Nonlinear Dynamics and Chaos

PHYMSCFUN12

Wladimir E. Banda Barragán

wbanda@yachaytech.edu.ec

MSc in Fundamental Physics

Yachay Tech University - 2025

Chaos

Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.

1. Aperiodic long-term behaviour: there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as $t \to \infty$

There is an open set of initial conditions leading to aperiodic trajectories, or perhaps that such trajectories should occur with nonzero probability, given a random initial condition.

2. Deterministic: the system has no random or noisy inputs or parameters. The irregular behaviour arises from the system's nonlinearity, rather than from noisy driving forces.

3. Sensitive dependence on initial conditions: nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent.

Attractor and Strange Attractor

Attractor: set to which all neighbouring trajectories converge. Stable fixed points and stable limit cycles are examples. More precisely, we define an ***attractor*** to be a closed set *A* with the following properties:

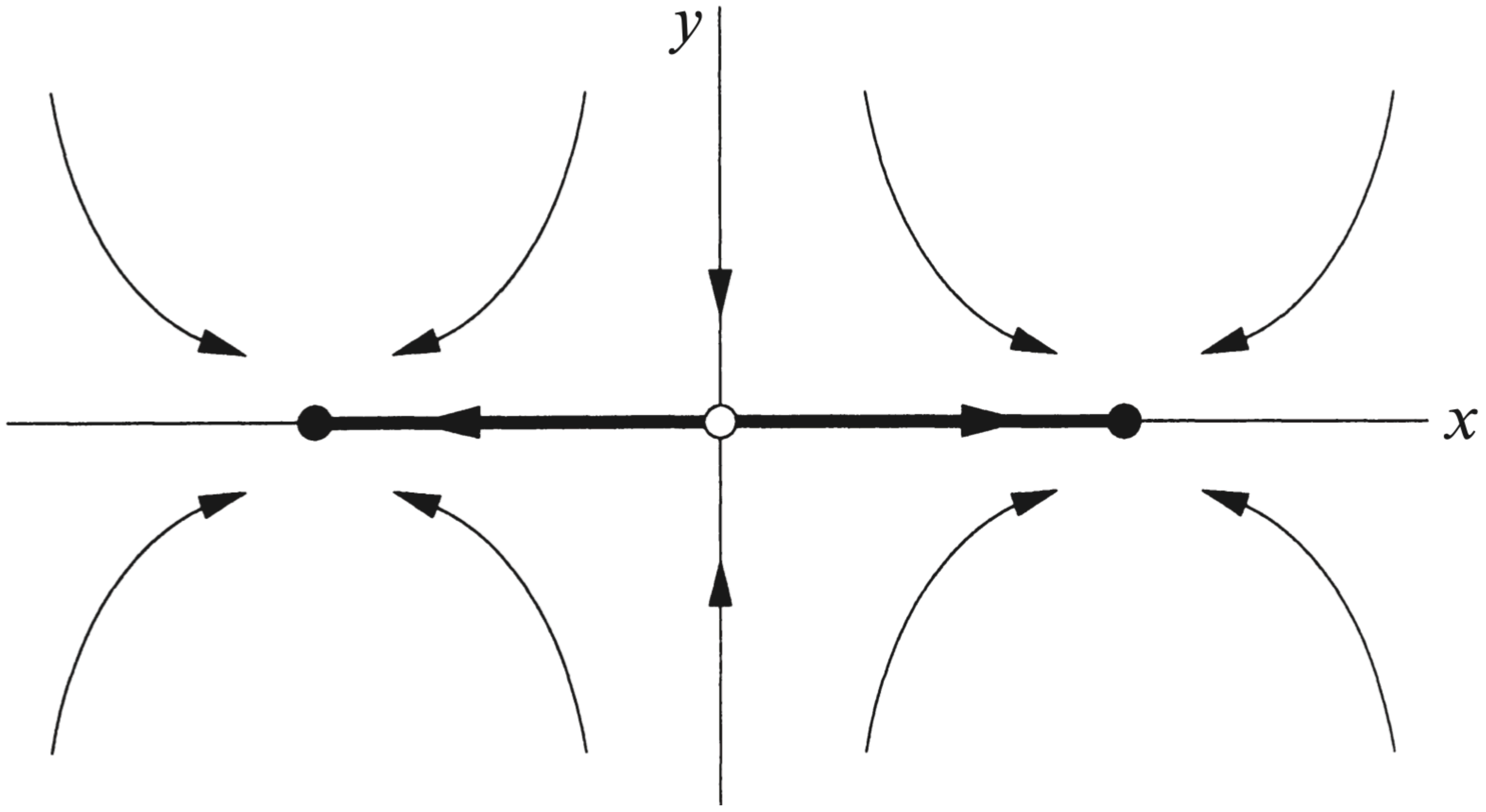
1. A is an *invariant set:* any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.

2. A *attracts an open set of initial conditions*: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction* of A.

3. A is *minimal:* there is no proper subset of A that satisfies conditions 1 and 2.

Attractor or not?

Consider the system: $\dot{x} = x - x^3$ and $\dot{y} = -y$. Let I denote the interval $-1 \leq x \leq 1$, $y = 0$. Is I an invariant set? Does it attract an open set of initial conditions? Is it an attractor?



1. I is an invariant set; any trajectory that starts in I stays in I forever.

2. I certainly attracts an open set of initial conditions—it attracts *all* trajectories in the xy plane.

3. I is *not* an attractor because it is not minimal. The stable fixed points $(\pm 1,0)$ are proper subsets of I that also satisfy properties 1 and 2.

Even if a certain set attracts all trajectories, it may fail to be an attractor because it may not be minimal—it may contain one or more smaller attractors.

Attractor and Strange Attractor

Strange attractor

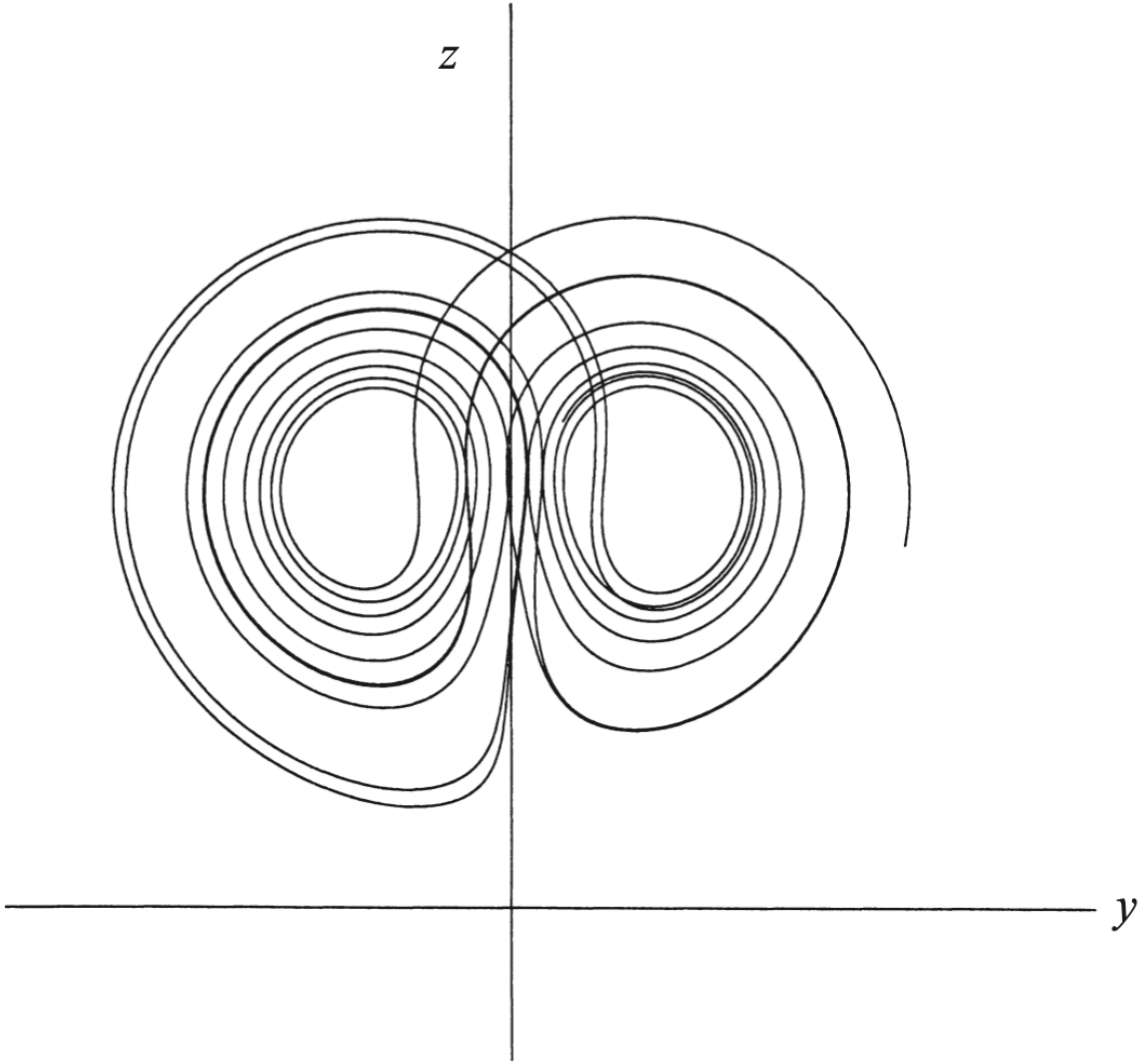
This is an attractor that exhibits sensitive dependence on initial conditions.

Strange attractors were originally called strange because they are often fractal sets.

Nowadays this geometric property is regarded as less important than the dynamical property of sensitive dependence on initial conditions.

The terms chaotic attractor and fractal attractor are used when one wishes to emphasise one or the other of those aspects.

Lorenz Map

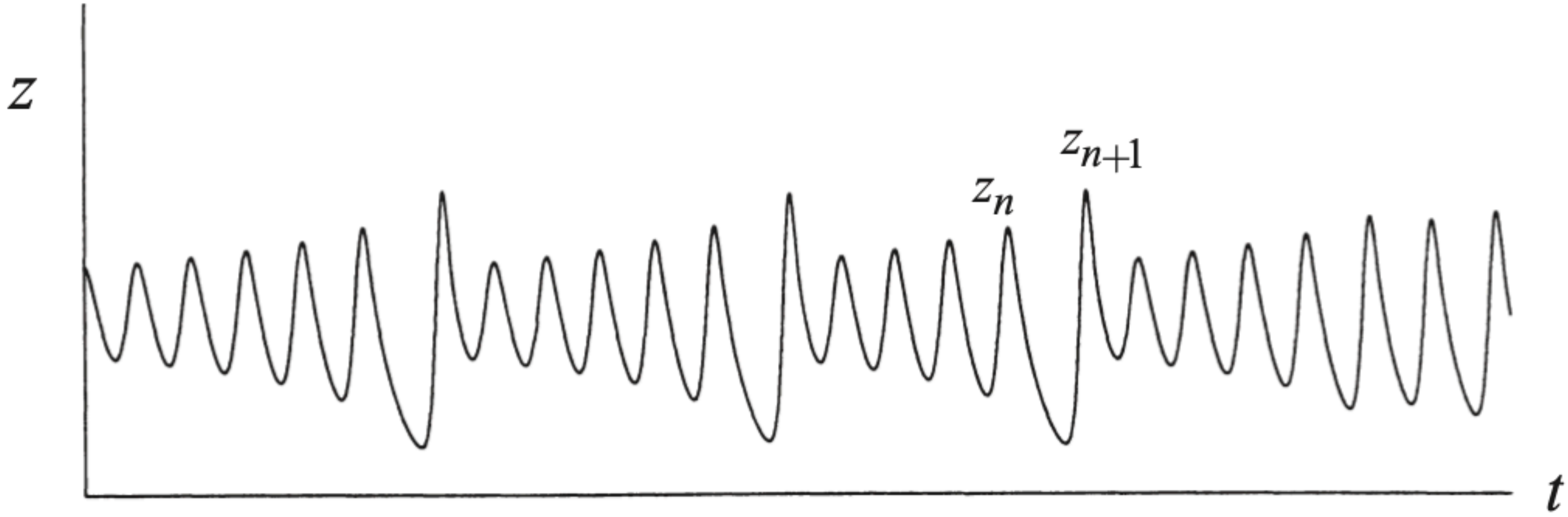


The trajectory apparently leaves one spiral only after exceeding some critical distance from the centre.

The extent to which this distance is exceeded appears to **determine the point at which the next spiral is entered**; this in turn seems to d**etermine the number of circuits to be executed** before changing spirals again.

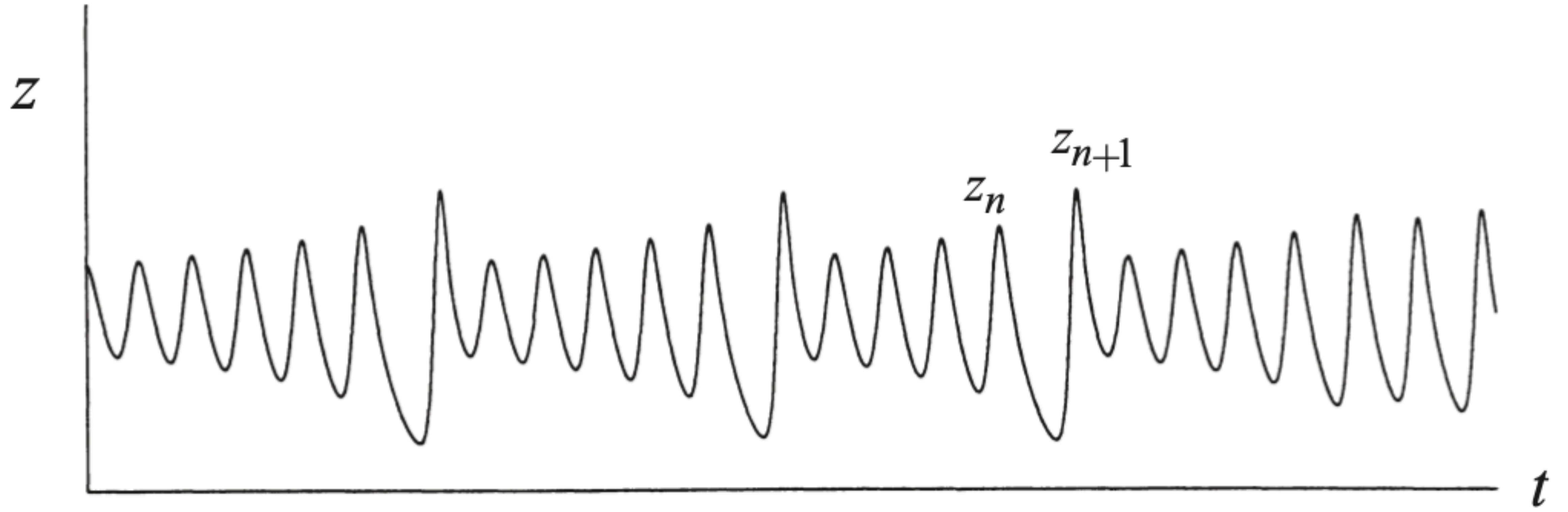
It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit.

Lorenz Map



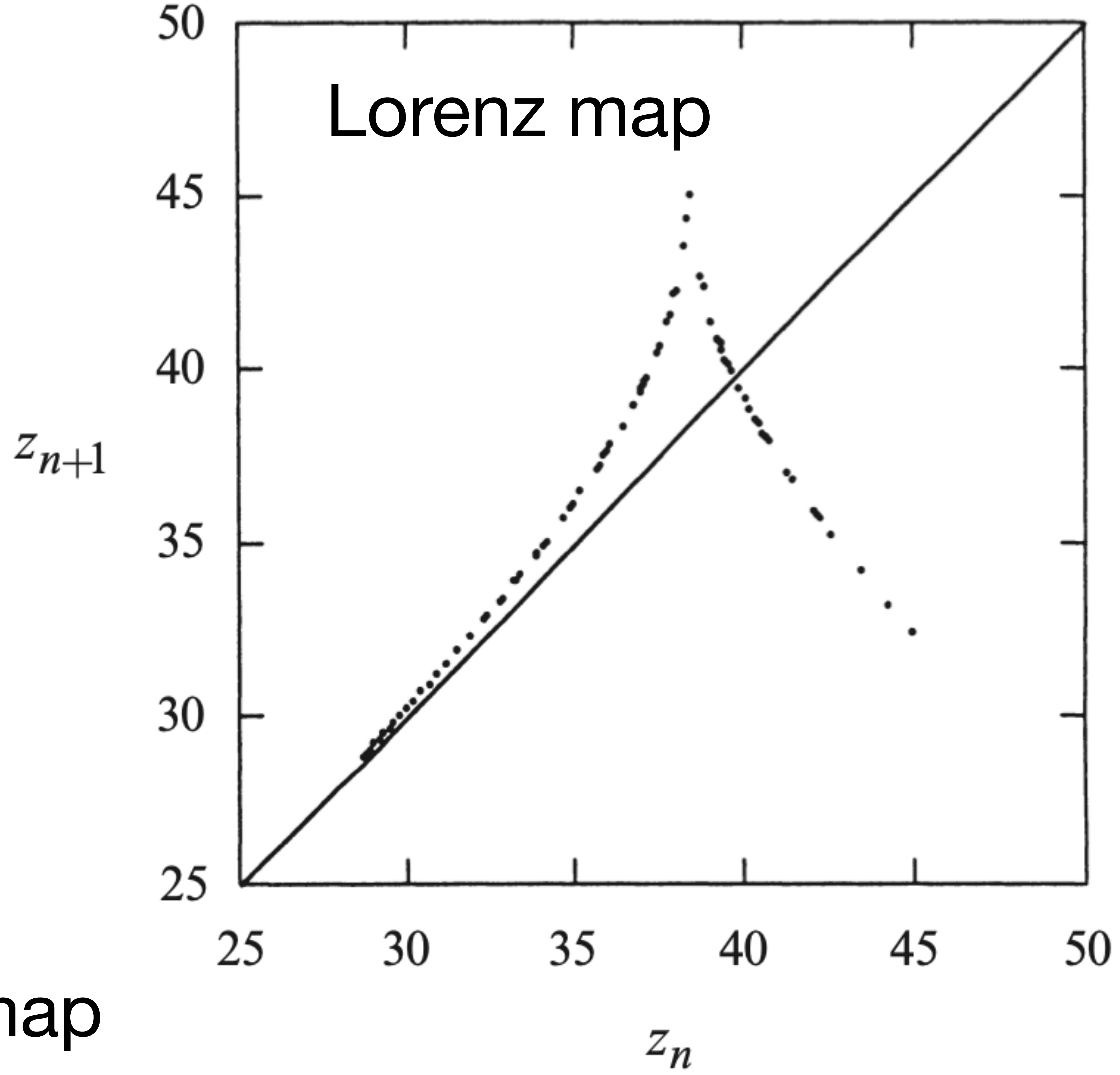
We should focus our attention on a single feature z_n, the nth local maximum of $z(t)$.

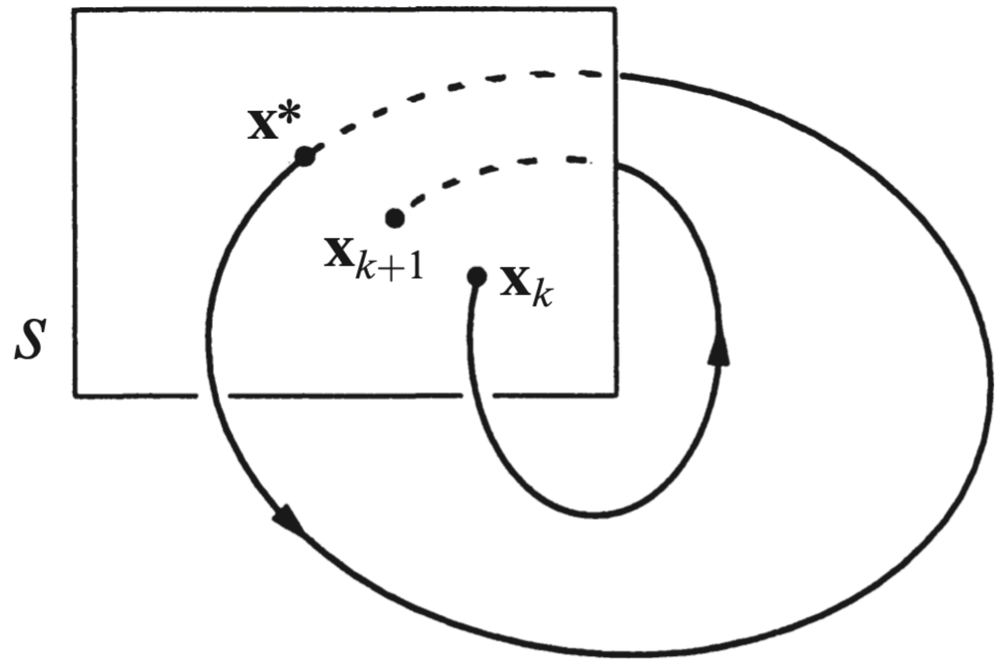
Lorenz Map



Lorenz's idea is that z_n should predict z_{n+1}. He numerically integrated the equations for a long time, then measured the local maxima of $z(t)$, and finally plotted z_{n+1} vs. z_n.

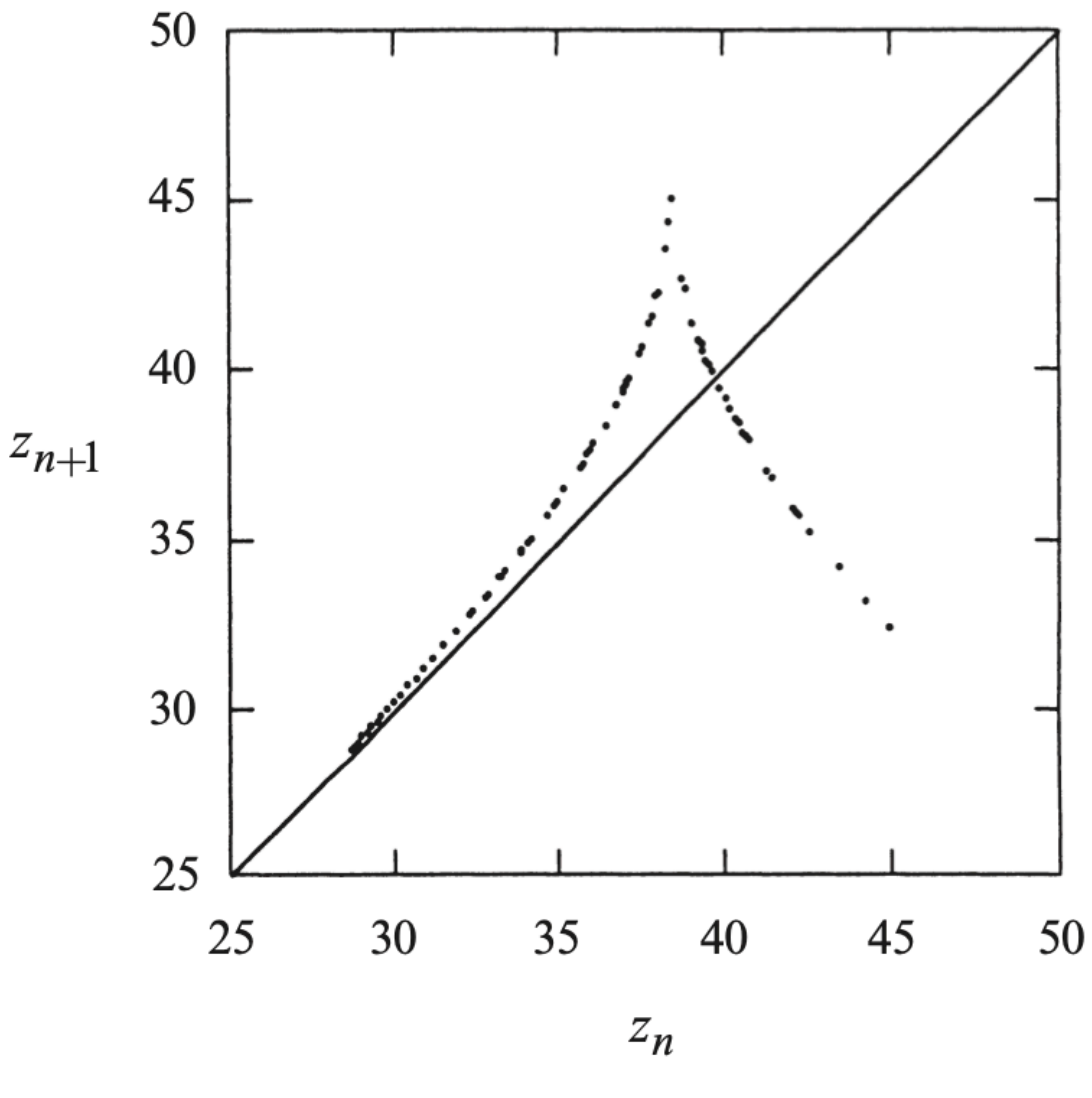
The data from the chaotic time series appear to fall neatly on a curve—there is almost no "thickness" to the graph!





Poincaré map

Lorenz Map



We can extract order from chaos.

Strictly speaking, $f(z)$ is not a well-defined function, because there can be more than one output z_{n+1} for a given input z_n.

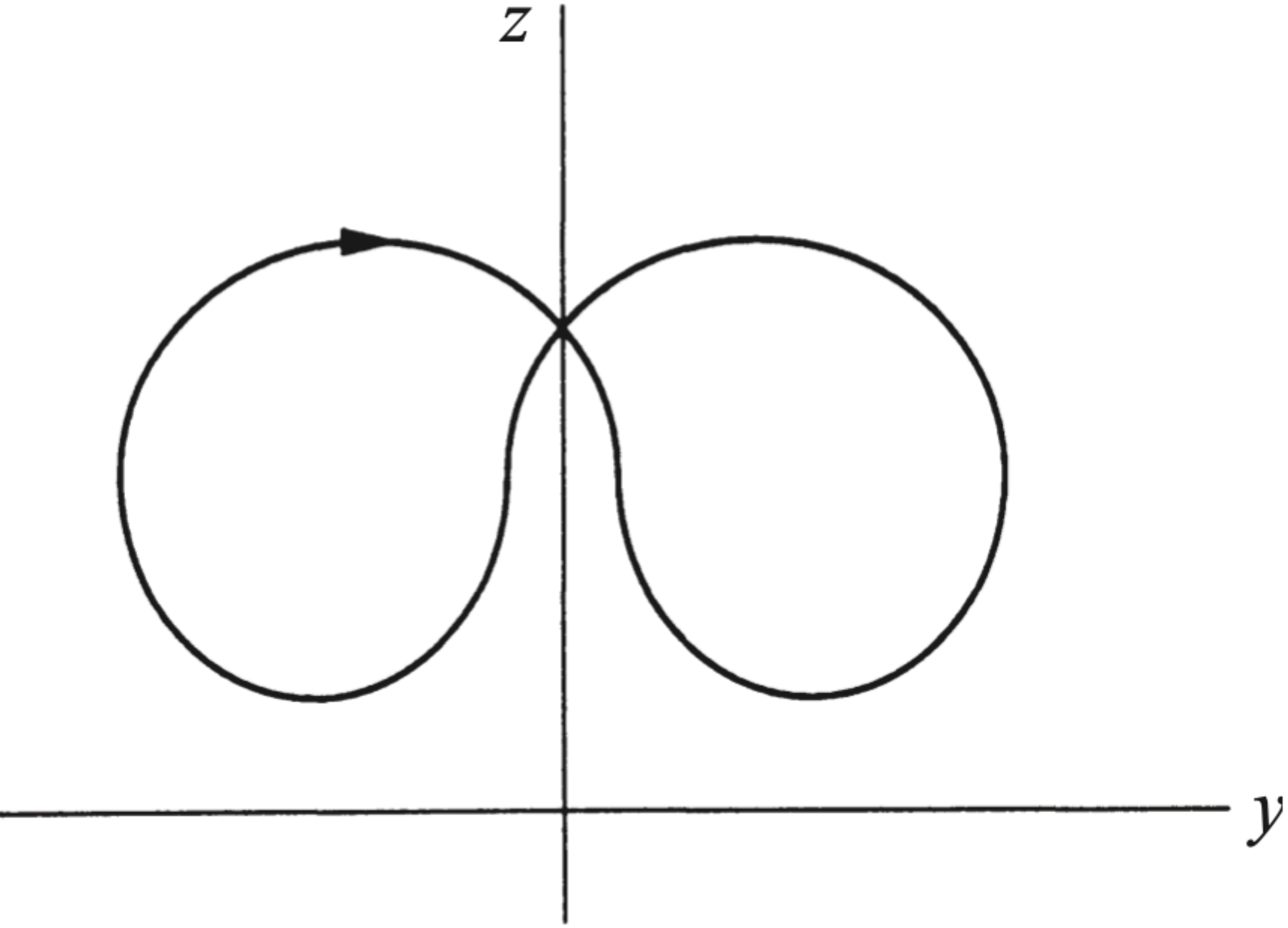
The thickness is so small, and there is so much to be gained by treating the graph as a curve, that we will simply make this approximation.

Stable Limit Cycles can be ruled out. If any limit cycles exist, they are necessarily unstable.

$$|f'(z)| > 1$$

$$|f'(z)| > 1$$

Lorenz Map



Why?

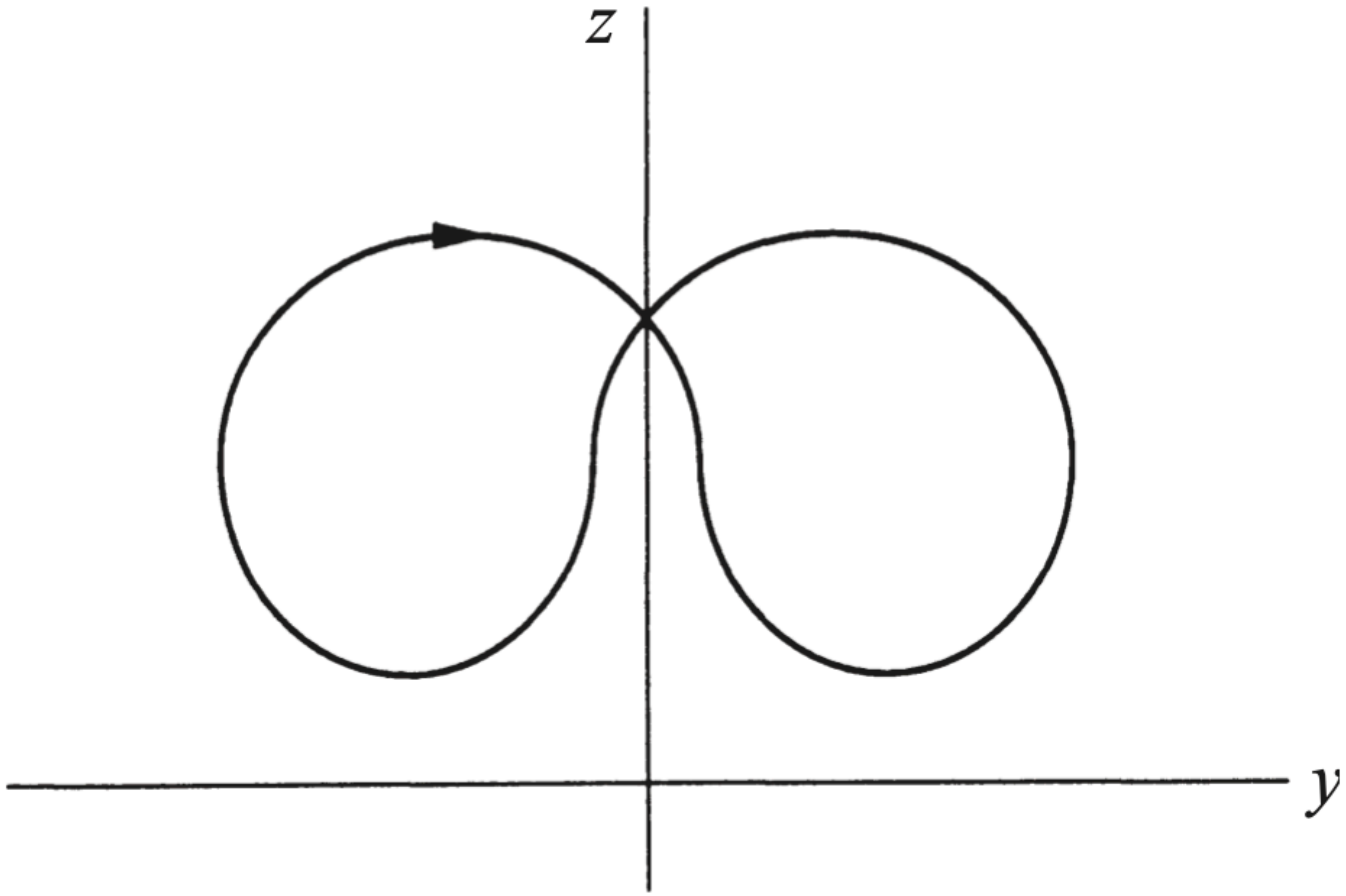
We need to analyse the fixed points on the map:

$$f(z^*) = z^*,$$

$$z_n = z_{n+1} = z_{n+2} = \ldots.$$

There is one fixed point, where the 45° diagonal intersects the graph. It represents a closed orbit.

Lorenz Map



To show that this closed orbit is unstable, consider a slightly perturbed trajectory that has:

$$z_n = z^* + \eta_n, \text{ where } \eta_n \text{ is small.}$$

After linearisation as usual, we find:

$$\eta_{n+1} \approx f'(z^*)\eta_n$$

$$|f'(z^*)| > 1 \quad \Rightarrow \quad |\eta_{n+1}| > |\eta_n|$$

The deviation grows in every iteration, so the original closed orbit is unstable.

All closed orbits are unstable.

Global behaviour

What happens if we change the parameters?

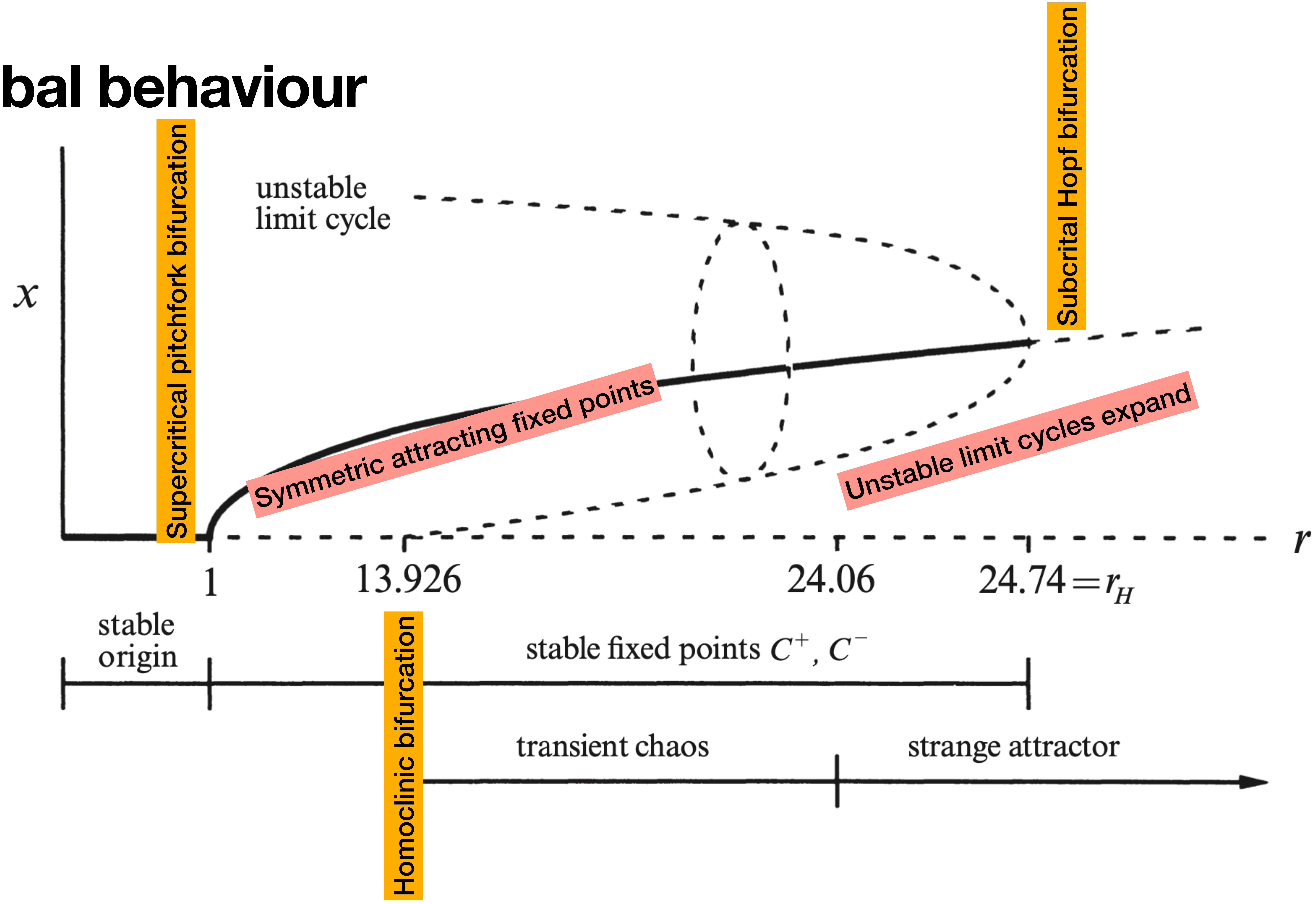
We can find exotic limit cycles tied in knots, pairs of limit cycles linked through each other, intermittent chaos, noisy periodicity, as well as strange attractors.

The origin is globally stable for $r < 1$.

At $r = 1$ the origin loses stability by a **supercritical pitchfork bifurcation**, and a symmetric pair of attracting fixed points is born.

At $r_H = 24.74$ the fixed points lose stability by absorbing an unstable limit cycle in a **subcritical Hopf bifurcation.**

Global behaviour



Global behaviour

As we decrease r from r_H, the unstable limit cycles expand and pass precariously close to the saddle point at the origin.

At $r \approx 13.926$ the cycles touch the saddle point and become homoclinic orbits; hence we have a **homoclinic bifurcation.**

Below $r = 13.926$, there are no limit cycles. Viewed in the other direction, we could say that a pair of unstable limit cycles are created as r increases through $r = 13.926$.

An invariant set is born at $r = 13.926$, along with the unstable limit cycles. This set is a thicket of infinitely many saddle-cycles and aperiodic orbits.

It is not an attractor and is not observable directly, but it generates sensitive dependence on initial conditions in its neighbourhood.

Global behaviour

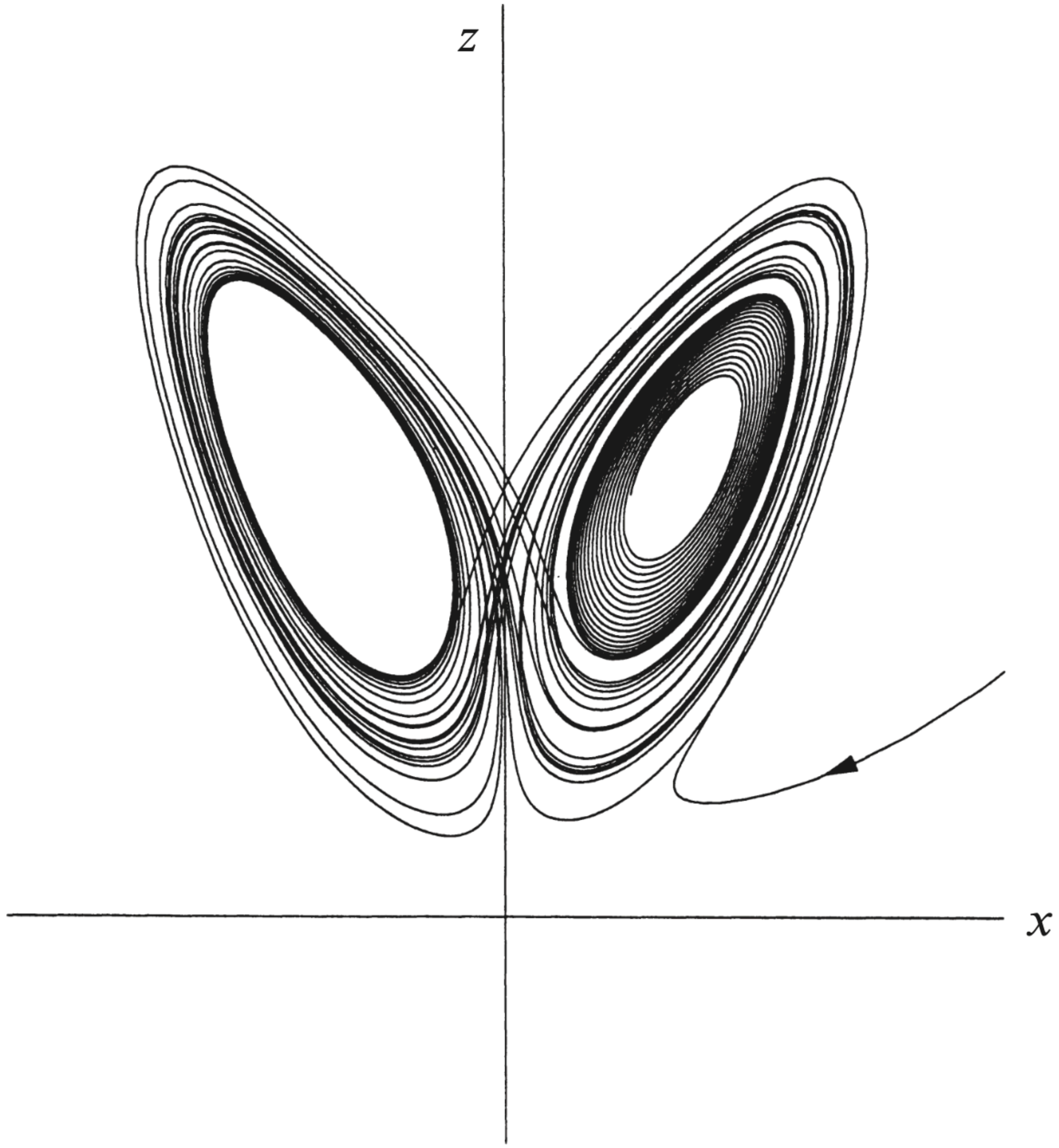
Trajectories can get hung up near this set, somewhat like wandering in a maze. Then they rattle around chaotically for a while, but eventually escape and settle down to C^+ or C^-.

The time spent wandering near the set gets longer and longer as *r* increases.

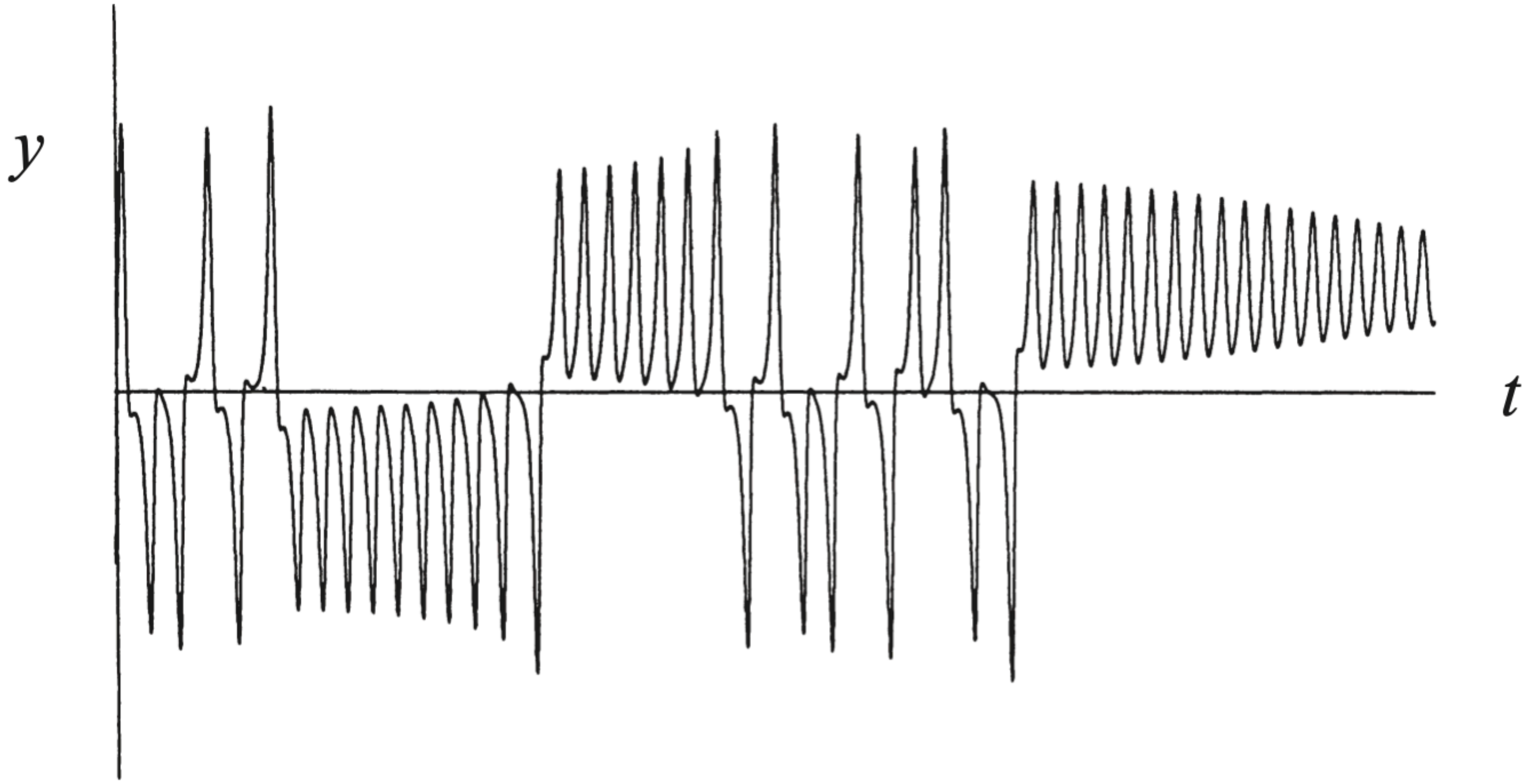
Finally, at $r = 24.06$ the time spent wandering becomes infinite and the set becomes a **strange attractor.**

Tutorial: transient or metastable chaos

Show numerically that the Lorenz equations can exhibit ***transient chaos*** when $r = 21$ (with $\sigma = 10$ and $b = \frac{8}{3}$ as usual).



At first the trajectory seems to be tracing out a strange attractor, but eventually it stays on the right and spirals down toward the stable fixed point C^+.



Transient or metastable chaos

The dynamics are not “chaotic,” because the long-term behaviour is not aperiodic.

The dynamics do exhibit sensitive dependence on initial conditions—if we had chosen a slightly different initial condition, the trajectory could easily have ended up at C^- instead of C^+.

Transient chaos shows that a deterministic system can be unpredictable, even if its final states are very simple. We don’t need strange attractors to generate effectively random behaviour.

Hysteresis between chaos and equilibrium

For $24.06 < r < 24.74$, there are two types of attractors: **fixed points and a strange attractor.**

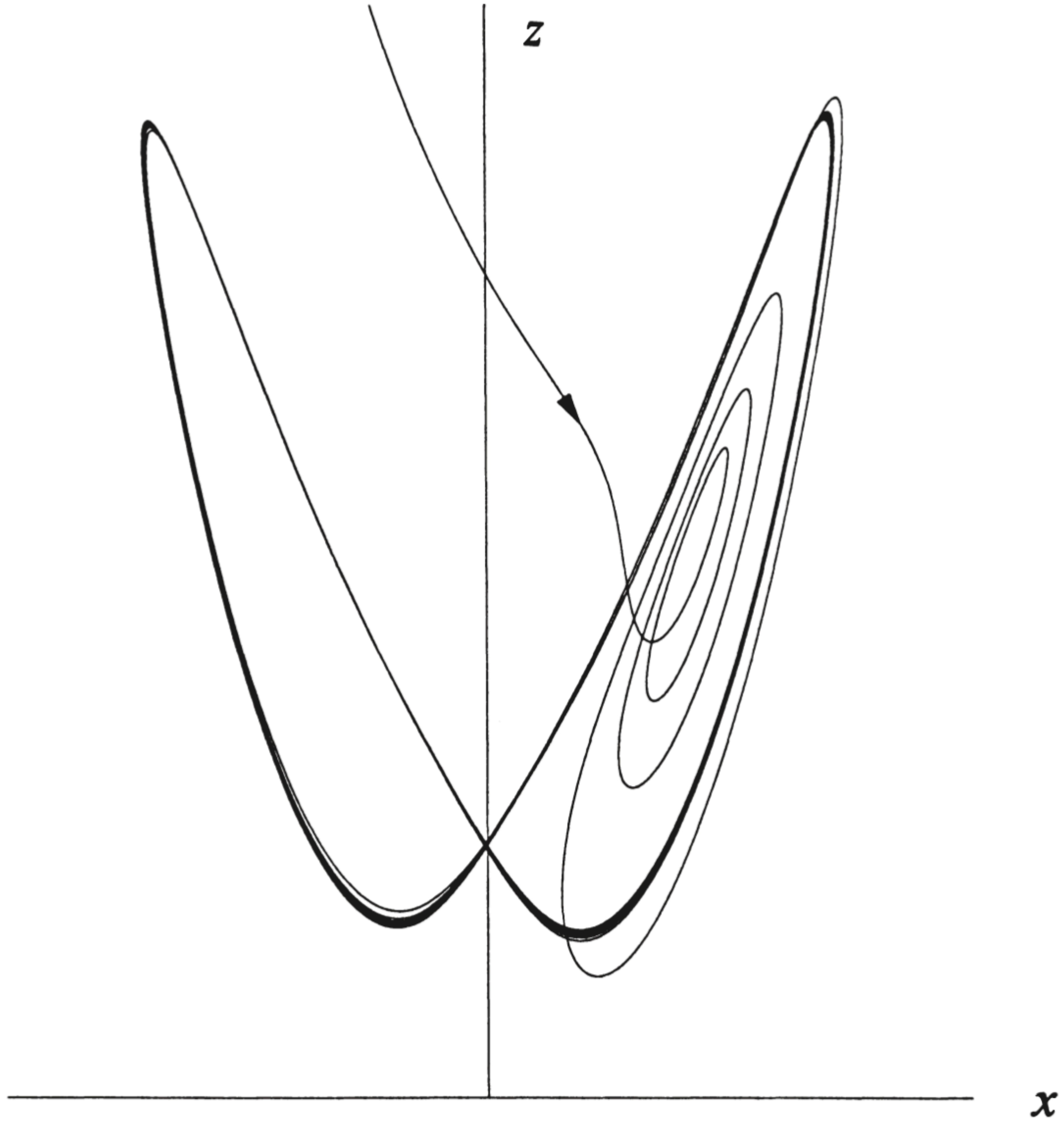
This coexistence means that we can have hysteresis between chaos and equilibrium by varying r slowly back and forth past these two endpoints.

It also means that a large enough perturbation can knock a steadily rotating waterwheel into permanent chaos.

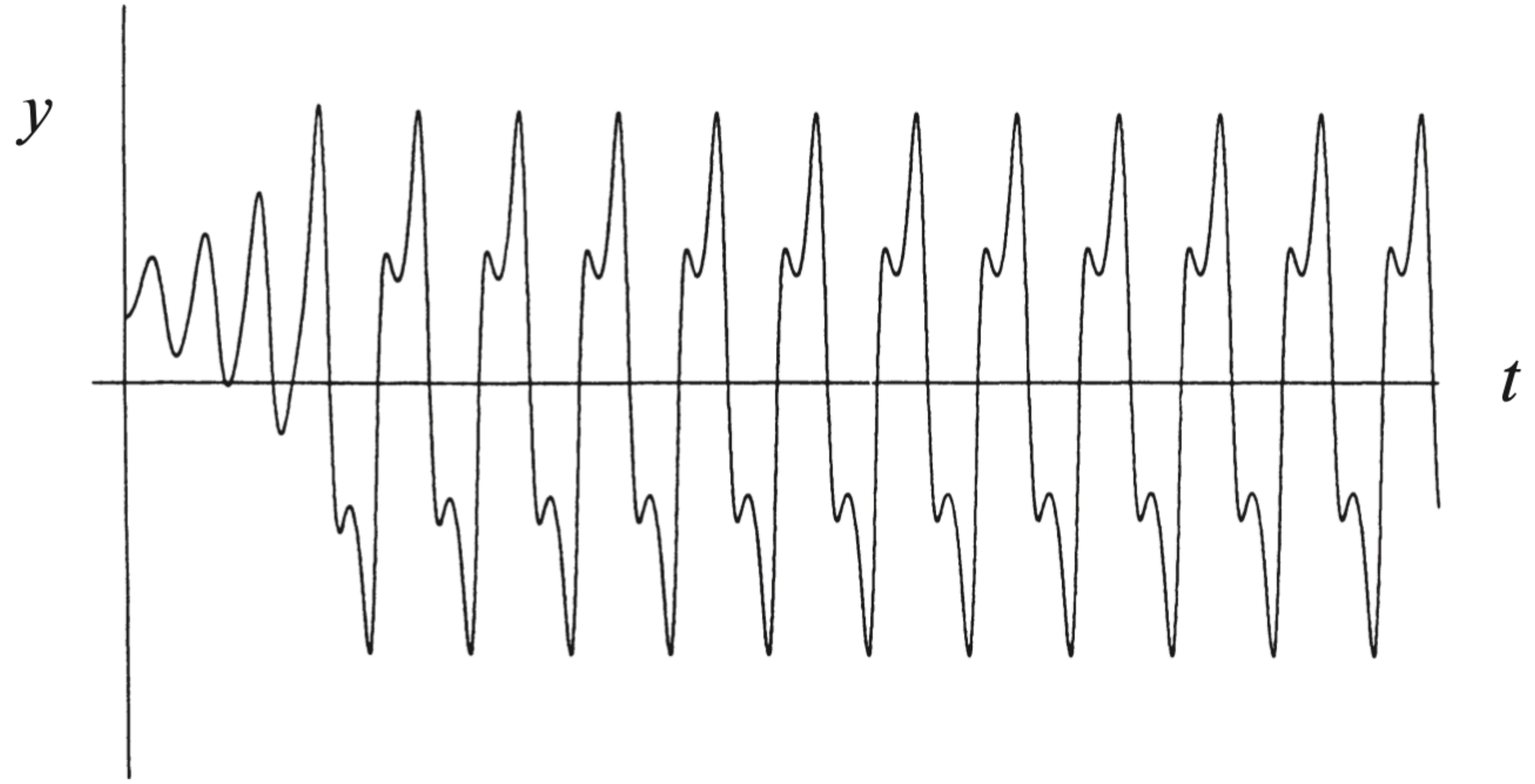
This is reminiscent of fluid flows that mysteriously become turbulent even though the basic laminar flow is still linearly stable.

Large Rayleigh Numbers

Numerical work shows that the system has a globally attracting limit cycle for all $r > 313$ (Sparrow 1982).



We approach to the limit cycle.



This solution predicts that the waterwheel should ultimately rock back and forth like a pendulum, turning once to the right, then back to the left, and so on. This is observed experimentally.

Tutorial: show solutions for large r values

Describe the long-term dynamics for large values of r, for $\sigma = 10$, $b = \frac{8}{3}$. Interpret the results in terms of the motion of the waterwheel of Section 9.1.