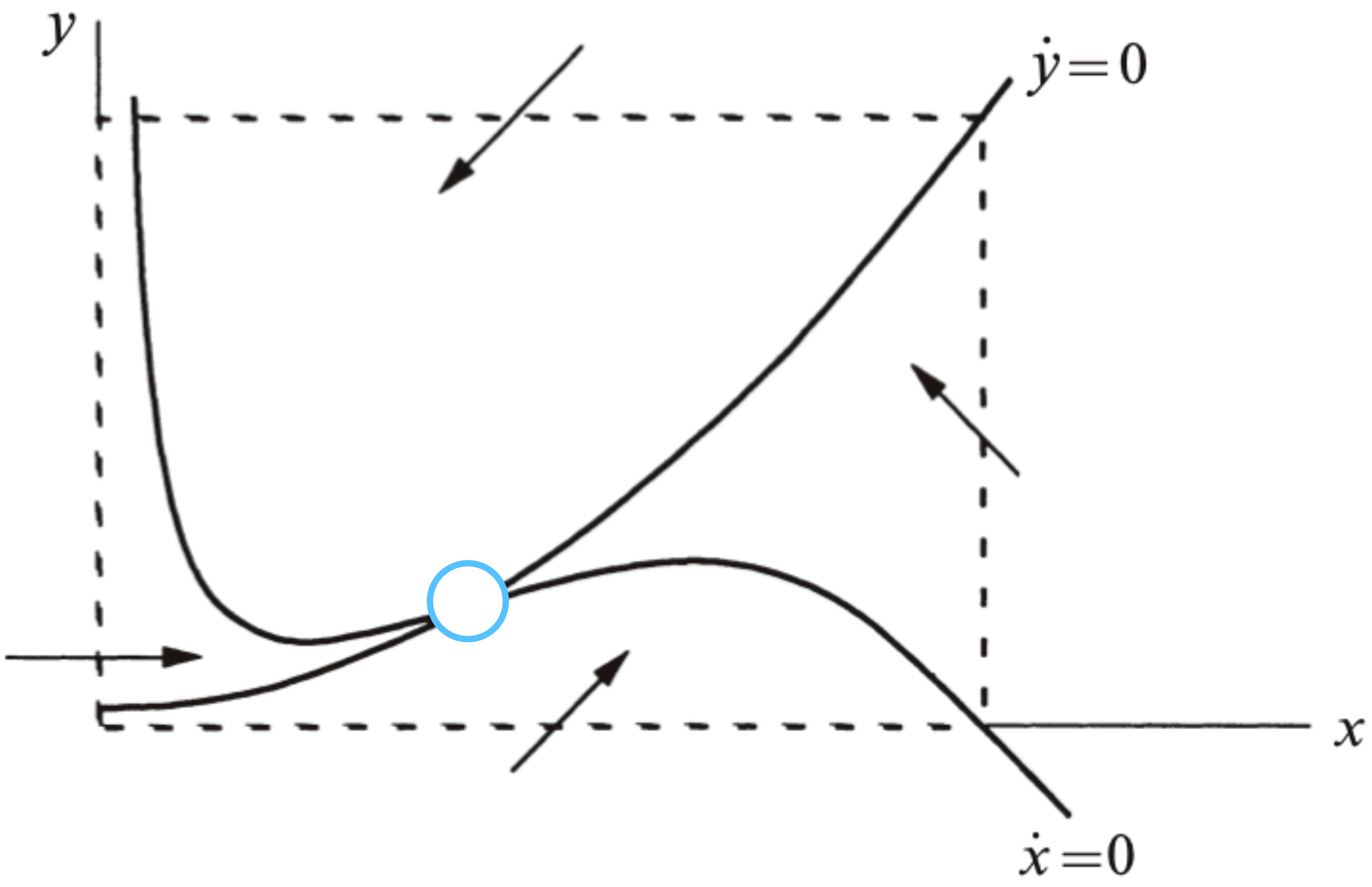
There is a **fixed point** inside the box at the intersection of the nullclines

$$x^* = a/5$$

$$y^* = 1 + (x^*)^2 = 1 + (a/5)^2$$

Hopf Bifurcation: Oscillating Chemical Reactions

Phase portrait: punctured box



If the fixed point turns out to be a repeller, we can apply the **Poincaré-Bendixson theorem.**

Then the trajectory must eventually approach a closed orbit.

We need to see under what conditions (if any) the fixed point is a **repeller.**

Hopf Bifurcation: Oscillating Chemical Reactions

The Jacobian at (x^*, y^*) is:

$$\frac{1}{1+(x^*)^2}\begin{pmatrix} 3(x^*)^2 - 5 & -4x^* \\ 2b(x^*)^2 & -bx^* \end{pmatrix}$$

The determinant and trace are given by:

$$\Delta = \frac{5bx^*}{1+(x^*)^2} > 0, \qquad \tau = \frac{3(x^*)^2 - 5 - bx^*}{1+(x^*)^2}$$

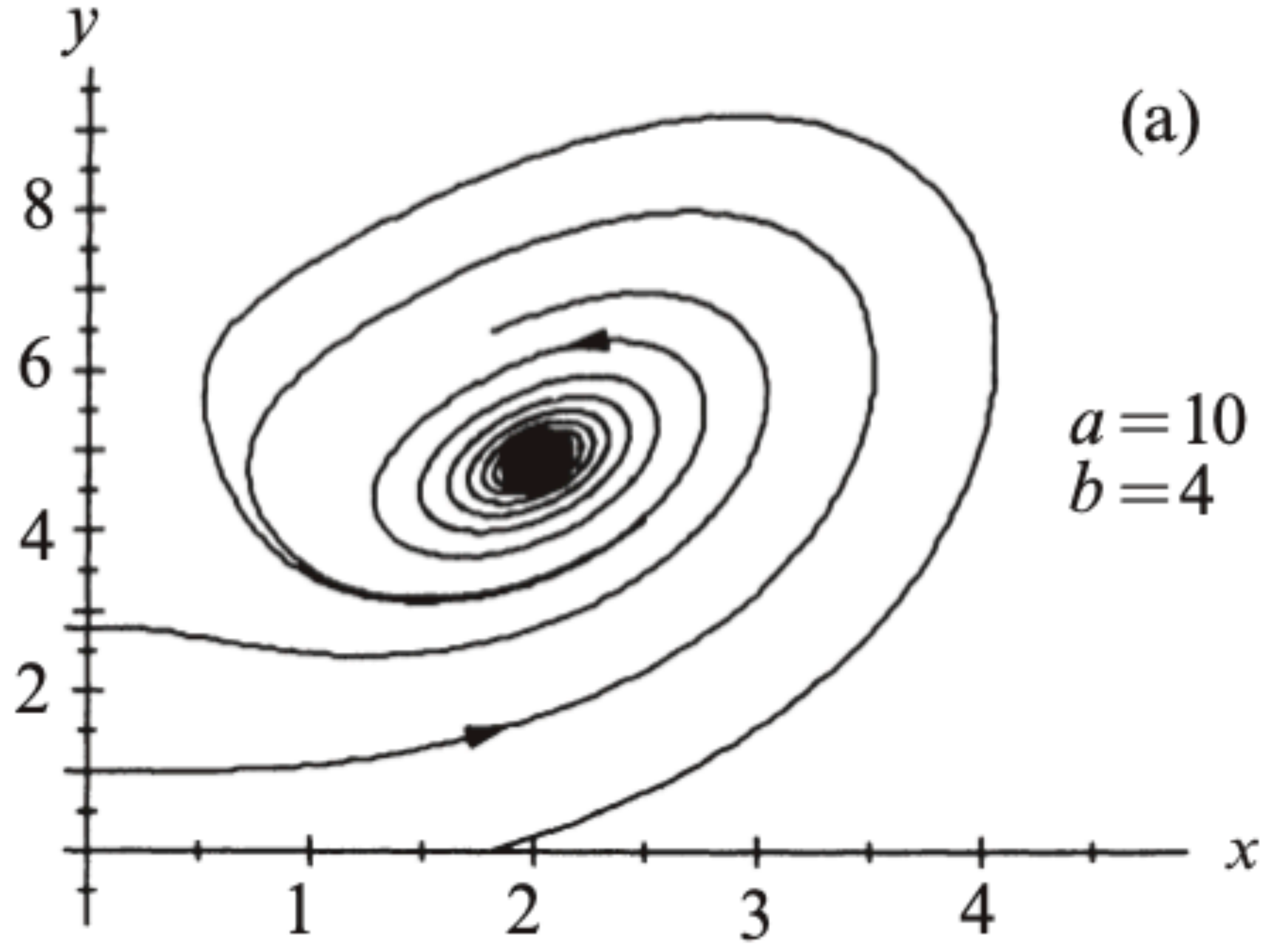
Since $\Delta > 0$, the fixed point is never a saddle, so (x^*, y^*) is a repeller if $\tau > 0$, i.e., if

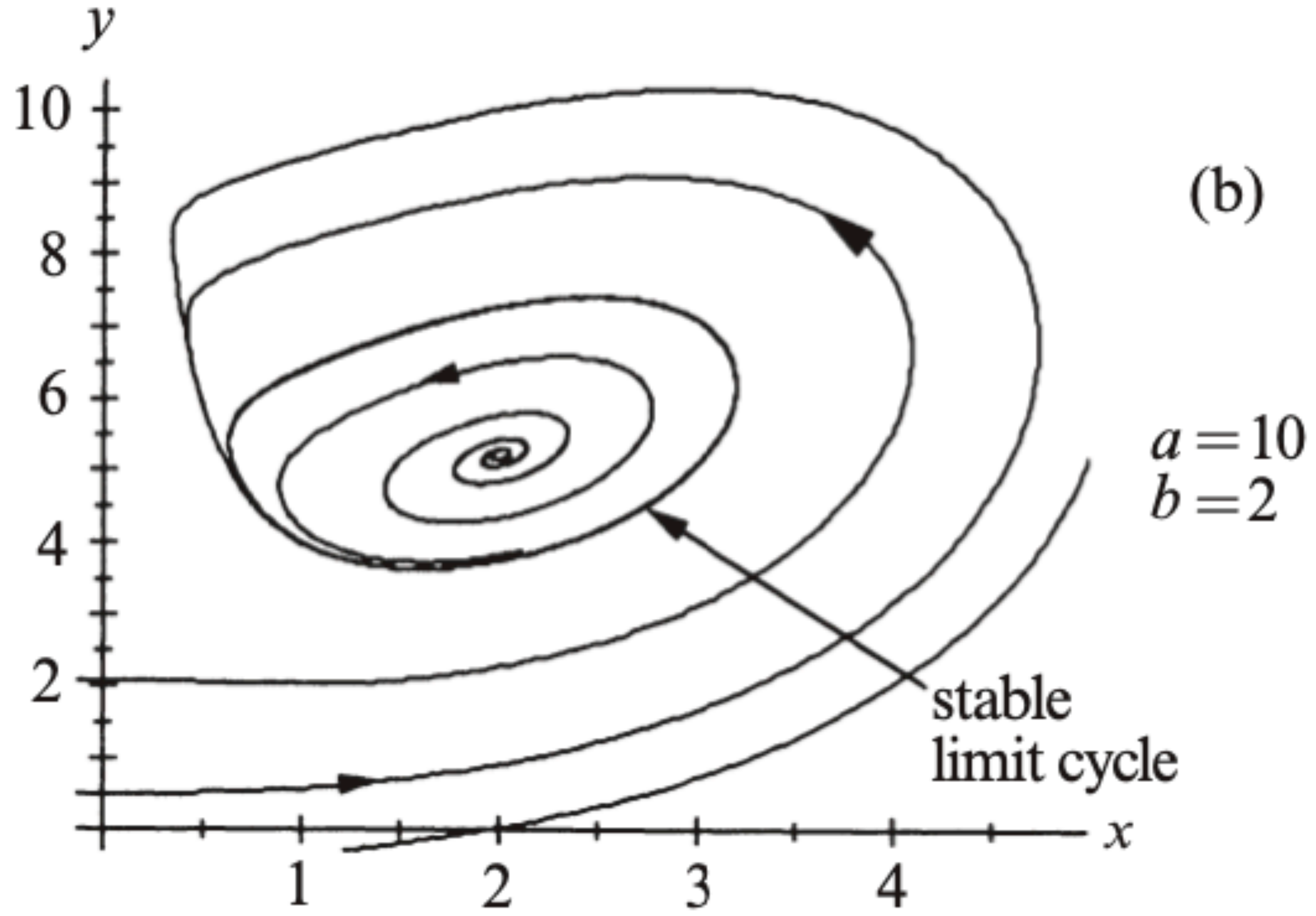
$$b < b_c \equiv 3a/5 - 25/a$$

Hopf Bifurcation: Oscillating Chemical Reactions

The Poincaré-Bendixson theorem implies the existence of a closed orbit somewhere in the punctured box.

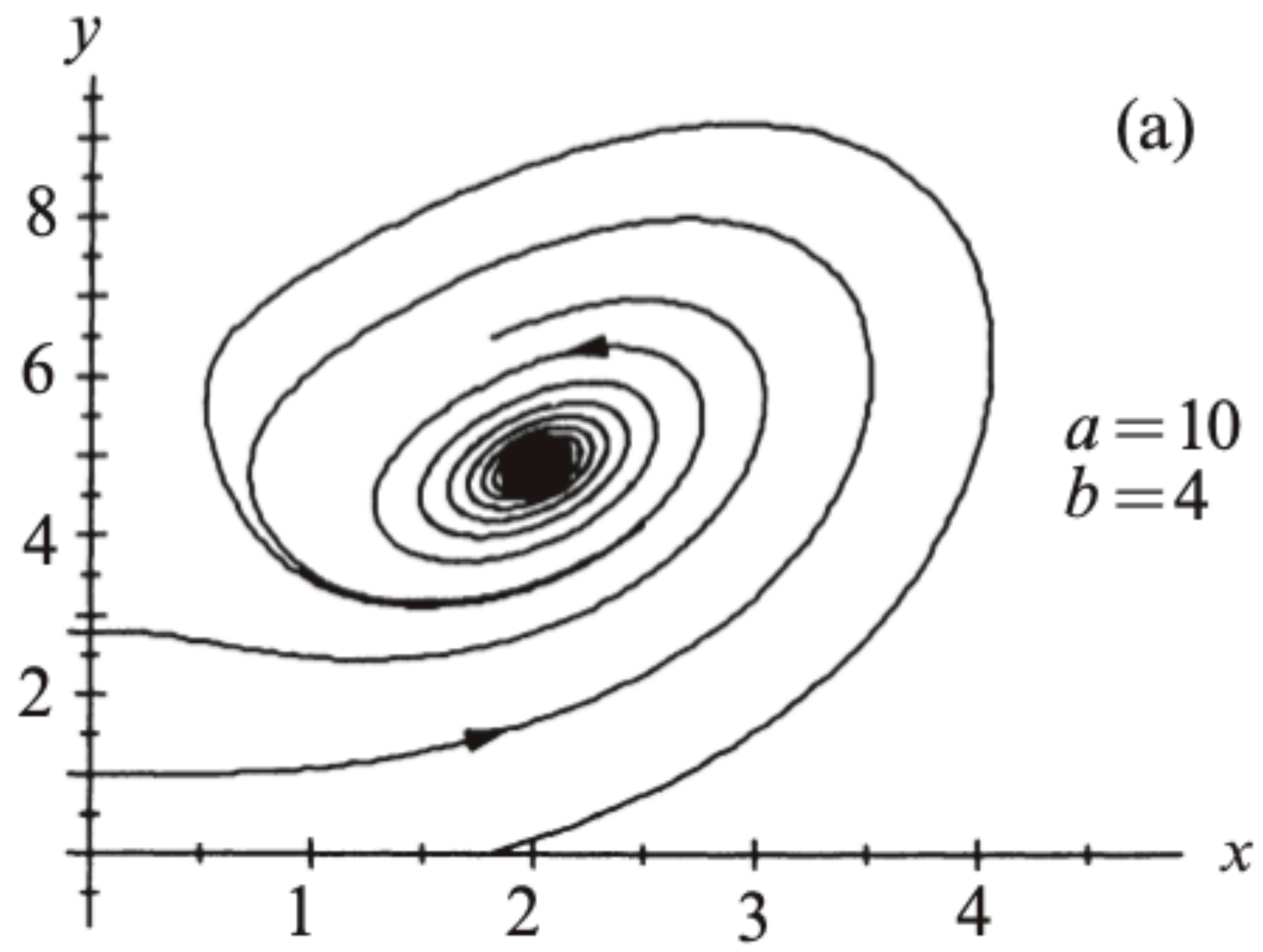
Using numerical integration, we can show that a Hopf bifurcation occurs at $b = b_c$ and decide whether the bifurcation is sub- or supercritical.

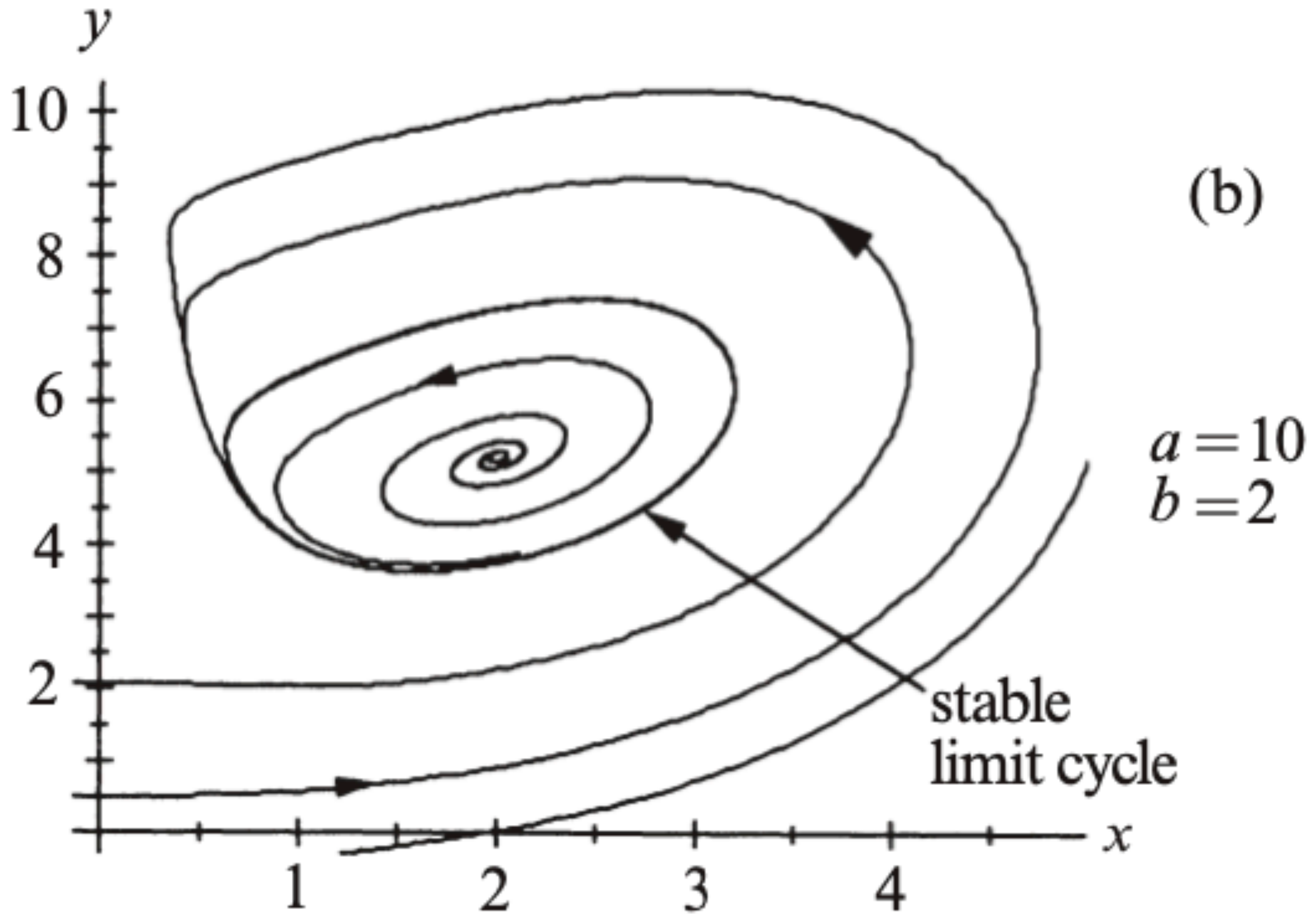




Hopf Bifurcation: Oscillating Chemical Reactions

As b decreases through b_c, the fixed point changes from a stable spiral to an unstable spiral > **Hopf bifurcation**



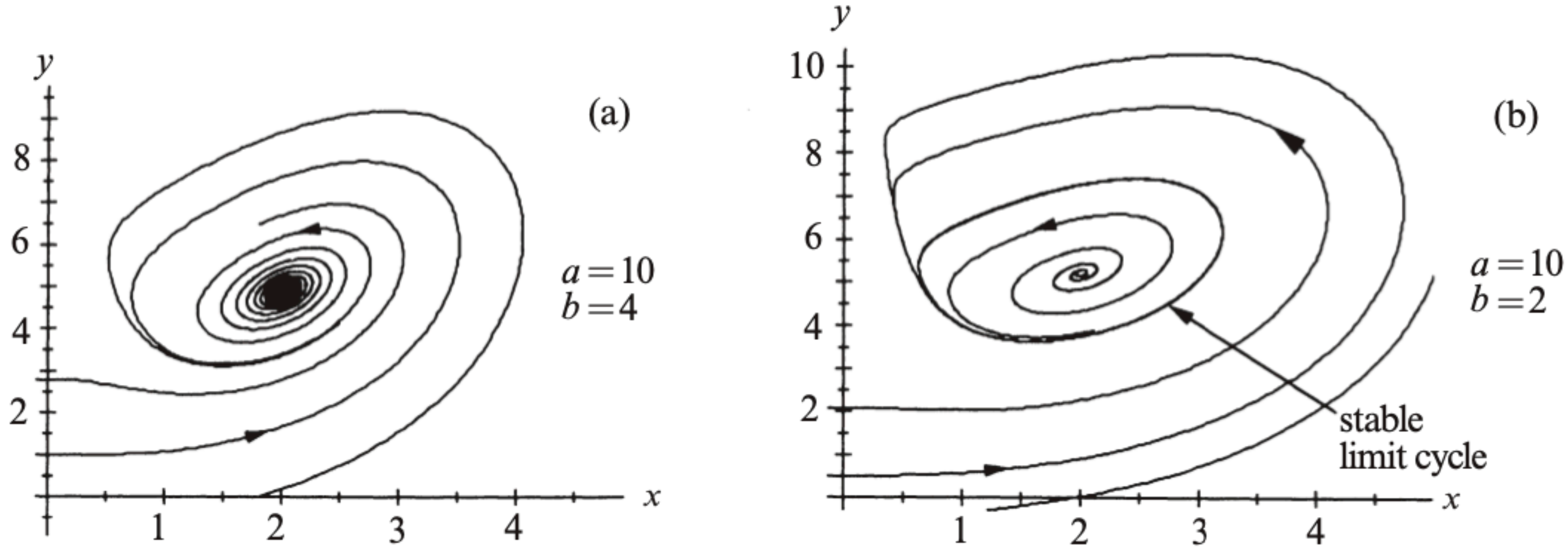


When $b > b_c$, all trajectories spiral into **the stable fixed point**

When $b < b_c$, they are attracted to a **stable limit cycle**.

Hopf Bifurcation: Oscillating Chemical Reactions

Hence the bifurcation is supercritical—after the fixed point loses stability, it is surrounded by a stable limit cycle.



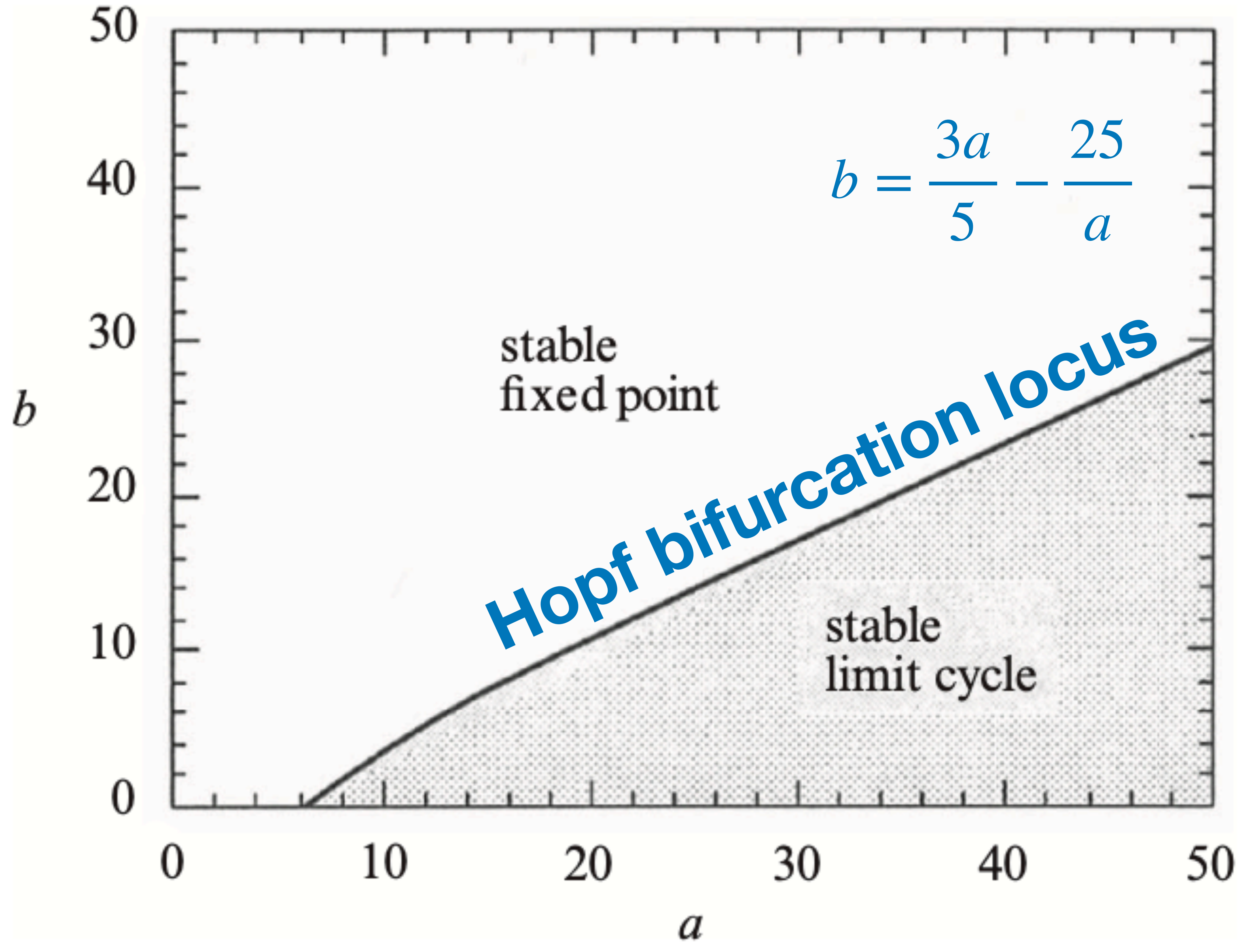
As $b \to b_c$ from below, the limit cycle shrinks continuously to a point.

Hopf Bifurcation: Oscillating Chemical Reactions

$$\dot{x} = a - x - \frac{4xy}{1+x^2}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

Stability diagram



What about the period of the limit cycle for $b \lesssim b_c$?

The frequency is approximated by the imaginary part of the eigenvalues at the bifurcation.

The eigenvalues satisfy:

$$\lambda^2 - \tau\lambda + \Delta = 0$$

Since $\tau = 0$ and $\Delta > 0$ at $b = b_c$, we find:

$$\lambda = \pm i\sqrt{\Delta}.$$

Hopf Bifurcation: Oscillating Chemical Reactions

At b_c:

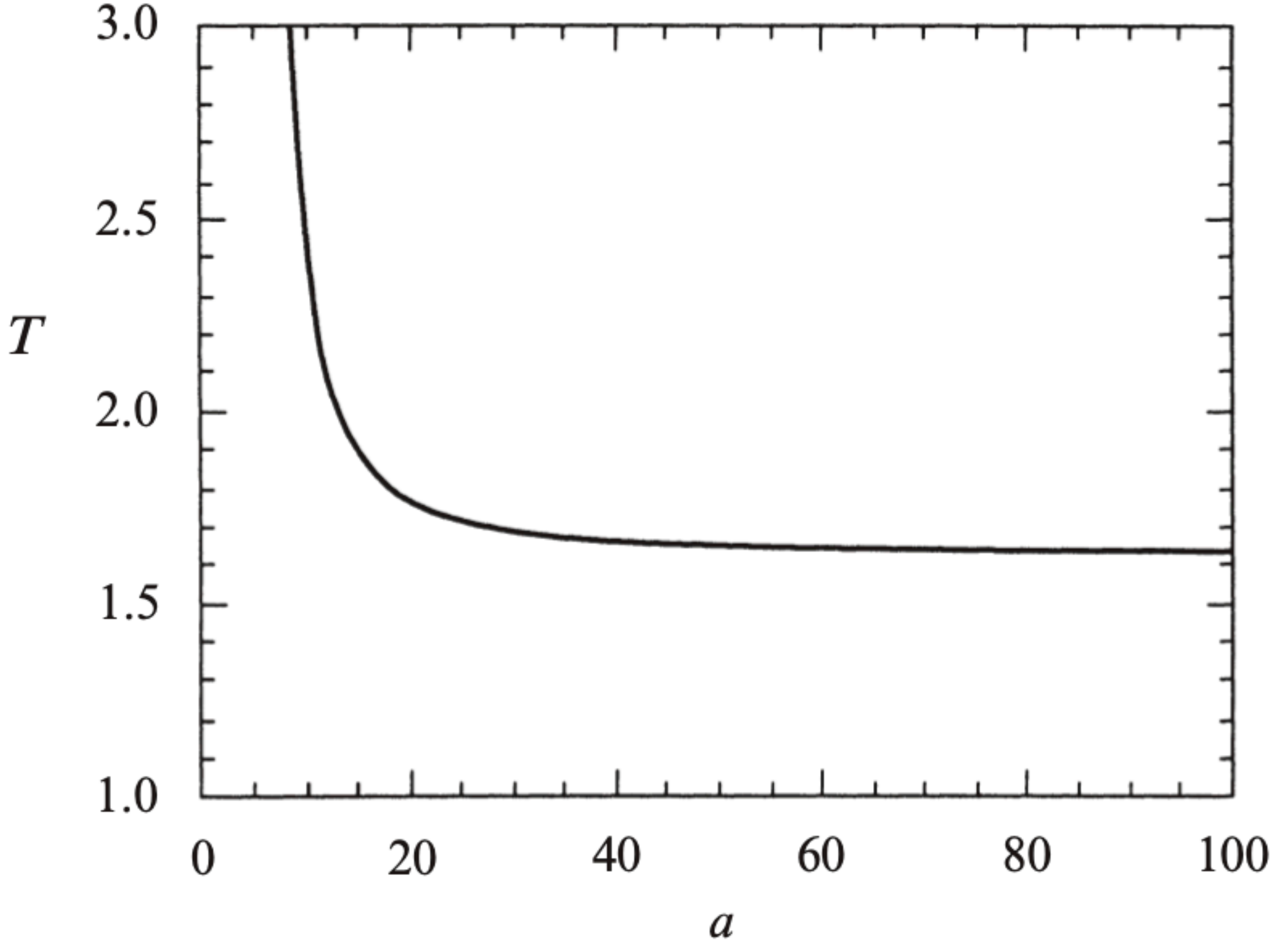
$$\Delta = \frac{5b_c x^*}{1+(x^*)^2} = \frac{5\left(\frac{3a}{5} - \frac{25}{a}\right)\left(\frac{a}{5}\right)}{1+(a/5)^2} = \frac{15a^2 - 625}{a^2 + 25}$$

Hence:

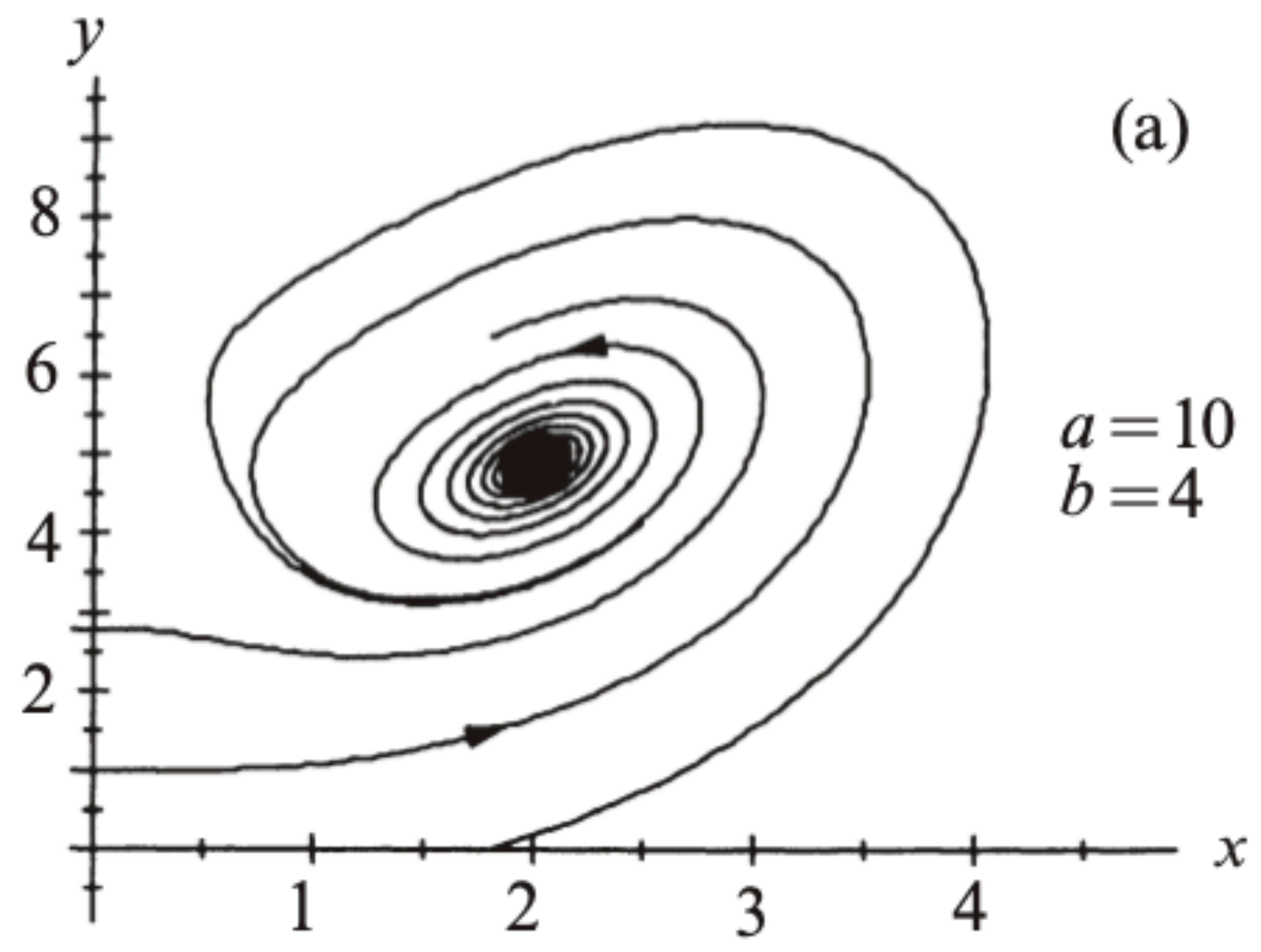
$$\omega \approx \Delta^{1/2} = \left[(15a^2 - 625)/(a^2 + 25)\right]^{1/2}$$

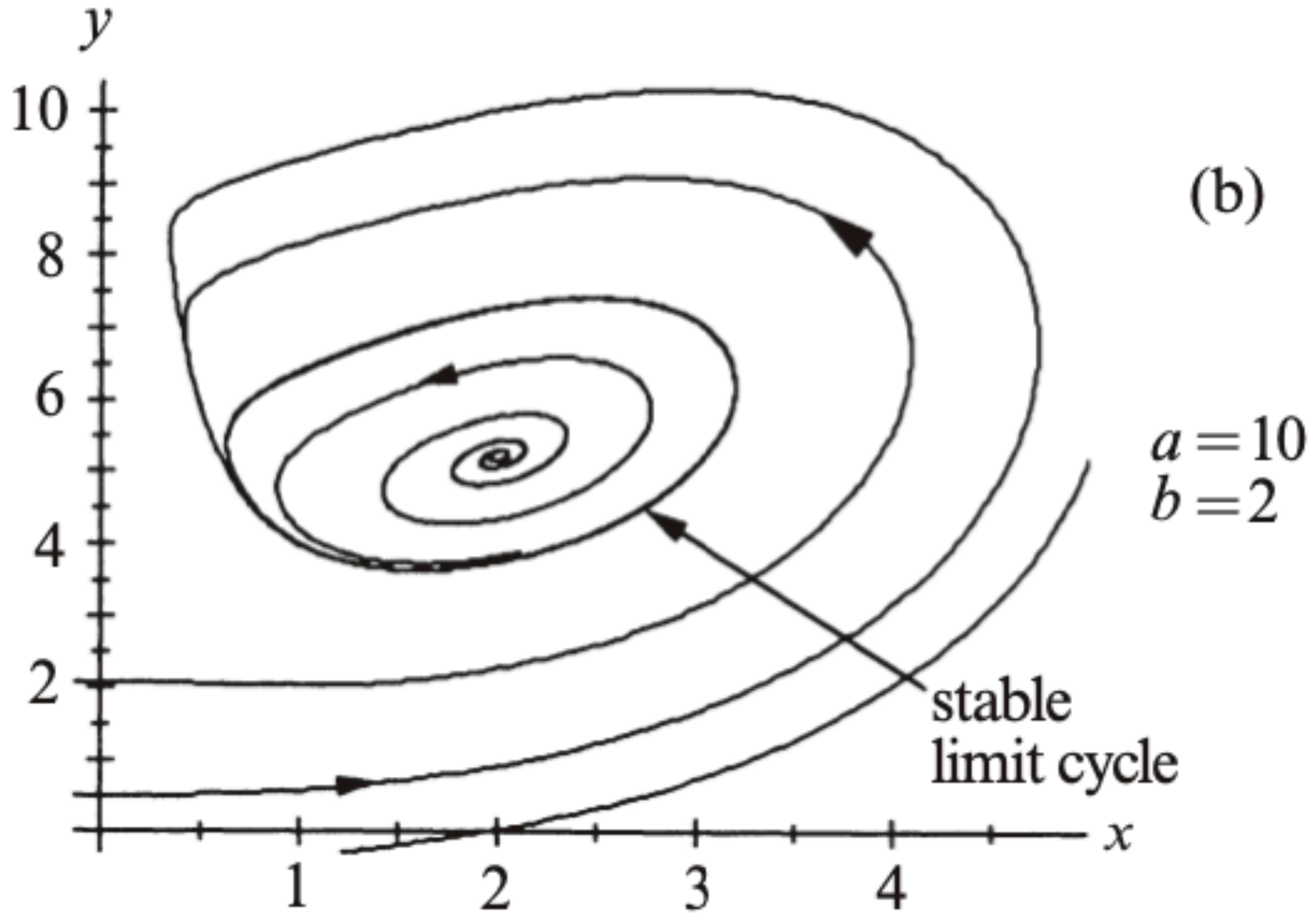
And the period:

$$\begin{aligned} T &= 2\pi/\omega \\ &= 2\pi\left[(a^2 + 25)/(15a^2 - 625)\right]^{1/2} \end{aligned}$$



Hopf Bifurcation: Oscillating Chemical Reactions





The reaction settles down to a steady concentration.

No oscillations.

The chemicals calm down and reach **equilibrium.**

The system does *not* settle to a point.

Instead, it settles into repeated oscillations forever.

That loop is the **oscillation cycle**.