Nonlinear Dynamics and Chaos

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Global Bifurcations of Cycles

In 2D there are **4 common ways** in which limit cycles are created or destroyed.

Hopf bifurcation

Global bifurcations

- **Saddle-node Bifurcation of Cycles**
- **Infinite-period Bifurcation**
- **Homoclinic Bifurcation**

These other 3 are harder to detect because they involve large regions of the phase plane rather than just the neighbourhood of a single fixed point.

Saddle-node Bifurcation of Cycles

This is a bifurcation in which two limit cycles coalesce and annihilate. It is called a **fold or saddle-node bifurcation of cycles.**

An example occurs in the system:

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + b r^2$$

We studied this before when we were interested in the subcritical Hopf bifurcation at $\mu = 0$; now we concentrate on the dynamics for $\mu < 0$.

We can regard the radial equation as a 1D system, which undergoes a saddle-node bifurcation of fixed points at $\mu_c = -1/4$.

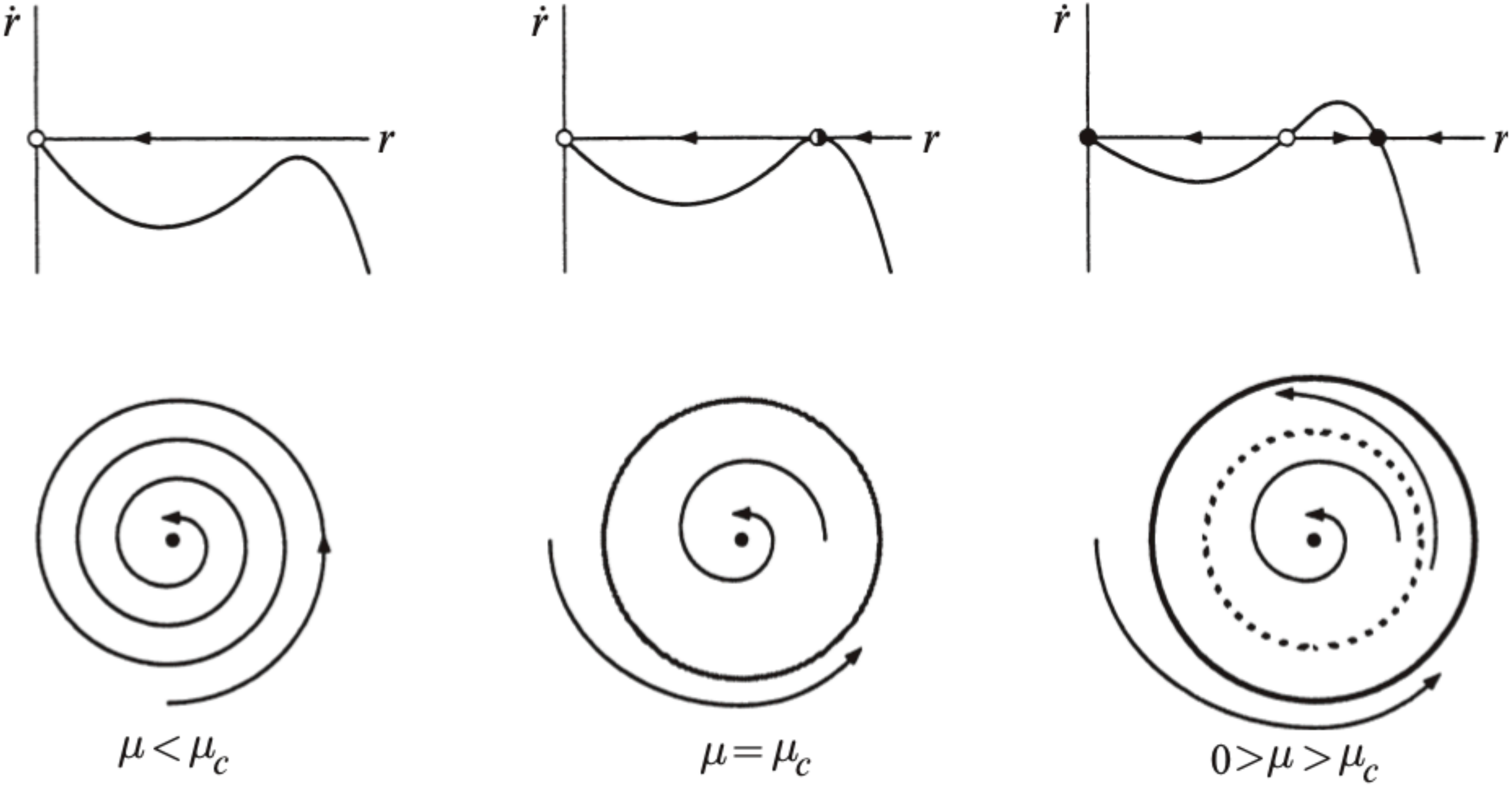
In 2D, these fixed points correspond to circular limit cycles.

Saddle-node Bifurcation of Cycles

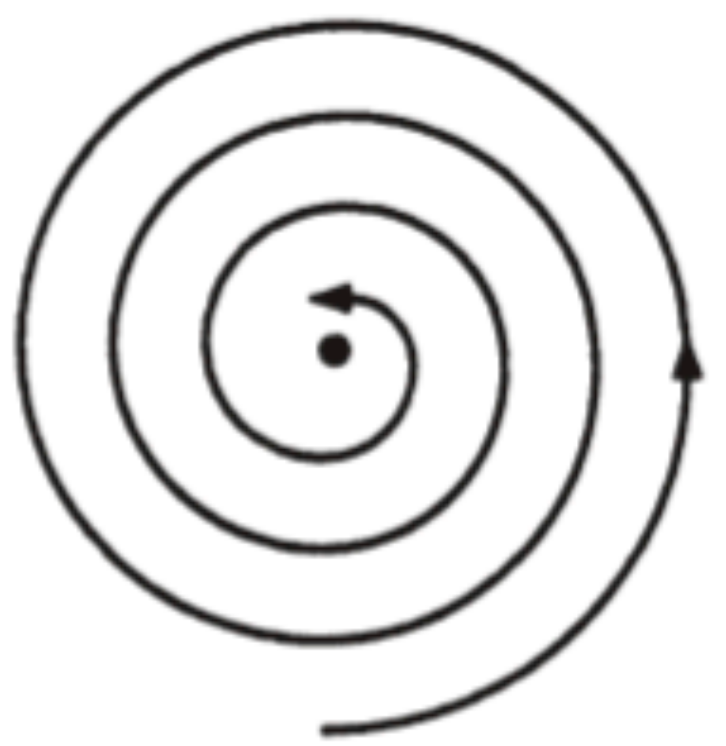
$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + b r^2$$

In 1D:

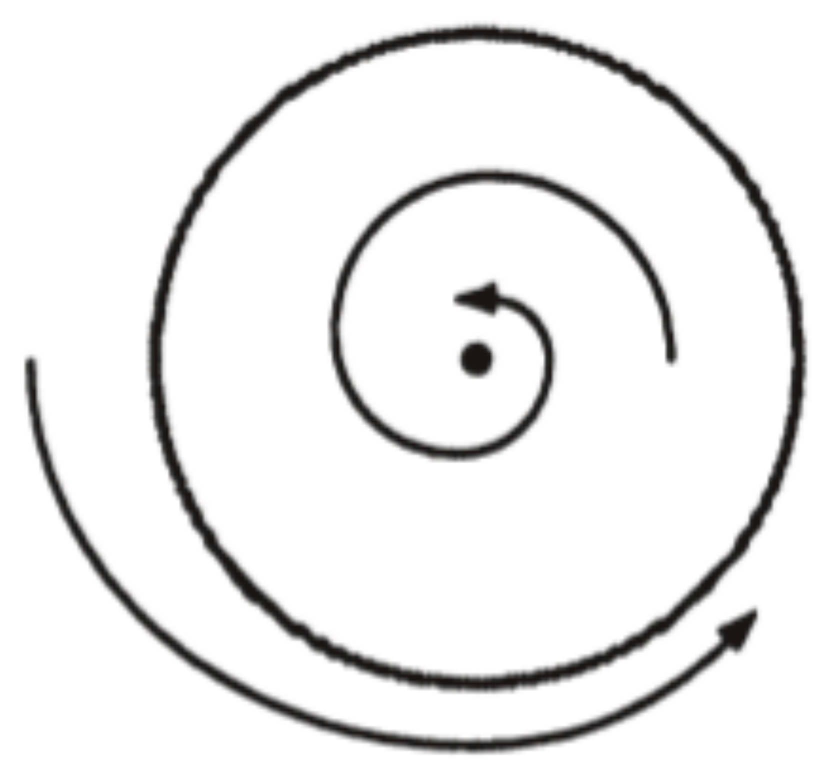
In 2D:



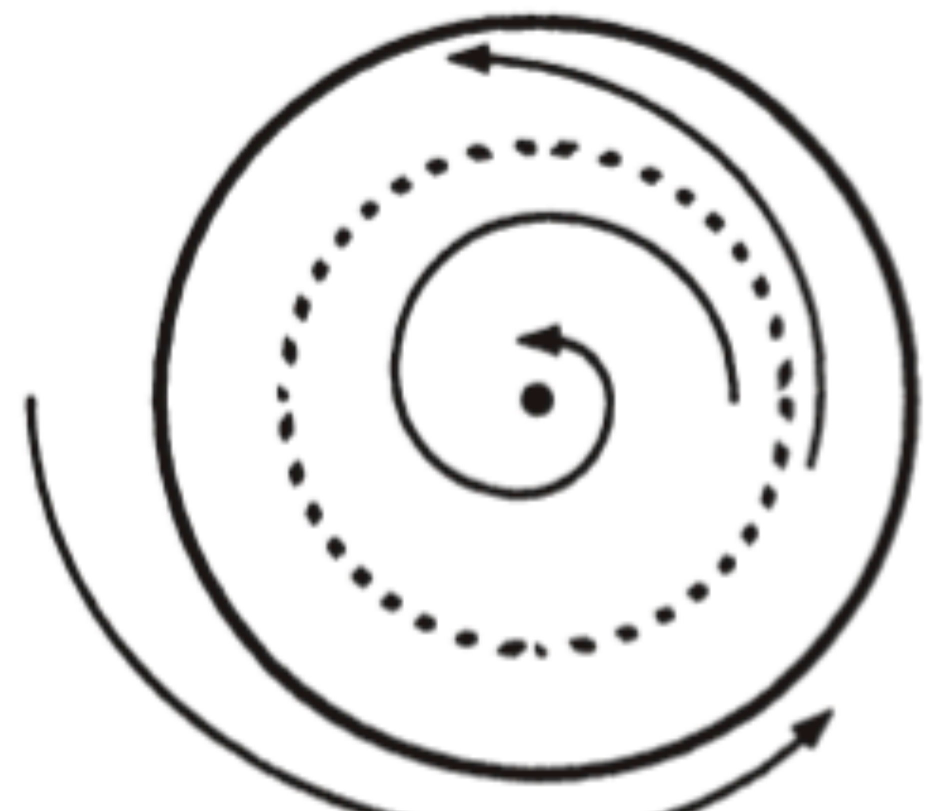
Saddle-node Bifurcation of Cycles



$\mu < \mu_c$



$\mu = \mu_c$



$0 > \mu > \mu_c$

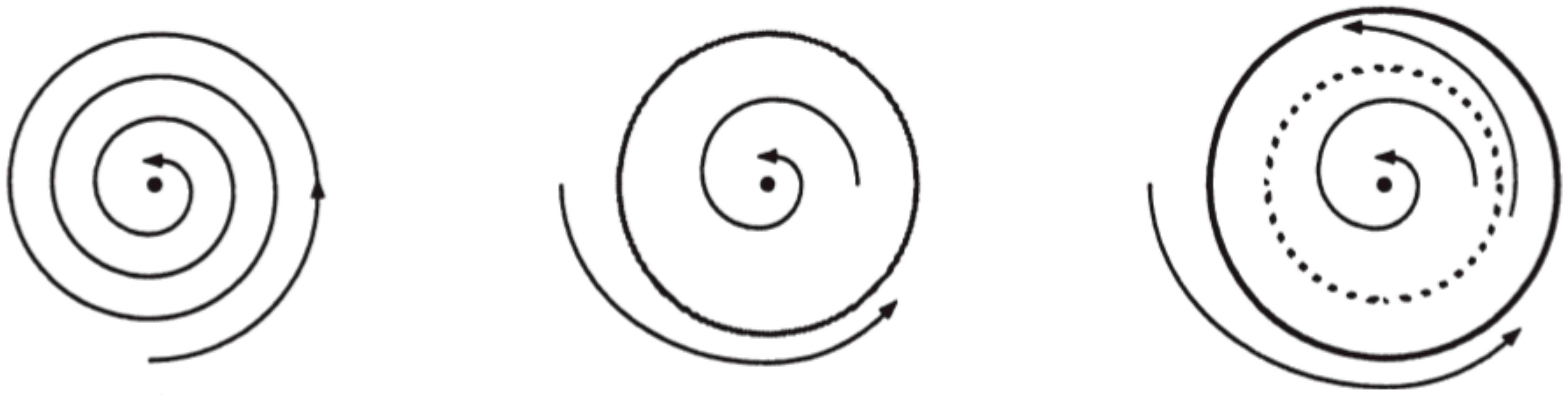
The origin remains stable throughout; it does not participate in this bifurcation.

At μ_c a half-stable cycle is born.

As μ increases it splits into a pair of limit cycles, one stable, one unstable.

In the other direction, a stable and unstable cycle collide and disappear as μ decreases through μ_c.

Saddle-node Bifurcation of Cycles



$\mu < \mu_c$ $\mu = \mu_c$ $0 > \mu > \mu_c$

At birth the cycle has $O(1)$ amplitude, in contrast to the Hopf bifurcation (limit cycle has small amplitude proportional to $(\mu - \mu_c)^{0.5}$

O(1) is relative and means:

The amplitud is comparable to the characteristic scale of the system.

It means not going to zero (not small) and not going to infinity.

Infinite-period Bifurcation

An example occurs in the system:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \mu - \sin\theta$$

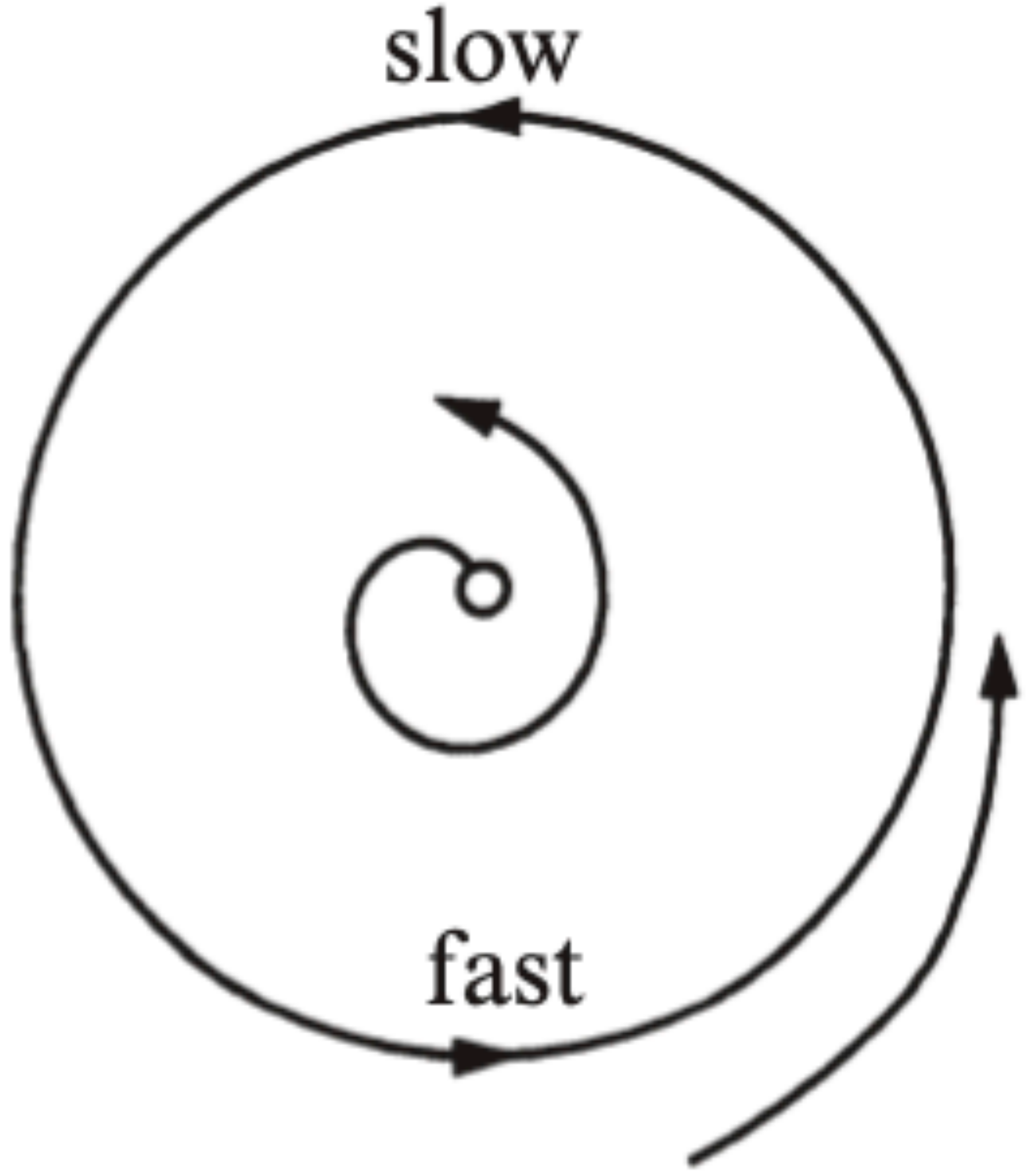
where $\mu \geq 0$.

In the radial direction, all trajectories (except $r^* = 0$) approach the unit circle monotonically as $t \to \infty$.

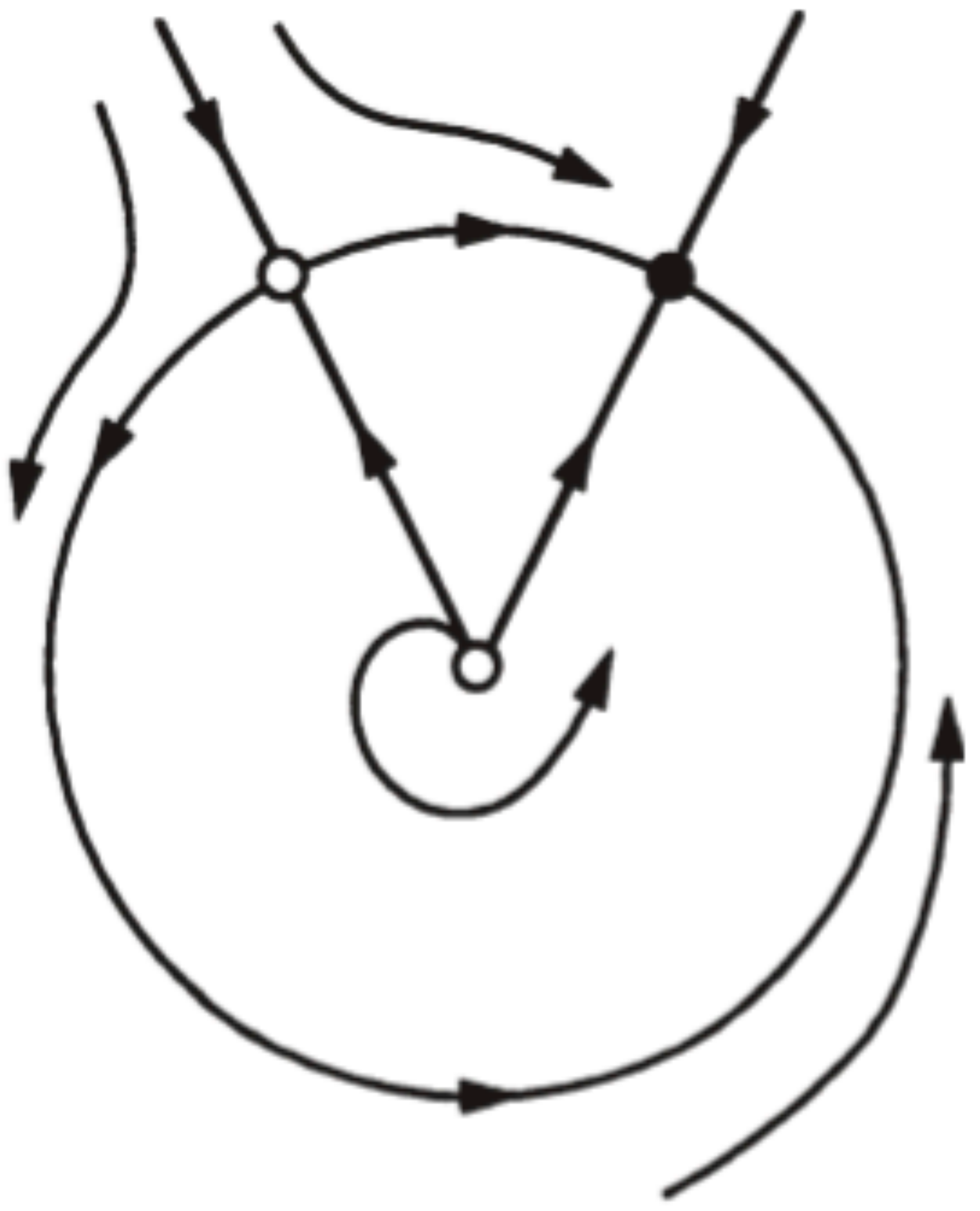
In the angular direction, the motion is everywhere counterclockwise if $\mu > 1$, whereas there are two invariant rays defined by $\sin(\theta) = \mu$ if $\mu < 1$.

Infinite-period Bifurcation

As μ decreases through $\mu_c = 1$:



$\mu > 1$

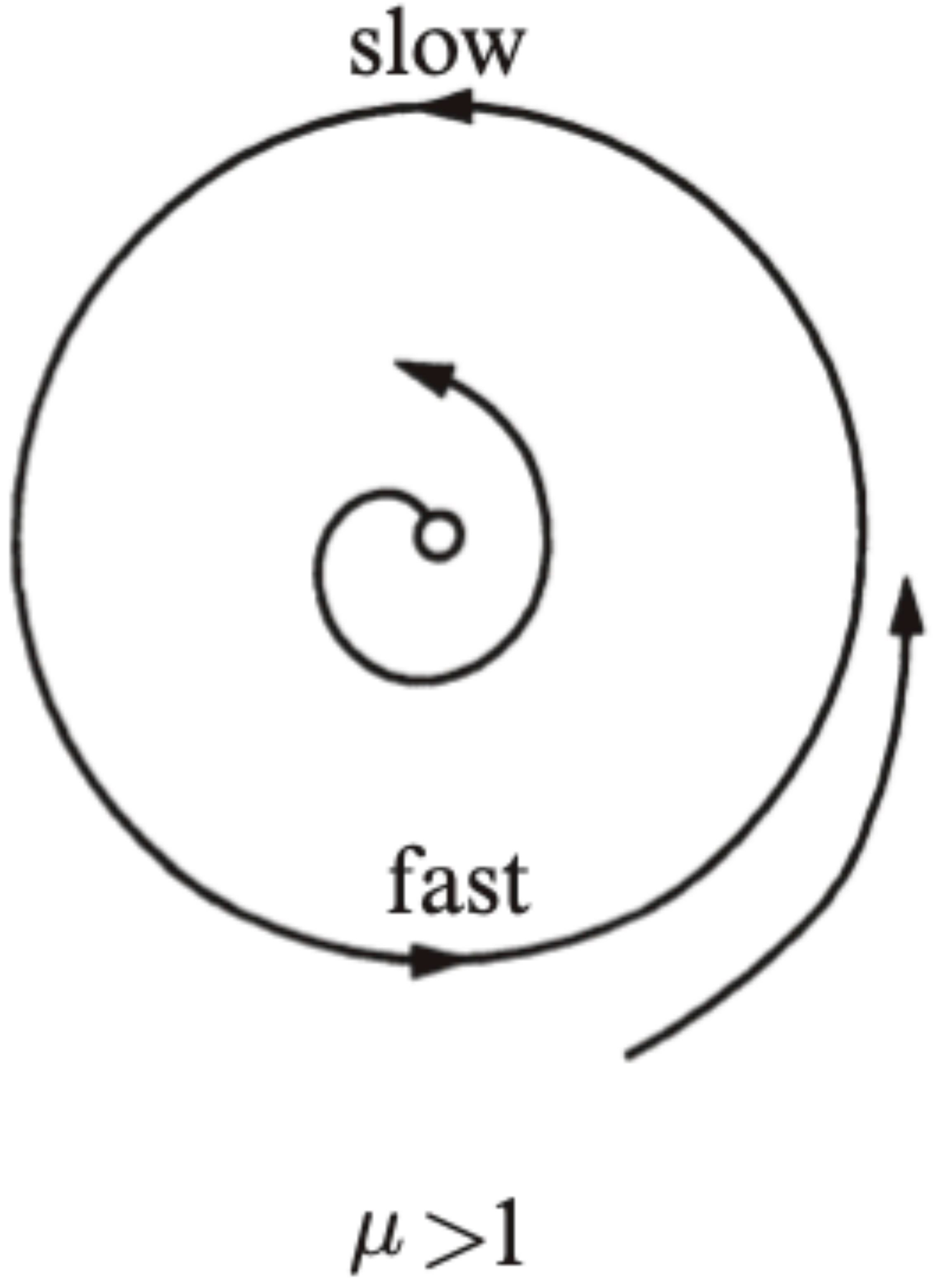


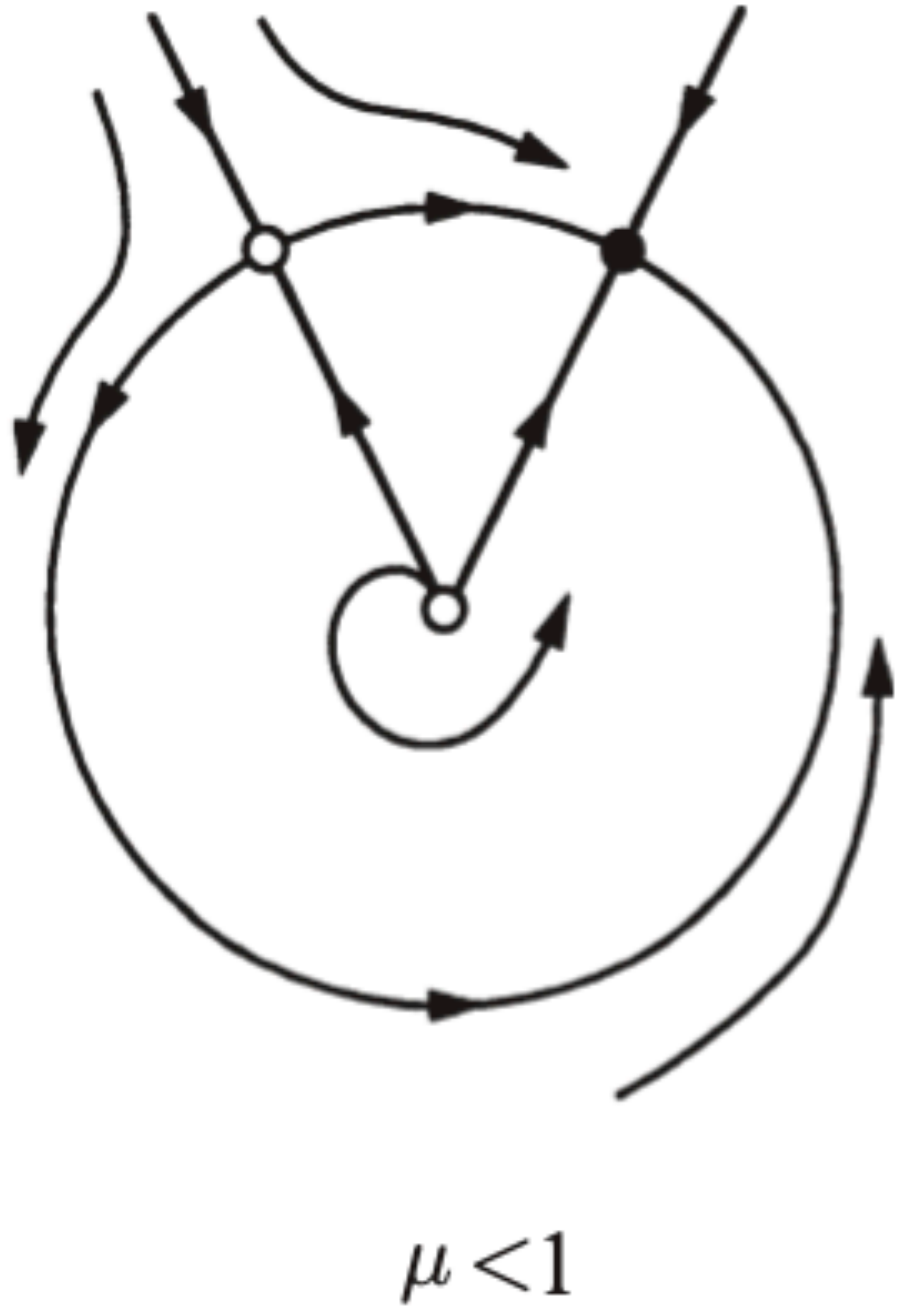
$\mu < 1$

As μ decreases, the limit cycle $r = 1$ develops a bottleneck at $\theta = \pi/2$ that becomes increasingly severe as $\mu \to 1^+$.

For $\mu < 1$, the fixed point splits into a saddle and a node.

Infinite-period Bifurcation





The oscillation period lengthens and finally becomes infinite at $\mu_c = 1$, when a fixed point appears on the circle.

As the bifurcation is approached, the amplitude of the oscillation stays $O(1)$ but the period increases like $(\mu - \mu_c)^{-0.5}$

Homoclinic Bifurcation

In this scenario, part of a limit cycle moves closer and closer to a saddle point.

At the bifurcation the cycle touches the saddle point and becomes a **homoclinic orbit** (the trajectories that start and end at the same fixed point).

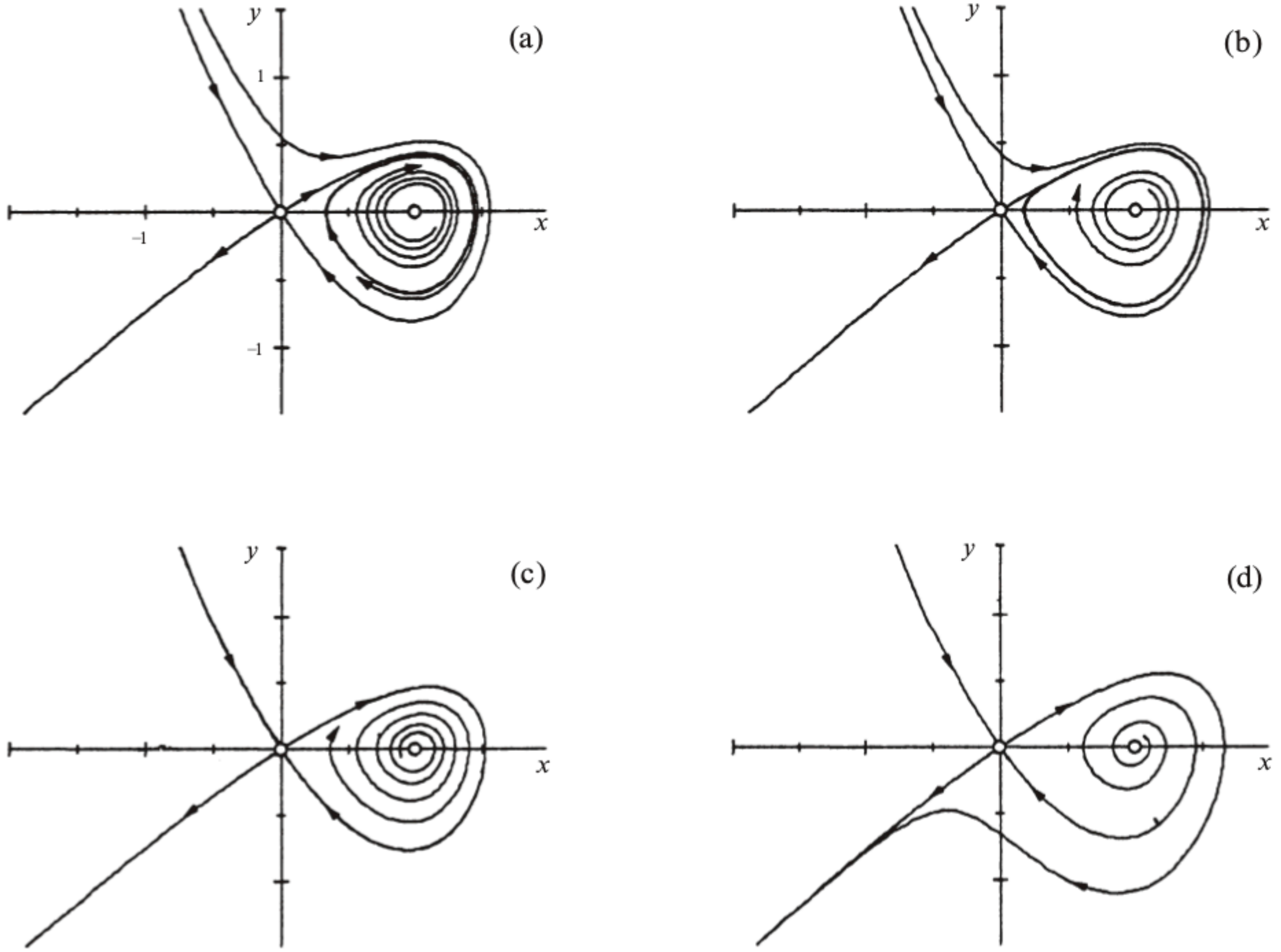
This is another kind of infinite-period bifurcation called **a saddle-loop or homoclinic bifurcation.**

Consider the computer-based analysis of the system:

$$\dot{x} = y$$

$$\dot{y} = \mu y + x - x^2 + xy.$$

Homoclinic Bifurcation

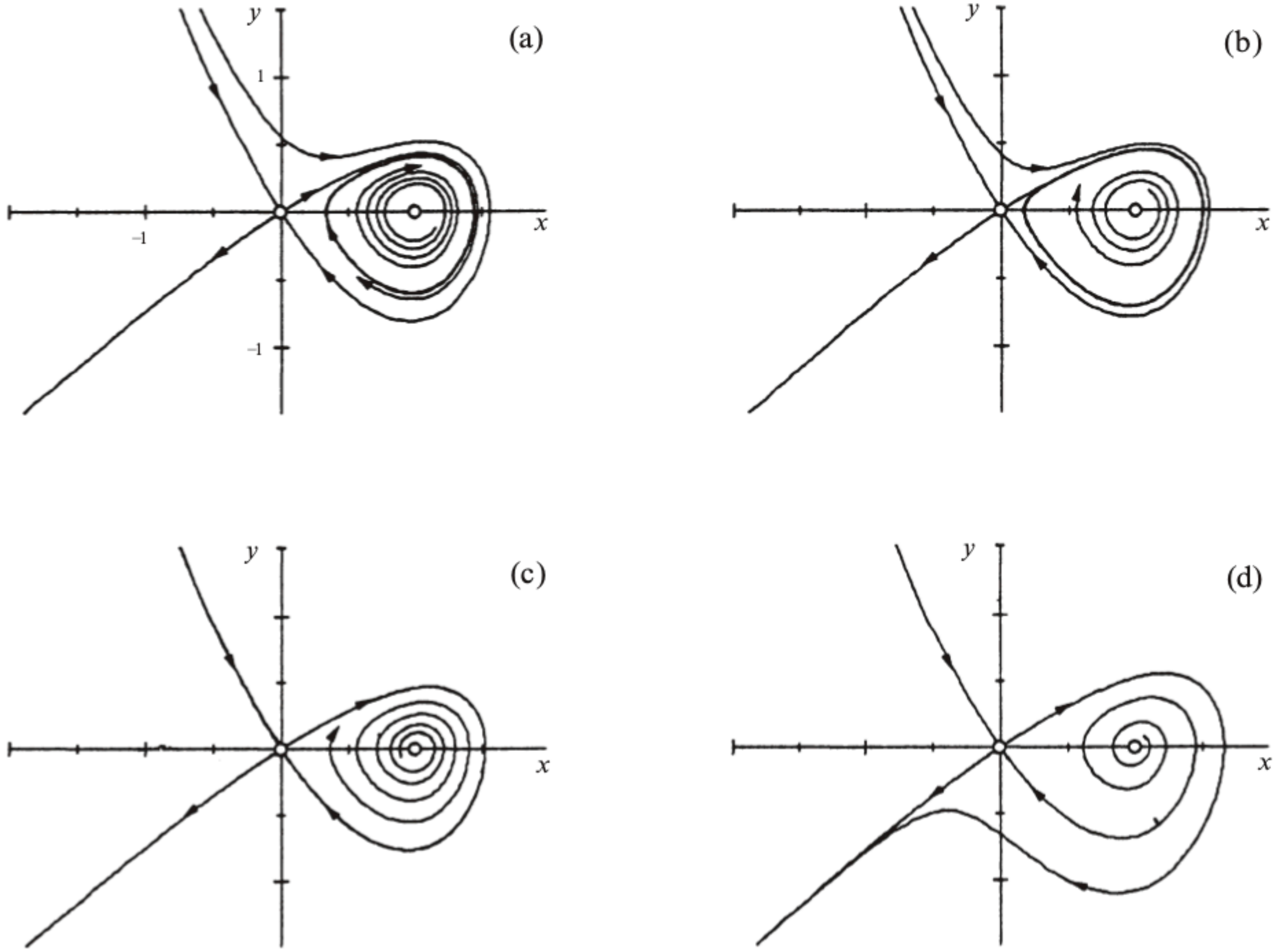


Phase portraits before, during, and after the bifurcation.

The bifurcation is numerically found to occur at $\mu_c \approx -0.8645$

(a) For $\mu < \mu_c$, say $\mu = -0.92$, a stable limit cycle passes close to a saddle point at the origin.

Homoclinic Bifurcation

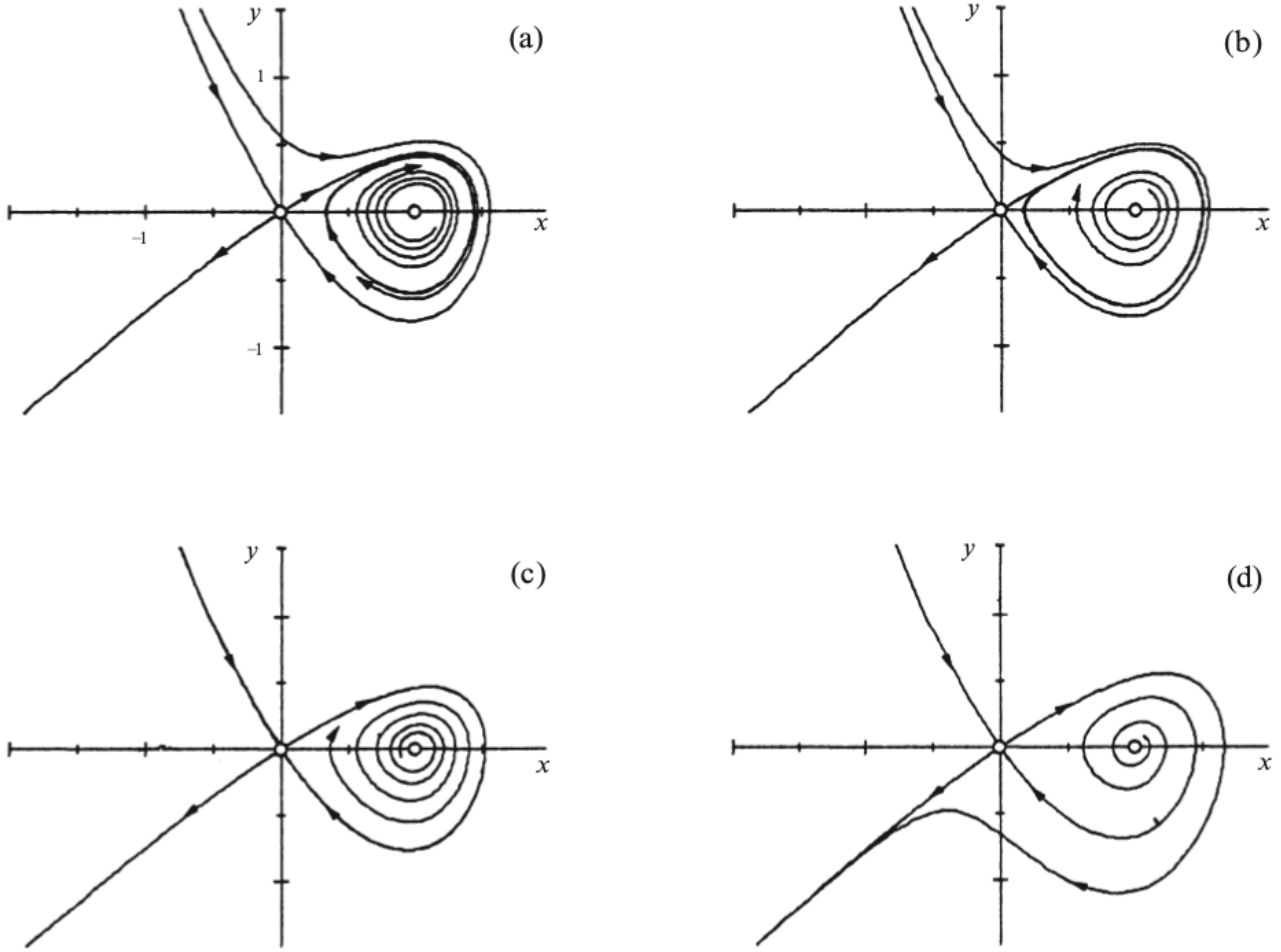


(b) As μ increases to μ_c, the limit cycle swells.

(c) Then, it collides into the saddle, creating a homoclinic orbit.

(d) Once $\mu > \mu_c$, the saddle connection breaks and the loop is destroyed

Homoclinic Bifurcation



The key to this bifurcation is the behaviour of the unstable manifold of the saddle.

After it loops around, it either hits the origin (c) or veers off to one side or the other (d).

The scaling of the period is obtained by estimating the time required for a trajectory to pass by a saddle point.

Scaling Laws

For each of the bifurcations, there are characteristic **scaling laws** that govern the amplitude and period of the limit cycle as the bifurcation is approached.

Let μ denote some dimensionless measure of the distance from the bifurcation, and assume that $\mu \ll 1$.

The generic scaling laws for bifurcations of cycles in 2D systems are given below:

	Amplitude of stable limit cycle	**Period of cycle**
Supercritical Hopf	$O(\mu^{1/2})$	$O(1)$
Saddle-node bifurcation of cycles	$O(1)$	$O(1)$
Infinite-period	$O(1)$	$O(\mu^{-1/2})$
Homoclinic	$O(1)$	$O(\ln \mu)$

Exceptions occur only if there is some symmetry that renders the problem nongeneric.

Scaling Laws

The amplitude of the limit cycle scales $\sqrt{\mu}$. As μ gets smaller (closer to the bifurcation, $\mu \to 0$), the amplitude shrinks to zero at a rate proportional to $\sqrt{\mu}$.

	Amplitude of stable limit cycle	Period of cycle
Supercritical Hopf	$O(\mu^{1/2})$	$O(1)$
Saddle-node bifurcation of cycles	$O(1)$	$O(1)$
Infinite-period	$O(1)$	$O(\mu^{-1/2})$
Homoclinic	$O(1)$	$O(\ln \mu)$

The amplitude of the limit cycle remains of order one (a finite, non-zero constant). As μ gets smaller (closer to the bifurcation, $\mu \to 0$), the amplitude does not shrink to zero but keeps a significant, macroscopic size.

Scaling Laws

In higher-dimensional phase spaces, the corresponding bifurcations obey the same scaling laws, but with two caveats:

(1) Many additional bifurcations of limit cycles become possible, so the table is no longer complete.

(2) The homoclinic bifurcation becomes much more subtle to analyse. It often creates chaotic dynamics in its aftermath.

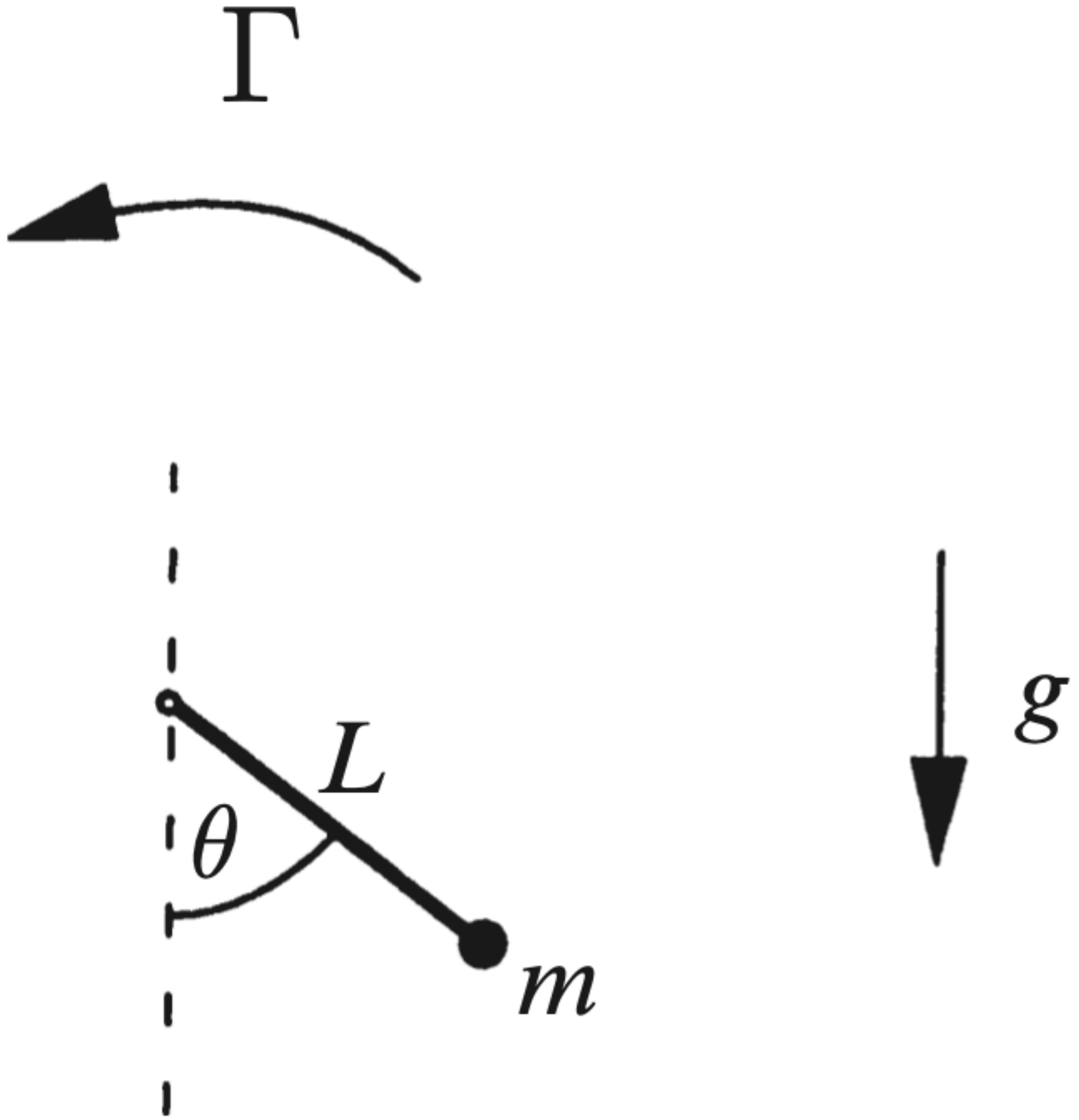
Why should we care about these scaling laws?

Suppose the system you study exhibits a stable limit cycle oscillation. Now suppose you change a control parameter and the oscillation stops.

By examining the scaling of the period and amplitude near this bifurcation, we can learn something about the system’s dynamics. **Possible models can be eliminated or supported.**

Overdamped Pendulum

We now consider a simple mechanical example of a nonuniform oscillator: **an over-damped pendulum driven by a constant torque.**



Let θ denote the angle between the pendulum and the downward vertical, and suppose that θ increases counterclockwise ($\Gamma > 0$).

Newton's law yields:

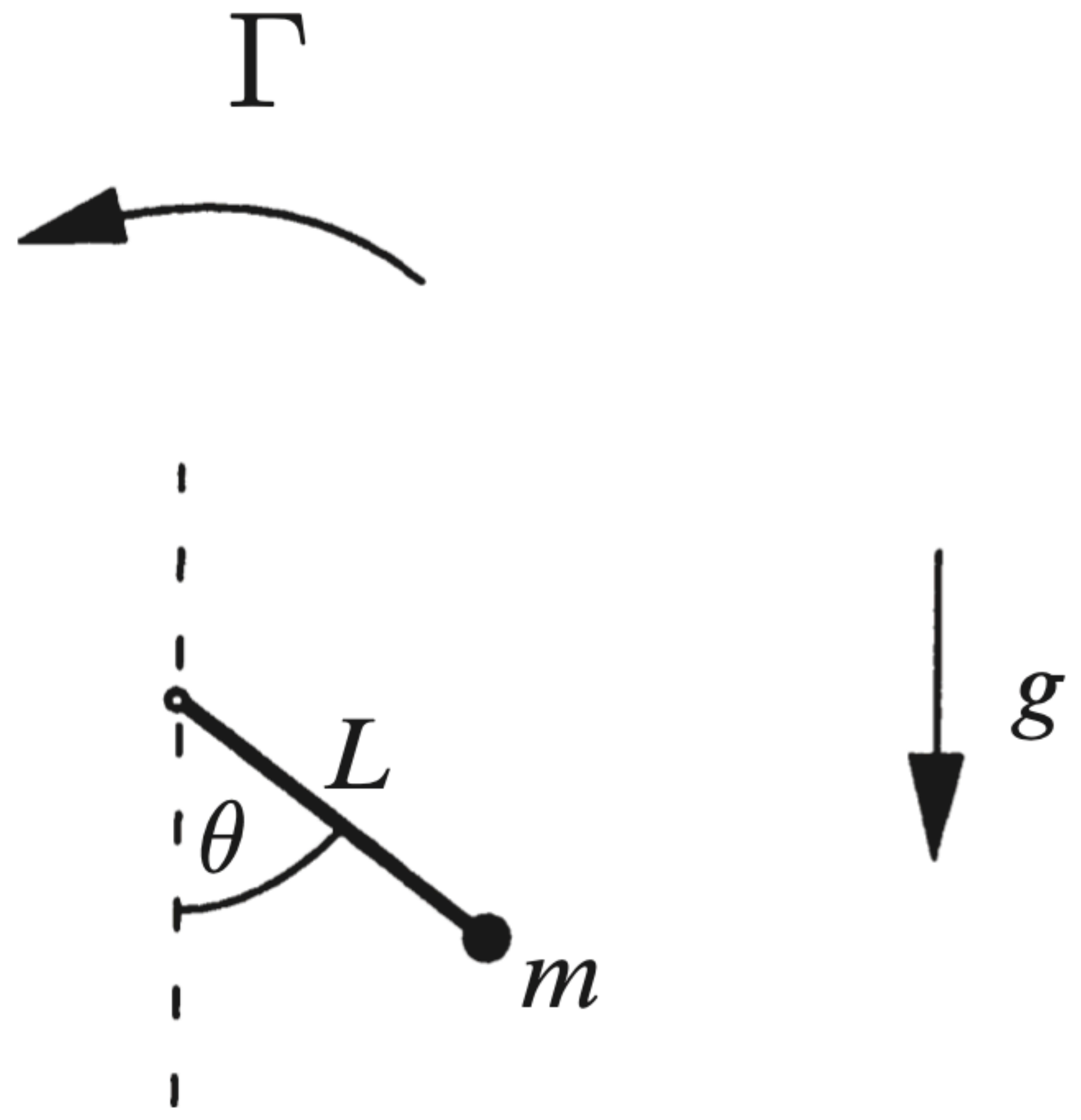
$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

where *m* is the mass and *L* is the length of the pendulum, *b* is a viscous damping constant, *g* is the acceleration due to gravity, and Γ is a constant applied torque.

Overdamped Pendulum

In the overdamped limit of extremely large b, it may be approximated by a first-order system:

$$b\dot{\theta} + mgL\sin\theta = \Gamma$$



We first non-dimensionalise it, dividing by mgL:

$$\frac{b}{mgL}\dot{\theta} = \frac{\Gamma}{mgL} - \sin\theta$$

$$\tau = \frac{mgL}{b}t, \qquad \gamma = \frac{\Gamma}{mgL}$$

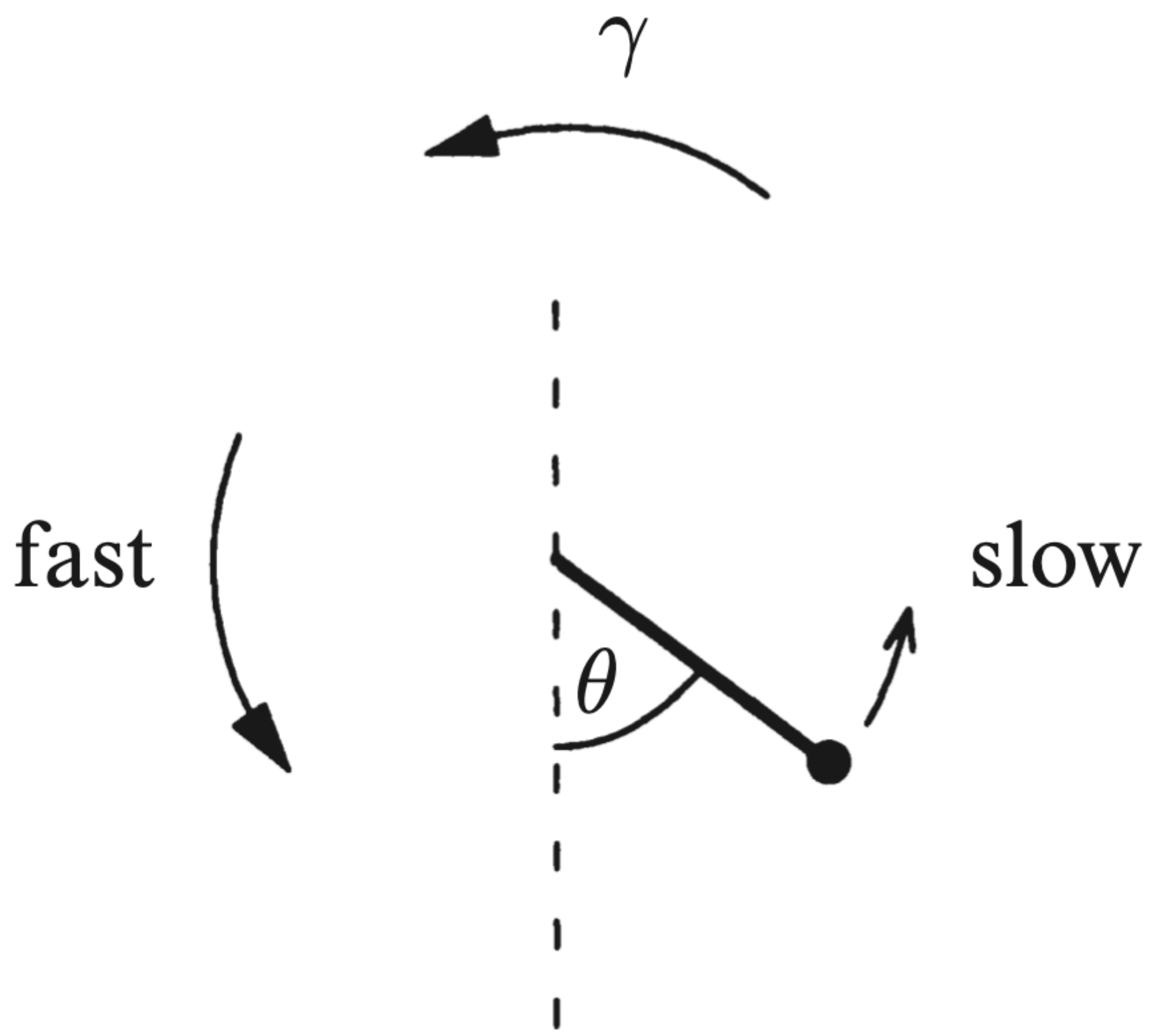
$$\theta' = \gamma - \sin\theta$$ where: $$\theta' = d\theta/d\tau$$

Here, γ is the ratio of the applied torque to the maximum gravitational torque.

Overdamped Pendulum

If $\gamma > 1$ then the applied torque can never be balanced by the gravitational torque and the pendulum will overturn continually.

The rotation rate is nonuniform, since gravity helps the applied torque on one side and opposes it on the other.



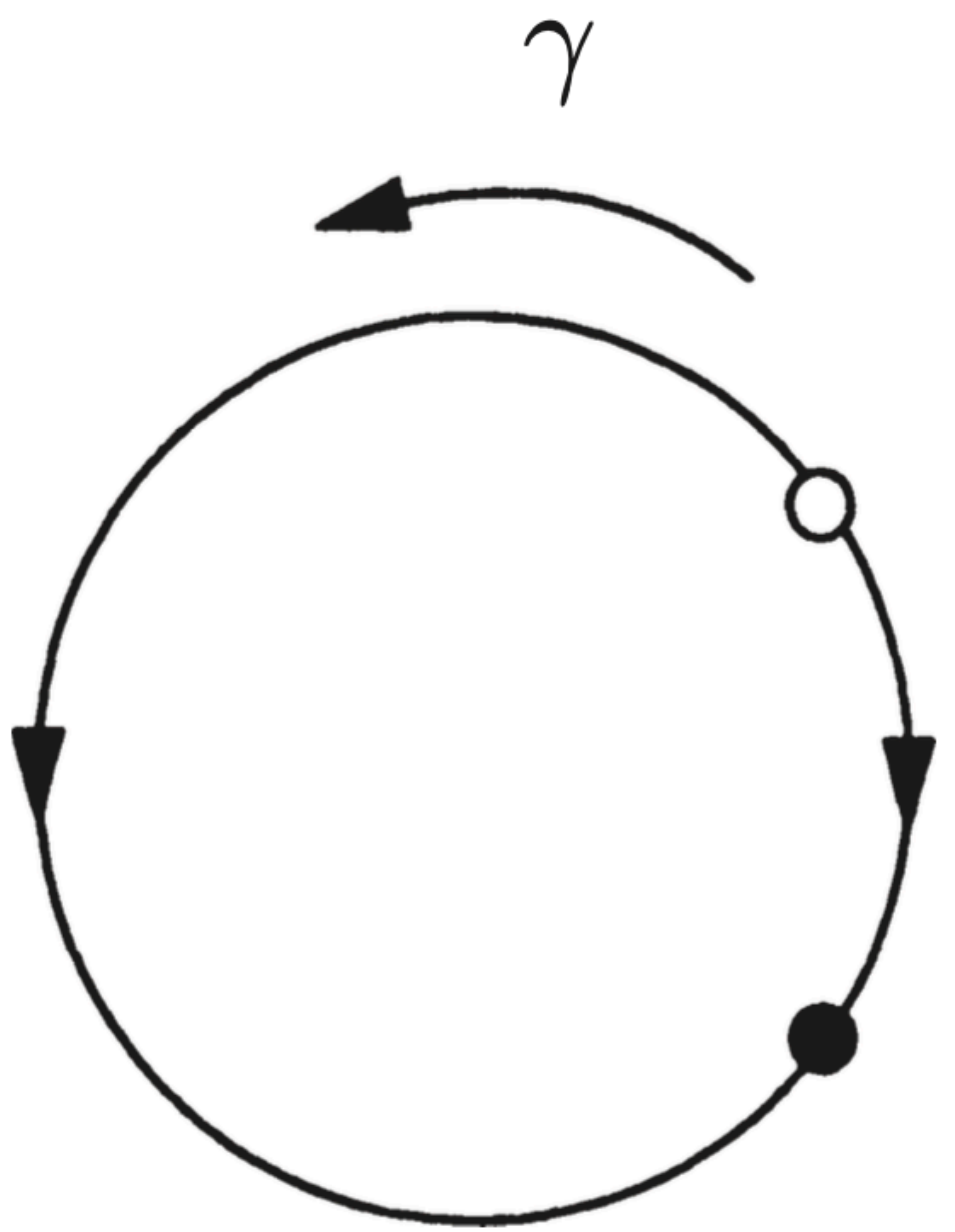
As $\gamma \to 1^+$, the pendulum takes longer and longer to climb past $\pi/2$ on the slow side.

When $\gamma = 1$ a fixed point appears at $\theta^* = \pi/2$, and then splits into two when $\gamma < 1$.

On physical grounds, it's clear that the lower of the two equilibrium positions is the stable one.

Overdamped Pendulum

As γ decreases, the two fixed points move farther apart.



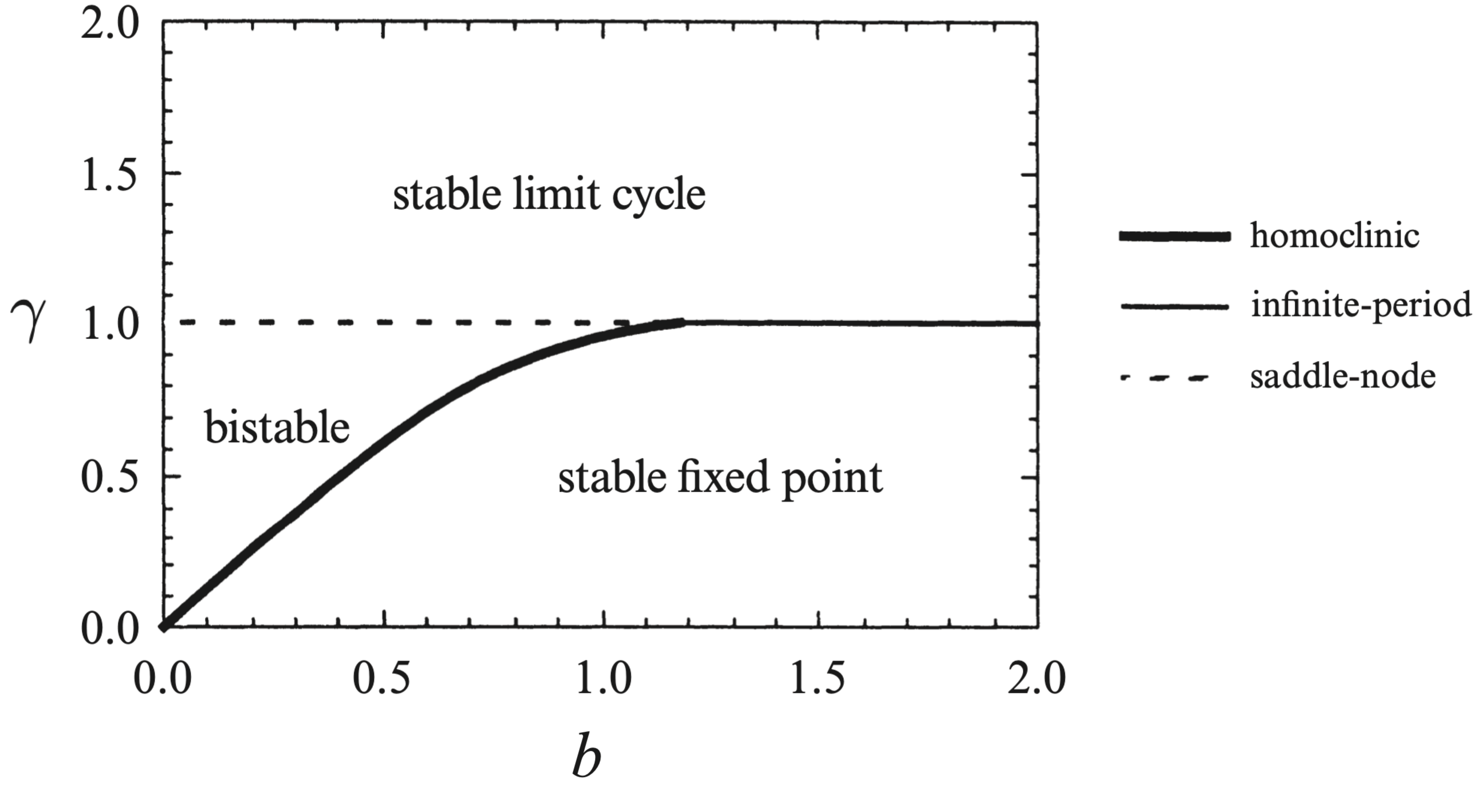
When $\gamma = 0$, the applied torque vanishes and there is an unstable equilibrium at the top (inverted pendulum) and a stable equilibrium at the bottom.

We are dealing with a physical problem in which both **homoclinic** and **infinite-period bifurcations** arise.

For weak damping the pendulum can exhibit intriguing **hysteresis effects**, thanks to the coexistence of a stable limit cycle and a stable fixed point.

Hysteresis in the Driven Pendulum

In physical terms, the pendulum can settle into either a rotating solution where it whirls over the top, or a stable rest state where gravity balances the applied torque.

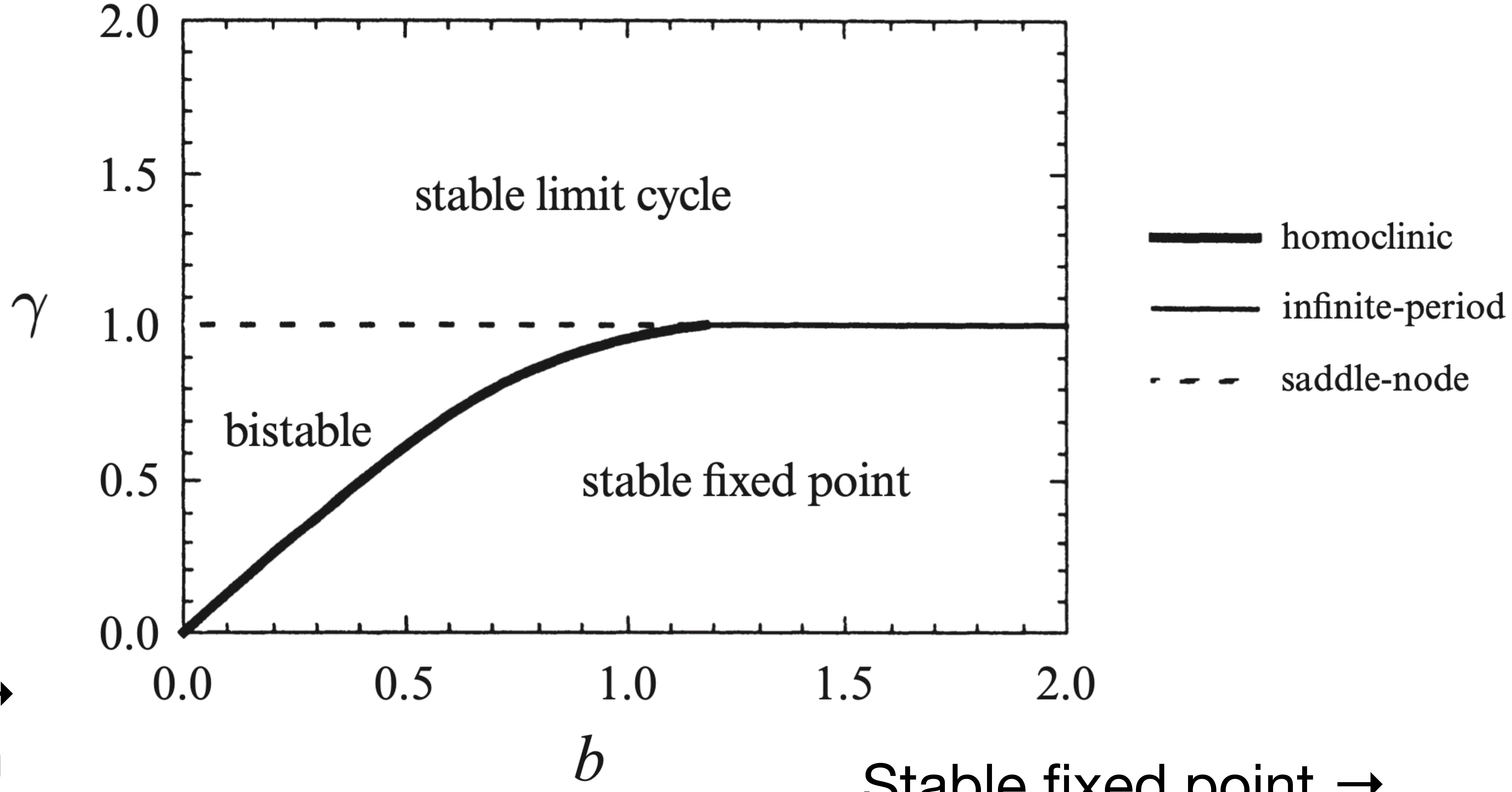


The final state depends on the initial conditions. Our goal now is to understand how this bistability comes about.

Hysteresis in the Driven Pendulum

stable limit cycle → pendulum rotates forever

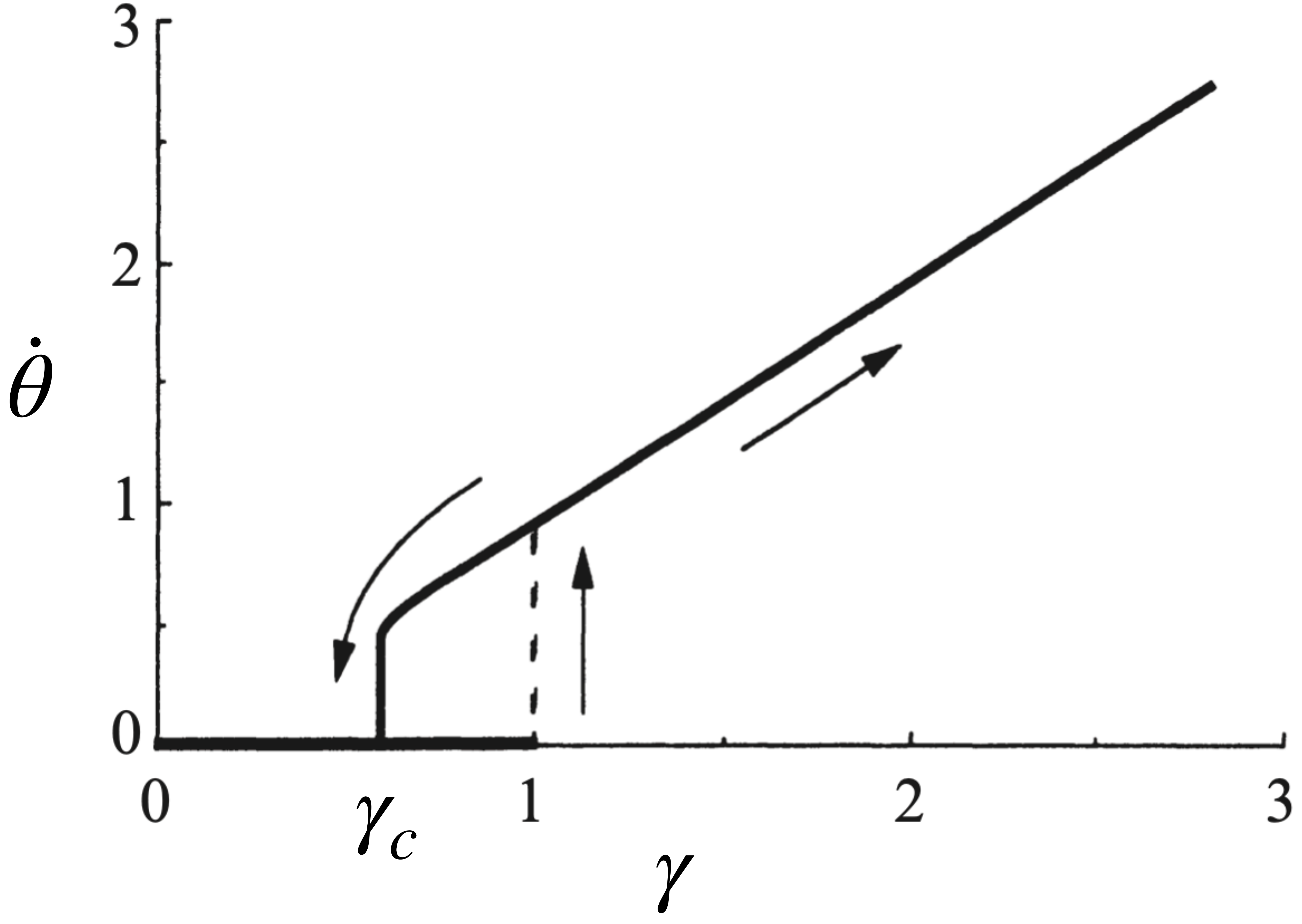
saddle-node bifurcation → point where the pendulum is forced to rotate



homoclinic curve → boundary between small & large swings

Stable fixed point → pendulum stays still

Hysteresis in the Driven Pendulum



This is the curve relating the rotation rate to the applied torque.

Coupled Oscillators and Quasiperiodicity

Besides the plane and the cylinder, another important two-dimensional phase space is the **torus**. It is the natural phase space for systems of the form:

$$\dot{\theta}_1 = f_1(\theta_1, \theta_2)$$
$$\dot{\theta}_2 = f_2(\theta_1, \theta_2)$$

where f_1 and f_2 are periodic in both arguments.

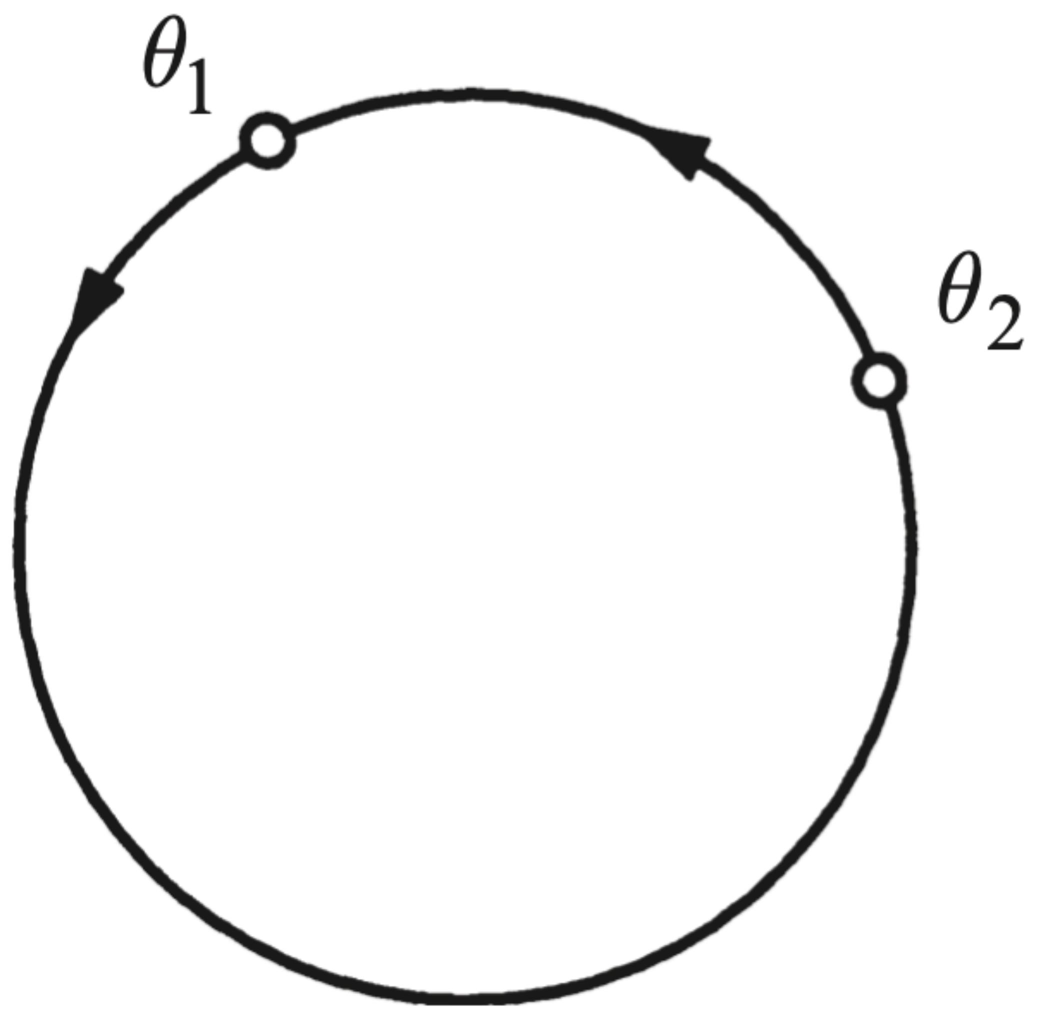
A simple model of coupled oscillators is given by:

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

where θ_1, θ_2 are the phases of the oscillators, $\omega_1, \omega_2 > 0$ are their natural frequencies, and $K_1, K_2 > 0$ are coupling constants.

Coupled Oscillators and Quasiperiodicity

We can imagine to particles moving along the circle:



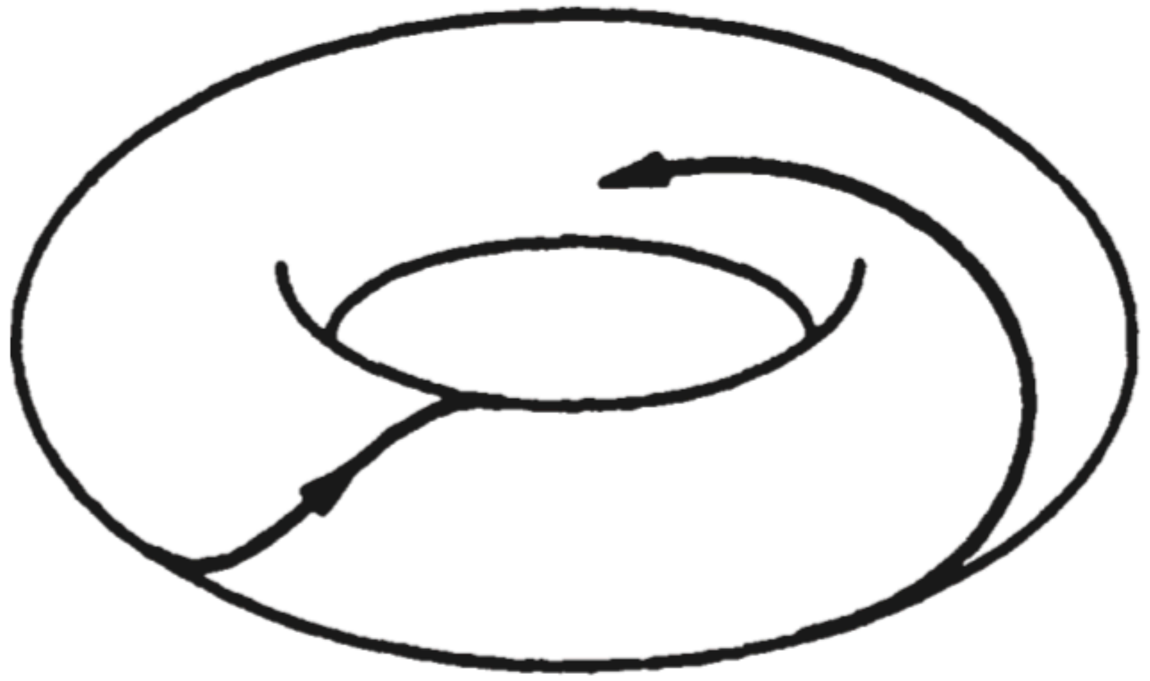
If they were uncoupled, then each would run at his or her preferred speed and the faster one would periodically overtake the slower one.

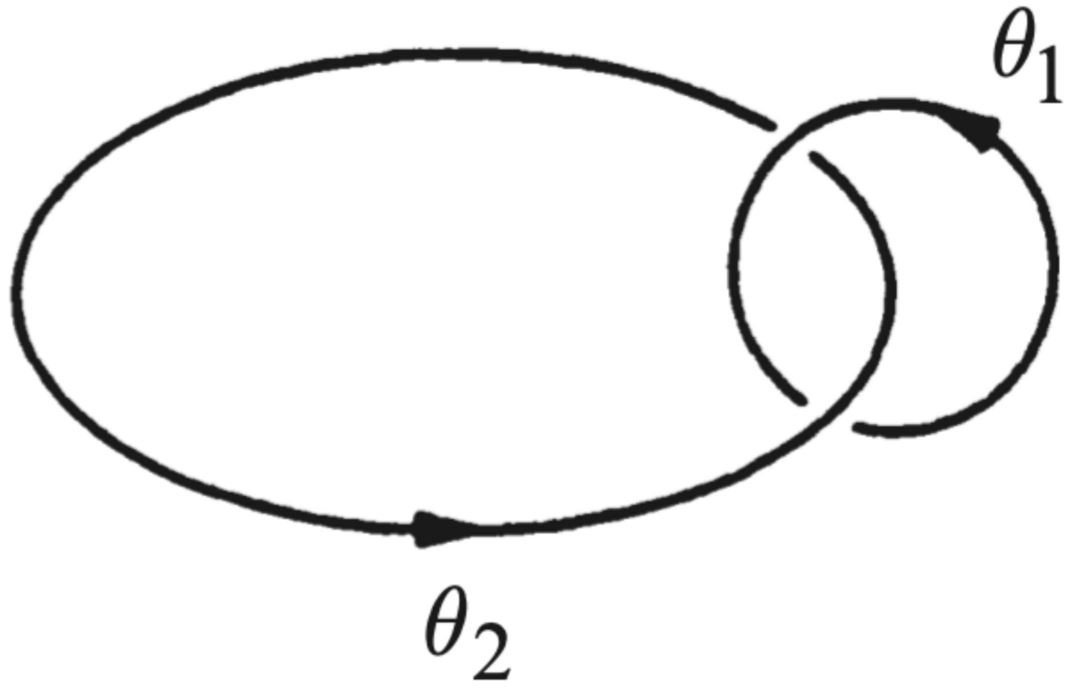
So they need to compromise, with each adjusting his or her speed as necessary. If their preferred speeds are too different, phase-locking will be impossible.

To visualise the flow, imagine 2 points running around a circle at instantaneous rates: $\dot{\theta}_1$ and $\dot{\theta}_2$

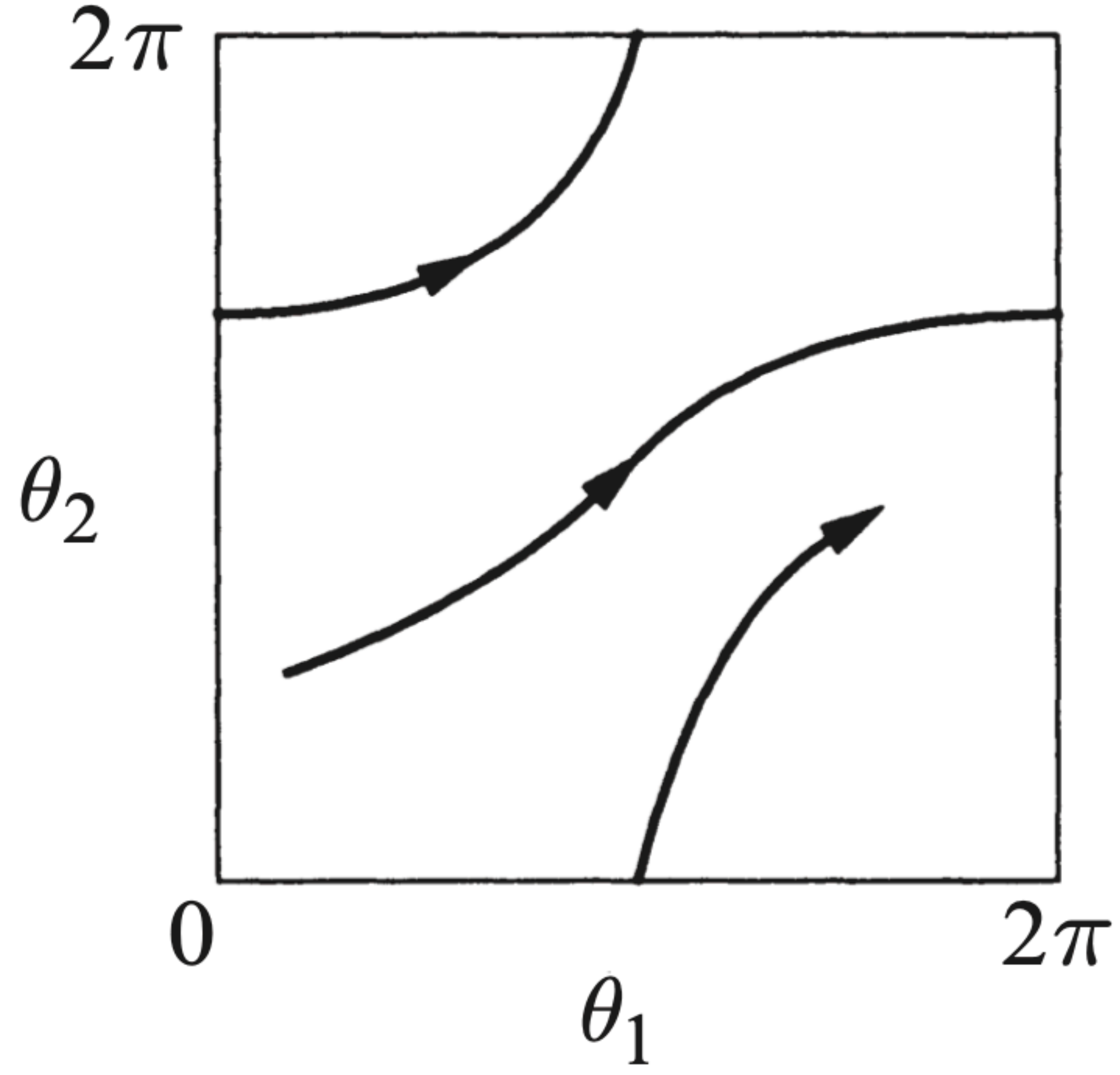
Coupled Oscillators and Quasiperiodicity

Alternatively, we could imagine a *single* point tracing out a trajectory on a torus with coordinates θ_1 and θ_2, analogous to latitude and longitude





coordinate system

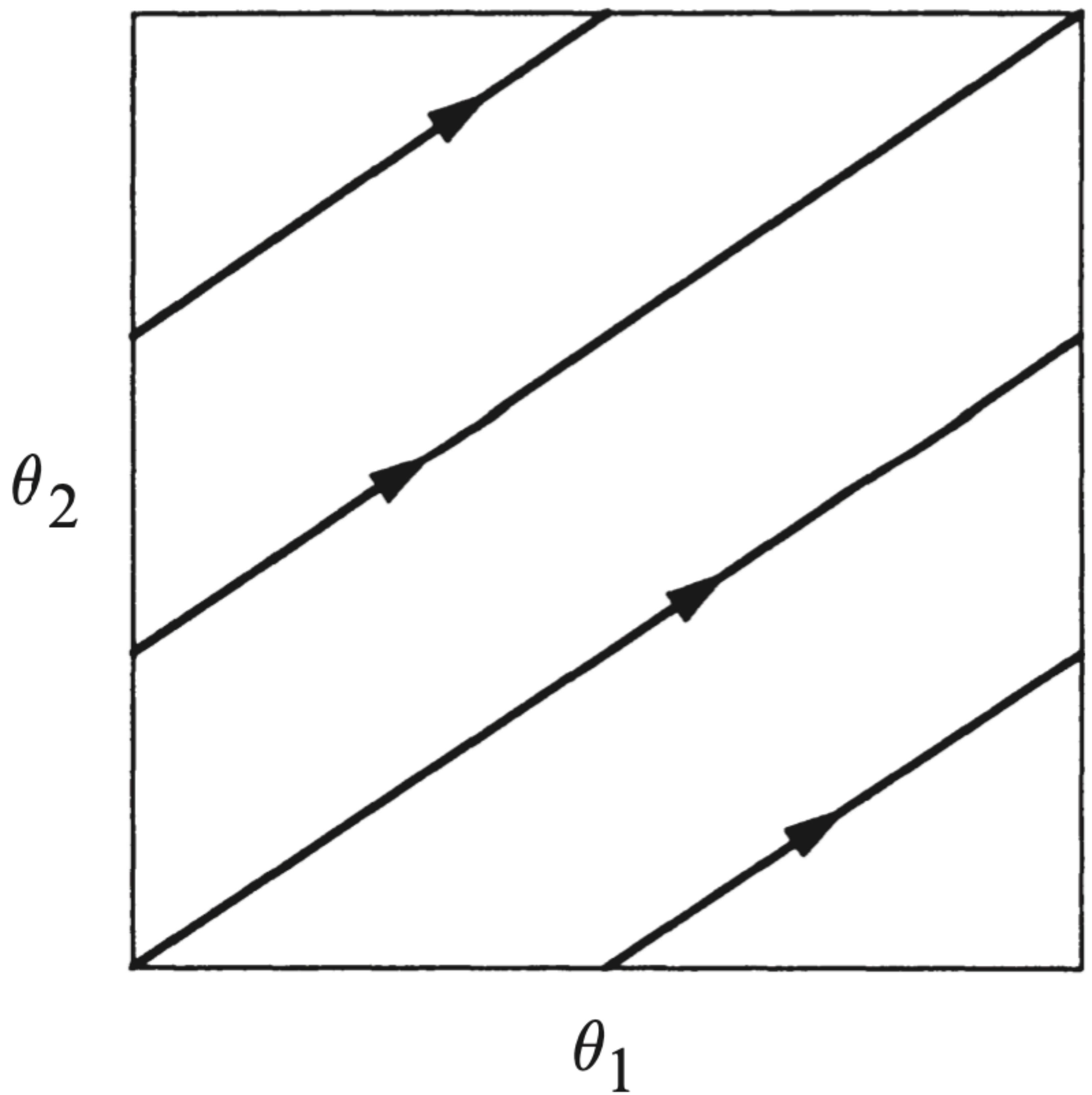


Since the curved surface of a torus makes it hard to draw phase portraits, we prefer to use an equivalent representation: **a square with periodic boundary conditions.**

Coupled Oscillators and Quasiperiodicity

Uncoupled System

$$K_1, K_2 = 0 \quad \Rightarrow \quad \dot{\theta}_1 = \omega_1, \;\; \dot{\theta}_2 = \omega_2$$



The corresponding trajectories on the square are straight lines with constant slope:

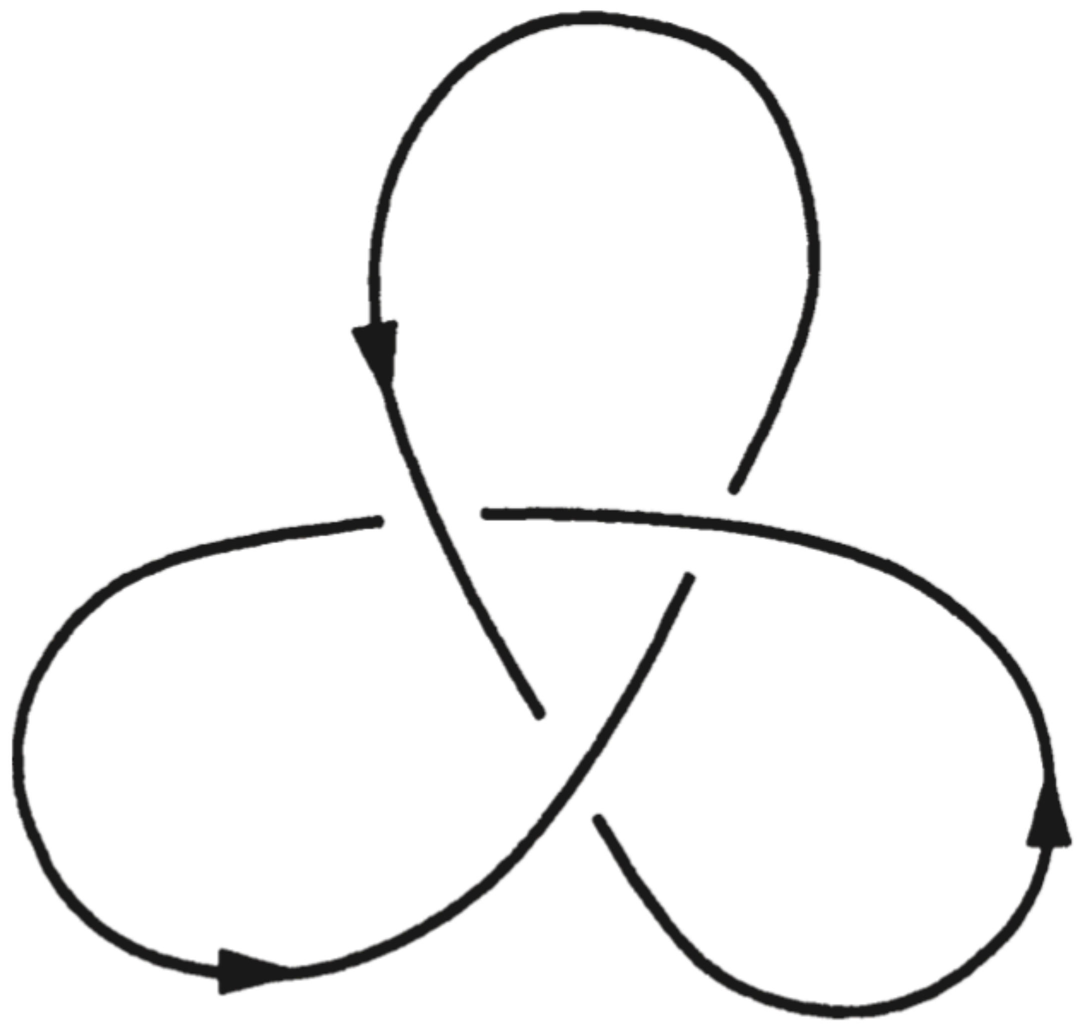
$$d\theta_2 / d\theta_1 = \omega_2 / \omega_1$$

If the slope is **rational**, then $\omega_1/\omega_2 = p/q$ for some integers *p*, *q* with no common factors.

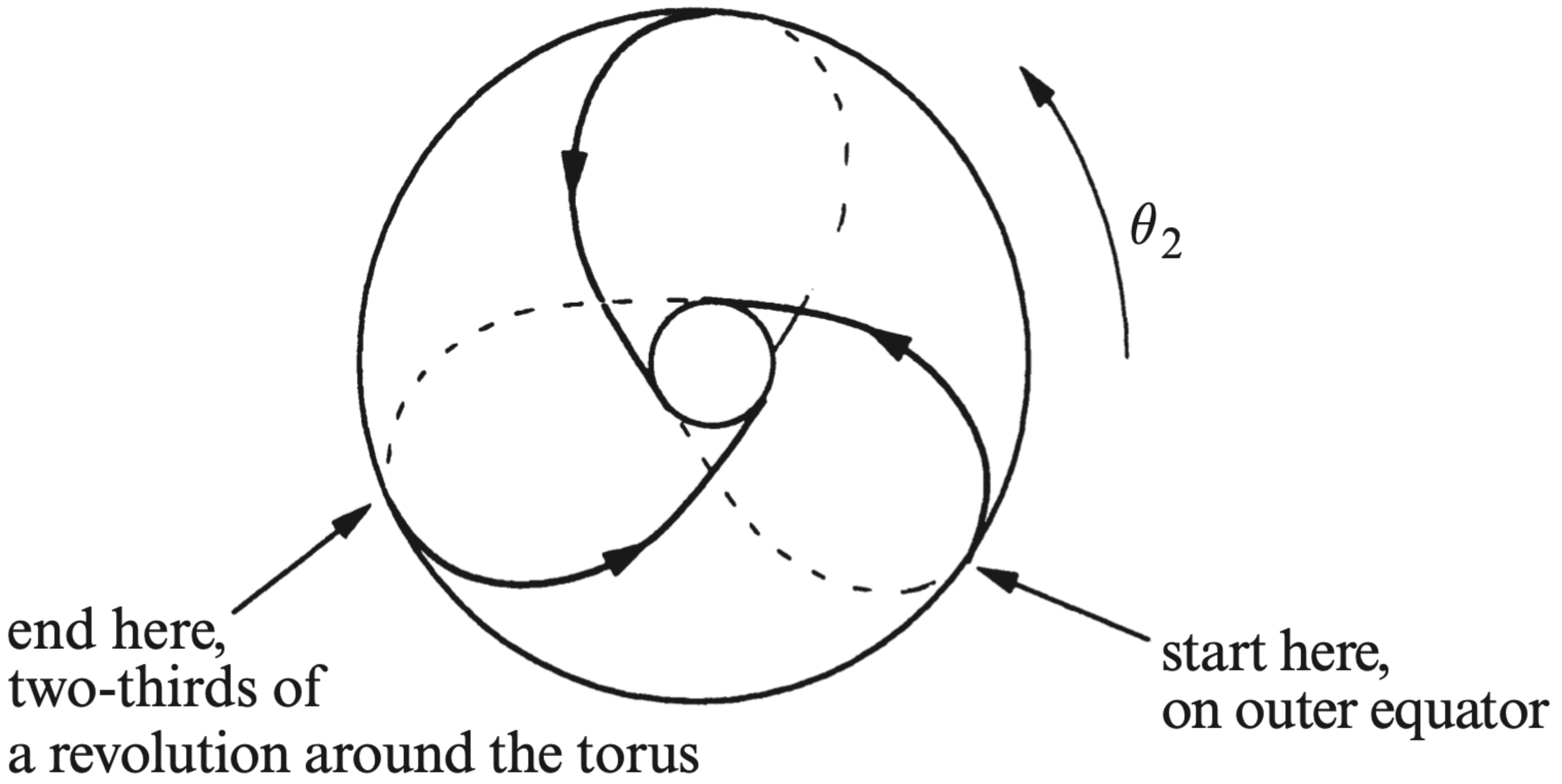
All trajectories are closed orbits on the torus, because θ_1 completes p revolutions in the same time that θ_2 completes q revolutions.

Coupled Oscillators and Quasiperiodicity

When plotted on the torus, the same trajectory gives . . . **a trefoil knot**



trefoil knot

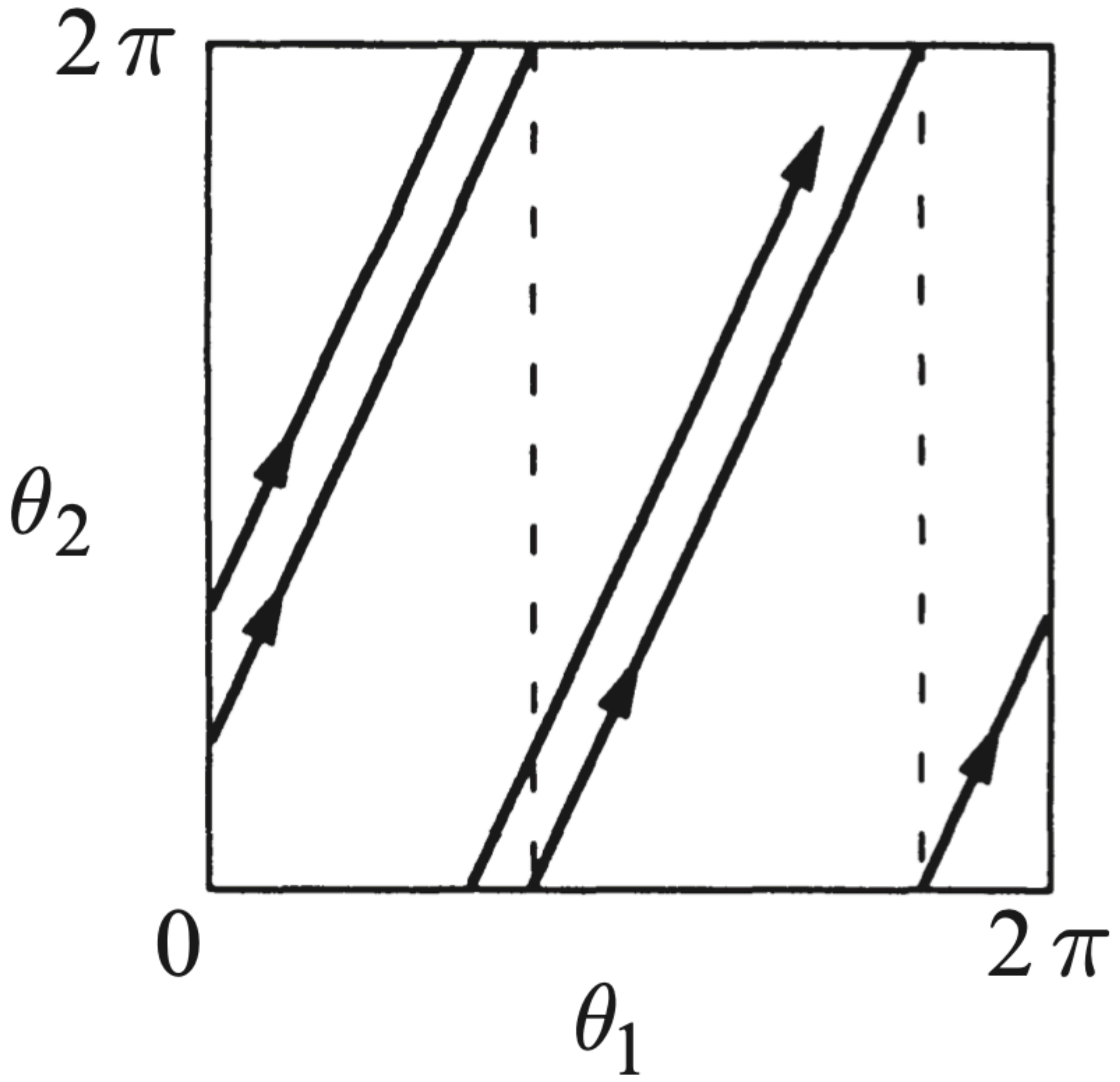


θ_2 makes *2/3* of a revolution while θ_1 makes 1 revolution; hence *p*=3, *q*=2.

The trajectories are always knotted if $p, q > 2$ have no common factors. **The resulting curves are called p:q torus knots.**

Coupled Oscillators and Quasiperiodicity

Coupled System $d\theta_2/d\theta_1 = \omega_2/\omega_1$



The second possibility is that the slope is **irrational**. Then the flow is said to be **quasiperiodic**.

Every trajectory winds around endlessly on the torus, never intersecting itself and yet never quite closing.

Each trajectory is **dense** on the torus: each trajectory comes arbitrarily close to any given point on the torus.

Quasiperiodicity is significant because it is a new type of long-term behaviour, that occurs only on the torus.

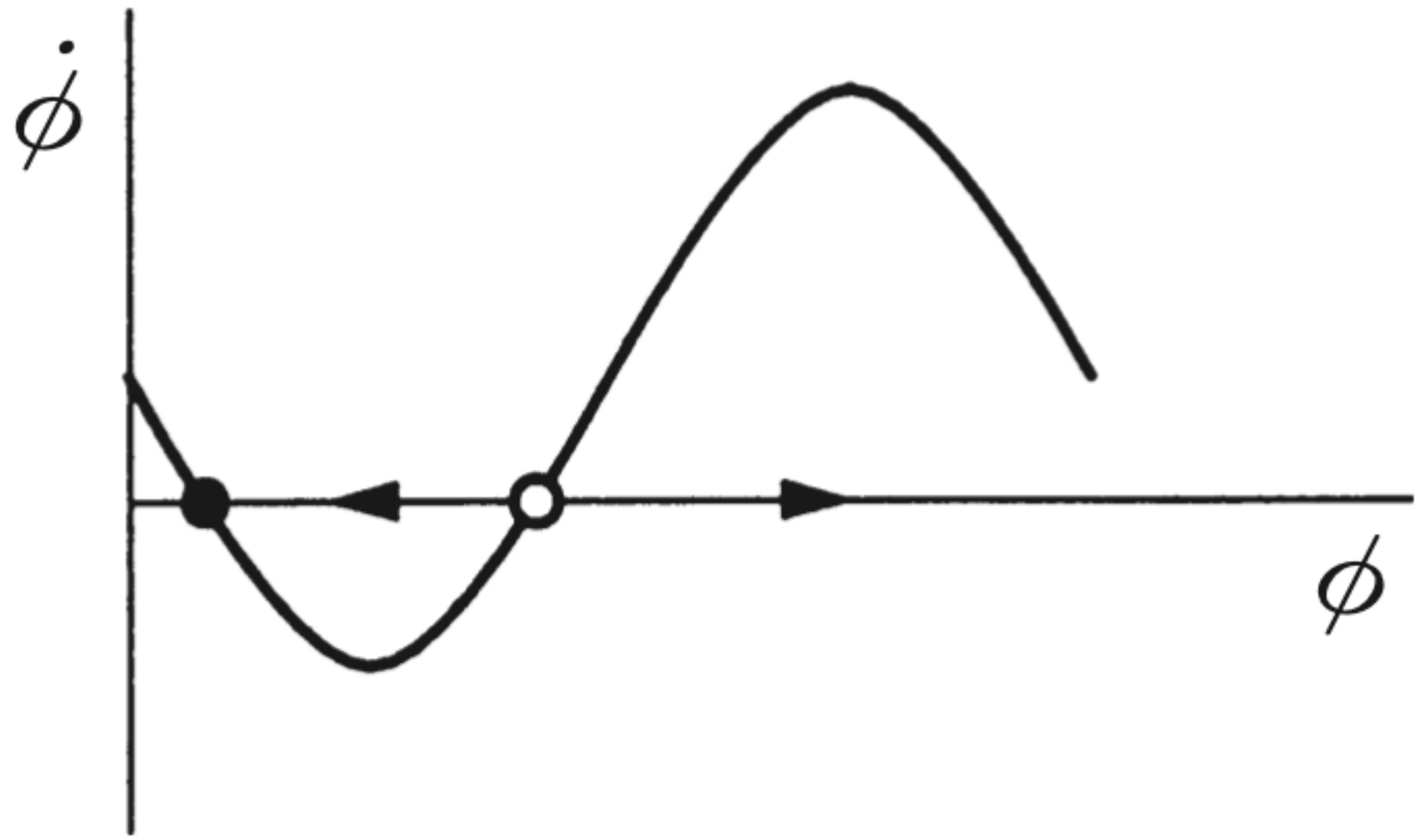
Coupled Oscillators and Quasiperiodicity

Coupled System

$$K_1, K_2 > 0$$

The dynamics can be deciphered by looking at the **phase difference** $\phi = \theta_1 - \theta_2$. Then:

$$\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2$$
$$= \omega_1 - \omega_2 - (K_1 + K_2)\sin\phi$$



which is just the **nonuniform oscillator.**

Coupled Oscillators and Quasiperiodicity

Coupled System

There are 2 fixed points:

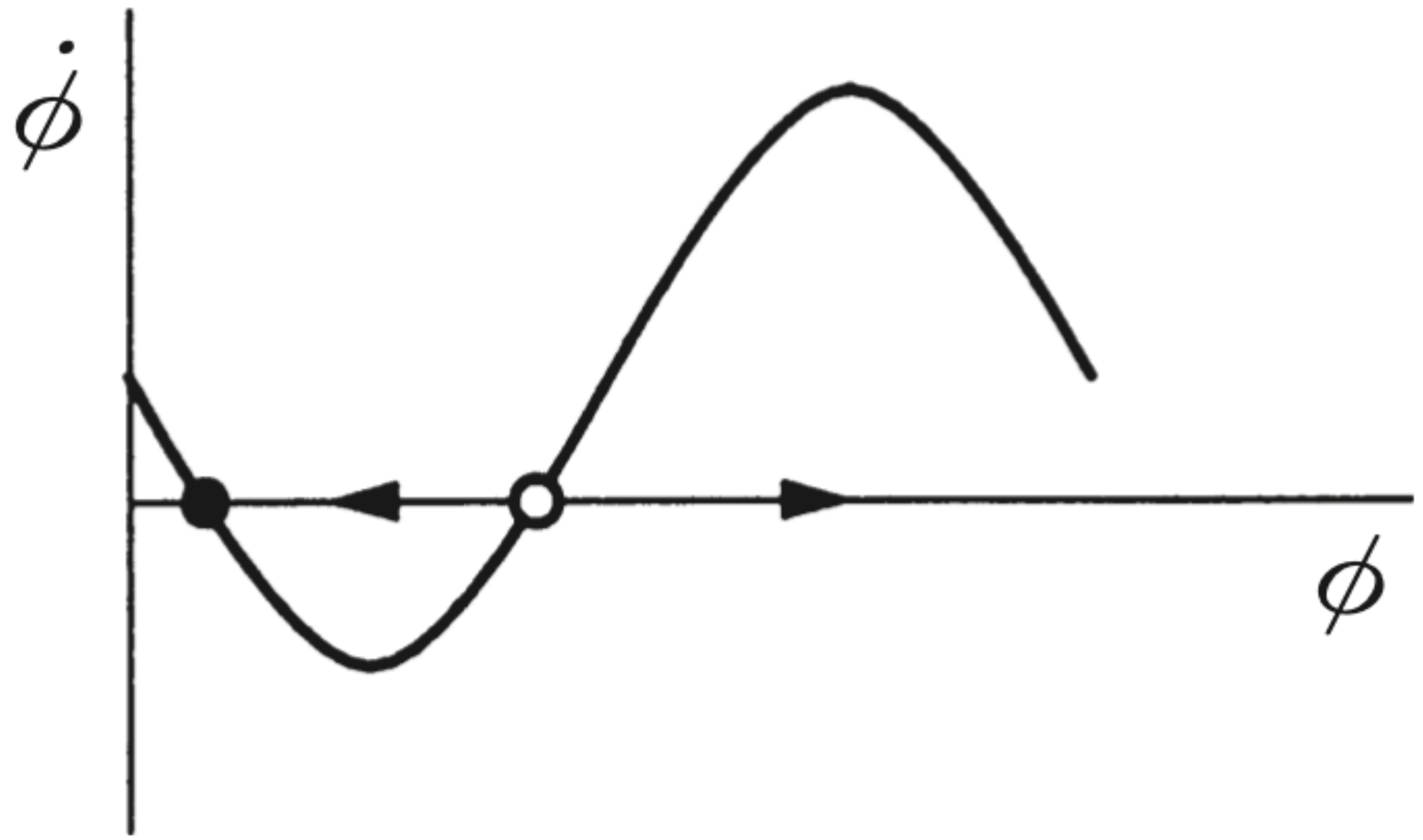
$$|\omega_1 - \omega_2| < K_1 + K_2$$

Or none if:

$$|\omega_1 - \omega_2| > K_1 + K_2$$

A **saddle-node bifurcation** occurs when:

$$|\omega_1 - \omega_2| = K_1 + K_2$$



Suppose for now that there are **2 fixed points,** defined implicitly by:

$$\sin \phi^* = \frac{\omega_1 - \omega_2}{K_1 + K_2}$$

Coupled Oscillators and Quasiperiodicity

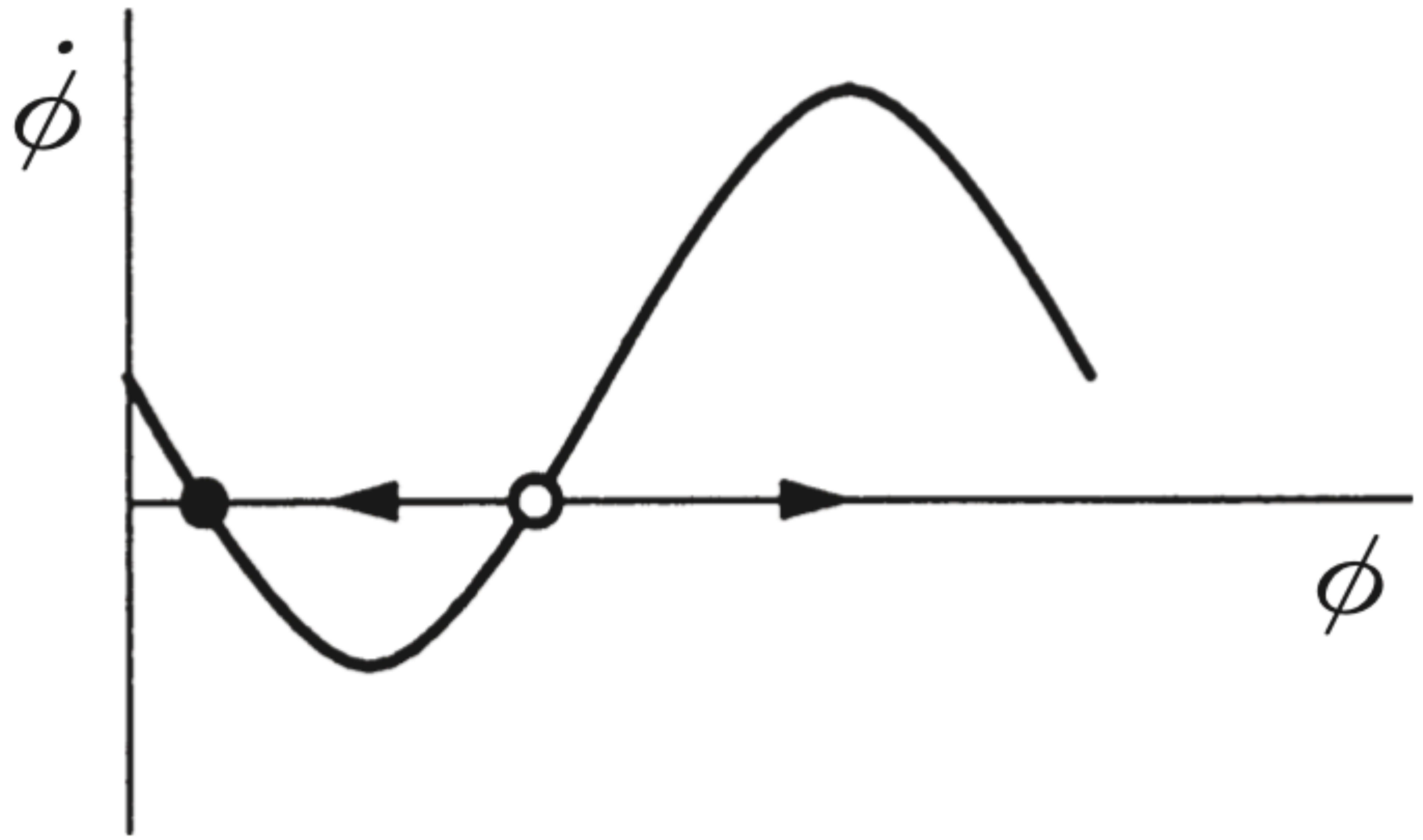
$$\sin\phi^* = \frac{\omega_1 - \omega_2}{K_1 + K_2}$$

Coupled System

All trajectories asymptotically approach the stable fixed point.

Back on the torus, the trajectories approach a stable **phase-locked** solution: the oscillators are separated by a **constant phase difference:** ϕ^*

The phase-locked solution is **periodic:** both oscillators run at a constant frequency given by:



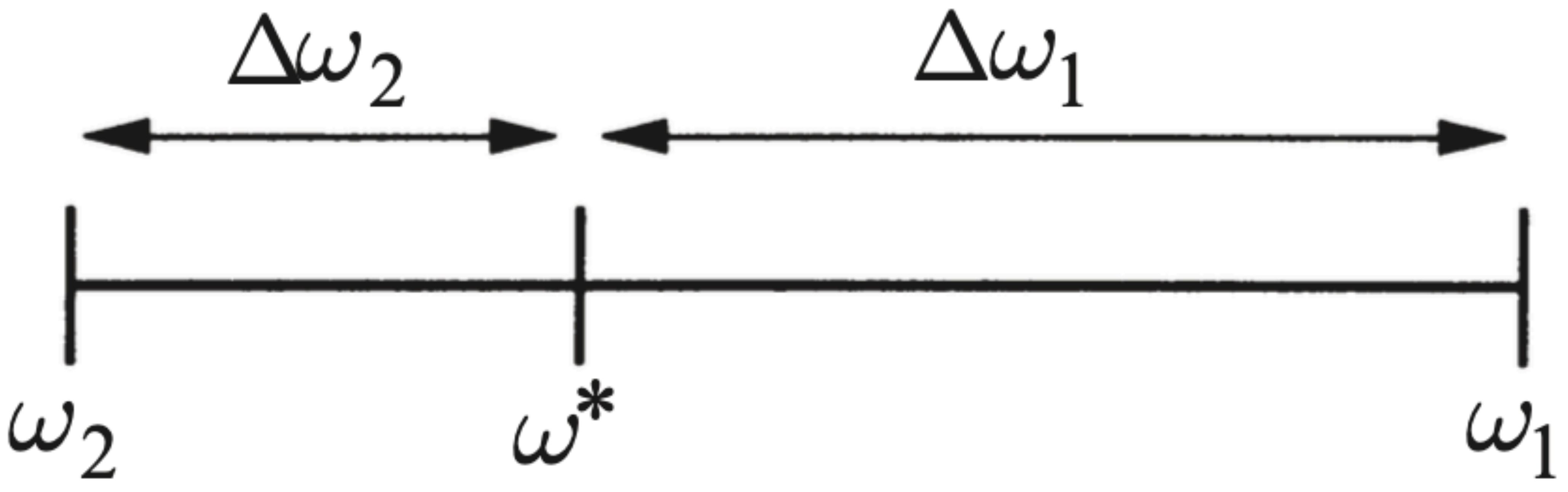
$$\omega^* = \dot{\theta}_1 = \dot{\theta}_2 = \omega_2 + K_2 \sin\phi^* \quad \Rightarrow \quad \omega^* = \frac{K_1\omega_2 + K_2\omega_1}{K_1 + K_2}$$

Coupled Oscillators and Quasiperiodicity

Coupled System

$$\omega^* = \frac{K_1\omega_2 + K_2\omega_1}{K_1 + K_2}$$

This is called the **compromise frequency** because it lies between the natural frequencies of the two oscillators.



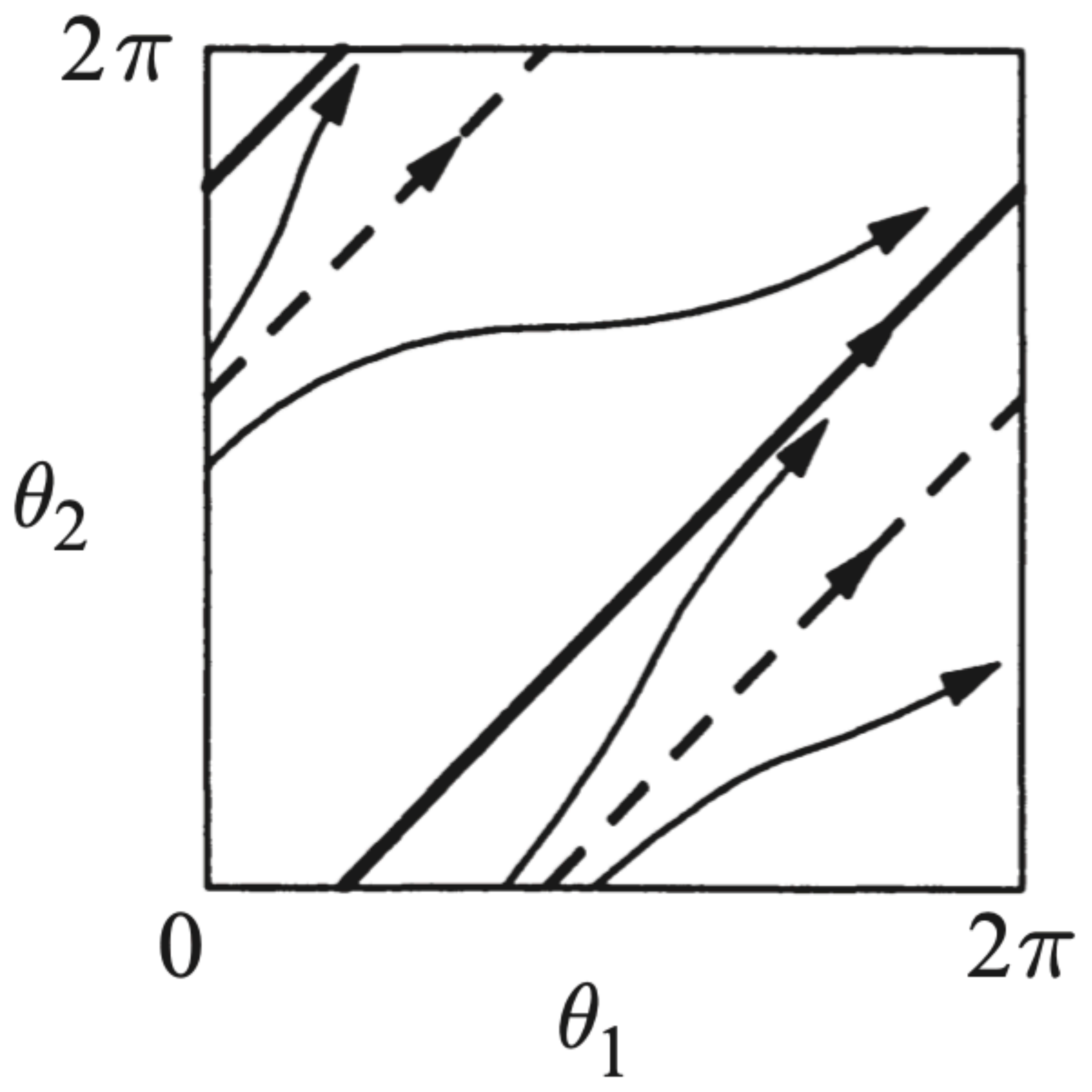
The compromise is not generally halfway.

The frequencies are shifted by an amount proportional to the coupling strengths, as shown by the identity:

$$\left|\frac{\Delta\omega_1}{\Delta\omega_2}\right| \equiv \left|\frac{\omega_1 - \omega^*}{\omega_2 - \omega^*}\right| = \left|\frac{K_1}{K_2}\right|$$

Coupled Oscillators and Quasiperiodicity

Phase portrait on the torus



The stable and unstable locked solutions appear as diagonal lines of slope 1:

$$\dot{\theta}_1 = \dot{\theta}_2 = \omega^*$$

If we pull the natural frequencies apart by detuning an oscillators, then the locked solutions approach each other and coalesce when:

$$|\omega_1 - \omega_2| = K_1 + K_2$$

The locked solution is destroyed in a **saddle-node bifurcation of cycles.**

After the bifurcation, the flow becomes the uncoupled case with either quasiperiodic or rational flow with curvy trajectories.

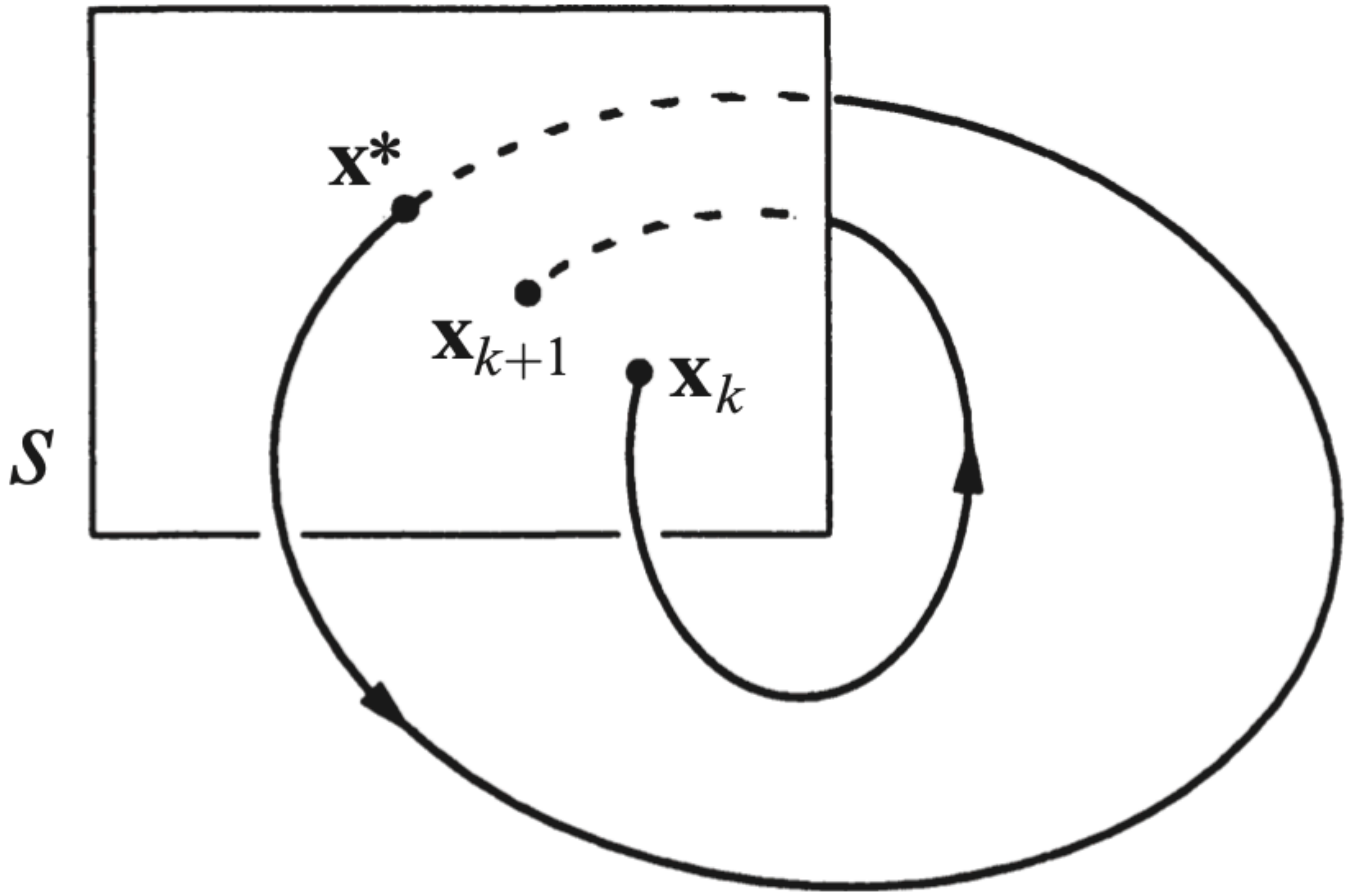
Poincaré Maps

Poincaré maps are useful for studying swirling flows, such as the flow near a periodic orbit (or as we'll see later, the flow in some chaotic systems).

Consider an n-dimensional system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Let S be an n-1 dimensional **surface of section.**

S is required to be transverse to the flow, i.e., all trajectories starting on S, flow through it, not parallel to it.

The **Poincaré map** *P* is a mapping from *S* to itself, obtained by following trajectories from one intersection with *S* to the next.

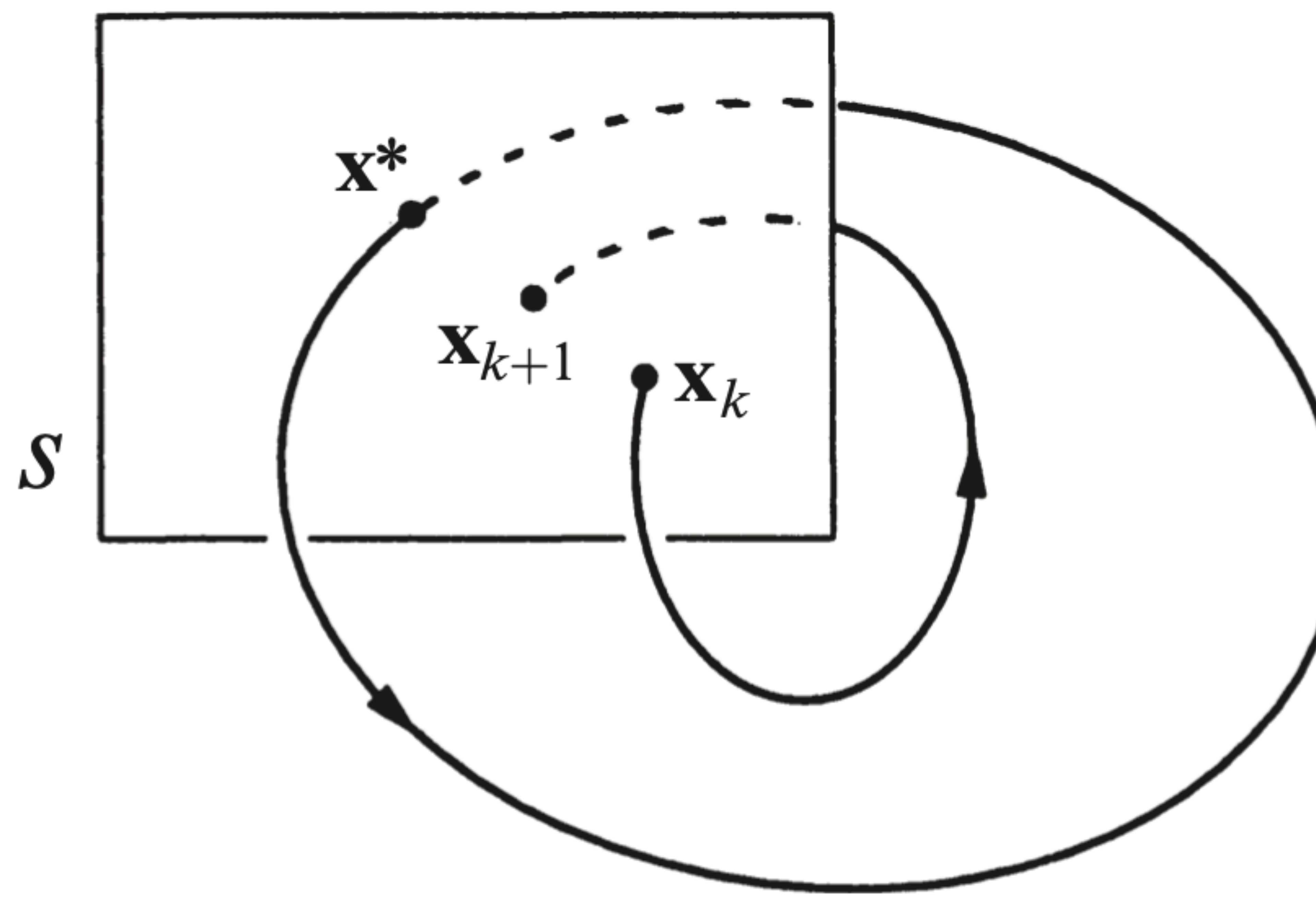


Poincaré Maps

If $\mathbf{x}_k \in S$ denotes the kth intersection, then the Poincaré map is defined by:

$$\mathbf{x}_{k+1} = P(\mathbf{x}_k).$$

Suppose that **x*** is a **fixed point** of P, i.e., P(**x***)=**x***. Then a trajectory starting at **x*** returns to **x*** after some time T, and is therefore a closed orbit for the original system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.



By looking at the behaviour of P near this fixed point, we can determine the stability of the closed orbit.

The Poincaré map converts problems about closed orbits into problems about fixed points of a mapping.

Example: Poincaré Maps

A sinusoidally forced RC-circuit can be written in dimensionless form as: $\dot{x} + x = A\sin\omega t$. Using a Poincaré map, show that this system has a unique, globally stable limit cycle.

We need to show that the map is a global contraction, which guarantees the global stability of the fixed point and the corresponding limit cycle

Here we introduce $\theta = \omega t$ and regard the system as a vector field on a cylinder:

$$\dot{\theta} = \omega$$

$$\dot{x} + x = A\sin\theta$$

Any vertical line on the cylinder is an appropriate section S with initial conditions:

$$S = \{(\theta, x)\colon \theta = 0 \bmod 2\pi\} \qquad \theta(0) = 0,\ x(0) = x_0$$

Example: Poincaré Maps

The time of flight between successive intersections is $t = 2\pi/\omega$. In physical terms, we strobe the system once per drive cycle and look at the consecutive values of x.

To compute *P*, we need to solve the differential equation. Its general solution is a sum of homogeneous and particular solutions:

$$x(t) = c_1 e^{-t} + c_2 \sin \omega t + c_3 \cos \omega t.$$

The constants c_2 and c_3 can be found explicitly, but the important point is that they depend on *A* and but *not* on the initial condition x_0; only c_1, depends on x_0:

$$t = 0 \quad \Rightarrow \quad x = x_0 = c_1 + c_3$$

Example: Poincaré Maps

Therefore:

$$x(t) = (x_0 - c_3)e^{-t} + c_2 \sin \omega t + c_3 \cos \omega t.$$

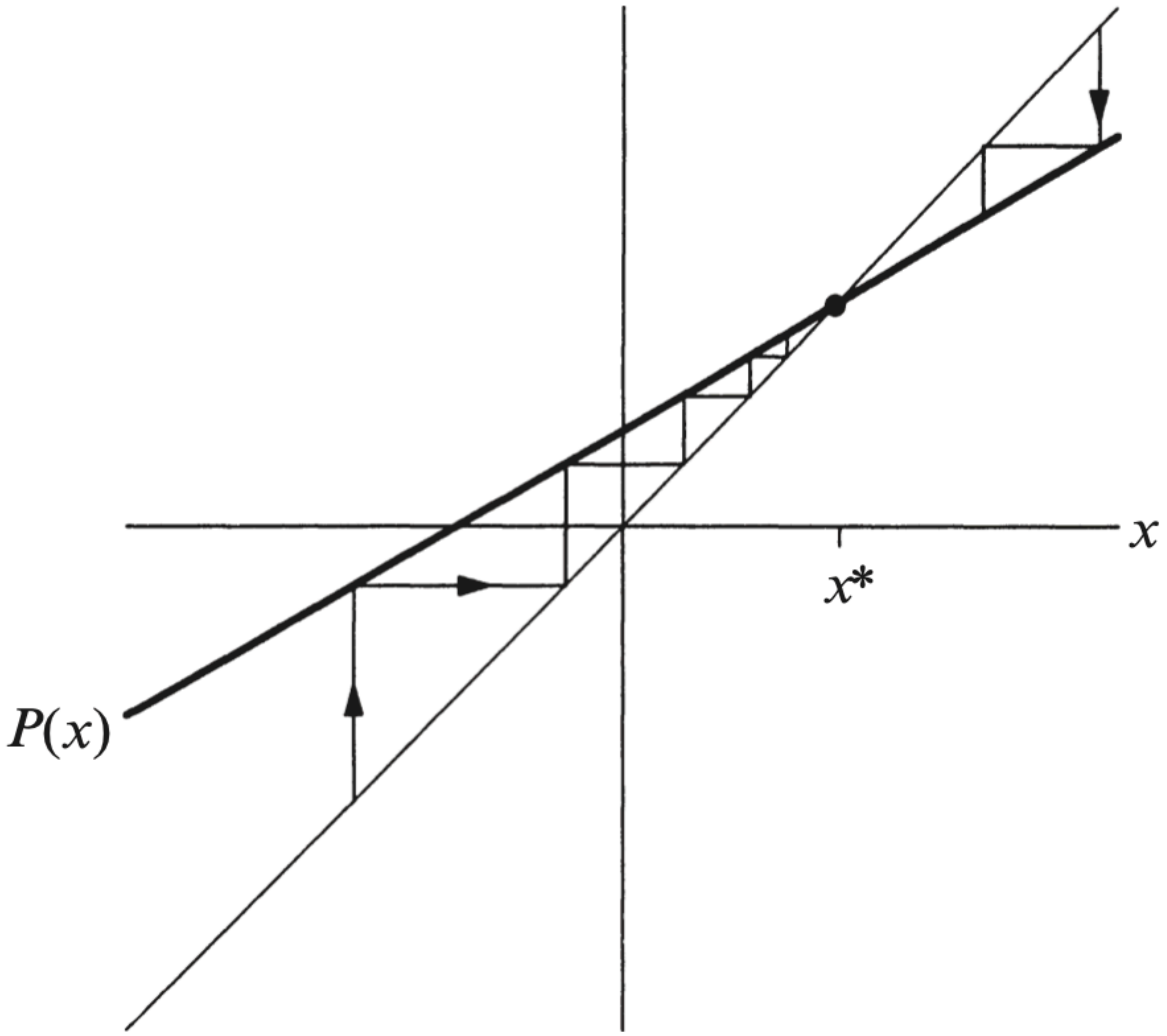
Then P is defined by:

$$x_1 = P(x_0) = x(2\pi / \omega)$$

$$P(x_0) = x(2\pi/\omega) = (x_0 - c_3)e^{-2\pi/\omega} + c_3$$
$$= x_0 e^{-2\pi/\omega} + c_4$$

where:

$$c_4 = c_3(1 - e^{-2\pi/\omega})$$



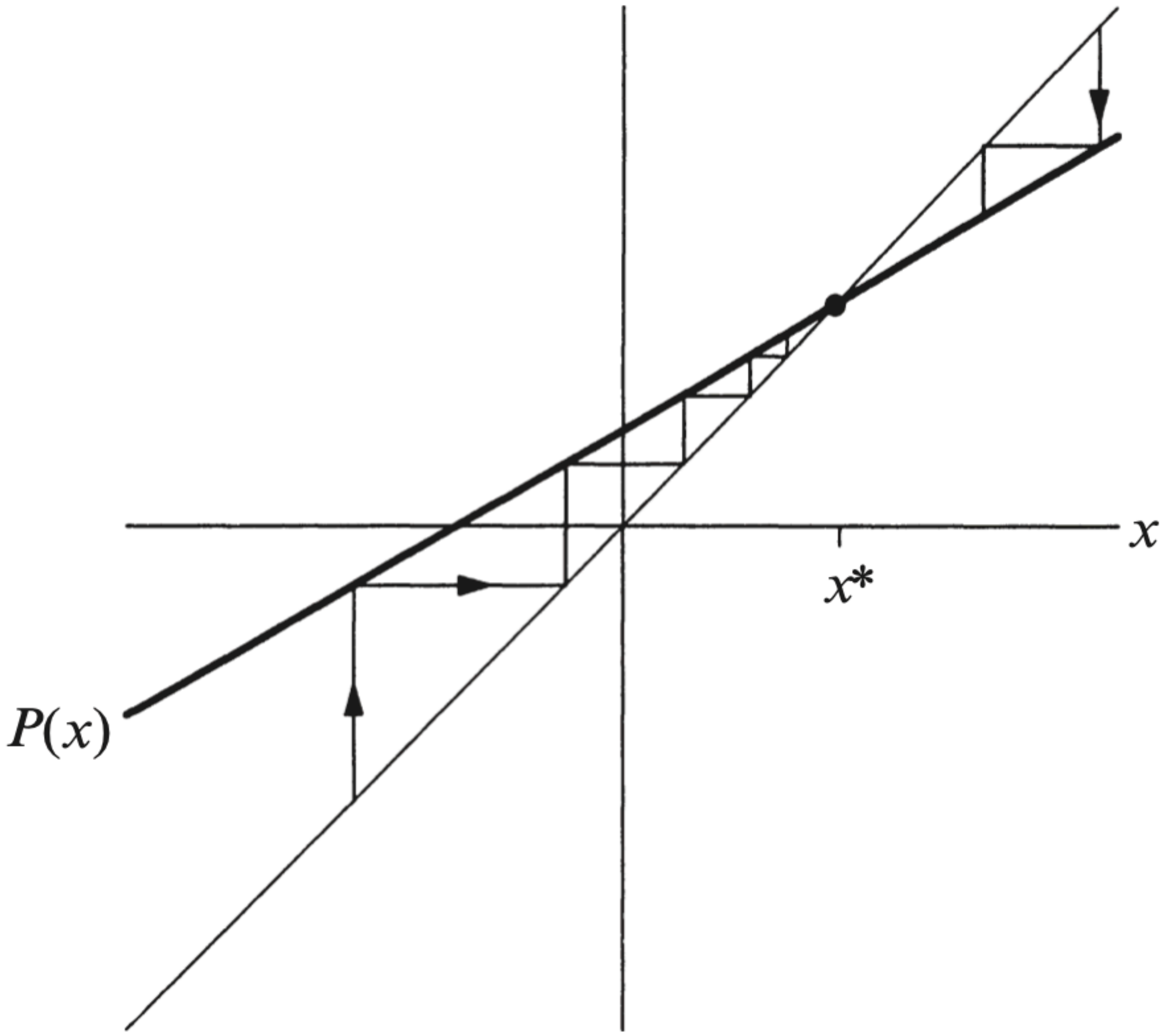
Example: Poincaré Maps

Since P has slope less than 1, it intersects the diagonal at a unique point.

The cobweb shows that the deviation of x_k from the fixed point is reduced by a constant factor with each iteration.

Hence the fixed point is unique and globally stable.

In physical terms, the circuit always settles into the same forced oscillation, regardless of the initial conditions.



Linear Stability of Periodic Orbits

Given a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a closed orbit, **how can we tell whether the orbit is stable is not?**

We ask whether the corresponding fixed point $\mathbf{x}^*$ of the Poincaré map is stable.

Let v_0 be an infinitesimal perturbation such that $\mathbf{x}^* + \mathbf{v_0}$ is in *S*.

Then after the first return to *S:*

$$\mathbf{x}^* + \mathbf{v}_1 = P(\mathbf{x}^* + \mathbf{v}_0)$$
$$= P(\mathbf{x}^*) + [DP(\mathbf{x}^*)]\mathbf{v}_0 + O\left(\|\mathbf{v}_0\|^2\right)$$

DP is a $(n-1) \times (n-1)$ matrix called the linearised Poincaré map at x*.

$$\mathbf{x}^* = P(\mathbf{x}^*) \quad \Rightarrow \quad \mathbf{v}_1 = [DP(\mathbf{x}^*)]\mathbf{v}_0$$

Linear Stability of Periodic Orbits

The desired stability criterion is expressed in terms of the eigenvalues of DP(**x***): The closed orbit is linearly stable if and only if $|\lambda_j| < 1$ for all $j = 1, \ldots, n-1$

To understand this criterion, consider the generic case where there are no repeated eigenvalues. Then there is a basis of eigenvectors $\{e_j\}$ and so we can write:

$$\mathbf{v}_0 = \sum_{j=1}^{n-1} v_j \mathbf{e}_j$$

for some scalars v_j. Hence:

$$\mathbf{v}_1 = \left(DP(\mathbf{x}^*)\right) \sum_{j=1}^{n-1} v_j \mathbf{e}_j = \sum_{j=1}^{n-1} v_j \lambda_j \mathbf{e}_j$$

Iterating the linearised map *k* times gives:

$$\mathbf{v}_k = \sum_{j=1}^{n-1} v_j (\lambda_j)^k \mathbf{e}_j$$

Linear Stability of Periodic Orbits

Hence, if all $|\lambda_j| < 1$, then $||\mathbf{v_k}|| \to \mathbf{0}$ geometrically fast.

This proves that **x*** is linearly stable.

Conversely, when $|\lambda_j| > 1$ for some j, then perturbations along $\mathbf{e_j}$ grow, so **x*** is unstable.

A borderline case occurs when the largest eigenvalue has magnitude $|\lambda_m| = 1$; this occurs at bifurcations of periodic orbits, and then a nonlinear stability analysis is required.

The λ_j are called the **characteristic** or **Floquet multipliers** of the periodic orbit.

In general, the characteristic multipliers can only be found by numerical integration.

Tutorial: Uncoupled oscillators