

UNIT-III

DYNAMIC PROGRAMMING

Dynamic Programming is a method to solve the given problem by taking sequence of decisions. In order to get the optimal solution of the given problem. We should write the possible decisions by using principal of optimality; we will select the optimal solution of the given problem.

The difference between greedy method and dynamic programming is, In greedy method we take only one decision to find out the solution. Whereas in dynamic programming we take sequence of decisions, which satisfy the condition and finally we get the optimal solution by using principal of optimality.

Principal of optimality:

The principle of optimality states that no matter what the first decision, the remaining decisions must be optimal with respect to the state that results from this first decision. This principle implies that an optimal decision sequence is comprised for some formulations of some problem.

Since the principle of optimality may not hold for some formulations of some problems, it is necessary to verify that it does not hold for the problem being solved.

Dynamic programming cannot be applied when this principle does not hold.

Steps for Dynamic programming :

1 : Characteristics the given problem by using mathematical equation which gives the solution (or)

Sub-solution for given problem.

2: Recursively identify the value of the optimal solution.

3 : By using backtracking calculate the optimal solution.

4 : Finalize the optimal solution from computing information.

Applications of Dynamic Programming

1 : Optimal Binary Search Tree (OBST)

2 : All Pairs Shortest Path Problem

3 : Travelling Sales Person Problem.

4 : 0 /1 Knapsack Problem -

5 : Reliability Design

OPTIMAL BINARY SEARCH TREE (OBST) : The given set of identifiers { a₁, a₂,..., a_n} with a₁<a₂<a₃...a_n. Let p(i) be the probability with which we can search for a_i. Let q(i) be the probability that the identifier x being searched. such that a_i < x < a_{i+1} and 0 ≤ i ≤ n. In other words p(i) is the probability of successful search and q(i) be the probability of unsuccessful search.

Clearly $\sum_{1 \leq i \leq n} p(i) + \sum_{1 \leq i \leq n} q(i)$ then obtain a tree with minimum cost. Such a tree with optimum Cost is

called optimal binary search tree.

To solve this problem using dynamic programming method by using following formulas.

$$1 : c(i, j) = \min_{i < k \leq j} \{ c(i, k-1) + c(k, j) + w(i, j) \}$$

$$2 : w(i, j) = p(j) + q(j) + w(i, j-1)$$

$$3 : r(i, j) = k$$

Example1 : Using algorithm OBST compute w(i,j), r(i,j) and c(i,j), 0 ≤ i ≤ j ≤ 4 for the identifier set (a₁, a₂, a₃, a₄) = (do, while, for, if) with (p₁, p₂, p₃, p₄) = (3, 3, 1, 1) and (q₀, q₁, q₂, q₃, q₄) = (2, 3, 1, 1, 1) using r(i, j) construct the optimal binary search tree.

Solution :

Successful Probability : (p₁, p₂, p₃, p₄) = (3, 3, 1, 1)

UnSuccessful Probability : (q₀, q₁, q₂, q₃, q₄) = (2, 3, 1, 1, 1)

identifier set : (a₁, a₂, a₃, a₄) = (end, goto, print, stop)

Initial Conditions :

$$w(i,j) = q(i)$$

$$c(i,j) = 0$$

$$r(i,j) = 0$$

Formulas :

$$1. w(i, j) = p(j) + q(j) + w(i, j-1)$$

$$2. c(i, j) = \min_{i < k \leq j} \{ c(i, k-1) + c(k, j) + w(i, j) \}$$

$$3. r(i, j) = k$$

Step1 : j - i = 0

$$w(i, j) = q(i)$$

$$w(0,0) = q(0) = 2 \quad c(0,0) = 0 \quad r(0,0) = 0$$

$$w(1,1) = q(1) = 3 \quad c(1,1) = 0 \quad r(1,1) = 0$$

$$w(2,2) = q(2) = 1 \quad c(2,2) = 0 \quad r(2,2) = 0$$

$$w(3,3) = q(3) = 1 \quad c(3,3) = 0 \quad r(3,3) = 0$$

$$w(4,4) = q(4) = 1 \quad c(4,4) = 0 \quad r(4,4) = 0$$

Step 2 : j - i = 1 , (i = 0, j = 1, k = 1)

$$w(i, j) = p(j) + q(j) + w(i, j - 1)$$

$$\begin{aligned} w(0, 1) &= p(1) + q(1) + w(0,0) \\ &= 3 + 3 + 2 = 8 \end{aligned}$$

$$\begin{aligned} c(i, j) &= \min \{ c(i, k - 1) + c(k, j) \} + w(i, j) \\ i &< k \leq j \end{aligned}$$

$$\begin{aligned} c(0,1) &= \min \{ c(0,0) + c(1,1) \} + w(0,1) \\ &= \min \{ 0 + 0 \} + 8 \\ &= 0 + 8 = 8 \end{aligned}$$

$$r(0,1) = 1$$

(i = 1, j = 2, k = 2)

$$\begin{aligned} w(1, 2) &= p(2) + q(2) + w(1,1) \\ &= 3 + 1 + 3 = 7 \end{aligned}$$

$$\begin{aligned} c(i, j) &= \min \{ c(i, k - 1) + c(k, j) \} + w(i, j) \\ i &< k \leq j \end{aligned}$$

$$\begin{aligned} c(1,2) &= \min \{ c(1,1) + c(2,2) \} + w(1,2) \\ &= \min \{ 0 + 0 \} + 7 = 7 \end{aligned}$$

$$r(1,2) = 2$$

(i = 2, j = 3, k = 3)

$$\begin{aligned} w(2, 3) &= p(3) + q(3) + w(2,2) \\ &= 1 + 1 + 1 = 3 \end{aligned}$$

$$\begin{aligned} c(i, j) &= \min \{ c(i, k - 1) + c(k, j) \} + w(i, j) \\ i &< k \leq j \end{aligned}$$

$$\begin{aligned} c(2,3) &= \min \{ c(2,2) + c(3,3) \} + w(2,3) \\ &= \min \{ 0 + 0 \} + 3 = 3 \end{aligned}$$

$$r(2,3) = 3$$

(i = 3, j = 4, k = 4)

$$w(3, 4) = p(4) + q(4) + w(3, 3)$$

$$= 1 + 1 + 1 = 3$$

$$c(i, j) = \min \{ c(i, k-1) + c(k, j) \} + w(i, j)$$

$$i < k \leq j$$

$$c(3, 4) = \min \{ c(3, 3) + c(4, 4) \} + w(3, 4)$$

$$= \min \{ 0 + 0 \} + 3 = 3$$

$$r(3, 4) = 4$$

Step 3 : j - i = 2 , (i = 0, j = 2, k = 1, 2)

$$w(0, 2) = p(2) + q(2) + w(0, 1)$$

$$= 3 + 1 + 8 = 12$$

$$c(0, 2) = \min \{ c(0, 0) + c(1, 2), c(0, 1) + c(2, 2) \} + w(0, 2)$$

$$= \min \{ 0 + 7, 8 + 0 \} + 12$$

$$= \min \{ 7, 8 \} + 12$$

$$= 7 + 12 = 19$$

$$r(0, 2) = 1$$

(i = 1, j = 3, k = 2, 3)

$$w(1, 3) = p(3) + q(3) + w(1, 2)$$

$$= 1 + 1 + 7 = 9$$

$$c(1, 3) = \min \{ c(1, 1) + c(2, 3), c(1, 2) + c(3, 3) \} + w(1, 3)$$

$$= \min \{ 0 + 3, 7 + 0 \} + 9$$

$$= \min \{ 3, 7 \} + 9$$

$$= 3 + 9 = 12$$

$$r(1, 3) = 2$$

(i = 2, j = 4, k = 3)

$$w(2, 4) = p(4) + q(4) + w(2, 3)$$

$$= 1 + 1 + 3 = 5$$

$$c(2, 4) = \min \{ c(2, 2) + c(3, 4) \} + w(2, 4)$$

$$= \min \{ 0 + 3 \} + 5 = 8$$

$$r(2, 4) = 3$$

Step 4 : $j - i = 3$, ($i=0, j= 3, k=1,2,3$)

$$\begin{aligned}w(0,3) &= p(3) + q(3) + w(0,2) \\&= 1 + 1 + 12 = 14\end{aligned}$$

$$\begin{aligned}c(0,3) &= \min \{ c(0,0)+c(1,3), c(0,1)+c(2,3), c(0,2) +c(3,3) \} + w(0,3) \\&= \min \{ 0 + 12, 8 + 3 , 19 + 0 \} + 14 \\&= \min \{ 12, 11, 19 \} + 14 \\&= 11 + 14 = 25\end{aligned}$$

$$r(0,3) = 2$$

($i=1, j= 4, k=2,3,4$)

$$\begin{aligned}w(1,4) &= p(4) + q(4) + w(1,3) \\&= 1 + 1 + 9 = 11\end{aligned}$$

$$\begin{aligned}c(1,4) &= \min \{ c(1,1)+c(2,4), c(1,2)+c(3,4), c(1,3)+c(4,4) \} + w(1,4) \\&= \min \{ 0 + 8, 7 + 3, 12 + 0 \} + 11 \\&= \min \{ 8, 10, 12 \} + 11 \\&= 8 + 11 = 19\end{aligned}$$

$$r(1,4) = 2$$


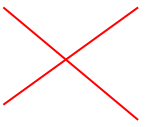
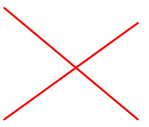

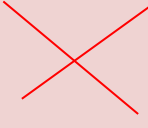
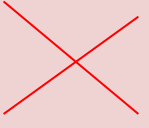
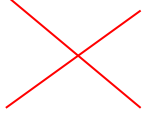



Step 5 : $j - i = 4$ ($i=0, j= 4, k=1,2,3,4$)

$$\begin{aligned}w(0,4) &= p(4) + q(4) + w(0,3) \\&= 1 + 1 + 14 = 16\end{aligned}$$

$$\begin{aligned}c(0,4) &= \min \{ c(0,0)+c(1,4), c(0,1)+c(2,4), c(0,2)+c(3,4), c(0,3)+c(4,4) \} + w(0,4) \\&= \min \{ 0 + 19, 8+8, 19+3 , 25+0 \} + 16 \\&= \min \{ 19, 16, 21, 25 \} + 16 \\&= 16 + 16 = 32\end{aligned}$$

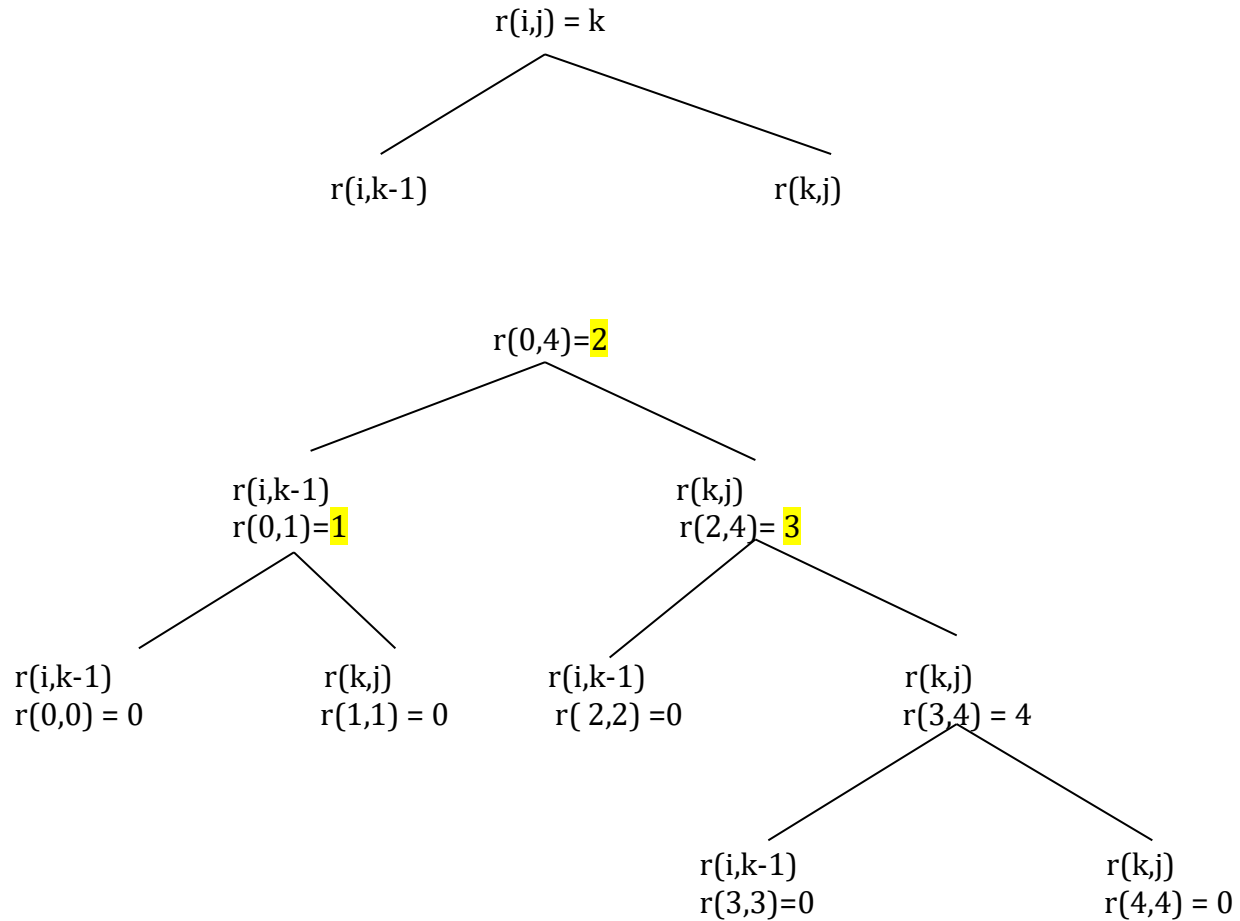
$$r(0,4) = 2$$

To build Optimal Binary Search Tree (OBST)

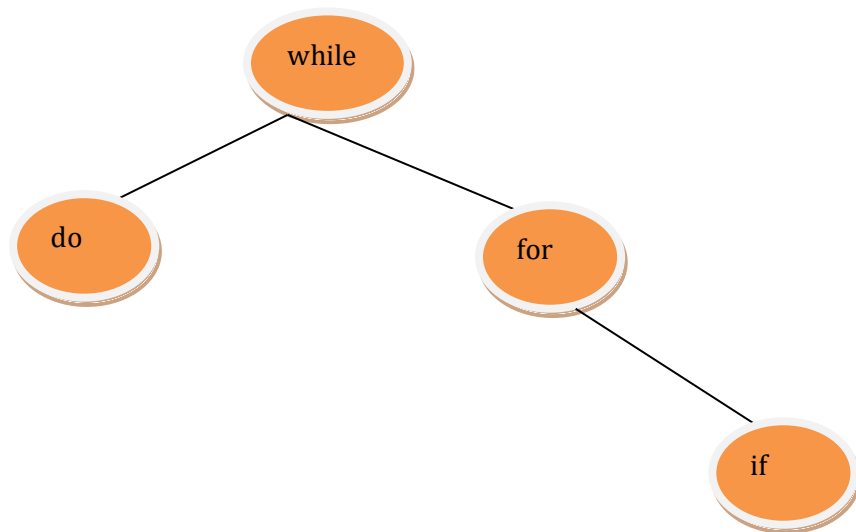
for a set(i, j)	$w(0,0) = 2$ $c(0,0) = 0$ $r(0,0) = 0$	$w(1,1) = 3$ $c(1,1) = 0$ $r(1,1) = 0$	$w(2,2) = 1$ $c(2,2) = 0$ $r(2,2) = 0$	$w(3,3) = 1$ $c(3,3) = 0$ $r(3,3) = 0$	$w(4,4) = 1$ $c(4,4) = 0$ $r(4,4) = 0$
$j - i = 1$	$w(0,1) = 8$ $c(0,1) = 8$ $r(0,1) = 1$	$w(1,2) = 7$ $c(1,2) = 7$ $r(1,2) = 2$	$w(2,3) = 3$ $c(2,3) = 3$ $r(2,3) = 3$	$w(3,4) = 3$ $c(3,4) = 3$ $r(3,4) = 4$	
$j - i = 2$	$w(0,2) = 12$ $c(0,2) = 19$ $r(0,2) = 1$	$w(1,3) = 9$ $c(1,3) = 12$ $r(1,3) = 2$	$w(2,4) = 5$ $c(2,4) = 8$ $r(2,4) = 3$		
$j - i = 3$	$w(0,3) = 14$ $c(0,3) = 25$ $r(0,3) = 2$	$w(1,4) = 11$ $c(1,4) = 19$ $r(1,4) = 2$			
$j - i = 4$	$w(0,4) = 16$ $c(0,4) = 32$ $r(0,4) = 2$				

To build OBST, $r(0,4) = 2, K = 2$.

Hence a_2 becomes root node.



$(a_1, a_2, a_3, a_4) = (\text{do}, \text{while}, \text{for}, \text{if})$



Optimal Binary Search Tree with cost = 32

Example 2 : Using the algorithm OBST, compute $W(i,j)$, $R(i,j)$ and $C(i,j)$, $0 \leq i < j \leq 4$ for the identifier set $(a_1, a_2, a_3, a_4) = (\text{end}, \text{goto}, \text{print}, \text{stop})$ with $p(1)=1/20$, $p(2)=1/5$, $p(3)=1/10$, $p(4)=1/20$; $q(0)=1/5$, $q(1)=1/10$, $q(2)=1/5$, $q(3)=1/20$ and $q(4)=1/20$. Using the $R(i,j)$'s construct the OBST.

Solution :

Successful Probability

$$P(1) = 1/20 * 20 = 1$$

$$P(2) = 1/5 * 20 = 4$$

$$P(3) = 1/10 * 20 = 2$$

$$P(4) = 1/20 * 20 = 1$$

$$(p_1, p_2, p_3, p_4) = (1, 4, 2, 1)$$

UnSuccessful Probability

$$q(0) = 1/5 * 20 = 4$$

$$q(1) = 1/10 * 20 = 2$$

$$q(2) = 1/5 * 20 = 4$$

$$q(3) = 1/20 * 20 = 1$$

$$q(4) = 1/20 * 20 = 1$$

$$(q_0, q_1, q_2, q_3, q_4) = (4, 2, 4, 1, 1)$$

$$(a_1, a_2, a_3, a_4) = (\text{end}, \text{goto}, \text{print}, \text{stop})$$

Initial Conditions :

$$w(i,j) = q(i)$$

$$c(i,j) = 0$$

$$r(i,j) = 0$$

Formulas :

1. $w(i, j) = p(j) + q(j) + w(i, j - 1)$
2. $c(i, j) = \min \{ c(i, k - 1) + c(k, j) + w(i, j) \}$
 $i < k \leq j$
3. $r(i, j) = k$

Step1 : $j - i = 0$

$$w(i, j) = q(i)$$

$$w(0,0) = q(0) = 4$$

$$c(0,0) = 0$$

$$r(0,0) = 0$$

$$w(1,1) = q(1) = 2$$

$$c(1,1) = 0$$

$$r(1,1) = 0$$

$$w(2,2) = q(2) = 4$$

$$c(2,2) = 0$$

$$r(2,2) = 0$$

$$w(3,3) = q(3) = 1$$

$$c(3,3) = 0$$

$$r(3,3) = 0$$

$$w(4,4) = q(4) = 1$$

$$c(4,4) = 0$$

$$r(4,4) = 0$$

Step 2 : $j - i = 1, (i=0, j=1, k=1)$

$$w(i, j) = p(j) + q(j) + w(i, j-1)$$

$$w(0, 1) = p(1) + q(1) + w(0,0)$$

$$= 1 + 2 + 4 = 7$$

$$c(i, j) = \min \{ c(i, k-1) + c(k, j) \} + w(i, j)$$

$$i < k \leq j$$

$$c(0,1) = \min \{ c(0,0) + c(1,1) \} + w(0,1)$$

$$= \min \{ 0 + 0 \} + 7$$

$$= 0 + 7 = 7$$

$$r(0,1) = 1$$

$(i=1, j=2, k=2)$

$$w(1, 2) = p(2) + q(2) + w(1,1)$$

$$= 4 + 4 + 2 = 10$$

$$c(i, j) = \min \{ c(i, k-1) + c(k, j) \} + w(i, j)$$

$$i < k \leq j$$

$$c(1,2) = \min \{ c(1,1) + c(2,2) \} + w(1,2)$$

$$= \min \{ 0 + 0 \} + 10 = 10$$

$$r(1,2) = 2$$

$(i=2, j=3, k=3)$

$$w(2, 3) = p(3) + q(3) + w(2,2)$$

$$= 2 + 1 + 4 = 7$$

$$c(i, j) = \min \{ c(i, k-1) + c(k, j) \} + w(i, j)$$

$$i < k \leq j$$

$$c(2,3) = \min \{ c(2,2) + c(3,3) \} + w(2,3)$$

$$= \min \{ 0 + 0 \} + 7 = 7$$

$$r(2,3) = 3$$

(i = 3, j = 4, k = 4)

$$w(3,4) = p(4) + q(4) + w(3,3)$$

$$= 1 + 1 + 1 = 3$$

$$c(i,j) = \min \{ c(i, k-1) + c(k, j) \} + w(i, j)$$

$$i < k \leq j$$

$$c(3,4) = \min \{ c(3,3) + c(4,4) \} + w(3,4)$$

$$= \min \{ 0 + 0 \} + 3 = 3$$

$$r(3,4) = 4$$

Step 3 : j - i = 2 , (i = 0, j = 2, k = 1, 2)

$$w(0,2) = p(2) + q(2) + w(0,1)$$

$$= 4 + 4 + 7 = 15$$

$$c(0,2) = \min \{ c(0,0) + c(1,2), c(0,1) + c(2,2) \} + w(0,2)$$

$$= \min \{ 0 + 10, 7 + 0 \} + 15$$

$$= \min \{ 10, 7 \} + 15$$

$$= 7 + 15 = 22$$

$$r(0,2) = 2$$

(i = 1, j = 3, k = 2, 3)

$$w(1,3) = p(3) + q(3) + w(1,2)$$

$$= 2 + 1 + 10 = 13$$

$$c(1,3) = \min \{ c(1,1) + c(2,3), c(1,2) + c(3,3) \} + w(1,3)$$

$$= \min \{ 0 + 7, 10 + 0 \} + 14$$

$$= \min \{ 7, 10 \} + 13$$

$$= 7 + 13 = 20$$

$$r(1,3) = 2$$

(i = 2, j = 4, k = 3)

$$w(2,4) = p(4) + q(4) + w(2,3)$$

$$= 1 + 1 + 7 = 9$$

$$c(2,4) = \min \{ c(2,2) + c(3,4) \} + w(2,4)$$

$$= \min \{ 0 + 3 \} + 9 = 12$$

$$r(2,4) = 3$$

Step 4 : j - i = 3 , (i = 0, j = 3, k = 1, 2, 3)

$$\begin{aligned}w(0,3) &= p(3) + q(3) + w(0,2) \\ &= 2 + 1 + 15 = 18\end{aligned}$$

$$\begin{aligned}c(0,3) &= \min \{ c(0,0)+c(1,3), c(0,1)+c(2,3), c(0,2) + c(3,3) \} + w(0,3) \\ &= \min \{ 0 + 20, 7 + 7, 22 + 0 \} + 18 \\ &= \min \{ 17, 14, 22 \} + 18 \\ &= 14 + 18 = 32\end{aligned}$$

$$r(0,3) = 2$$

(i = 1, j = 4, k = 2, 3, 4)

$$\begin{aligned}w(1,4) &= p(4) + q(4) + w(1,3) \\ &= 1 + 1 + 13 = 15\end{aligned}$$

$$\begin{aligned}c(1,4) &= \min \{ c(1,1)+c(2,4), c(1,2)+c(3,4), c(1,3)+c(4,4) \} + w(1,4) \\ &= \min \{ 0 + 12, 10 + 3, 20 + 0 \} + 15 \\ &= \min \{ 12, 13, 20 \} + 16 \\ &= 12 + 15 = 27\end{aligned}$$

$$r(1,4) = 2$$

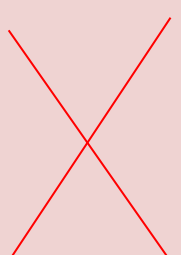
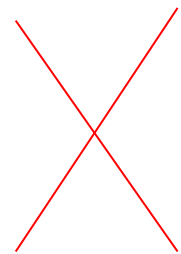
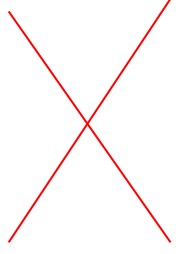
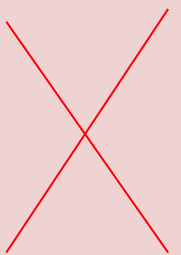
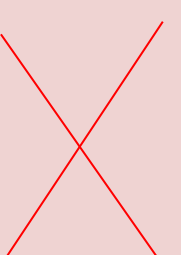
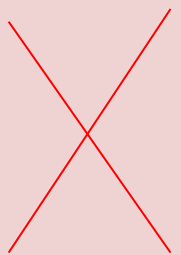
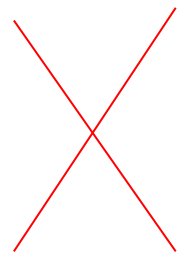
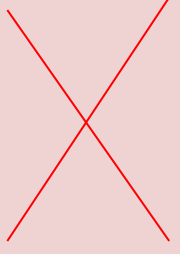
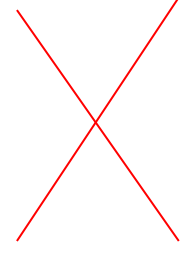
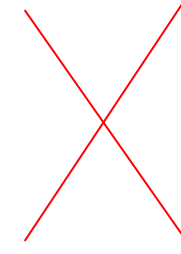
Step 5 : j - i = 4 (i = 0, j = 4, k = 1, 2, 3, 4)

$$\begin{aligned}w(0,4) &= p(4) + q(4) + w(0,3) \\ &= 1 + 1 + 18 = 20\end{aligned}$$

$$\begin{aligned}c(0,4) &= \min \{ c(0,0)+c(1,4), c(0,1)+c(2,4), c(0,2)+c(3,4), c(0,3)+c(4,4) \} + w(0,4) \\ &= \min \{ 0 + 27, 7 + 12, 22 + 3, 32 + 0 \} + 21 \\ &= \min \{ 27, 19, 25, 32 \} + 20 \\ &= 19 + 20 = 39\end{aligned}$$

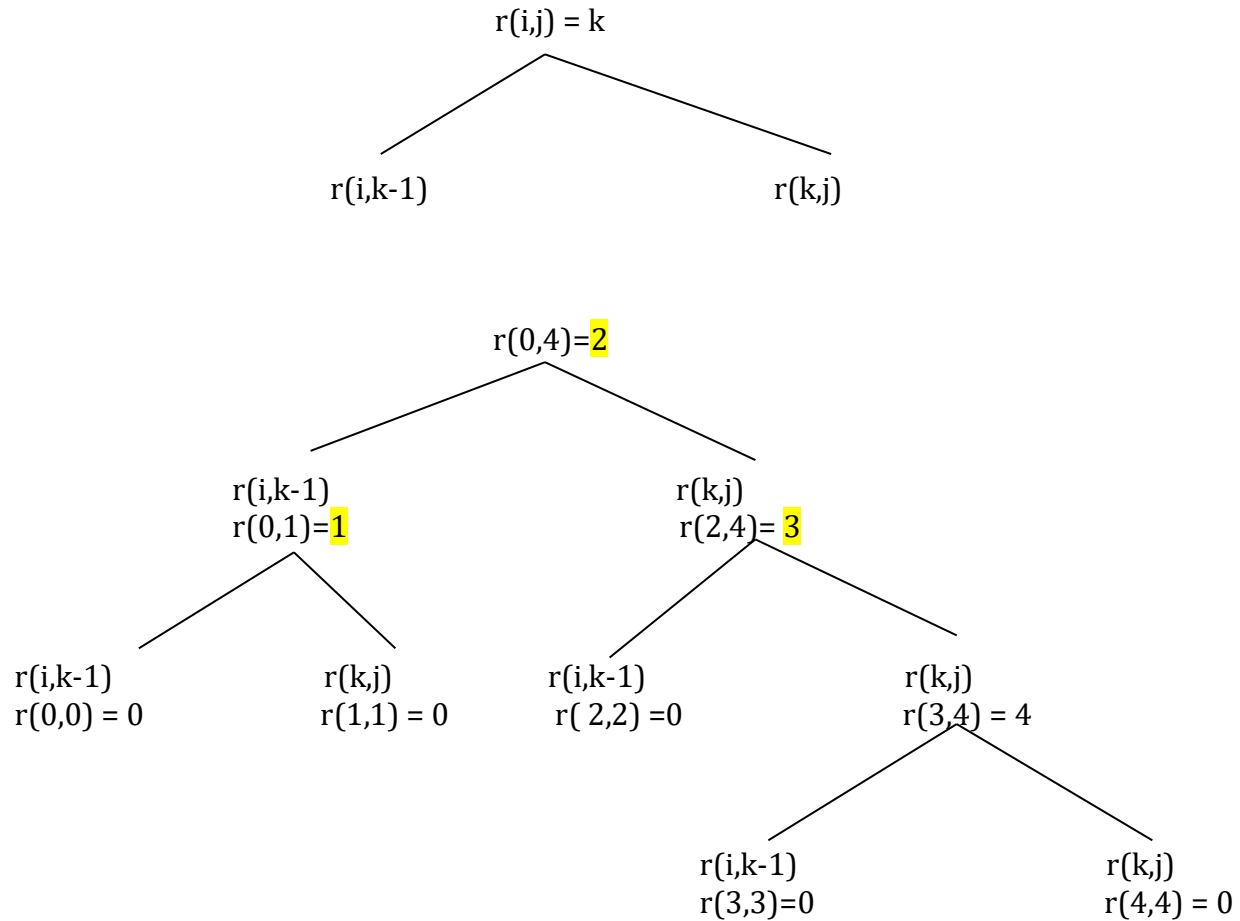
$$r(0,4) = 2$$

To build Optimal Binary Search Tree (OBST)

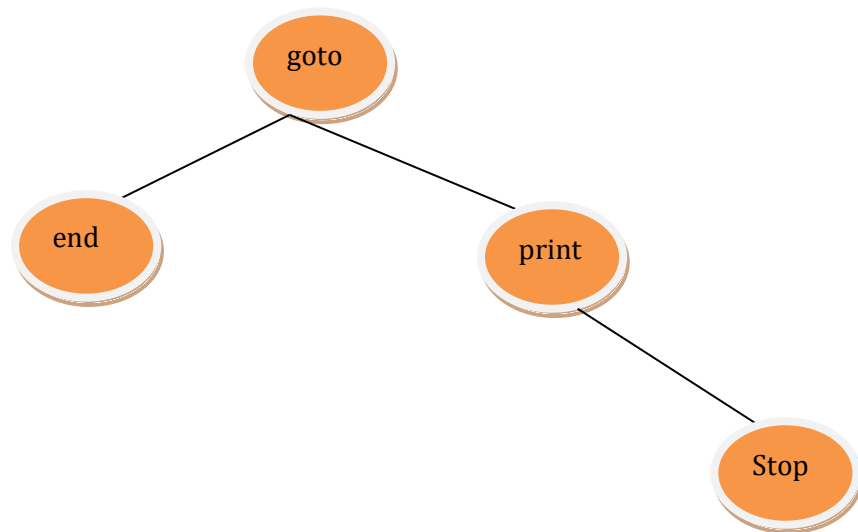
for a set(i, j) j - i = 0	w (0, 0) = 4 c (0, 0) = 0 r (0, 0) = 0	w (1, 1) = 2 c (1, 1) = 0 r (1, 1) = 0	w (2, 2) = 4 c (2, 2) = 0 r (2, 2) = 0	w (3, 3) = 1 c (3, 3) = 0 r (3, 3) = 0	w (4, 4) = 1 c (4, 4) = 0 r (4, 4) = 0
j - i = 1	w (0, 1) = 7 c (0, 1) = 7 r (0, 1) = 1	w (1, 2) = 10 c (1, 2) = 10 r (1, 2) = 2	w (2, 3) = 7 c (2, 3) = 7 r (2, 3) = 3	w (3, 4) = 3 c (3, 4) = 3 r (3, 4) = 4	
j - i = 2	w (0, 2) = 15 c (0, 2) = 22 r (0, 2) = 2	w (1, 3) = 13 c (1, 3) = 20 r (1, 3) = 2	w (2, 4) = 9 c (2, 4) = 12 r (2, 4) = 3		
j - i = 3	w (0, 3) = 18 c (0, 3) = 21 r (0, 3) = 2	w (1, 4) = 15 c (1, 4) = 27 r (1, 4) = 2			
j - i = 4	w (0, 4) = 20 c (0, 4) = 39 r (0, 4) = 2				

To build OBST, $r(0,4) = 2, K = 2$.

Hence a_2 becomes root node.



$(a_1, a_2, a_3, a_4) = (\text{end}, \text{goto}, \text{print}, \text{stop})$



Optimal Binary Search Tree with cost = 39

ALL PAIRS SHORTEST PATHS PROBLEM: Let $G = (V, E)$ be a directed graph consisting of n vertices and each edge is associated with a weight. The problem of finding the shortest path between all pairs of vertices in a graph is called all pairs shortest path problem.

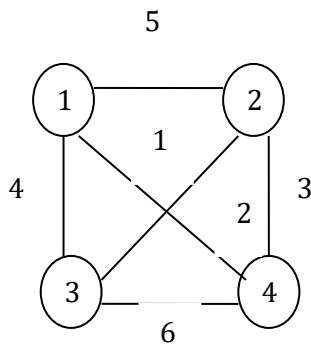
This problem can be solved by using Dynamic Programming technique. The all pairs shortest path problem is to determine a matrix A such that $A(i, j)$ is the length of a shortest path from vertex i to vertex j . Assume that this path contains no cycles. If k is an intermediate vertex on this path, then the sub paths from i to k and from k to j are the shortest paths from i to k and k to j respectively. Otherwise the path i to j is not shortest path. If k is intermediate vertex with highest index, then the path i to k is the shortest path going through no vertex with index greater than $k - 1$. Similarly the path k to j is the shortest path going through no vertex with index greater than $k - 1$.

The shortest path can be computed using following recursive method.

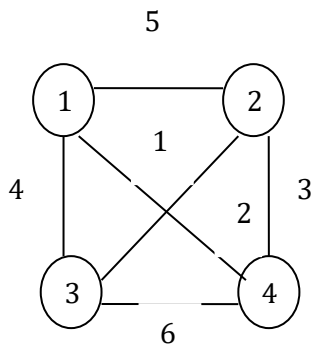
$$A^k(i, j) = w(i, j), \quad \text{if } k = 0$$

$$= \min \{ A^{k-1}(i, j), A^{k-1}(i, k) + A^{k-1}(k, j) \}, \quad \text{if } k \geq 1.$$

Ex : find the shortest path between all pairs of node in the following graph.



Sol :



Weight matrix for this undirected graph is as follows.

$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 5 & 4 & 1 \\ 5 & 0 & 2 & 3 \\ 4 & 2 & 0 & 6 \\ 1 & 3 & 6 & 0 \end{pmatrix} = A^0$$

The formula for finding the all pairs shortest path problem as follows,

$$A^k(i, j) = \min \{ A^{k-1}(i, j), A^{k-1}(i, k) + A^{k-1}(k, j) \}$$

Where if $k \geq 1$.

A^k , number of iterations to be taken .

These iterations depends on the number of weights of vertices given, in our problem, we have 5 vertices.

we will take 4 iterations and we will always start from A^0 is the given matrix. So we have to find A^1, A^2, A^3, A^4 for the shortest path. According to the formula

K = 1

$$\begin{aligned}A^1(1, 1) &= \min \{ A^{1-1}(1, 1), A^{1-1}(1, 1) + A^{1-1}(1, 1) \} \\&= \min \{ 0, 0 + 0 \} = 0\end{aligned}$$

$$\begin{aligned}A^1(1, 2) &= \min \{ A^{1-1}(1, 2), A^{1-1}(1, 1) + A^{1-1}(1, 2) \} \\&= \min \{ 5, 0 + 5 \} = 5\end{aligned}$$

$$\begin{aligned}A^1(1, 3) &= \min \{ A^{1-1}(1, 3), A^{1-1}(1, 1) + A^{1-1}(1, 3) \} \\&= \min \{ 4, 0 + 4 \} = 4\end{aligned}$$

$$\begin{aligned}A^1(1, 4) &= \min \{ A^{1-1}(1, 4), A^{1-1}(1, 1) + A^{1-1}(1, 4) \} \\&= \min \{ 1, 0 + 1 \} = 1\end{aligned}$$

$$\begin{aligned}A^1(2, 1) &= \min \{ A^{1-1}(2, 1), A^{1-1}(2, 1) + A^{1-1}(1, 1) \} \\&= \min \{ 5, 5 + 0 \} = 5\end{aligned}$$

$$\begin{aligned}A^1(2, 2) &= \min \{ A^{1-1}(2, 2), A^{1-1}(2, 1) + A^{1-1}(1, 2) \} \\&= \min \{ 0, 5 + 5 \} = 0\end{aligned}$$

$$\begin{aligned}A^1(2, 3) &= \min \{ A^{1-1}(2, 3), A^{1-1}(2, 1) + A^{1-1}(1, 3) \} \\&= \min \{ 2, 5 + 4 \} = 2\end{aligned}$$

$$\begin{aligned}A^1(2, 4) &= \min \{ A^{1-1}(2, 4), A^{1-1}(2, 1) + A^{1-1}(1, 4) \} \\&= \min \{ 3, 5 + 1 \} = 3\end{aligned}$$

$$\begin{aligned}A^1(3, 1) &= \min \{ A^{1-1}(3, 1), A^{1-1}(3, 1) + A^{1-1}(1, 1) \} \\&= \min \{ 4, 4 + 0 \} = 4\end{aligned}$$

$$\begin{aligned}A^1(3, 2) &= \min \{ A^{1-1}(3, 2), A^{1-1}(3, 1) + A^{1-1}(1, 2) \} \\&= \min \{ 2, 4 + 5 \} = 2\end{aligned}$$

$$\begin{aligned}A^1(3, 3) &= \min \{ A^{1-1}(3, 3), A^{1-1}(3, 1) + A^{1-1}(1, 3) \} \\&= \min \{ 0, 4 + 4 \} = 0\end{aligned}$$

$$\begin{aligned}A^1(3, 4) &= \min \{ A^{1-1}(3, 4), A^{1-1}(3, 1) + A^{1-1}(1, 4) \} \\&= \min \{ 6, 4 + 1 \} = 5\end{aligned}$$

$$\begin{aligned}A^1(4, 1) &= \min \{ A^{1-1}(4, 1), A^{1-1}(4, 1) + A^{1-1}(1, 1) \} \\&= \min \{ 1, 1 + 0 \} = 1\end{aligned}$$

$$\begin{aligned}A^1(4, 2) &= \min \{ A^{1-1}(4, 2), A^{1-1}(4, 1) + A^{1-1}(1, 2) \} \\&= \min \{ 3, 1 + 5 \} = 3\end{aligned}$$

$$\begin{aligned}A^1(4, 3) &= \min \{ A^{1-1}(4, 3), A^{1-1}(4, 1) + A^{1-1}(1, 3) \} \\&= \min \{ 6, 1 + 4 \} = 5\end{aligned}$$

$$\begin{aligned}A^1(4, 4) &= \min \{ A^{1-1}(4, 4), A^{1-1}(4, 1) + A^{1-1}(1, 4) \} \\&= \min \{ 0, 1 + 1 \} = 0\end{aligned}$$

$$A^1 = \begin{pmatrix} 0 & 5 & 4 & 1 \\ 5 & 0 & 2 & 3 \\ 4 & 2 & 0 & 5 \\ 1 & 3 & 5 & 0 \end{pmatrix}$$

K = 2

$$\begin{aligned} A^2(1, 1) &= \min \{ A^{2-1}(1, 1), A^{2-1}(1, 2) + A^{2-1}(2, 1) \} \\ &= \min \{ 0, 5 + 5 \} = 0 \end{aligned}$$

$$\begin{aligned} A^2(1, 2) &= \min \{ A^{2-1}(1, 2), A^{2-1}(1, 2) + A^{2-1}(2, 2) \} \\ &= \min \{ 5, 5 + 0 \} = 5 \end{aligned}$$

$$\begin{aligned} A^2(1, 3) &= \min \{ A^{2-1}(1, 3), A^{2-1}(1, 2) + A^{2-1}(2, 3) \} \\ &= \min \{ 4, 5 + 2 \} = 4 \end{aligned}$$

$$\begin{aligned} A^2(1, 4) &= \min \{ A^{2-1}(1, 4), A^{2-1}(1, 2) + A^{2-1}(2, 4) \} \\ &= \min \{ 1, 5 + 3 \} = 1 \end{aligned}$$

$$\begin{aligned} A^2(2, 1) &= \min \{ A^{2-1}(2, 1), A^{2-1}(2, 2) + A^{2-1}(2, 1) \} \\ &= \min \{ 5, 0 + 5 \} = 5 \end{aligned}$$

$$\begin{aligned} A^2(2, 2) &= \min \{ A^{2-1}(2, 2), A^{2-1}(2, 2) + A^{2-1}(2, 2) \} \\ &= \min \{ 0, 0 + 0 \} = 0 \end{aligned}$$

$$\begin{aligned} A^2(2, 3) &= \min \{ A^{2-1}(2, 3), A^{2-1}(2, 2) + A^{2-1}(2, 3) \} \\ &= \min \{ 2, 0 + 2 \} = 2 \end{aligned}$$

$$\begin{aligned} A^2(2, 4) &= \min \{ A^{2-1}(2, 4), A^{2-1}(2, 2) + A^{2-1}(2, 4) \} \\ &= \min \{ 3, 0 + 3 \} = 3 \end{aligned}$$

$$\begin{aligned} A^2(3, 1) &= \min \{ A^{2-1}(3, 1), A^{2-1}(3, 2) + A^{2-1}(2, 1) \} \\ &= \min \{ 4, 2 + 5 \} = 4 \end{aligned}$$

$$\begin{aligned} A^2(3, 2) &= \min \{ A^{2-1}(3, 2), A^{2-1}(3, 2) + A^{2-1}(2, 2) \} \\ &= \min \{ 2, 2 + 0 \} = 2 \end{aligned}$$

$$\begin{aligned} A^2(3, 3) &= \min \{ A^{2-1}(3, 3), A^{2-1}(3, 2) + A^{2-1}(2, 3) \} \\ &= \min \{ 0, 2 + 2 \} = 0 \end{aligned}$$

$$\begin{aligned} A^2(3, 4) &= \min \{ A^{2-1}(3, 4), A^{2-1}(3, 2) + A^{2-1}(2, 4) \} \\ &= \min \{ 5, 2 + 3 \} = 5 \end{aligned}$$

$$A^2(4, 1) = \min \{ A^{2-1}(4, 1), A^{2-1}(4, 2) + A^{2-1}(2, 1) \}$$

$$= \min \{ 1, 3 + 5 \} = 1$$

$$A^2(4, 2) = \min \{ A^{2-1}(4, 2), A^{2-1}(4, 2) + A^{2-1}(2, 2) \}$$

$$= \min \{ 3, 3 + 0 \} = 3$$

$$A^2(4, 3) = \min \{ A^{2-1}(4, 3), A^{2-1}(4, 2) + A^{2-1}(2, 3) \}$$

$$= \min \{ 5, 3 + 2 \} = 5$$

$$A^2(4, 4) = \min \{ A^{2-1}(4, 4), A^{2-1}(4, 2) + A^{2-1}(2, 4) \}$$

$$= \min \{ 0, 3 + 3 \} = 0$$

$$A^2 = \begin{pmatrix} 0 & 5 & 4 & 1 \\ 5 & 0 & 2 & 3 \\ 4 & 2 & 0 & 5 \\ 1 & 3 & 5 & 0 \end{pmatrix}$$

K = 3

$$A^3(1, 1) = \min \{ A^{3-1}(1, 1), A^{3-1}(1, 3) + A^{3-1}(3, 1) \}$$

$$= \min \{ 0, 4 + 4 \} = 0$$

$$A^3(1, 2) = \min \{ A^{3-1}(1, 2), A^{3-1}(1, 3) + A^{3-1}(3, 2) \}$$

$$= \min \{ 5, 5 + 0 \} = 5$$

$$A^3(1, 3) = \min \{ A^{3-1}(1, 3), A^{3-1}(1, 3) + A^{3-1}(3, 3) \}$$

$$= \min \{ 4, 5 + 2 \} = 4$$

$$A^3(1, 4) = \min \{ A^{3-1}(1, 4), A^{3-1}(1, 3) + A^{3-1}(3, 4) \}$$

$$= \min \{ 1, 4 + 5 \} = 1$$

$$A^3(2, 1) = \min \{ A^{3-1}(2, 1), A^{3-1}(2, 3) + A^{3-1}(3, 1) \}$$

$$= \min \{ 5, 4 + 4 \} = 5$$

$$A^3(2, 2) = \min \{ A^{3-1}(2, 2), A^{3-1}(2, 3) + A^{3-1}(3, 2) \}$$

$$= \min \{ 0, 2 + 2 \} = 0$$

$$A^3(2, 3) = \min \{ A^{3-1}(2, 3), A^{3-1}(2, 3) + A^{3-1}(3, 3) \}$$

$$= \min \{ 2, 2 + 0 \} = 2$$

$$A^3(2, 4) = \min \{ A^{3-1}(2, 4), A^{3-1}(2, 3) + A^{3-1}(3, 4) \}$$

$$= \min \{ 3, 2 + 5 \} = 3$$

$$A^3(3,1) = \min \{ A^{3-1}(3,1), A^{3-1}(3,3) + A^{3-1}(3,1) \}$$

$$= \min \{ 4, 0 + 4 \} = 4$$

$$A^3(3,2) = \min \{ A^{3-1}(3,2), A^{3-1}(3,3) + A^{3-1}(3,2) \}$$

$$= \min \{ 2, 0 + 2 \} = 2$$

$$A^3(3,3) = \min \{ A^{3-1}(3,3), A^{3-1}(3,3) + A^{3-1}(3,3) \}$$

$$= \min \{ 0, 0 + 0 \} = 0$$

$$A^3(3,4) = \min \{ A^{3-1}(3,4), A^{3-1}(3,3) + A^{3-1}(3,4) \}$$

$$= \min \{ 5, 0 + 5 \} = 5$$

$$A^3(4,1) = \min \{ A^{3-1}(4,1), A^{3-1}(4,3) + A^{3-1}(3,1) \}$$

$$= \min \{ 1, 5 + 4 \} = 1$$

$$A^3(4,2) = \min \{ A^{3-1}(4,2), A^{3-1}(4,3) + A^{3-1}(3,2) \}$$

$$= \min \{ 3, 5 + 2 \} = 3$$

$$A^3(4,3) = \min \{ A^{3-1}(4,3), A^{3-1}(4,3) + A^{3-1}(3,3) \}$$

$$= \min \{ 5, 5 + 0 \} = 5$$

$$A^3(4,4) = \min \{ A^{3-1}(4,4), A^{3-1}(4,3) + A^{3-1}(3,4) \}$$

$$= \min \{ 0, 5 + 5 \} = 0$$

$$A^3 = \begin{pmatrix} 0 & 5 & 4 & 1 \\ 5 & 0 & 2 & 3 \\ 4 & 2 & 0 & 5 \\ 1 & 3 & 5 & 0 \end{pmatrix}$$

K = 4

$$A^4(1,1) = \min \{ A^{4-1}(1,1), A^{4-1}(1,4) + A^{4-1}(4,1) \}$$

$$= \min \{ 0, 1 + 1 \} = 0$$

$$A^4(1,2) = \min \{ A^{4-1}(1,2), A^{4-1}(1,4) + A^{4-1}(4,2) \}$$

$$= \min \{ 5, 1 + 3 \} = 4$$

$$A^4(1,3) = \min \{ A^{4-1}(1,3), A^{4-1}(1,4) + A^{4-1}(4,3) \}$$

$$= \min \{ 4, 1 + 5 \} = 4$$

$$A^4(1,4) = \min \{ A^{4-1}(1,4), A^{4-1}(1,4) + A^{4-1}(4,4) \}$$

$$= \min \{ 1, 1 + 0 \} = 1$$

$$A^4(2, 1) = \min \{ A^{4-1}(2, 1), A^{4-1}(2, 4) + A^{4-1}(4, 1) \}$$

$$= \min \{ 5, 3 + 1 \} = 4$$

$$A^4(2, 2) = \min \{ A^{4-1}(2, 2), A^{4-1}(2, 4) + A^{4-1}(4, 2) \}$$

$$= \min \{ 0, 3 + 3 \} = 0$$

$$A^4(2, 3) = \min \{ A^{4-1}(2, 3), A^{4-1}(2, 4) + A^{4-1}(4, 3) \}$$

$$= \min \{ 2, 3 + 5 \} = 2$$

$$A^4(2, 4) = \min \{ A^{4-1}(2, 4), A^{4-1}(2, 4) + A^{4-1}(4, 4) \}$$

$$= \min \{ 3, 3 + 0 \} = 3$$

$$A^4(3, 1) = \min \{ A^{4-1}(3, 1), A^{4-1}(3, 4) + A^{4-1}(4, 1) \}$$

$$= \min \{ 4, 5 + 1 \} = 4$$

$$A^4(3, 2) = \min \{ A^{4-1}(3, 2), A^{4-1}(3, 4) + A^{4-1}(4, 2) \}$$

$$= \min \{ 2, 5 + 3 \} = 2$$

$$A^4(3, 3) = \min \{ A^{4-1}(3, 3), A^{4-1}(3, 4) + A^{4-1}(4, 3) \}$$

$$= \min \{ 0, 5 + 5 \} = 0$$

$$A^4(3, 4) = \min \{ A^{4-1}(3, 4), A^{4-1}(3, 4) + A^{4-1}(4, 4) \}$$

$$= \min \{ 5, 5 + 0 \} = 5$$

$$A^4(4, 1) = \min \{ A^{4-1}(4, 1), A^{4-1}(4, 4) + A^{4-1}(4, 1) \}$$

$$= \min \{ 1, 0 + 1 \} = 1$$

$$A^4(4, 2) = \min \{ A^{4-1}(4, 2), A^{4-1}(4, 4) + A^{4-1}(4, 2) \}$$

$$= \min \{ 3, 0 + 3 \} = 3$$

$$A^4(4, 3) = \min \{ A^{4-1}(4, 3), A^{4-1}(4, 4) + A^{4-1}(4, 3) \}$$

$$= \min \{ 5, 0 + 5 \} = 5$$

$$A^4(4, 4) = \min \{ A^{4-1}(4, 4), A^{4-1}(4, 4) + A^{4-1}(4, 4) \}$$

$$= \min \{ 0, 0 + 0 \} = 0$$

$$A^4 = \begin{pmatrix} 0 & 4 & 4 & 1 \\ 4 & 0 & 2 & 3 \\ 4 & 2 & 0 & 5 \\ 1 & 3 & 5 & 0 \end{pmatrix}$$

Travelling Salesperson Problems : If there are n cities and cost of traveling from one city to other city is given. A salesman has to start from any one of the city and has to visit all the cities exactly once and has to return to the starting place with shortest distance or minimum cost.

Travelling Sales person problem can be computed following recursive method.

$$1 : g(i, \Phi) = C_{i,1}$$

$$2 : g(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$$

Here **g(i, S)** means i is starting node and the nodes in S are to be traversed. **min** is considered as the intermediate node **g(j, S - {j})** means j is already traversed. So next we have to traverse S - {j} with j as starting point.

Example 1 : Construct an optimal travelling sales person tour using Dynamic Programming.

$$\left\{ \begin{array}{cccc} \infty & 12 & 5 & 7 \\ 11 & \infty & 13 & 6 \\ 4 & 9 & \infty & 18 \\ 10 & 3 & 2 & \infty \end{array} \right\}$$

Solution : The formula for solving this problem is,

$$1 : g(i, \Phi) = C_{i,1}$$

$$2 : g(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$$

Step1 : Consider set of 0 elements, such that

$$S = \Phi$$

$$g(i, \Phi) = C_{i,1}$$

$$g(1, \Phi) = C_{1,1} = 0$$

$$g(2, \Phi) = C_{2,1} = 5$$

$$g(3, \Phi) = C_{3,1} = 6$$

$$g(4, \Phi) = C_{4,1} = 8$$

Step2 : Consider set of 1 elements, such that

$$S = 1, \{2\}, \{3\}, \{4\}$$

$$g(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$$

$$\begin{aligned} g(2, \{3\}) &= \min \{ c_{23} + g(3, \{3\} - \{3\}) \} \\ &= \min \{ c_{23} + g(3, \Phi) \} \\ &= \min \{ 13 + 4 \} = 17. \end{aligned}$$

$$\begin{aligned} g(2, \{4\}) &= \min \{ c_{24} + g(4, \{4\} - \{4\}) \} \\ &= \min \{ c_{24} + g(4, \Phi) \} \\ &= \min \{ 6 + 10 \} = 16. \end{aligned}$$

$$\begin{aligned} g(3, \{4\}) &= \min \{ c_{34} + g(4, \{4\} - \{4\}) \} \\ &= \min \{ c_{34} + g(4, \Phi) \} \\ &= \min \{ 18 + 10 \} = 28. \end{aligned}$$

$$\begin{aligned} g(3, \{2\}) &= \min \{ c_{32} + g(2, \{2\} - \{2\}) \} \\ &= \min \{ c_{32} + g(2, \Phi) \} \\ &= \min \{ 9 + 11 \} \\ &= 20. \end{aligned}$$

$$\begin{aligned} g(4, \{2\}) &= \min \{ c_{42} + g(2, \{2\} - \{2\}) \} \\ &= \min \{ c_{42} + g(2, \Phi) \} \\ &= \min \{ 3 + 11 \} = 14. \end{aligned}$$

$$\begin{aligned} g(4, \{3\}) &= \min \{ c_{43} + g(3, \{3\} - \{3\}) \} \\ &= \min \{ c_{43} + g(3, \Phi) \} \\ &= \min \{ 2 + 4 \} = 6. \end{aligned}$$

Step 3 : Consider set of 2 elements, such that

$$S = 2, \{2,3\}, \{2,4\}, \{3,4\}$$

$$g(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$$

$$\begin{aligned} g(2, \{3,4\}) &= \min \{ c_{23} + g(3, \{3,4\} - \{3\}), c_{24} + g(4, \{3,4\} - \{4\}) \} \\ &= \min \{ c_{23} + g(3, \{4\}), c_{24} + g(4, \{3\}) \} \\ &= \min \{ 13 + 28, 6 + 6 \} \\ &= \min \{ 41, 12 \} \\ &= 12. \end{aligned}$$

$$\begin{aligned} g(3, \{2,4\}) &= \min \{ c_{32} + g(2, \{2,4\} - \{2\}), c_{34} + g(4, \{2,4\} - \{4\}) \} \\ &= \min \{ 9 + g(2, \{4\}), 18 + g(4, \{2\}) \} \\ &= \min \{ 9 + 16, 18 + 14 \} \\ &= \min \{ 25, 32 \} \\ &= 25. \end{aligned}$$

$$\begin{aligned}
 g(4, \{2,3\}) &= \min \{ c_{42} + g(2, \{2,3\} - \{2\}), c_{43} + g(3, \{2,3\} - \{3\}) \} \\
 &= \min \{ 3 + g(2, \{3\}), 2 + g(3, \{2\}) \} \\
 &= \min \{ 3 + 17, 2 + 20 \} \\
 &= \min \{ 20, 22 \} \\
 &= 20.
 \end{aligned}$$

Step 4: Consider set of 4 elements, such that

$$S = 3, \{2,3,4\}$$

$$\begin{aligned}
 g(1, \{2,3,4\}) &= \min \{ c_{12} + g(2, \{2,3,4\} - \{2\}), c_{13} + g(3, \{2,3,4\} - \{3\}), c_{14} + g(4, \{2,3,4\} - \{4\}) \} \\
 &= \min \{ c_{12} + g(2, \{3,4\}), 5 + g(3, \{2,4\}), 7 + g(4, \{2,3\}) \} \\
 &= \min \{ 12 + 12, 5 + 25, 7 + 20 \} \\
 &= \min \{ 24, 30, 27 \} \\
 &= 24
 \end{aligned}$$

The optimal Solution is : $c_{12} + g(2, \{3,4\})$,
 $c_{12} + c_{24} + g(4, \{3\})$,
 $c_{12} + c_{24} + c_{43} + g(3, \Phi)$,
 $c_{12} + c_{24} + c_{43} + c_{31}$

The optimal Cost : $1-2-4-3-1 = 12 + 6 + 2 + 4 = 24$

Example 2 : Construct an optimal travelling sales person tour using Dynamic Programming.

$$\left\{ \begin{array}{cccc} 0 & 10 & 15 & 20 \\ 5 & 0 & 9 & 10 \\ 6 & 13 & 0 & 12 \\ 8 & 8 & 9 & 0 \end{array} \right\}$$

Solution : The formula for solving this problem is,

$$1 : g(i, \Phi) = C_{i1}$$

$$2 : g(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$$

Step1 : Consider set of 0 elements, such that

$$S = \Phi$$

$$g(i, \Phi) = C_{i1}$$

$$g(1, \Phi) = C_{11} = 0$$

$$g(2, \Phi) = C_{21} = 5$$

$$g(3, \Phi) = C_{31} = 6$$

$$g(4, \Phi) = C_{41} = 8$$

Step2 : Consider set of 1 elements, such that

$$S = 1, \{2\}, \{3\}, \{4\}$$

$$g(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$$

$$\begin{aligned} g(2, \{3\}) &= \min \{ c_{23} + g(3, \{3\} - \{3\}) \} \\ &= \min \{ c_{23} + g(3, \Phi) \} \\ &= \min \{ 9 + 6 \} = 15 \end{aligned}$$

$$\begin{aligned} g(2, \{4\}) &= \min \{ c_{24} + g(4, \{4\} - \{4\}) \} \\ &= \min \{ c_{24} + g(4, \Phi) \} \\ &= \min \{ 10 + 8 \} = 18 \end{aligned}$$

$$\begin{aligned} g(3, \{2\}) &= \min \{ c_{32} + g(2, \{2\} - \{2\}) \} \\ &= \min \{ c_{32} + g(2, \Phi) \} \\ &= \min \{ 13 + 5 \} = 18 \end{aligned}$$

$$\begin{aligned} g(3, \{4\}) &= \min \{ c_{34} + g(4, \{4\} - \{4\}) \} \\ &= \min \{ c_{34} + g(4, \Phi) \} \\ &= \min \{ 12 + 8 \} = 20 \end{aligned}$$

$$\begin{aligned} g(4, \{2\}) &= \min \{ c_{42} + g(2, \{2\} - \{2\}) \} \\ &= \min \{ c_{42} + g(2, \Phi) \} \\ &= \min \{ 8 + 5 \} = 13 \end{aligned}$$

$$\begin{aligned} g(4, \{3\}) &= \min \{ c_{43} + g(3, \{3\} - \{3\}) \} \\ &= \min \{ c_{43} + g(3, \Phi) \} \\ &= \min \{ 9 + 6 \} = 15 \end{aligned}$$

Step 3 : Consider set of 2 elements, such that

$$S = 2, \{2,3\}, \{2,4\}, \{3,4\}$$

$$g(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$$

$$\begin{aligned} g(2, \{3,4\}) &= \min \{ c_{23} + g(3, \{3,4\} - \{3\}), c_{24} + g(4, \{3,4\} - \{4\}) \} \\ &= \min \{ c_{23} + g(3, \{4\}), c_{24} + g(4, \{3\}) \} \\ &= \min \{ 9 + 20, 10 + 15 \} \\ &= \min \{ 29, 25 \} \\ &= 25 \end{aligned}$$

$$\begin{aligned} g(3, \{2,4\}) &= \min \{ c_{32} + g(2, \{2,4\} - \{2\}), c_{34} + g(4, \{2,4\} - \{4\}) \} \\ &= \min \{ c_{32} + g(2, \{4\}), c_{34} + g(4, \{2\}) \} \\ &= \min \{ 13 + 18, 12 + 13 \} \\ &= \min \{ 31, 25 \} \\ &= 25 \end{aligned}$$

$$\begin{aligned} g(4, \{2,3\}) &= \min \{ c_{42} + g(2, \{2,3\} - \{2\}), c_{43} + g(3, \{2,3\} - \{3\}) \} \\ &= \min \{ c_{42} + g(2, \{3\}), c_{43} + g(3, \{2\}) \} \\ &= \min \{ 8 + 15, 9 + 18 \} \\ &= \min \{ 23, 27 \} \\ &= 23 \end{aligned}$$

Step 4: Consider set of 3 elements, such that

$$S = 3, \{2,3,4\}$$

$$g(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$$

$$\begin{aligned} g(1, \{2,3,4\}) &= \min \{ c_{12} + g(2, \{2,3,4\} - \{2\}), c_{13} + g(3, \{2,3,4\} - \{3\}), c_{14} + g(4, \{2,3,4\} - \{4\}) \} \\ &= \min \{ c_{12} + g(2, \{3,4\}), c_{13} + g(3, \{2,4\}), c_{14} + g(4, \{2,3\}) \} \\ &= \min \{ 10 + 25, 15 + 25, 20 + 23 \} \\ &= \min \{ 35, 40, 43 \} \\ &= 35 \end{aligned}$$

The optimal Solution is : $c_{12} + g(2, \{3,4\}),$
 $c_{12} + c_{24} + g(4, \{3\}),$
 $c_{12} + c_{24} + c_{43} + g(3, \Phi),$
 $c_{12} + c_{24} + c_{43} + c_{31}$

The optimal Cost: $1-2-4-3-1 = 10 + 10 + 9 + 6 = 35$

0 /1 Knapsack Problem: If we are given n objects and a knapsack or a bag in which the object i that has weight w_i is to be placed. The knapsack has a capacity W . Then the profit that can be earned is p_i . The objective is to obtain filling of knapsack with maximum profit earned. Maximized $p_i x_i$. Subject to constraint $w_i x_i \leq W$ Where $1 \leq i \leq n$ and n is total no. of objects and $x_i = 0$ or 1 .

0 /1 Knapsack Problem can be computed following recursive method.

1. Initially $s^0 = \{(0,0)\}$ (P, W)

2. Merging Operation

$$S^{i+1} = S^i + S_{i_1}$$

3. Purging rule (OR) dominance rule

If S^{i+1} contains (P_j, W_j) and (P_k, W_k) these two pairs such that $P_j \leq P_k$ and $W_j \geq W_k$, then (P_j, W_j) can be eliminated. This purging rule is also called as dominance rule. In purging rule basically the dominated tuples gets purged. In short remove the pair with less profit and more weight.

Example 1 : Consider the following 0 / 1 Knapsack problem using dynamic programming $m = 6$, $n = 3$, $(w_1, w_2, w_3) = (2, 3, 4)$, $(p_1, p_2, p_3) = (1, 2, 5)$.

Solution : we have to build the sequence of decisions S^0, S^1, S^2, S^3

Initially $S^0 = \{ (0,0) \}$

S^{0_1} = Select next (p_1, w_1) pair and add it with S^0

$$= (1, 2) + \{ (0, 0) \}$$

$$= \{ (1+0), (2+0) \} = \{ (1, 2) \}$$

$$S^{i+1} = S^i \cup S^{i_1}$$

$$S^1 = S^0 + S^{0_1}$$

$$= \{ (0, 0) \} \cup \{ (1, 2) \}$$

$$= \{ (0, 0), (1, 2) \}$$

To apply Purging Rule : There will be no deleted

S^{1_1} = Select next (P_2, W_2) pair and add it with S^1

$$= (2, 3) + \{ (0, 0), (1, 2) \}$$

$$= \{ (2+0, 3+0), (2+1, 3+2) \}$$

$$= \{ (2, 3), (3, 5) \}$$

$$S^2 = S^1 \cup S^{1_1}$$

$$= \{ (0, 0), (1, 2) \} \cup \{ (2, 3), (3, 5) \}$$

$$= \{ (0, 0), (1, 2), (2, 3), (3, 5) \}$$

To apply Purging Rule : There will be no deleted

S^{2_1} = Select next (P_3, W_3) pair and add it with S^2

$$= (5, 4) + \{ (0, 0), (1, 2), (2, 3), (3, 5) \}$$

$$= \{ (5+0, 4+0), (5+1, 4+2), (5+2, 4+3), (5+3, 4+5) \}$$

$$= \{ (5, 4), (6, 6), (7, 7), (8, 9) \}$$

$$S^3 = S^2 \cup S^{2_1}$$

$$= \{ (0, 0), (1, 2), (2, 3), (3, 5) \} \cup \{ (5, 4), (6, 6), (7, 7), (8, 9) \}$$

$$= \{ (0, 0), (1, 2), (2, 3), (3, 5), (5, 4), (6, 6), (7, 7), (8, 9) \}$$

To apply Purging Rule

$$(3, 5) \text{ and } (5, 4)$$

$$(p_j, w_j) \text{ and } (p_k, w_k)$$

$$p_j \leq p_k \text{ and } w_j \geq w_k$$

$$3 \leq 5 \text{ and } 5 \geq 4 \text{ -- True, } (p_j, w_j) \text{ pair can be deleted } (3, 5)$$

$$S^3 = \{ (0, 0), (1, 2), (2, 3), (5, 4), (6, 6), (7, 7), (8, 9) \}$$

Here Capacity of the Knapsack is 6, Now we have to remove the pairs, in which $w_i > m$, i.e $w_i > 6$

$$S^3 = \{ (0,0), (1,2), (2,3), (5,4), (6,6) \}$$

Here $M = 6$, we will find the tuple, denoting the weight ' 6 '

i.e $(6,6)$ belongs to S^3

$(6,6)$ does not belongs to S^2

Therefore , We must set $x_3 = 1$

The pair $(6,6)$ came from the pair $(6-p_3, 6-w_3)$

$$(6-5, 6-4) = (1,2)$$

Here $(1,2)$ belongs to S^2

$(1,2)$ belongs to S^1

Therefore , We must set $x_2 = 0$

$(1,2)$ does not belongs to S^0

Therefore , We must set $x_1 = 1$

Hence an Optimal Solution is $(x_1, x_2, x_3) = (1, 0, 1)$

$$\begin{aligned} \text{Maximum Profit} &= p_1x_1 + p_2x_2 + p_3x_3 \\ &= 1*1 + 2*0 + 5*1 \\ &= 1 + 0 + 5 = 6 \end{aligned}$$

$$\begin{aligned} \text{Maximum Weight} &= w_1x_1 + w_2x_2 + w_3x_3 \\ &= 2*1 + 3*0 + 4*1 \\ &= 2 + 0 + 4 = 6 \end{aligned}$$

Example 2 : Consider the following 0 / 1 Knapsack problem using dynamic programming $m = 8$, $n = 4$.

i	p_i	w_i
1	1	2
2	2	3
3	5	4
4	6	5

Solution : we have to build the sequence of decisions S^0, S^1, S^2, S^3, S^4

Initially $S^0 = \{ (0,0) \}$

S^{0_1} = Select next (p_1, w_1) pair and add it with S^0

$$= \{ (1,2) \} + \{ (0,0) \}$$

$$= \{ (1+0, 2+0) \} = \{ (1,2) \}$$

$$S^{i+1} = S^i \cup S^{i_1}$$

$$S^1 = S^0 \cup S^{0_1}$$

$$= \{ (0,0) \} \cup \{ (1,2) \}$$

$$= \{ (0,0), (1,2) \}$$

To apply Purging Rule : There will be no deleted

S^{1_1} = Select next (P_2, W_2) pair and add it with S^1

$$= \{ (2,3) \} + \{ (0,0), (1,2) \}$$

$$= \{ (2+0, 3+0), (2+1, 3+2) \}$$

$$= \{ (2,3), (3,5) \}$$

$$S^2 = S^1 \cup S^{1_1}$$

$$= \{ (0,0), (1,2) \} \cup \{ (2,3), (3,5) \}$$

$$= \{ (0,0), (1,2), (2,3), (3,5) \}$$

To apply Purging Rule : There will be no deleted

S^{2_1} = Select next (P_3, W_3) pair and add it with S^2

$$= \{ (5,4) \} + \{ (0,0), (1,2), (2,3), (3,5) \}$$

$$= \{ (5+0, 4+0), (5+1, 4+2), (5+2, 4+3), (5+3, 4+5) \}$$

$$= \{ (5,4), (6,6), (7,7), (8,9) \}$$

$$S^3 = S^2 \cup S^2_1$$

$$= \{ (0,0), (1,2), (2,3), (3,5) \} \cup \{ (5,4), (6,6), (7,7), (8,9) \}$$

$$= \{ (0,0), (1,2), (2,3), (3,5), (5,4), (6,6), (7,7), (8,9) \}$$

To apply Purging Rule

$$\begin{aligned} & (3,5) \text{ and } (5,4) \\ & (p_j, w_j) \text{ and } (p_k, w_k) \end{aligned}$$

$$p_j \leq p_k \text{ and } w_j > w_k$$

$$3 \leq 5 \text{ and } 5 > 4 \text{ -- True, } (p_j, w_j) \text{ pair can be deleted } (3,5)$$

$$S^3 = \{ (0,0), (1,2), (2,3), (5,4), (6,6), (7,7), (8,9) \}$$

S^3_1 = Select next (P4, W4) pair and add it with S^3

$$= (6,5) + \{ (0,0), (1,2), (2,3), (5,4), (6,6), (7,7), (8,9) \}$$

$$= \{ (6+0, 5+0), (6+1, 5+2), (6+2, 5+3), (6+5, 5+4), (6+6, 5+6), (6+7, 5+7), (6+8, 5+9) \}$$

$$= \{ (6,5), (7,7), (8,8), (11,9), (12,11), (13,12), (14,14) \}$$

$$S^4 = S^3 \cup S^3_1$$

$$= \{ (0,0), (1,2), (2,3), (5,4), (6,6), (7,7), (8,9) \} \cup \{ (6,5), (7,7), (8,8), (11,9), (12,11), (13,12), (14,14) \}$$

$$= \{ (0,0), (1,2), (2,3), (5,4), (6,5), (6,6), (7,7), (8,8), (8,9), (11,9), (12,11), (13,12), (14,14) \}$$

To apply Purging Rule : There will be no deleted

Here Capacity of the Knapsack is 8, Now we have to remove the pairs, in which $w_i > m$, i.e $w_i > 8$

$$\text{Therefore } S^4 = \{ (0,0), (1,2), (2,3), (5,4), (6,5), (6,6), (7,7), (8,8) \}$$

Here $M = 8$, we will find the tuple, denoting the weight '8'

$$\text{i.e } (8,8) \text{ belongs to } S^4$$

$$(8,8) \text{ does not belongs to } S^3$$

$$\text{Therefore, We must set } x_4 = 1$$

$$\text{The pair } (8,8) \text{ came from the pair } (8-p_4, 8-w_4)$$

$$(8-6, 8-5) = (2,3)$$

$$\text{Here } (2,3) \text{ belongs to } S^3$$

$$(2,3) \text{ belongs to } S^2$$

Therefore , We must set $x_3 = 0$

$(2,3)$ belongs to S^2

$(2,3)$ does not belongs to S^1

Therefore , We must set $x_2 = 1$

The pair $(2,3)$ came from the pair $(2-p_2, 3-w_2)$

$$(2-2, 3-3) = (0,0)$$

Here $(0,0)$ belongs to S^1

$(0,0)$ belongs to S^0

Therefore , We must set $x_1 = 0$

Hence an Optimal Solution is $(x_1, x_2, x_3, x_4) = (0, 1, 0, 1)$

Maximum Profit $= p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4$
 $= 1.0 + 2.1 + 5.0 + 6.1$
 $= 0 + 2 + 0 + 6 = 8$

Maximum Weight $= w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4$
 $= 2.0 + 3.1 + 4.0 + 5.1$
 $= 0 + 3 + 0 + 5 = 8$