Nonlinear Dynamics and Chaos

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1 Dimensional Systems / First-order Systems

General dynamical systems:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots\ldots, x_n) \\ \cdots\cdots \\ \dot{x}_n = f_n(x_1, \ldots\ldots, x_n) \end{cases}$$

1D dynamical system:

$$n = 1 \quad \Rightarrow \dot{x} = f(x) \qquad\qquad x \in \mathbb{R}$$

Notes:

- $n = 1$ dynamical system.
- Non-autonomous systems, $f(x, t)$, systems are 2D.

A Geometric Way of Thinking

Example:

Non-linear ODE: $\dot{x} = \sin(x)$

Variable Separation: $dt = \dfrac{dx}{\sin(x)}$

Analytical Solution: $t = \ln\left|\dfrac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$

Questions:

Q1. Suppose $x_0 = \pi/4$, describe the qualitative features of the solution $x(t)$ for all $t > 0$. In particular, what happens as $t \to \infty$?

Q2. For an *arbitrary* initial condition x_0, what is the behavior of $x(t)$ as $t \to \infty$?

Not obvious!

A Geometric Way of Thinking

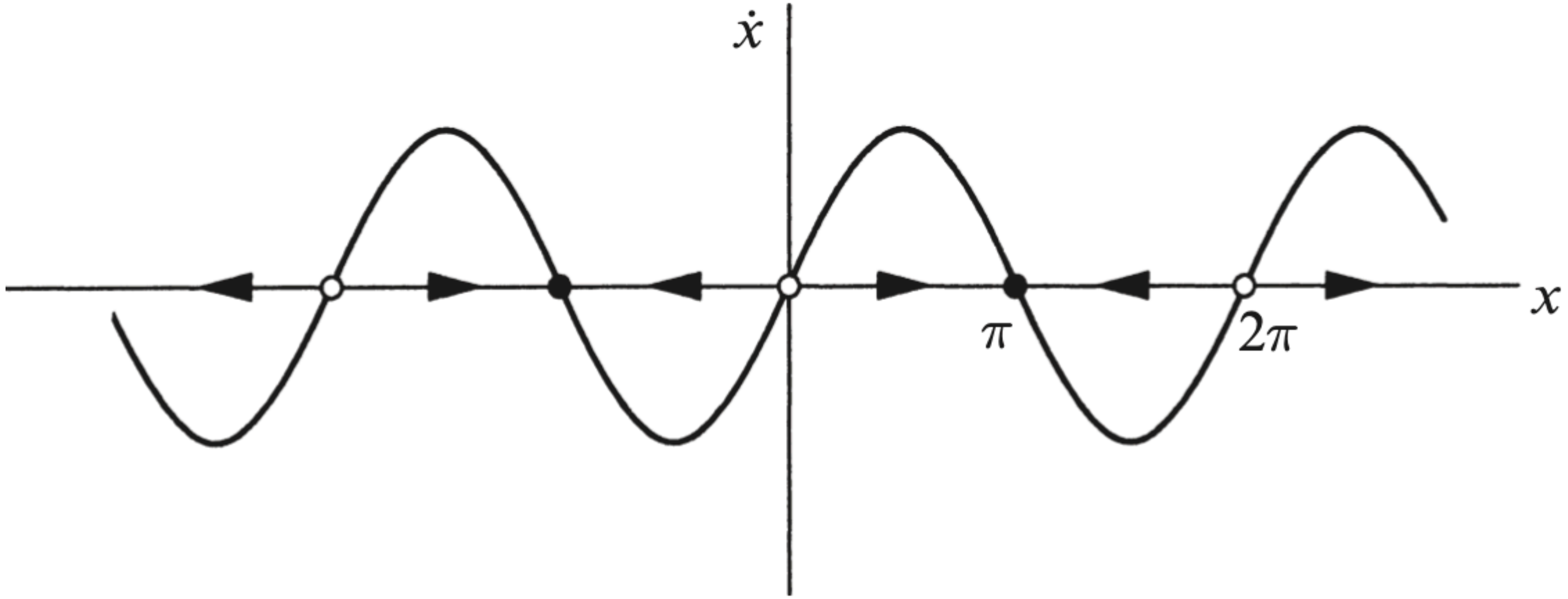
Example:

Non-linear ODE: $\dot{x} = \sin(x)$

Velocity vector field on the X axis.

$x \equiv$ position of an imaginary particle.

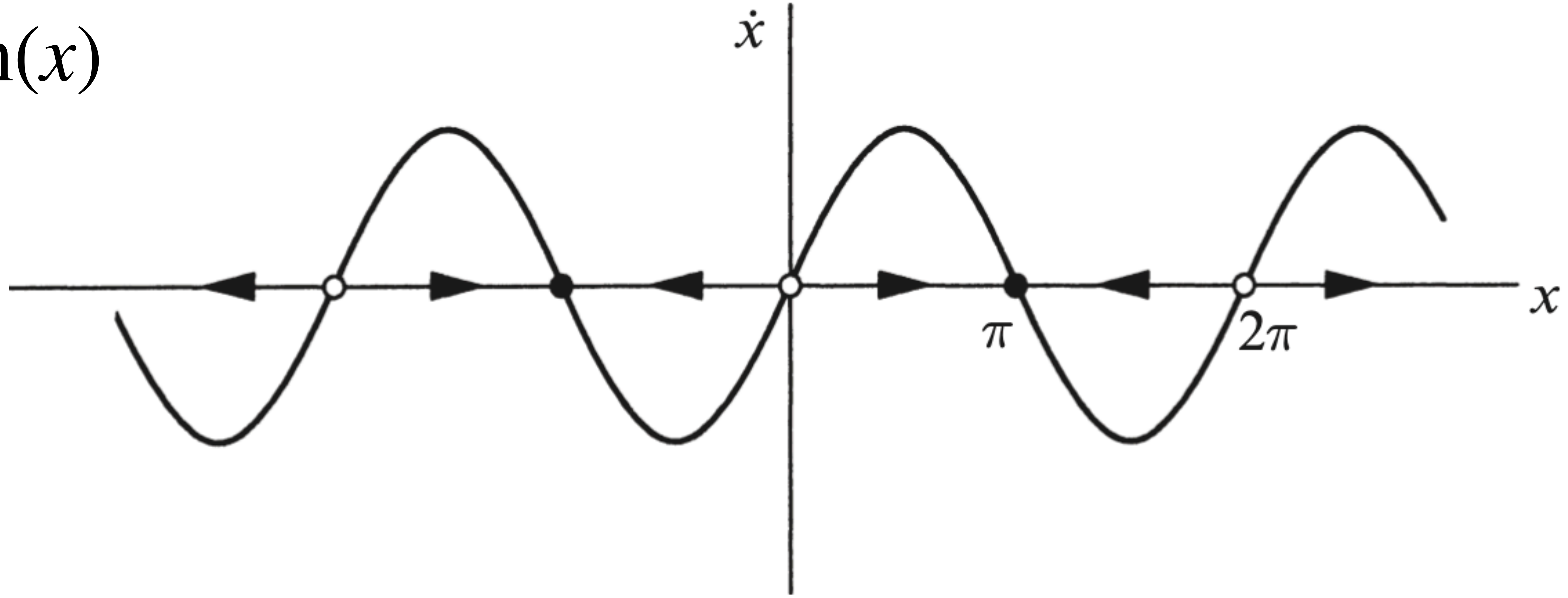
$\dot{x} \equiv$ velocity of the particle.



Fixed points (there is no "flow" at x^*), where $\dot{x} = 0$

A Geometric Way of Thinking

$$\dot{x} = \sin(x)$$



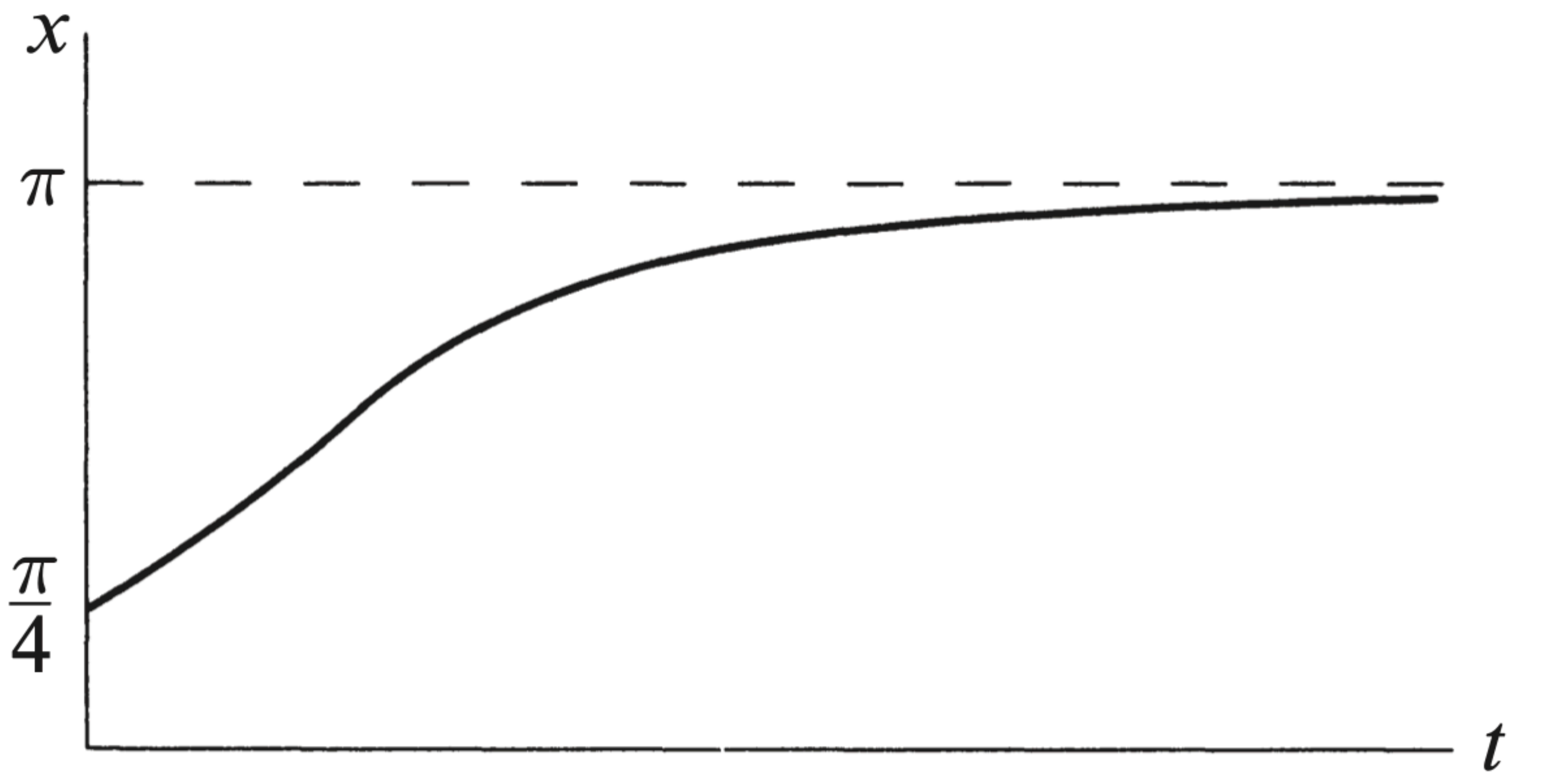
Fixed points (there is no “flow” at x^*), where $\dot{x} = 0$

Solid black dots represent ***stable*** fixed points (*attractors* or *sinks).*

Open circles represent ***unstable*** fixed points (*repellers* or *sources*).

A Geometric Way of Thinking

$x(t)$



Q1. Suppose $x_0 = \pi/4$, describe the qualitative features of the solution $x(t)$ for all $t > 0$. In particular, what happens as $t \to \infty$?

A particle starting at $x_0 = \pi/4$ moves to the right faster and faster until it crosses $x = \pi/2$ (where $\sin(x)$ reaches its maximum). The particle starts slowing down and eventually approaches the stable fixed point *x* from the left.

A Geometric Way of Thinking

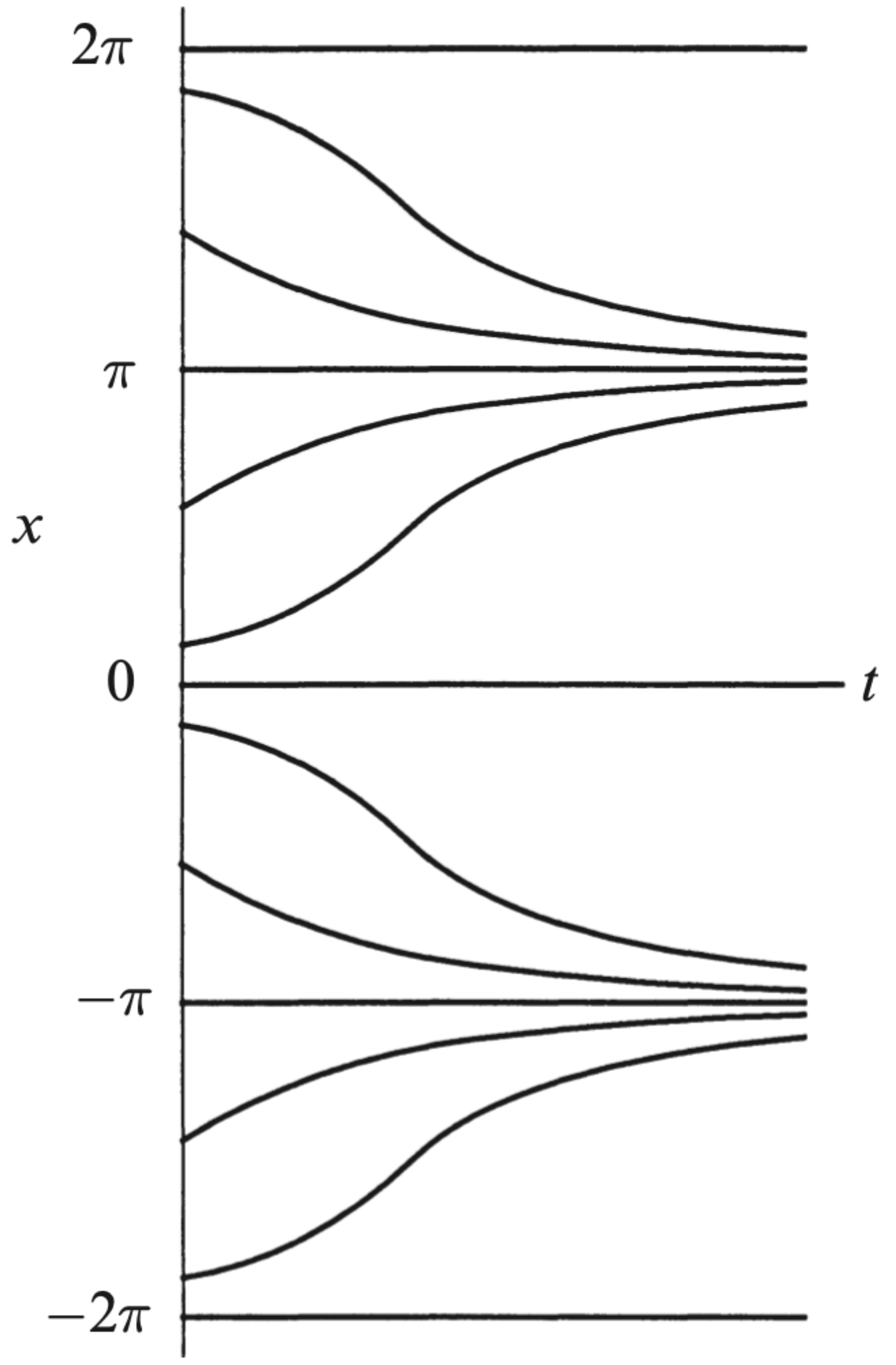
Q2. For an *arbitrary* initial condition x_0, what is the behavior of $x(t)$ as $t \to \infty$?

if $\dot{x} > 0$ initially, the particle heads to the right and asymptotically approaches the nearest stable fixed point.

Similarly, if $\dot{x} < 0$ initially, the particle approaches the nearest stablefixed point to its left.

If $\dot{x} = 0$, then x remains constant.

Qualitative information is what we care about.



Population Growth

Example:

Linear ODE: $\dot{x} = rx$

Exponential growth: $x = x_0\, e^{rt}$

(No overcrowding)

$r > 0$

Growth rate: r

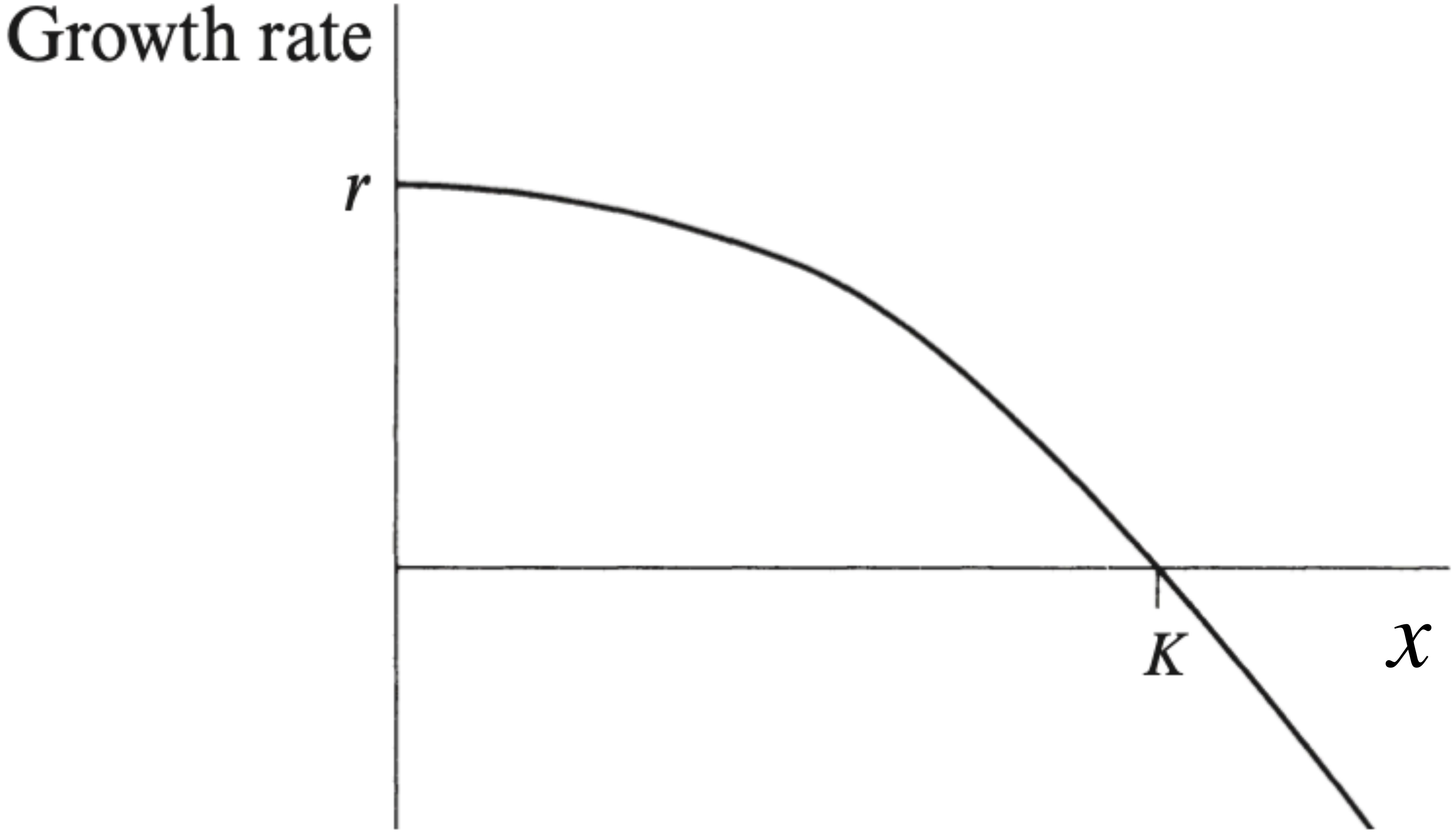
With overcrowding:

Population size: x

Population growth rate: $\dot{x}$

Per-capita growth rate: $\dot{x}/x$

Population Growth

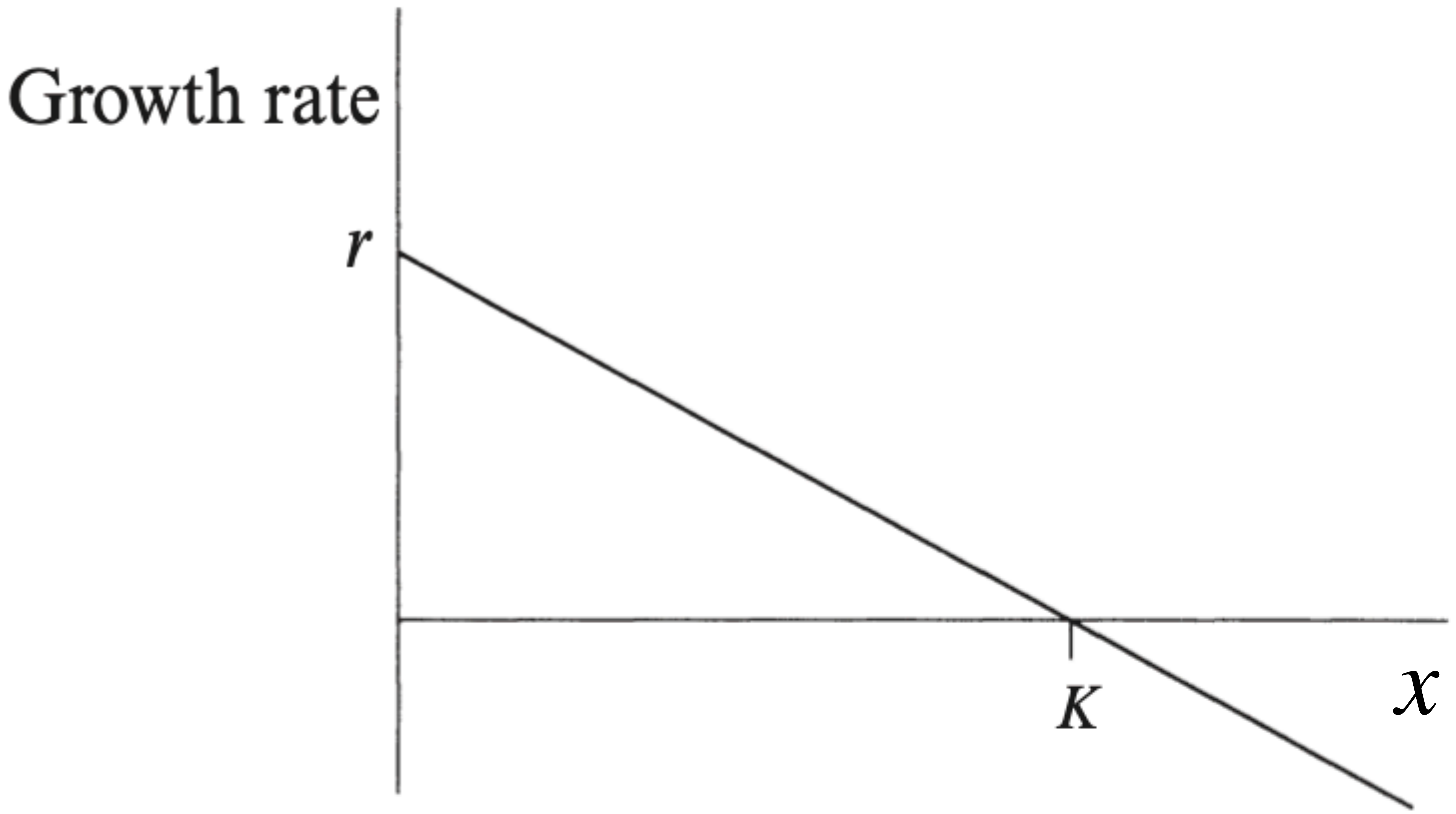


For small x, the growth rate equals r.

For populations larger than a certain ***carrying capacity*** K, the growth rate becomes negative: the death rate is higher than the birth rate.

$K > 0$ Carrying capacity: K

Population Growth



We can assume that the per capita growth rate decreases linearly with N.

With overcrowding: $\dot{x} = rx\left(1 - \frac{x}{K}\right)$

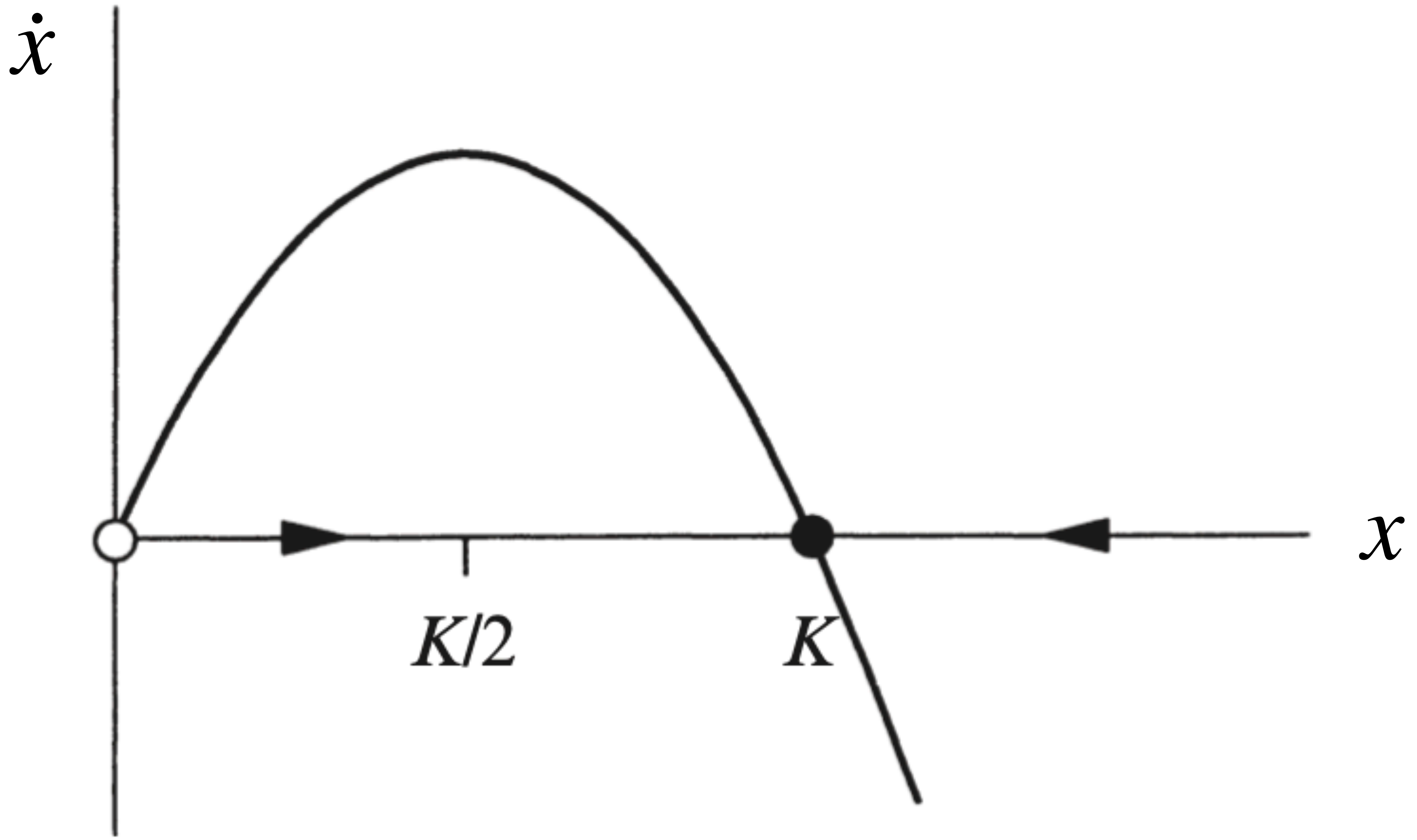
Verhulst (1838)

Logistic equation

Population Growth

Logistic equation

With overcrowding: $\dot{x} = rx\left(1 - \dfrac{x}{K}\right)$



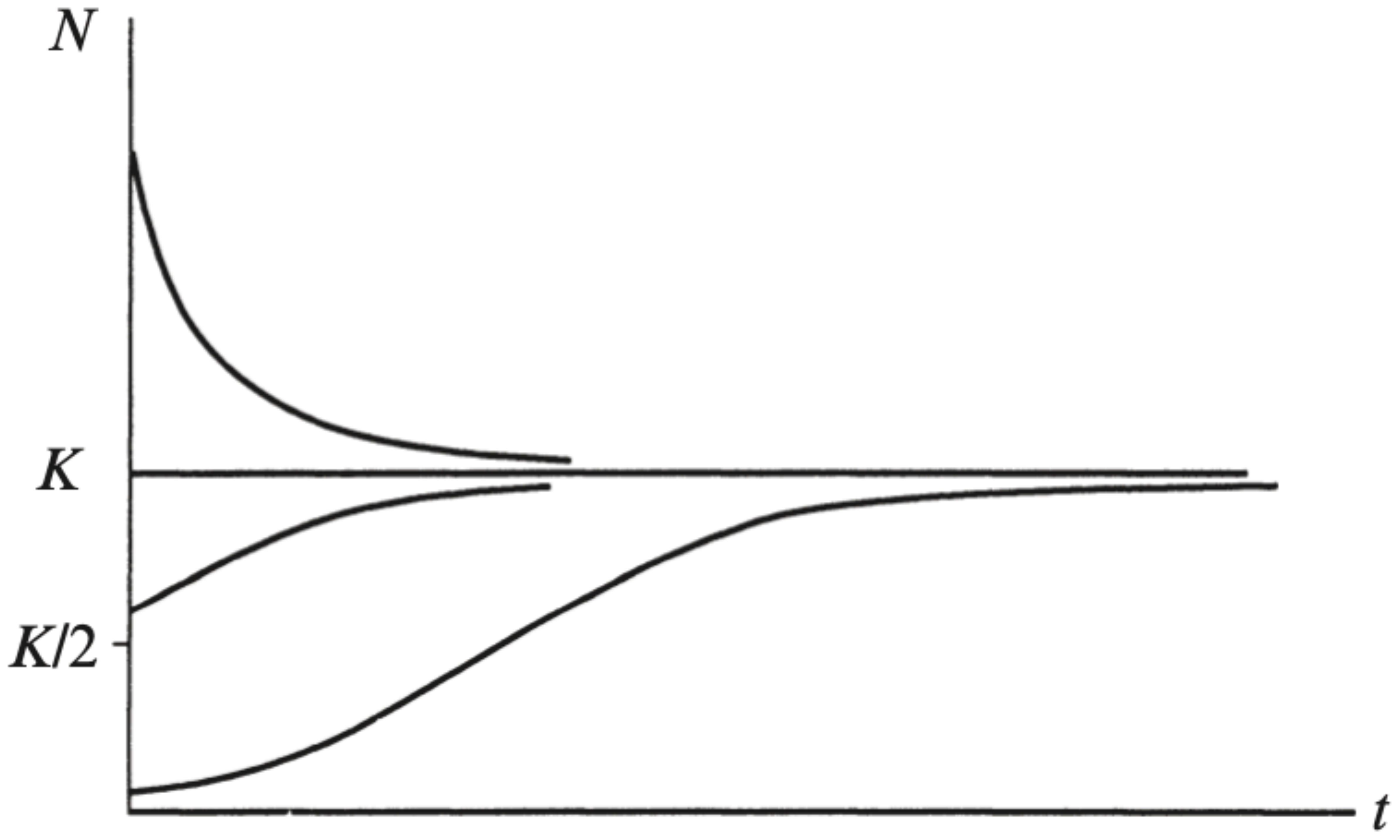
$x = 0$ is an unstable equilibrium.

The population always approaches the carrying capacity.

Population Growth

Logistic equation

With overcrowding: $\dot{x} = rx\left(1 - \frac{x}{k}\right)$



If $x_0 < K/2$, the phase point moves faster and faster until it crosses $x = K/2$.

Then the phase point slows down and eventually creeps toward $x = K$.

The graph of $x(t)$ is S-shaped or *sigmoid* for $x_0 = K/2$.

If x_0 lies between *K / 2* and *K*; now the solutions are decelerating from the start.

If x_0 exceeds the carrying capacity, *x*(*t*) decreases to $x = K$

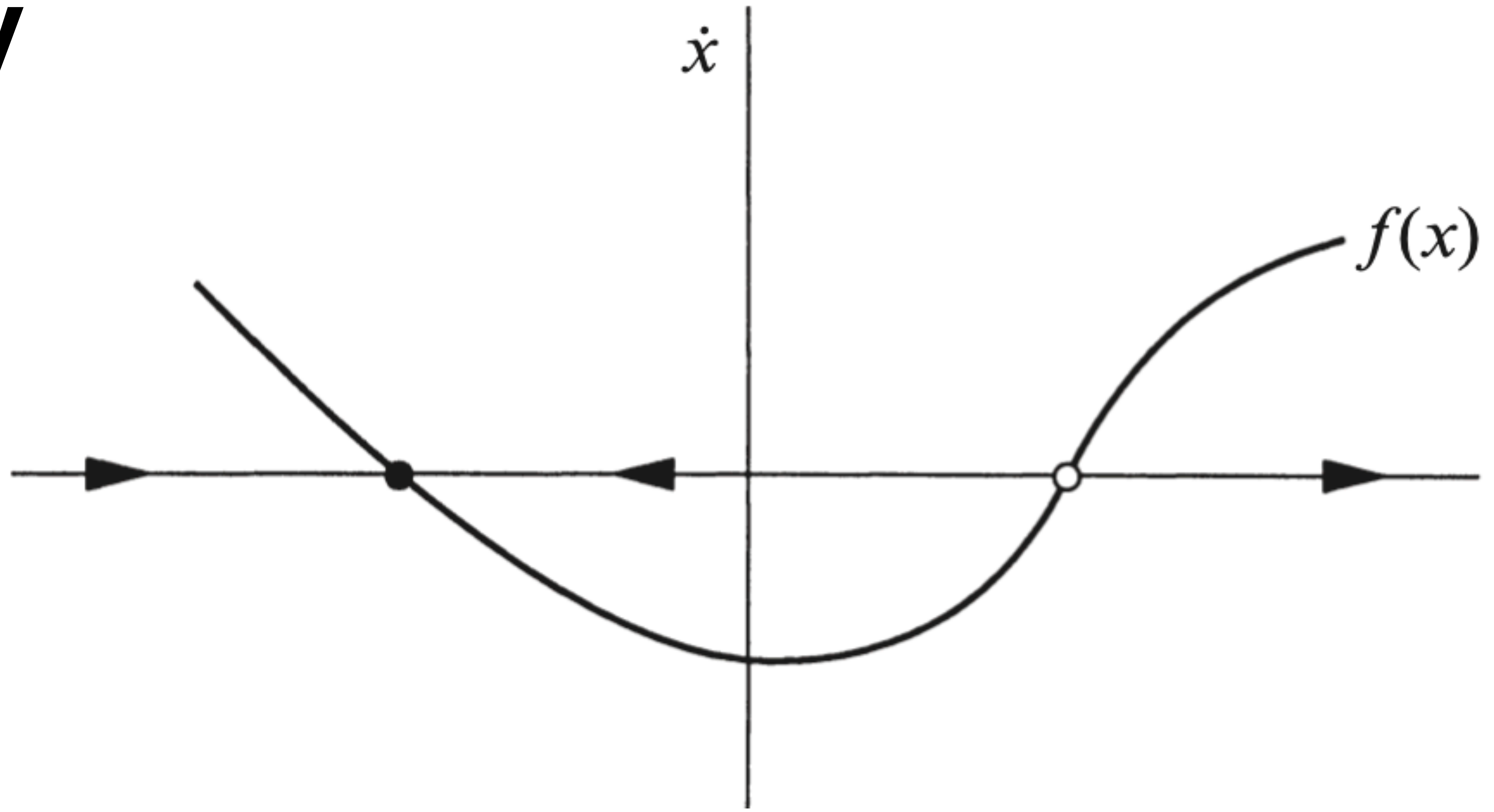
If $x_0 = 0$ *or* $x_0 = K$, then x = constant.

Tutorial Time:

You should go to the course GitHub repository and click on **Python 1D systems notebook.**

https://github.com/MSc-Fundamental-Physics/nonlinear-dynamics-chaos

Fixed Points and Stability



$$\dot{x} = f(x)$$

Phase portrait: sketch of the vector field on the real line.

A fluid is flowing along the real line with a local velocity $f(x)$.

This imaginary fluid is called the **phase fluid**, and the real line is the **phase space**.

To find the solution to $\dot{x} = f(x)$ starting from an arbitrary initial condition x_0, we place an imaginary particle (***phase point***) at x and see how it is carried by the flow.

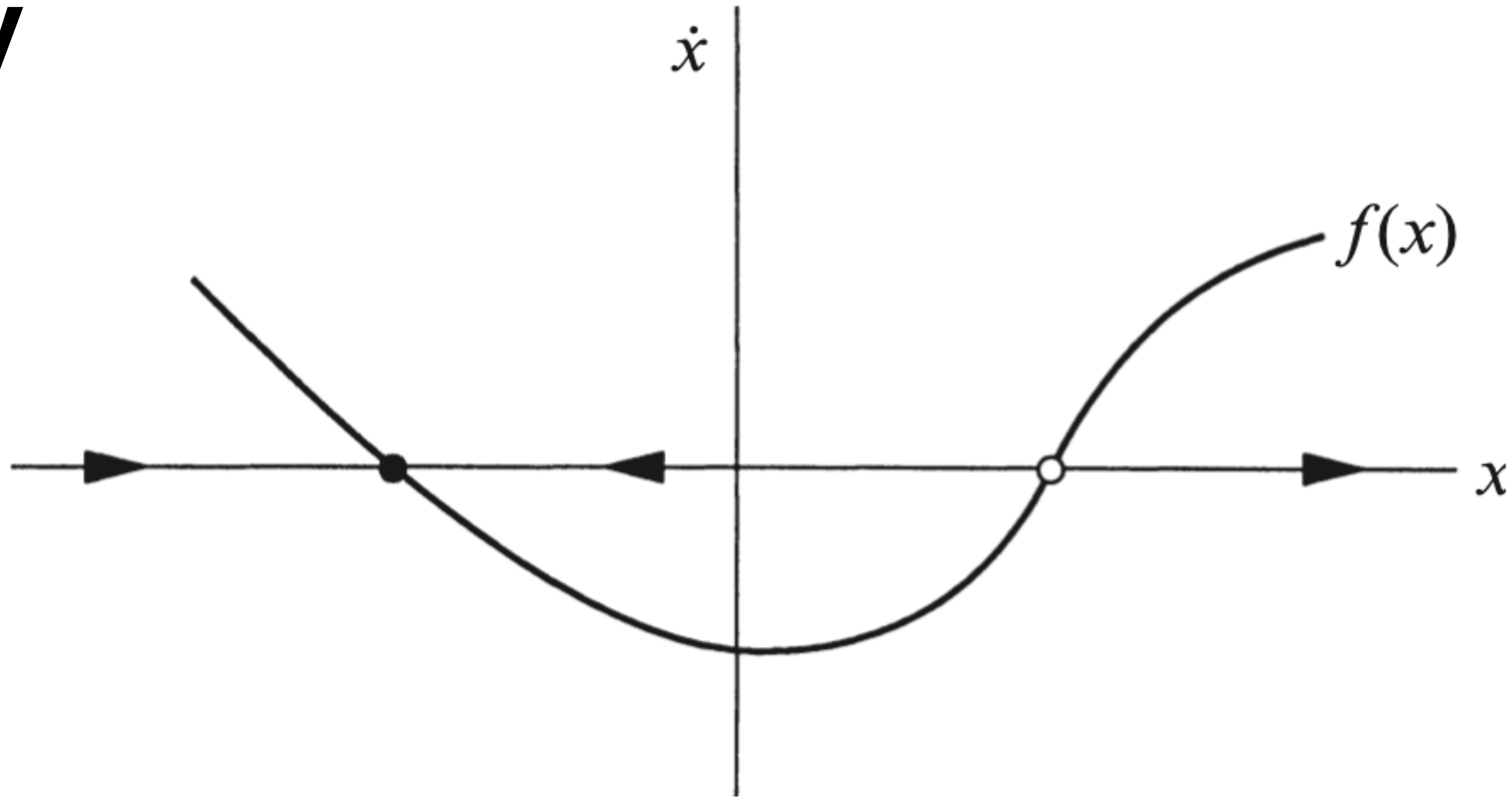
$x(t)$ is the trajectory based at x_0 .

Fixed Points and Stability

$\dot{x} = f(x)$

Fixed points: $x*$ defined by
$f(x^*) = 0$

Fixed points are stagnation points of the flow.



Fixed points represent ***equilibrium*** solutions (steady, constant, or rest solutions).

If $x = x*$ initially, then $x(t) = x*$ for all time

Stable equilibrium: all small disturbances away from it damp out in time.

Unstable equilibrium: disturbances grow in time.

Linear Stability Analysis

We want to have a quantitative measure of stability, such as the rate of decay to a stable fixed point (linearising about a fixed point).

Small perturbation away from *x**: $\eta(t) = x(t) - x^*$

Does the perturbation grows or decays? We need an ODE:

$$\dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x} \quad \Rightarrow \quad \dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$$

Taylor expanding this: $f(x^* + \eta) = \cancel{f(x^*)} + \eta f'(x^*) + O(\eta^2)$

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2) \quad \Rightarrow \quad \dot{\eta} \approx \eta f'(x^*)$$

(Linearisation about x*)

Linear Stability Analysis

Linearisation about a fixed point:

$$\dot{\eta} \approx \eta f'(x^*)$$

The perturbation η grows exponentially if $f'(x^*) > 0$ and decays if $f'(x^*) < 0$.

If $f'(x^*) = 0$, non-linear stability analysis is required.

The slope $f'(x^*)$ at the fixed point determines its stability (magnitude of f′(x*)).

The slope is always negative at a stable fixed point.

Characteristic time scale: the time required for *x*(*t*) to vary significantly in the neighbourhood of *x**.

$$1/|f'(x^*)|$$

Existence and Uniqueness of Solutions to ODEs

Consider the initial value problem:

$$\dot{x} = f(x), \qquad x(0) = x_0$$

If f(x) and f'(x) are continuous (f is continuously differentiable), the solutions exist and are unique.

If f(x) is smooth enough, then solutions exist and are unique. Even so, there's no guarantee that solutions exist forever.

Example: $\dot{x} = 1 + x^2, \; x(0) = x_0$

Do solutions exist for all time? No, solutions blow up.

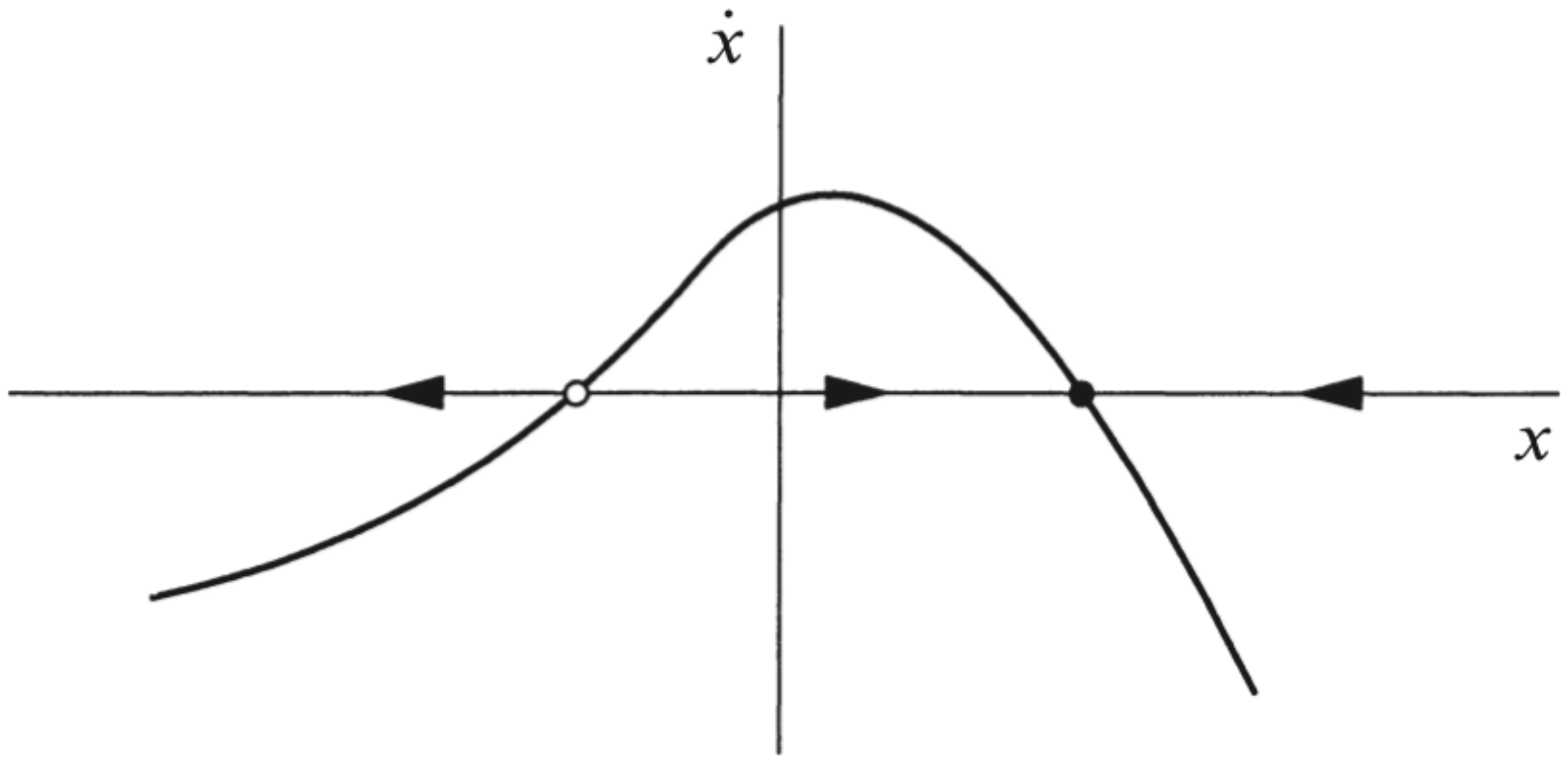
$$\int \frac{dx}{1+x^2} = \int dt, \qquad \tan^{-1} x = t + C$$

$$x(t) = \tan t$$

$$-\pi/2 < t < \pi/2$$

Impossibility of Oscillations

Fixed points dominate the dynamics of first-order systems.



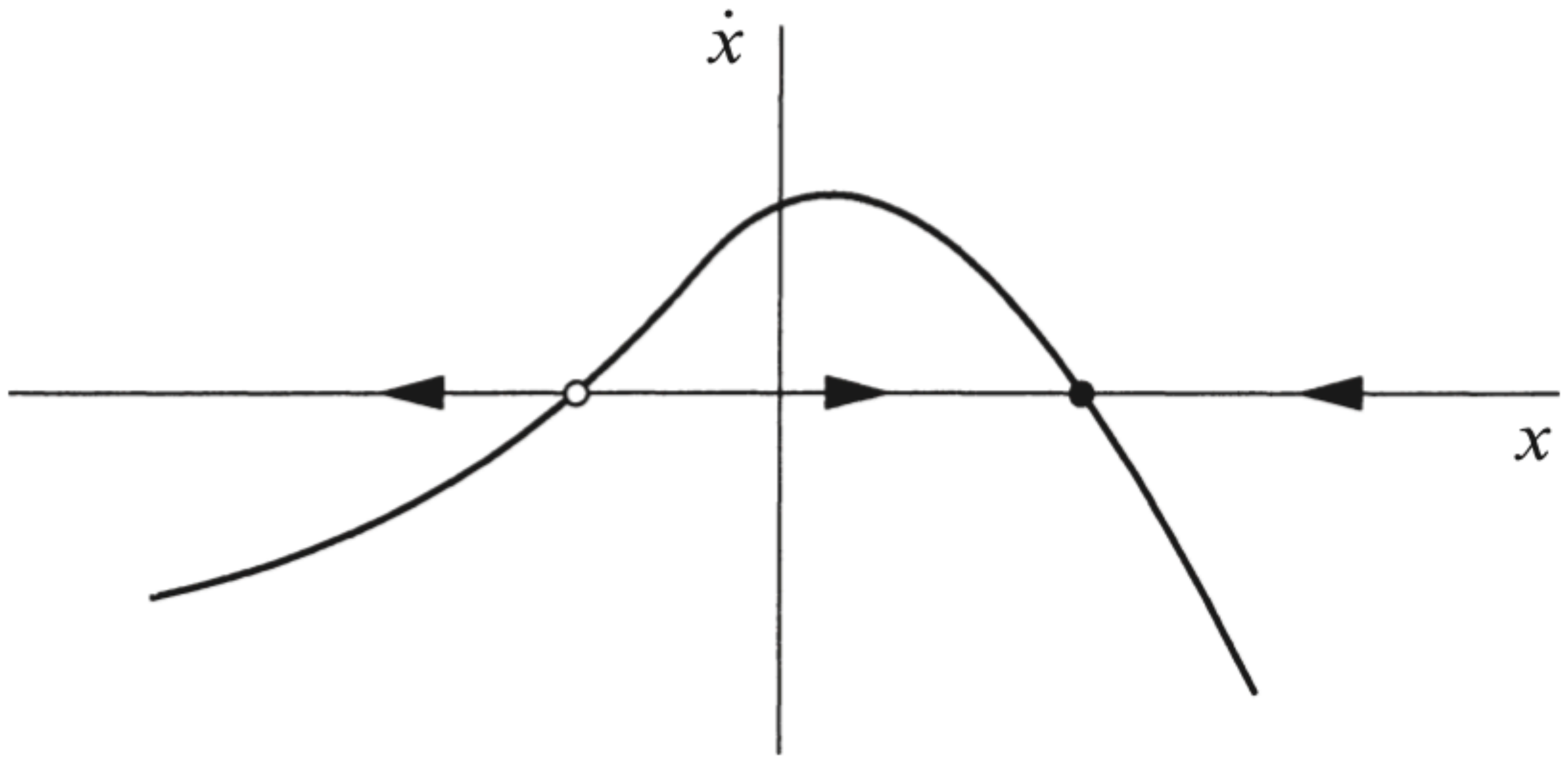
All trajectories either approached a fixed point, or diverged to infinity. Trajectories are forced to increase or decrease monotonically, or remain constant.

Those are the only things that can happen for a vector field on the real line.

Geometrically, the phase point never reverses direction.

Impossibility of Oscillations

There are NO oscillations in these systems! No chaos!



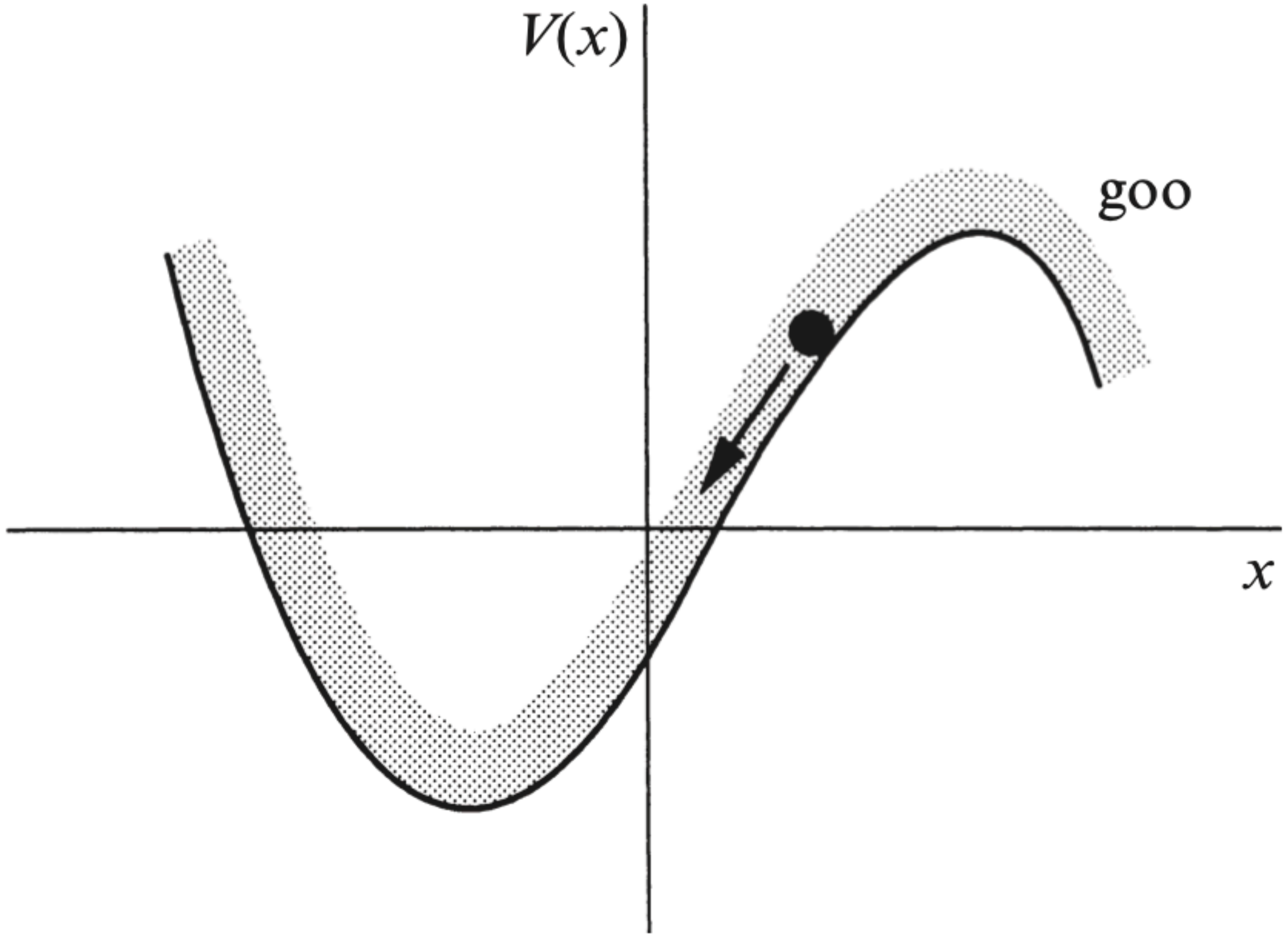
The approach to equilibrium is always *monotonic*—overshoot and damped oscillations can never occur in a first-order system.

Undamped oscillations, periodic solutions are impossible.

These general results are **fundamentally topological** in origin: If the particle flows monotonically on a line, it will never come back to its starting place.

Impossibility of Oscillations: Potentials

Over-damped motion: the inertia is completely negligible compared to the damping force and the force due to the potential



$$\dot{x} = f(x)$$

$$f(x) = -\frac{dV}{dx}$$

$$\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$

Local minima of $V(x)$ correspond to stable fixed points.

Local maxima correspond to unstable fixed points.