Nonlinear Dynamics and Chaos

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Lorenz equations

We begin our study of chaos with the **Lorenz equations:**

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx-y-xz\\ \dot{z} &= xy-bz.\end{aligned}$$

Here σ, r, $b>0$ are parameters.

Ed Lorenz (1963) derived this 3D system from a drastically simplified model of convection rolls in the atmosphere. The same equations also arise in models of lasers and dynamos.

They *exactly* describe the motion of a certain **waterwheel**.

Lorenz equations: a strange attractor and fractal

Lorenz discovered that this simple-looking deterministic system could have extremely erratic dynamics:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

Over a wide range of parameters, the solutions oscillate irregularly, never exactly repeating but always remaining in a bounded region of phase space.

When he plotted the trajectories in three dimensions, he discovered that they settled onto a complicated set, **now called a strange attractor.**

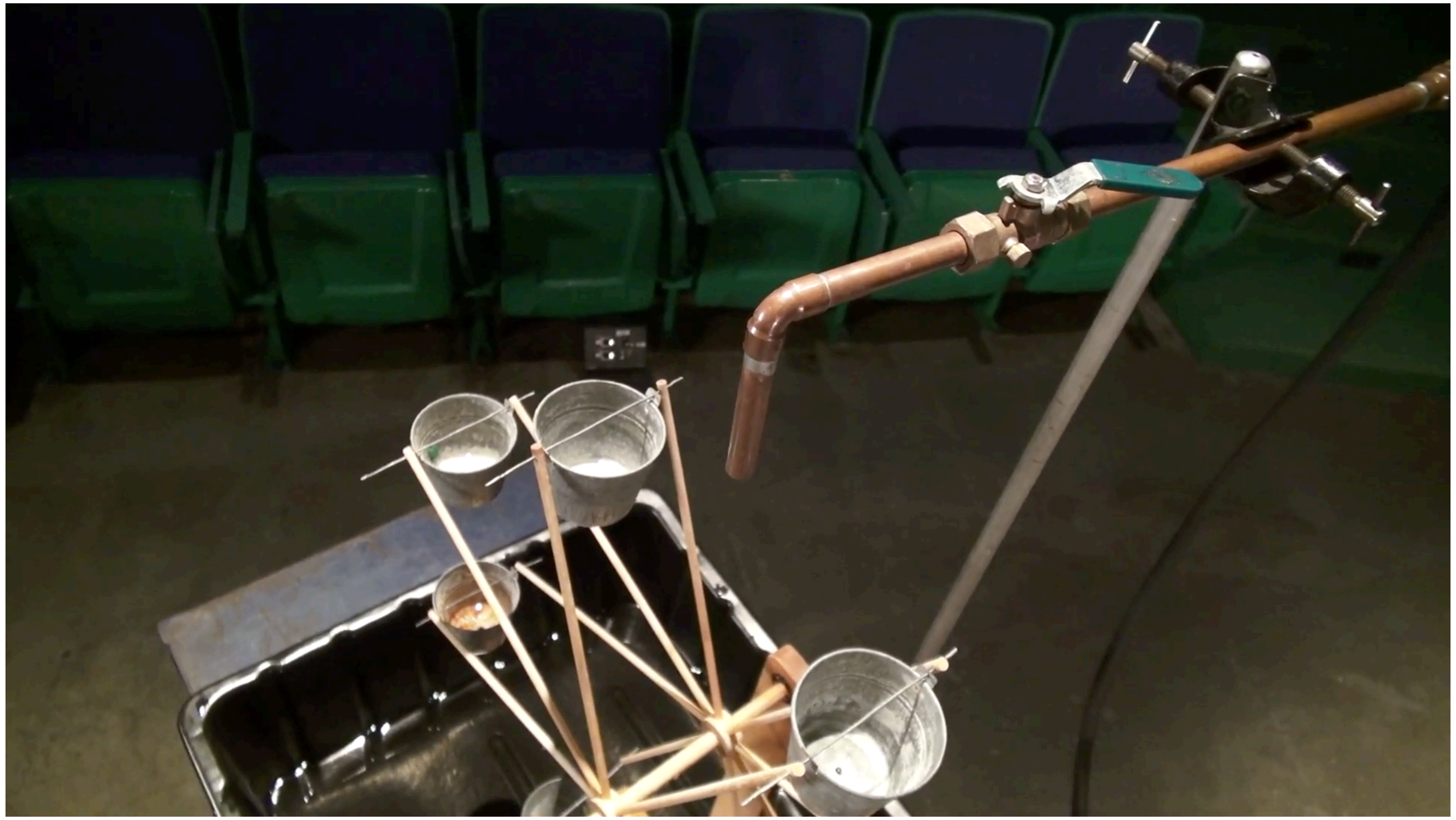
Unlike stable fixed points and limit cycles, the strange attractor is not a point or a curve or even a surface. It’s a **fractal**, with a fractional dimension between 2 and 3.

A Chaotic Waterwheel

A mechanical model of the Lorenz equations was invented by Malkus and Howard at MIT in the 1970s

The simplest version is a toy waterwheel with leaky paper cups suspended from its rim.

A Chaotic Waterwheel



https://www.youtube.com/watch?v=Lx8gMBJBlP8

Water is poured in steadily from the top. If the flow rate is too slow, the top cups never fill up enough to overcome friction, so the wheel remains motionless.

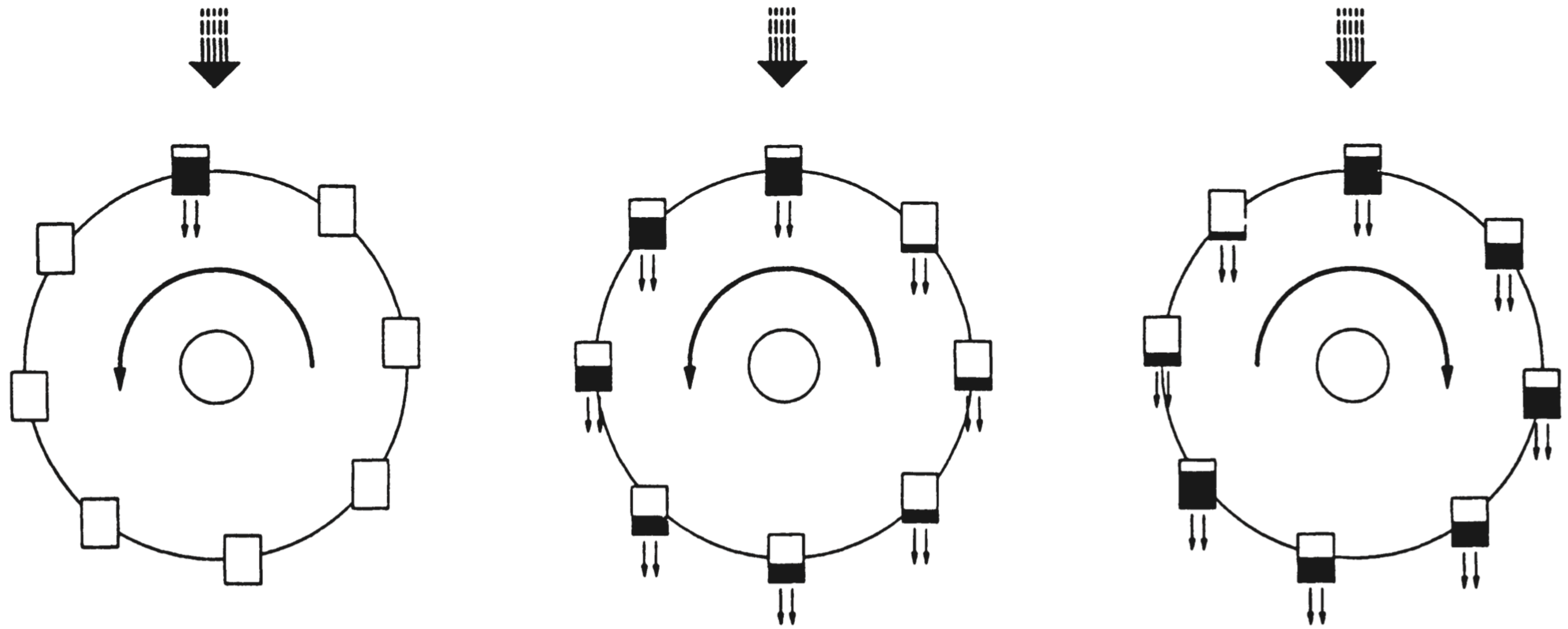
For faster inflow, the top cup gets heavy enough to start the wheel turning.

Eventually the wheel settles into a steady rotation in one direction or the other.

By symmetry, rotation in either direction is equally possible; the outcome depends on the initial conditions.

A Chaotic Waterwheel

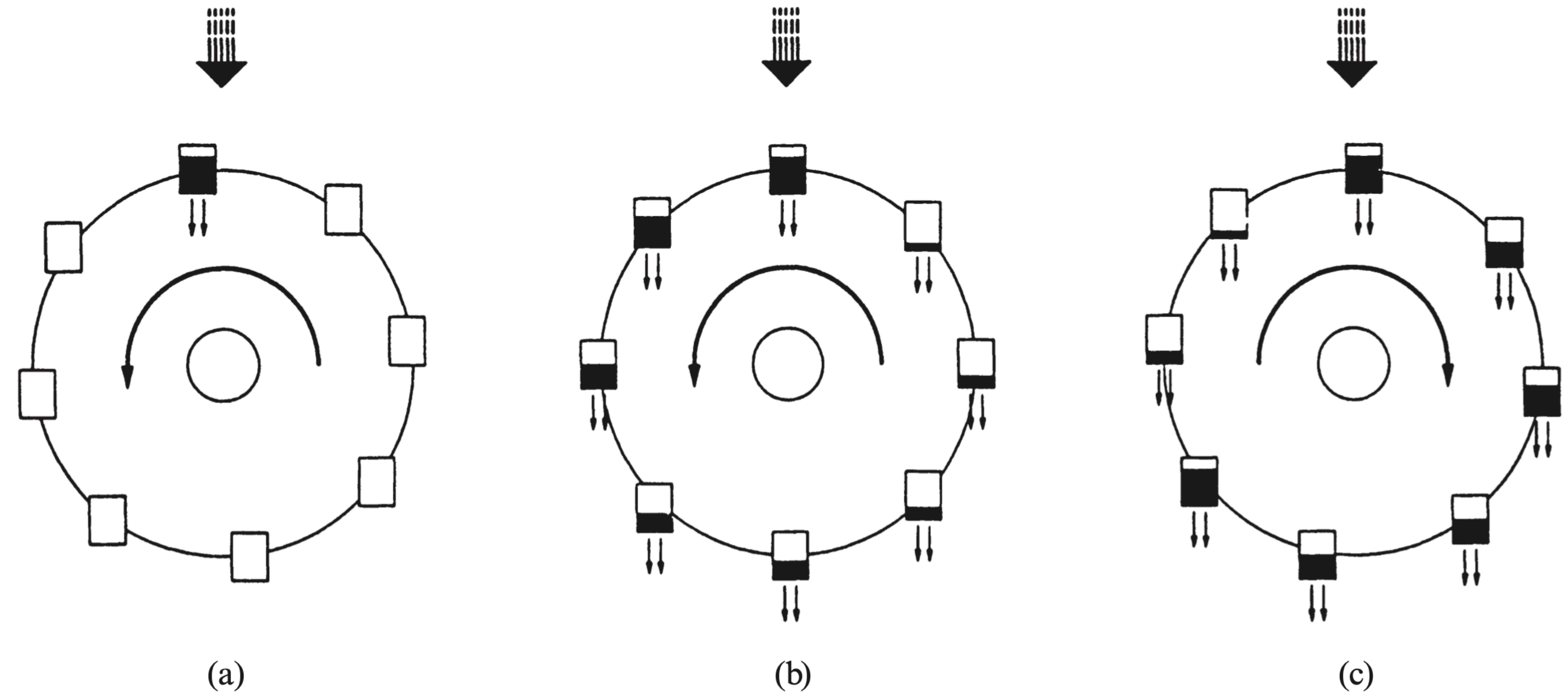
By increasing the flow rate still further, we can destabilise the steady rotation.



(a) (b) (c)

The motion becomes chaotic: the wheel rotates one way for a few turns, then some of the cups get too full and the wheel doesn't have enough inertia to carry them over the top, so the wheel slows down and may even reverse its direction.

A Chaotic Waterwheel



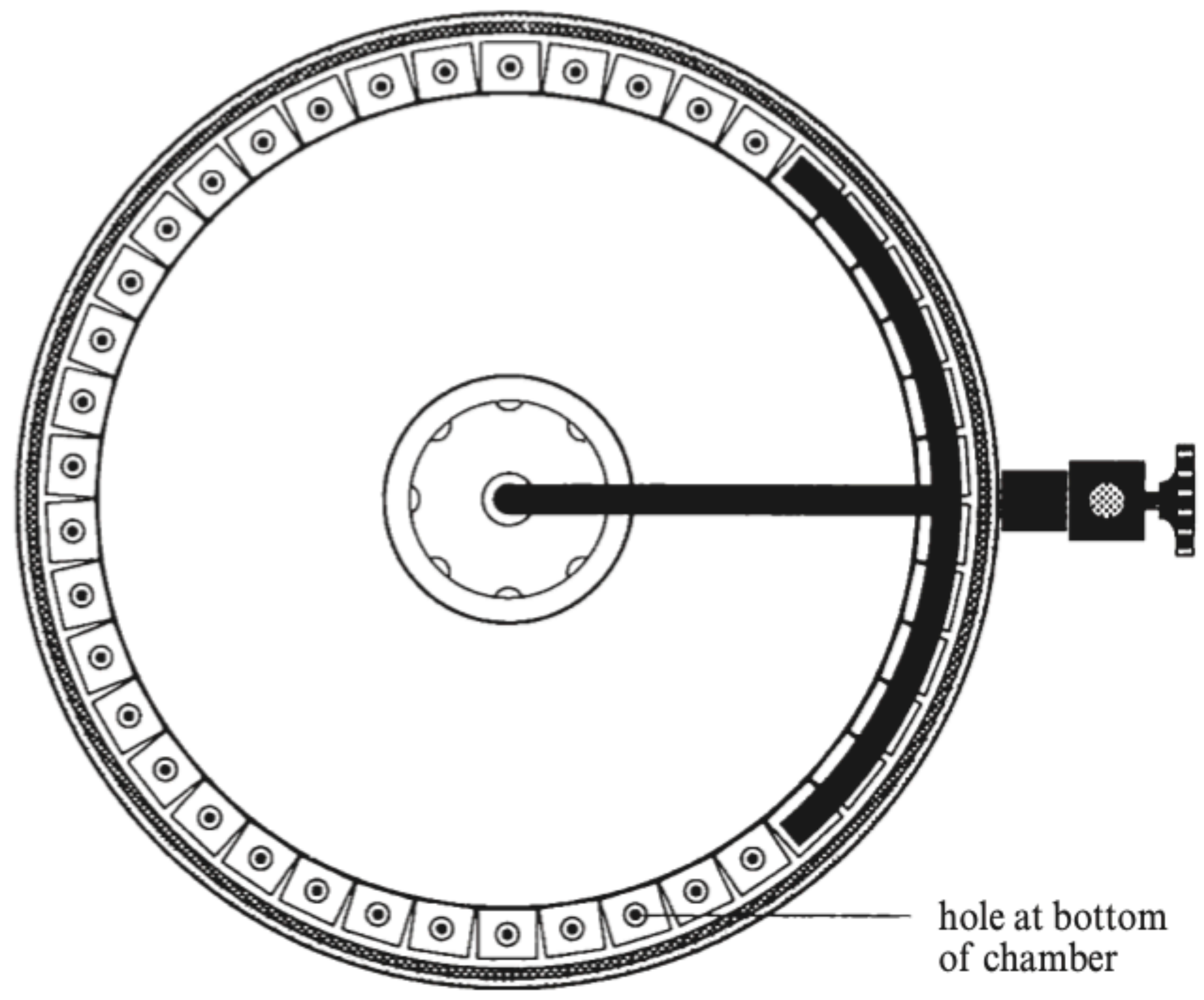
It spins the other way for a while. The wheel keeps changing direction erratically.

Malkus's Chaotic Waterwheel

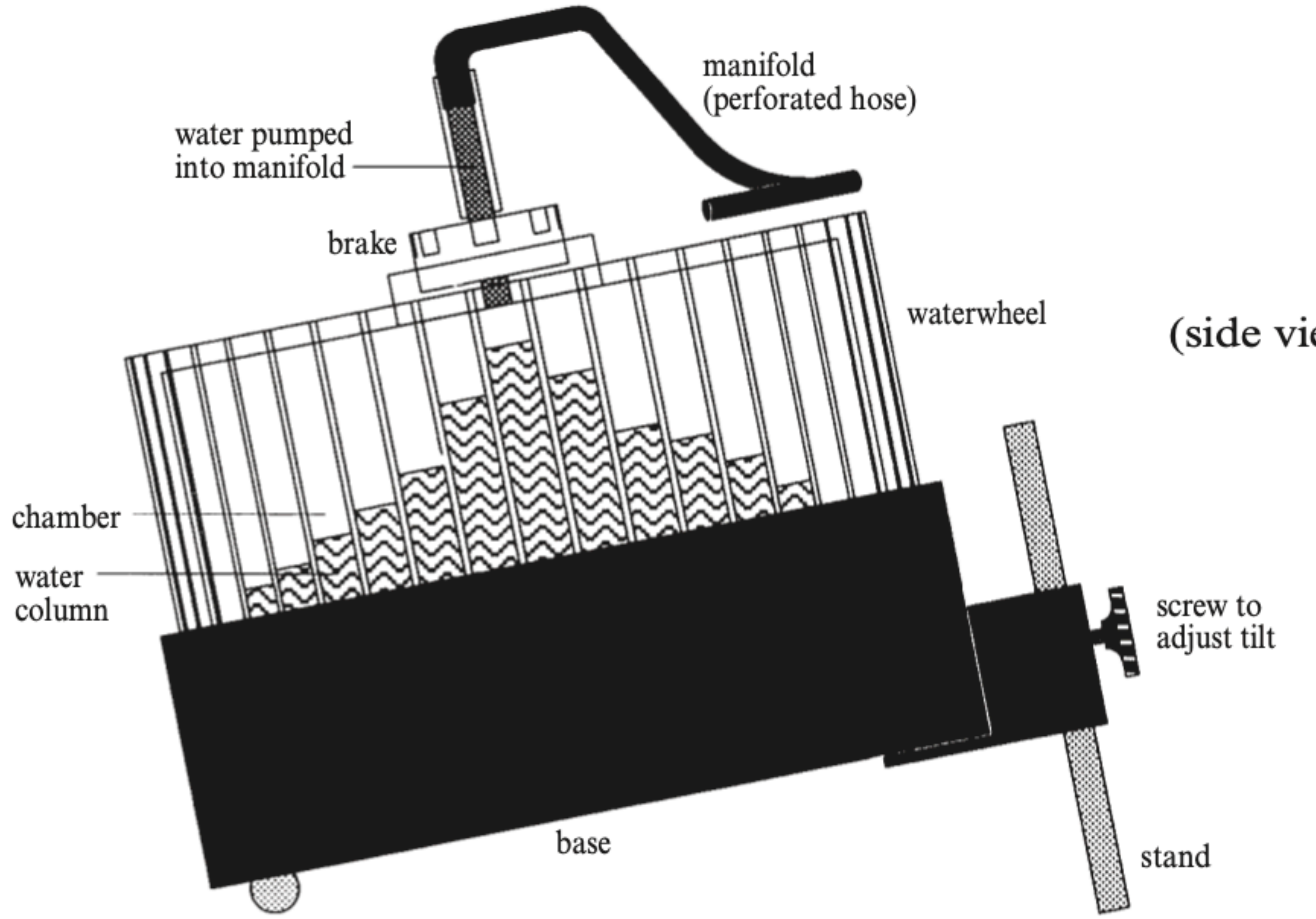
The wheel sits on a table top. It rotates in a plane that is tilted slightly from the horizontal.

Water is pumped up into an overhanging manifold and then sprayed out through dozens of small nozzles. The nozzles direct the water into separate chambers around the rim of the wheel.

The chambers are transparent, and the water has food colouring in it, so the distribution of water around the rim is easy to see.



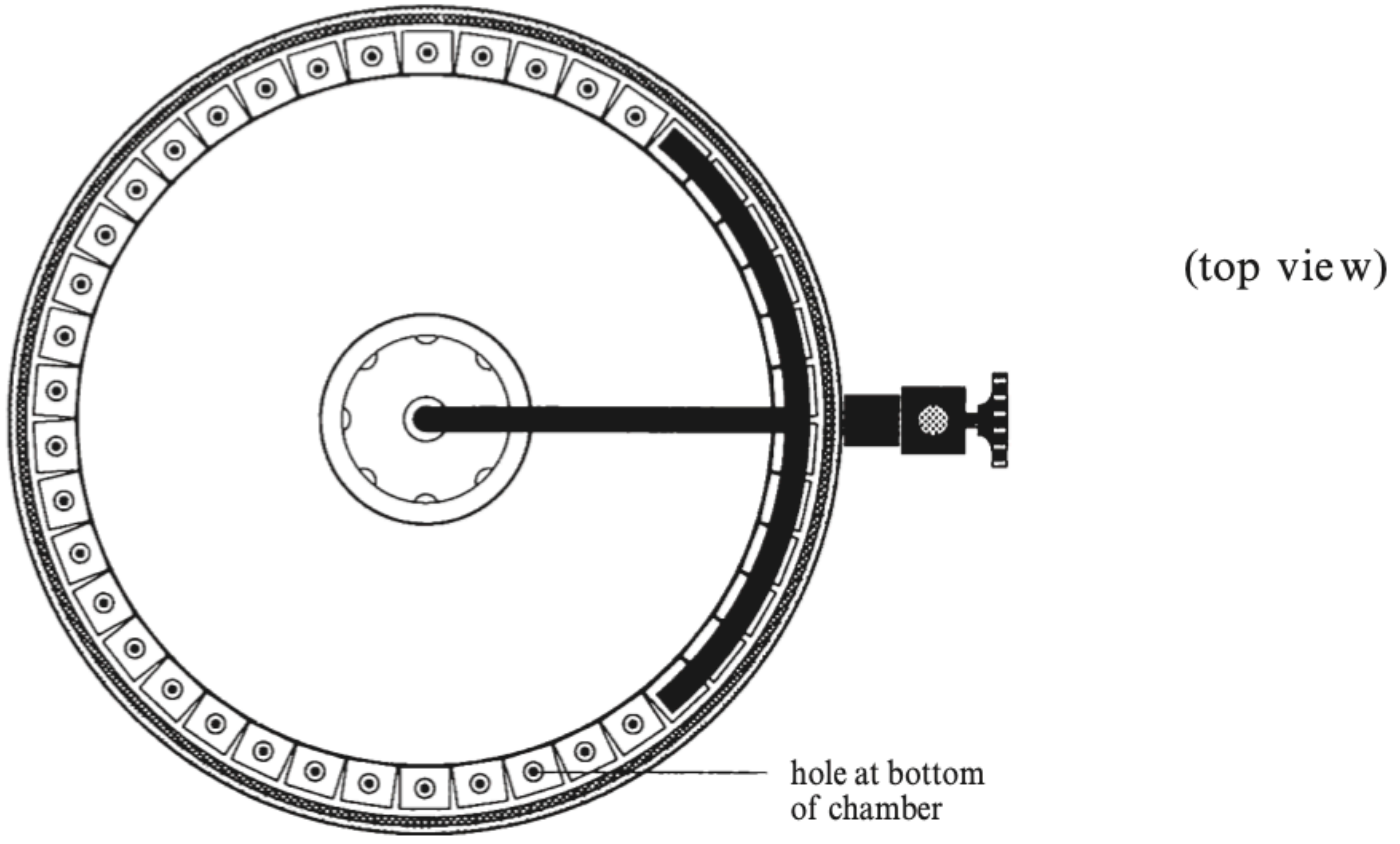
(top view)



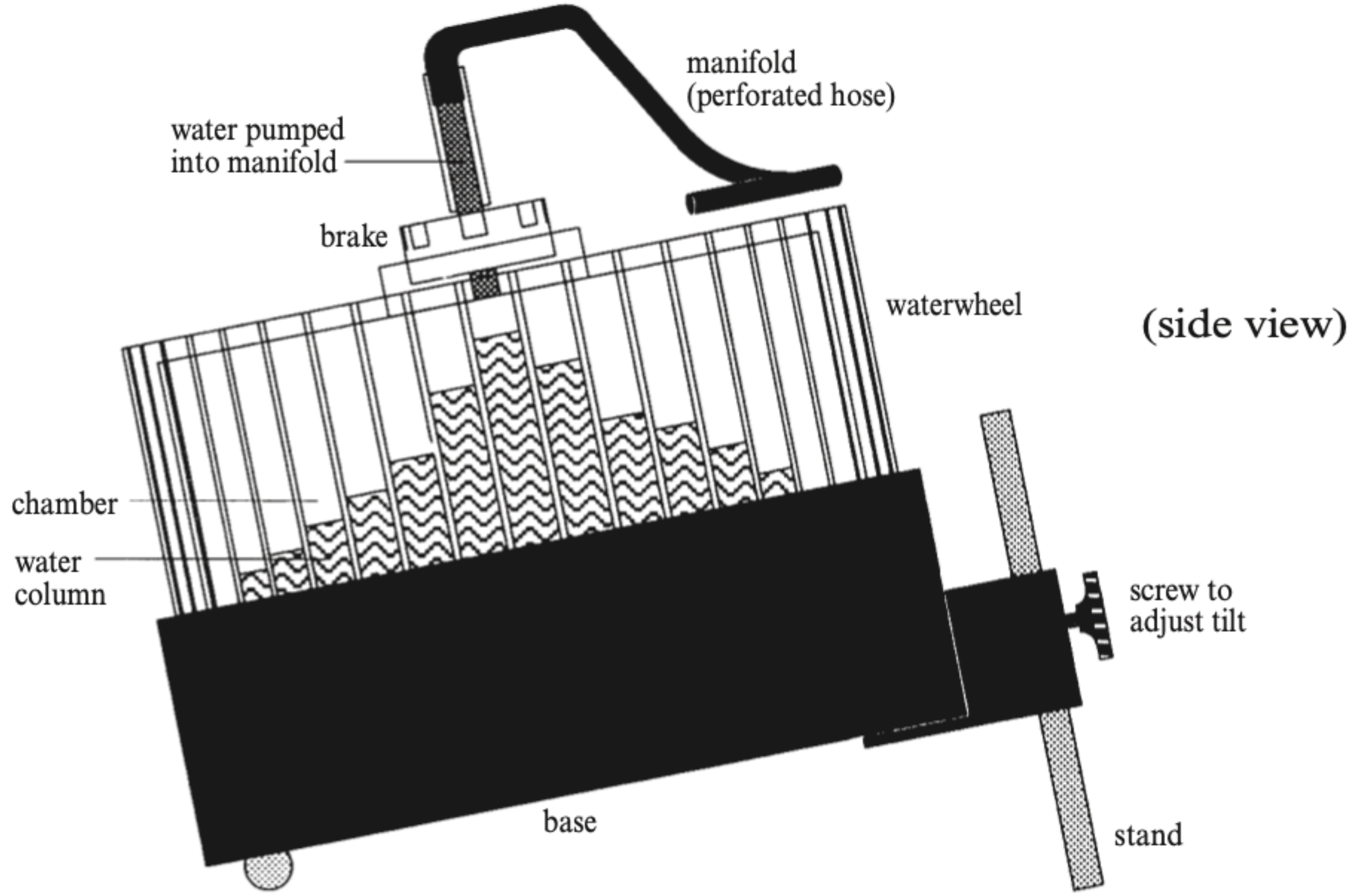
(side view)

Malkus's Chaotic Waterwheel

The water leaks out through a small hole at the bottom of each chamber, and then collects underneath the wheel, where it is pumped back up through the nozzles. This system provides a steady input of water.



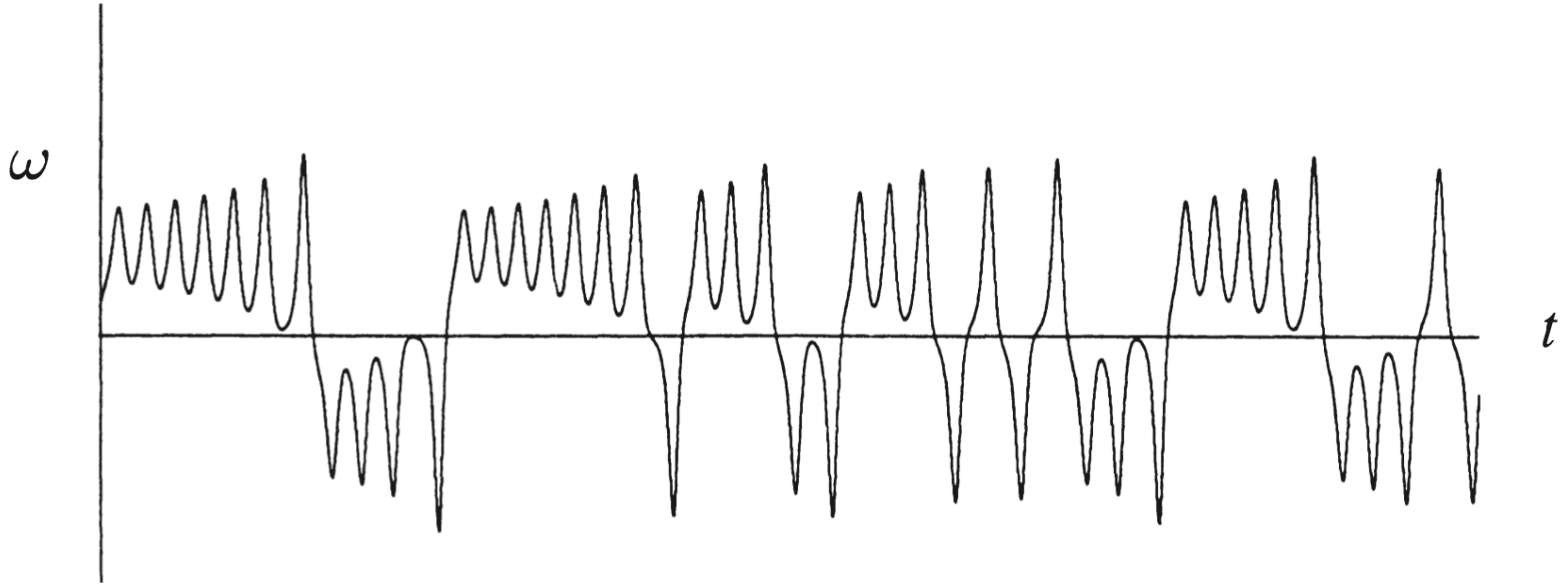
The parameters can be changed in two ways. A **brake on the wheel** can be adjusted to add more or less friction. The **tilt of the wheel** can be varied by turning a screw that props the wheel up; this alters the effective strength of gravity.



Malkus's Chaotic Waterwheel

A sensor measures the wheel's angular velocity $\omega(t)$, and sends the data to a strip chart recorder which then plots $\omega(t)$ in real time.

The Figure on the right shows a record of $\omega(t)$ when the wheel is rotating chaotically. Notice once again the irregular sequence of reversals.



Malkus's Chaotic Waterwheel

We want to explain where this chaos comes from, and to understand the bifurcations that cause the wheel to go from static equilibrium to steady rotation to irregular reversals.

Conservation of mass

Torque balance

Amplitude Equations

Fourier series

The resulting equations are in a closed system:

$$\dot{a}_1 = \omega b_1 - K a_1$$
$$\dot{b}_1 = -\omega a_1 - K b_1 + q_1$$
$$\dot{\omega} = (-\nu\omega + \pi g r a_1)/I$$

Equivalent to:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

Malkus's Chaotic Waterwheel

$$\dot{a}_1 = \omega b_1 - K a_1$$
$$\dot{b}_1 = -\omega a_1 - K b_1 + q_1$$
$$\dot{\omega} = (-\nu\omega + \pi g r a_1)/I$$

$r =$ radius of the wheel
$K =$ leakage rate
$\nu =$ rotational damping rate
$I =$ moment of inertia of the wheel

a_1: First Fourier coefficient of the **mass distribution** of water around the wheel It measures the **cosine component** of how much water is on the wheel. Physically, how much heavier the right side is compared to the left.

b_1: Second (sine) Fourier coefficient of the **mass distribution** Measures the **sine component** of water imbalance. Physically, how much heavier the *top* is compared to the bottom.

q_1: Fourier component of the **water inflow rate** (captures how water is being poured in at the top).

Chaotic Waterwheel: Fixed points

$$a_1 = \omega b_1 / K$$
$$\omega a_1 = q_1 - K b_1$$
$$a_1 = v\omega / \pi g r.$$

$$\Rightarrow \quad b_1 = \frac{K q_1}{\omega^2 + K^2}$$

$$\omega b_1 / K = v\omega / \pi g r. \quad \Rightarrow \quad \omega = 0, \quad b_1 = K v / \pi g r.$$

There are 2 kinds of fixed points:

1) $\omega = 0$ $\quad a_1 = 0, \quad b_1 = q_1 / K.$ $\quad \Rightarrow \quad (a_1^*, b_1^*, \omega^*) = (0,\ q_1/K, 0)$

This corresponds to a state of **no rotation.** The wheel is at rest, with inflow balanced by leakage.

Chaotic Waterwheel: Fixed points

2) $\omega \neq 0$ ➡ $b_1 = Kq_1/(\omega^2 + K^2) = Kv/\pi gr.$

$K \neq 0$ ➡ $q_1/(\omega^2 + K^2) = v/\pi gr.$

➡ $(\omega^*)^2 = \dfrac{\pi g r q_1}{v} - K^2$

If the RHS is positive, there are two solutions: $\pm\omega^*$ corresponding to steady rotation in either direction.

Condition -> Rayleigh number $\dfrac{\pi g r q_1}{K^2 v} > 1$

Chaotic Waterwheel: Rayleigh number

Rayleigh number $$\frac{\pi g r q_1}{K^2 v} > 1$$

Measurement of how hard we're driving the system, relative to the dissipation.

This ratio expresses a competition between g and q_1 (gravity and inflow, which tend to spin the wheel), and K and v (leakage and damping, which tend to stop the wheel).

Steady rotation is possible only if the Rayleigh number is large enough.

Rayleigh number and analogy to convection

The Rayleigh number appears in other parts of **fluid mechanics**, notably convection, in which a layer of fluid is heated from below. It is proportional to the difference in temperature from bottom to top.

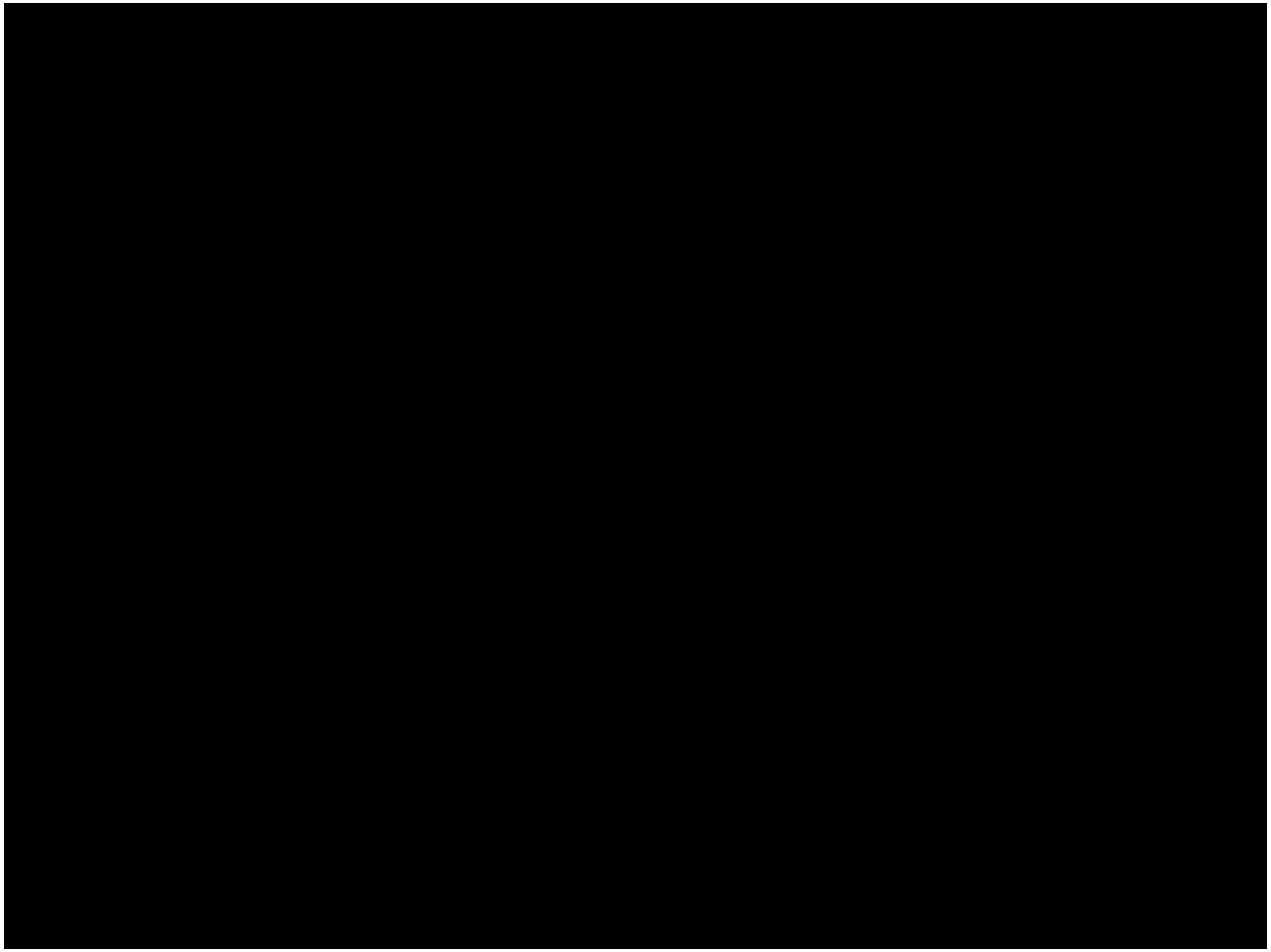
For small temperature gradients, heat is conducted vertically but the fluid remains motionless.

When the Rayleigh number increases past a critical value, an instability occurs: **the hot fluid is less dense and begins to rise, while the cold fluid on top begins to sink.**

This sets up a pattern of **convection rolls**, completely analogous to the steady rotation of our waterwheel.

With further increases of the Rayleigh number, **the rolls become wavy and eventually chaotic.**

Rayleigh number and analogy to convection



The analogy to the waterwheel breaks down at still higher Rayleigh numbers, when **turbulence develops and the convective motion becomes complex** in space as well as time.

The waterwheel settles into a **pendulum-like pattern of reversals**, turning once to the left, then back to the right, and so on indefinitely.

https://www.youtube.com/watch?v=M-yXOEsDivs