Mixture Models & EM

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based on the slides of Nick Rouzzi and Vibhav Gogate

Previously…

- We looked at k-means and hierarchical clustering as mechanisms for unsupervised learning

  - k-means was simply a block coordinate descent scheme for a specific objective function

- Today: how to learn probabilistic models for unsupervised learning problems

EM: Soft Clustering

- Clustering (e.g., k-means) typically assumes that each instance is given a "hard" assignment to exactly one cluster

- Does not allow uncertainty in class membership or for an instance to belong to more than one cluster

  - Problematic because data points that lie roughly midway between cluster centers are assigned to one cluster

- Soft clustering gives probabilities that an instance belongs to each of a set of clusters
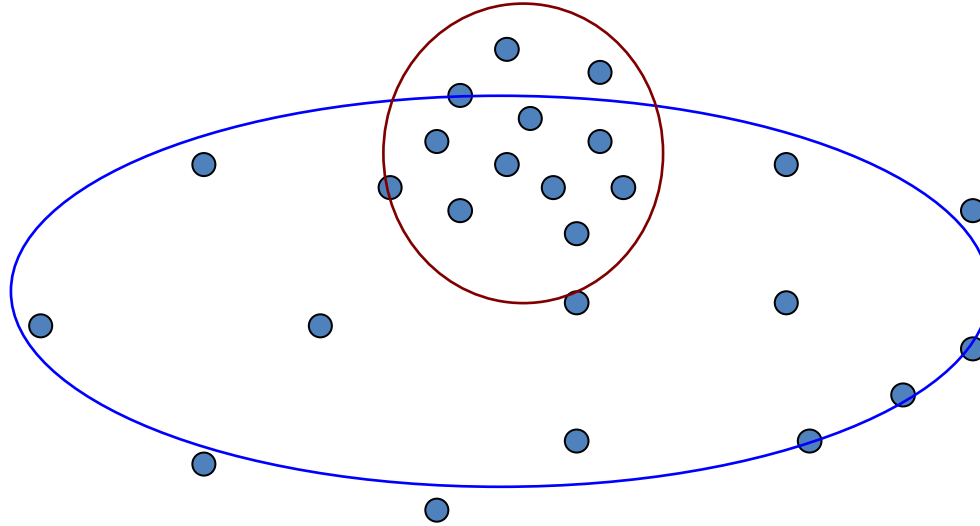
Probabilistic Clustering

- Try a probabilistic model!

  - Allows overlaps, clusters of different size, etc.

- Can tell a generative story for data

  - $p(x|y)\, p(y)$

- Challenge: we need to estimate model parameters without labeled y's (i.e., in the unsupervised setting)

Z	X$_1$	X$_2$
??	0.1	2.1
??	0.5	-1.1
??	0.0	3.0
??	-0.1	-2.0
??	0.2	1.5
…	…	…

Probabilistic Clustering

- Clusters of different shapes and sizes

- Clusters can overlap! (k-means doesn't allow this)

Finite Mixture Models

- Given a dataset: $x^{(1)}, \dots, x^{(N)}$

- **Mixture model**: $\Theta = \{\lambda_1, \dots, \lambda_k, \theta_1, \dots, \theta_k\}$

$$p(x|\Theta) = \sum_{y=1}^{k} \lambda_y p_y(x|\theta_y)$$

  - $p_y(x|\theta_y)$ is a mixture component from some family of probability distributions parameterized by θ_y and $\lambda \geq 0$ such that $\sum_y \lambda_y = 1$ are the mixture weights

  - We can think of $\lambda_y = p(Y = y|\Theta)$ for some random variable Y that takes values in $\{1, \dots, k\}$

Finite Mixture Models
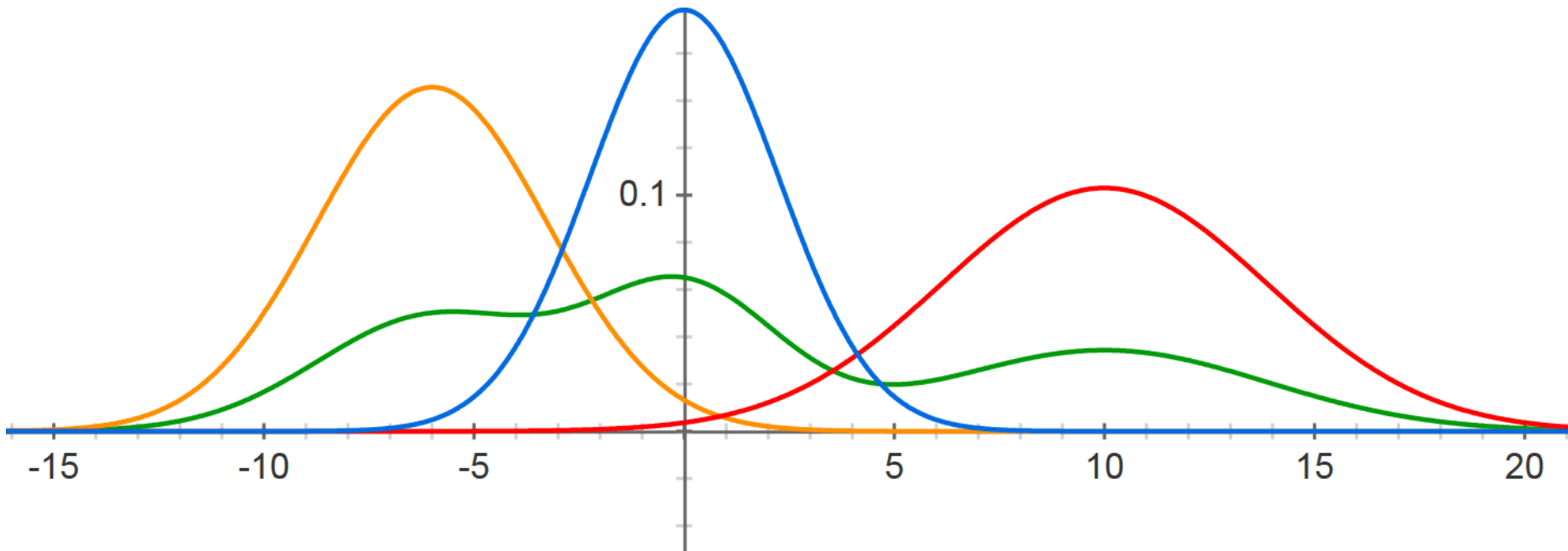
Uniform mixture of 3 Gaussians

Multivariate Gaussian

- A d-dimensional multivariate Gaussian distribution is defined by a $d \times d$ covariance matrix Σ and a mean vector μ

$$p(x|\mu, \Sigma) = \frac{1}{\sqrt{(2\pi)^{\mathrm{d}}\det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$$

- The covariance matrix describes the degree to which pairs of variables vary together

  - The diagonal elements correspond to variances of the individual variables

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- The covariance matrix must be a symmetric positive definite matrix in order for the above to make sense

  - Positive definite: all eigenvalues are positive & matrix is invertible

  - Ensures that the quadratic form is concave

Gaussian Mixture Models (GMMs)

- We can define a GMM by choosing the kth component of the mixture to be a Gaussian density with parameters

$$\theta_k = \{\mu_k, \Sigma_k\}$$

$$p(x|\mu_k, \Sigma_k) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma_k)}} \exp\left(-\frac{1}{2}(x - \mu_k)^T \Sigma_k^{-1}(x - \mu_k)\right)$$

We could cluster by fitting a mixture of k Gaussians to our data

How do we learn these kinds of models?

Learning Gaussian Parameters

- MLE for supervised univariate Gaussian

$$\mu_{MLE} = \frac{1}{N} \sum_{i=1}^{N} x^{(i)}$$

$$\sigma^2_{MLE} = \frac{1}{N} \sum_{i=1}^{N} \left(x^{(i)} - \mu_{MLE} \right)^2$$

- MLE for supervised multivariate Gaussian

$$\mu_{MLE} = \frac{1}{N} \sum_{i=1}^{N} x^{(i)}$$

$$\Sigma_{\mathrm{MLE}} = \frac{1}{N} \sum_{i=1}^{N} \left(x^{(i)} - \mu_{MLE} \right)\left(x^{(i)} - \mu_{MLE} \right)^T$$

Learning Gaussian Parameters

- MLE for supervised multivariate mixture of k Gaussian distributions

$$\mu_{MLE}^k = \frac{1}{|M_k|} \sum_{i \in M_k} x^{(i)}$$

$$\Sigma_{MLE}^k = \frac{1}{|M_k|} \sum_{i \in M_k} \left(x^{(i)} - \mu_{MLE}^k \right) \left(x^{(i)} - \mu_{MLE}^k \right)^T$$

Sums are over the observations that were generated by the kth mixture component (this requires that we know which points were generated by which distribution!)

The Unsupervised Case

- What if our observations do not include information about which of the k mixture components generated them?

- Consider a joint probability distribution over data points, $x^{(i)}$, and mixture assignments, $y \in \{1, \dots, k\}$

$$\arg \max_{\Theta} \prod_{i=1}^{N} p(x^{(i)}|\Theta) = \arg \max_{\Theta} \prod_{i=1}^{N} \sum_{y=1}^{k} p(x^{(i)}, Y = y|\Theta)$$

$$= \arg \max_{\Theta} \prod_{i=1}^{N} \sum_{y=1}^{k} p(x^{(i)}|Y = y, \Theta) p(Y = y|\Theta)$$

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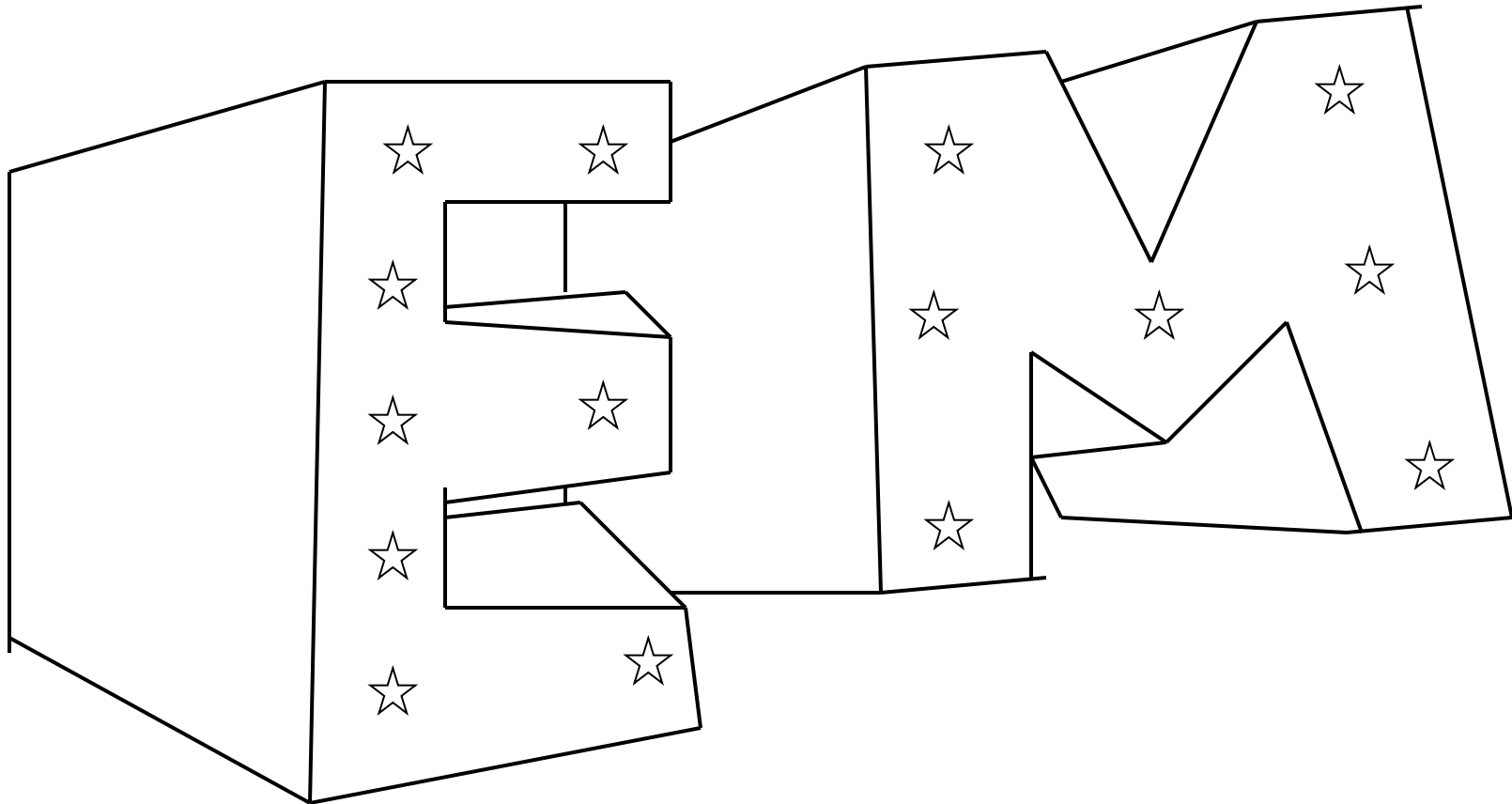We only know how to compute the probabilities for each mixture component

The Unsupervised Case

- In the case of a Gaussian mixture model

$$p\left(x^{(i)}\middle|Y = y, \Theta\right) = N(x^{(i)}|\mu_y, \Sigma_y)$$

$$p(Y = y|\Theta) = \lambda_y$$

- Differentiating the MLE objective yields a system of equations that is difficult to solve in general

  - The solution: modify the objective to make the optimization easier

Jensen's Inequality

For a convex function $f: \mathbb{R}^n \to \mathbb{R}$, any $a_1, \dots, a_k \in [0,1]$ such that $a_1 + \cdots + a_k = 1$, and any $x^{(1)}, \dots, x^{(k)} \in \mathbb{R}^n$,

$$a_1 f\left(x^{(1)}\right) + \cdots + a_k f\left(x^{(k)}\right) \geq f\left(a_1 x^{(1)} + \cdots + a_k x^{(k)}\right)$$

Inequality is reversed for concave functions

EM Algorithm

$$\log \ell(\Theta) = \sum_{i=1}^{N} \log \sum_{y=1}^{k} p\big(x^{(i)}, Y = y \big| \Theta\big)$$

$$= \sum_{i=1}^{N} \log \sum_{y=1}^{k} \frac{q_i(y)}{q_i(y)} p\big(x^{(i)}, Y = y \big| \Theta\big)$$

$$\geq \sum_{i=1}^{N} \sum_{y=1}^{k} q_i(y) \log \frac{p\big(x^{(i)}, Y = y \big| \Theta\big)}{q_i(y)}$$

$$\equiv F(\Theta, q)$$

EM Algorithm

$$\log \ell(\Theta) = \sum_{i=1}^{N} \log \sum_{y=1}^{k} p\big(x^{(i)}, Y = y \big| \Theta\big)$$

$$= \sum_{i=1}^{N} \log \sum_{y=1}^{k} \frac{q_i(y)}{q_i(y)} p\big(x^{(i)}, Y = y \big| \Theta\big)$$

$q_i(y)$ is an arbitrary positive probability distribution

$$\geq \sum_{i=1}^{N} \sum_{y=1}^{k} q_i(y) \log \frac{p\big(x^{(i)}, Y = y \big| \Theta\big)}{q_i(y)}$$

$$\equiv F(\Theta, q)$$

EM Algorithm

$$\log \ell(\Theta) = \sum_{i=1}^{N} \log \sum_{y=1}^{k} p\big(x^{(i)}, Y = y \big| \Theta\big)$$

$$= \sum_{i=1}^{N} \log \sum_{y=1}^{k} \frac{q_i(y)}{q_i(y)} p\big(x^{(i)}, Y = y \big| \Theta\big)$$

$$\geq \sum_{i=1}^{N} \sum_{y=1}^{k} q_i(y) \log \frac{p\big(x^{(i)}, Y = y \big| \Theta\big)}{q_i(y)} \qquad \text{Jensen's ineq.}$$

$$\equiv F(\Theta, q)$$

EM Algorithm

$$\arg \max_{\Theta, q_1, .., q_N} \sum_{i=1}^{N} \sum_{y=1}^{k} q_i(y) \log \frac{p\left(x^{(i)}, Y = y \middle| \Theta\right)}{q_i(y)}$$

- This objective is not jointly concave in Θ and $q_1, \ldots, q_N$

  - Best we can hope for is a local maxima (and there could be **A LOT** of them)

- The EM algorithm is a block coordinate ascent scheme that finds a local optimum of this objective

  - Start from an initialization Θ^0 and $q_1^0, \ldots, q_N^0$

EM Algorithm

- E step: with the θ's fixed, maximize the objective over q

$$q^{t+1} \in \arg \max_{q_1,..,q_N} \sum_{i=1}^{N} \sum_{y=1}^{k} q_i(y)\log\frac{p\left(x^{(i)}, Y = y\middle|\Theta^t\right)}{q_i(y)}$$

- Using the method of Lagrange multipliers for the constraint that $\sum_y q_i(y) = 1$ gives

$$q_i^{t+1}(y) = p\left(Y = y\middle|X = x^{(i)}, \Theta^t\right)$$

EM Algorithm

- M step: with the q's fixed, maximize the objective over Θ

$$\theta^{t+1} \in \arg\max_{\Theta} \sum_{i=1}^{N} \sum_{y=1}^{k} q_i^{t+1}(y) \log \frac{p\left(x^{(i)}, Y = y \mid \theta\right)}{q_i^{t+1}(y)}$$

- For the case of GMM, we can compute this update in closed form

  - This is not necessarily the case for every model

  - May require gradient ascent

EM Algorithm

- Start with random parameters

- E-step maximizes a lower bound on the log-sum for fixed parameters

- M-step solves a MLE estimation problem for fixed probabilities

- Iterate between the E-step and M-step until convergence

EM for Gaussian Mixtures

- E-step:

$$q_i^t(y) = \frac{\lambda_y^t \cdot p\left(x^{(i)} \middle| \mu_y^t, \Sigma_y^t\right)}{\sum_{y'} \lambda_{y'}^t \cdot p\left(x^{(i)} \middle| \mu_{y'}^t, \Sigma_{y'}^t\right)}$$

Probability of $x^{(i)}$ under the appropriate multivariate normal distribution

- M-step:

$$\mu_y^{t+1} = \frac{\sum_{i=1}^{N} q_i^t(y) x^{(i)}}{\sum_{i=1}^{N} q_i^t(y)}$$

$$\Sigma_y^{t+1} = \frac{\sum_{i=1}^{N} q_i^t(y)\left(x^{(i)} - \mu_y^{t+1}\right)\left(x^{(i)} - \mu_y^{t+1}\right)^T}{\sum_{i=1}^{N} q_i^t(y)}$$

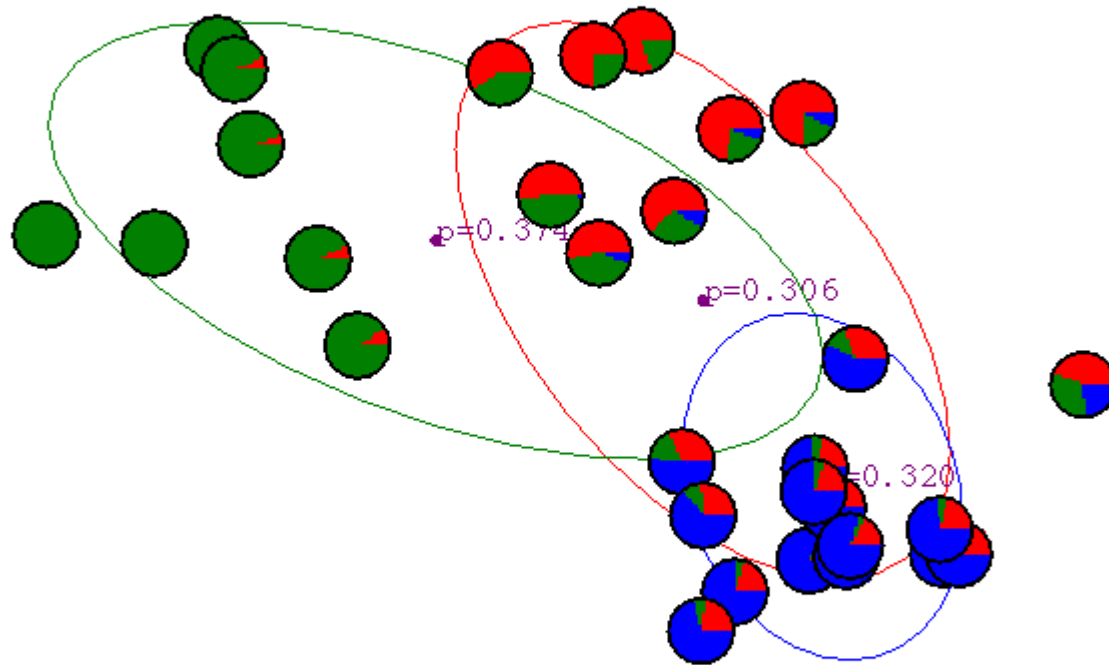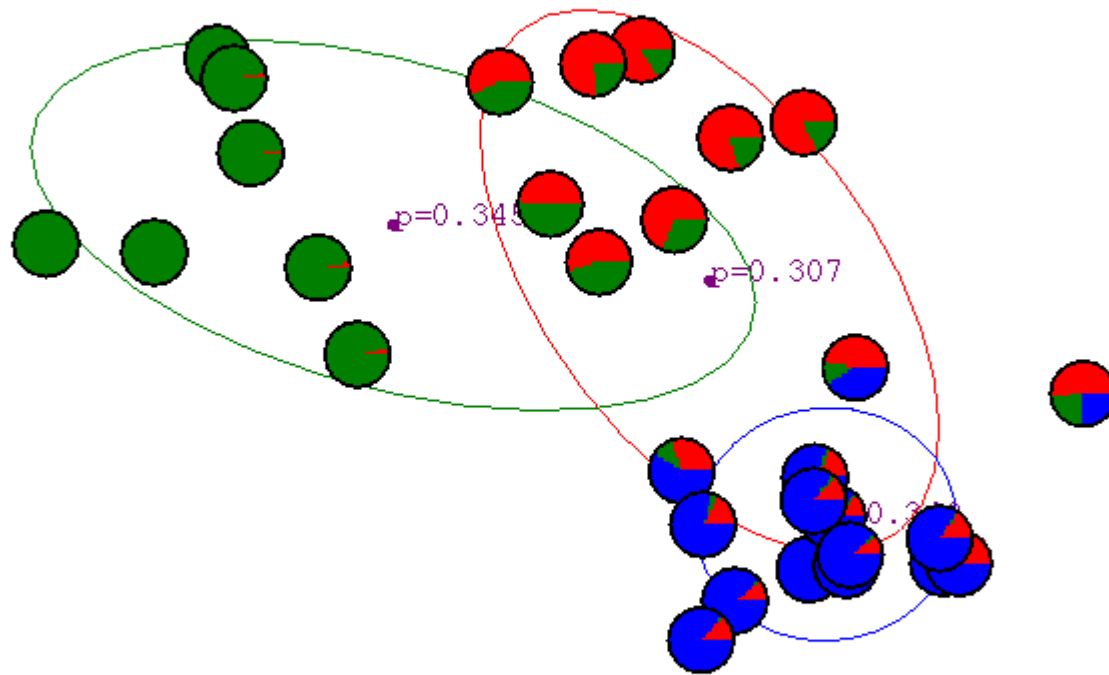$$\lambda_y^{t+1} = \frac{1}{N}\sum_{i=1}^{N} q_i^t(y)$$

EM for Gaussian Mixtures

- E-step:

$$q_i^t(y) = \frac{\lambda_y^t \cdot p\left(x^{(i)} \middle| \mu_y^t, \Sigma_y^t\right)}{\sum_{y'} \lambda_{y'}^t \cdot p\left(x^{(i)} \middle| \mu_{y'}^t, \Sigma_{y'}^t\right)}$$

Probability of $x^{(i)}$ under the mixture model

- M-step:

$$\mu_y^{t+1} = \frac{\sum_{i=1}^N q_i^t(y) x^{(i)}}{\sum_{i=1}^N q_i^t(y)}$$

$$\Sigma_y^{t+1} = \frac{\sum_{i=1}^N q_i^t(y)\left(x^{(i)} - \mu_y^{t+1}\right)\left(x^{(i)} - \mu_y^{t+1}\right)^T}{\sum_{i=1}^N q_i^t(y)}$$

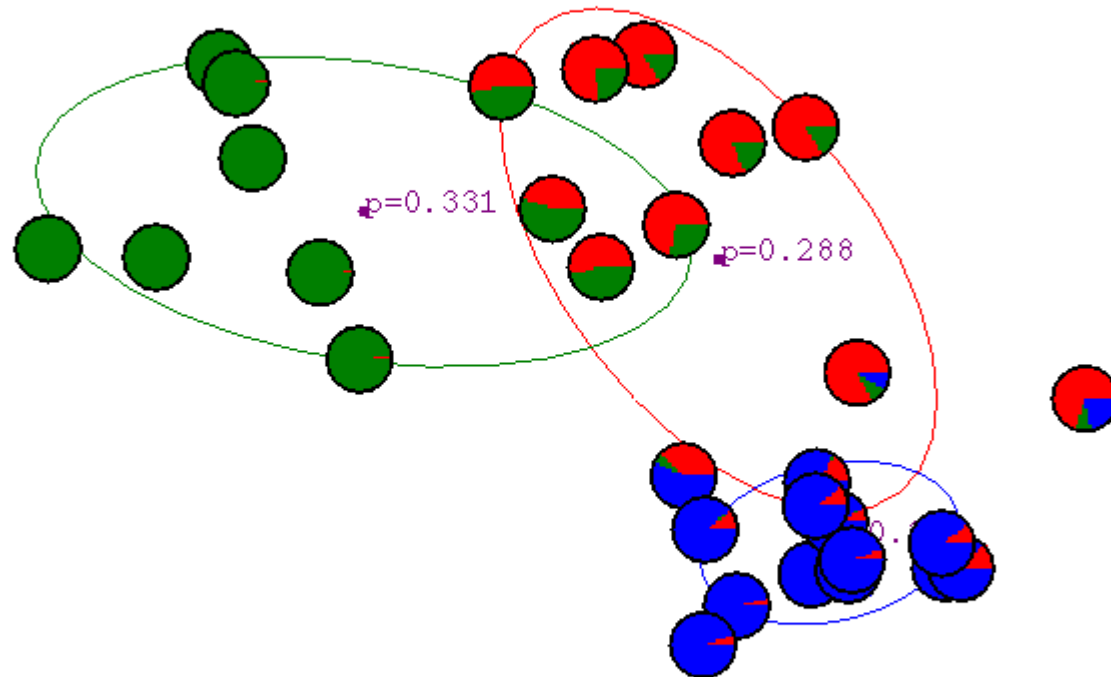$$\lambda_y^{t+1} = \frac{1}{N}\sum_{i=1}^N q_i^t(y)$$
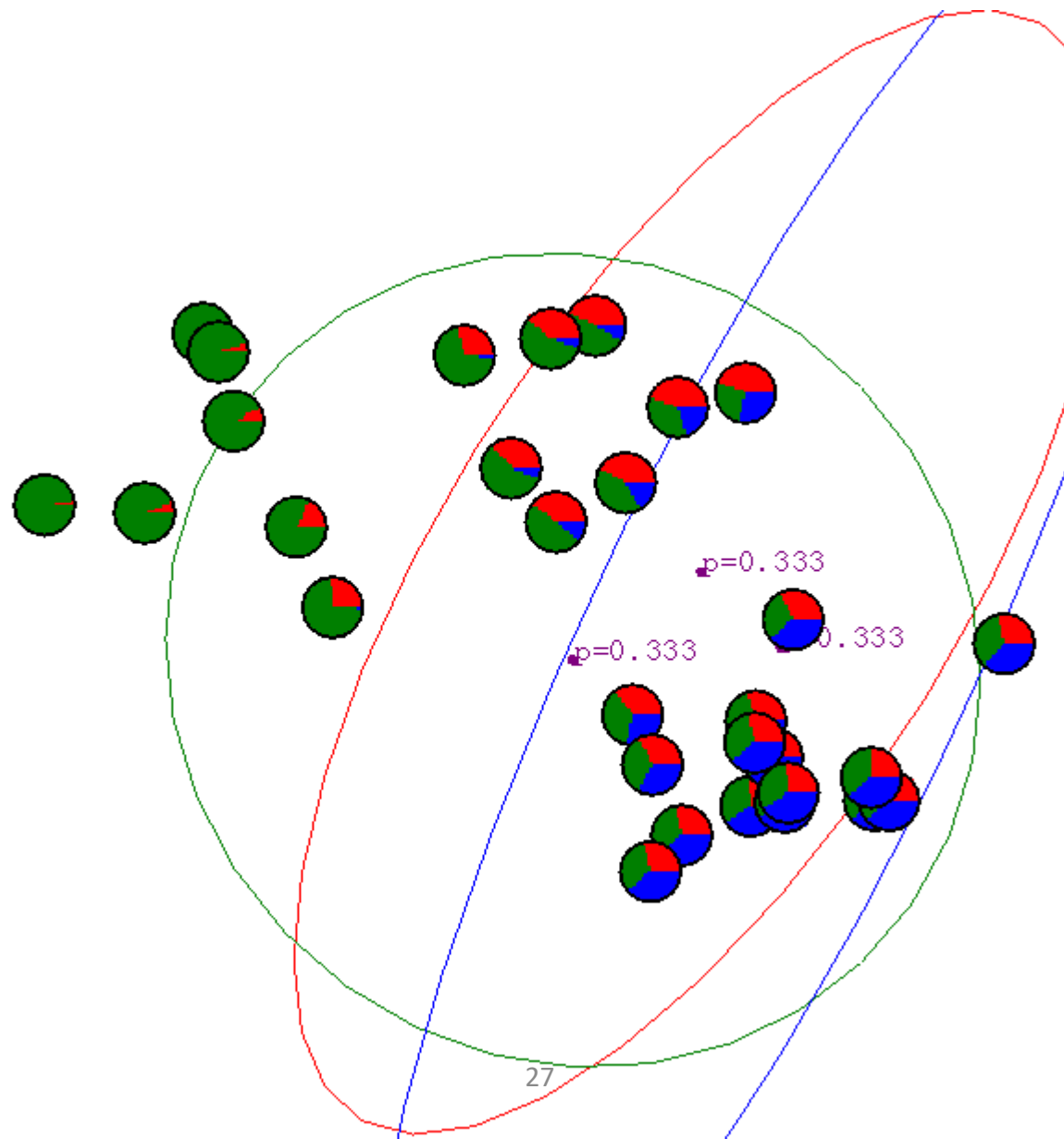
After first iteration
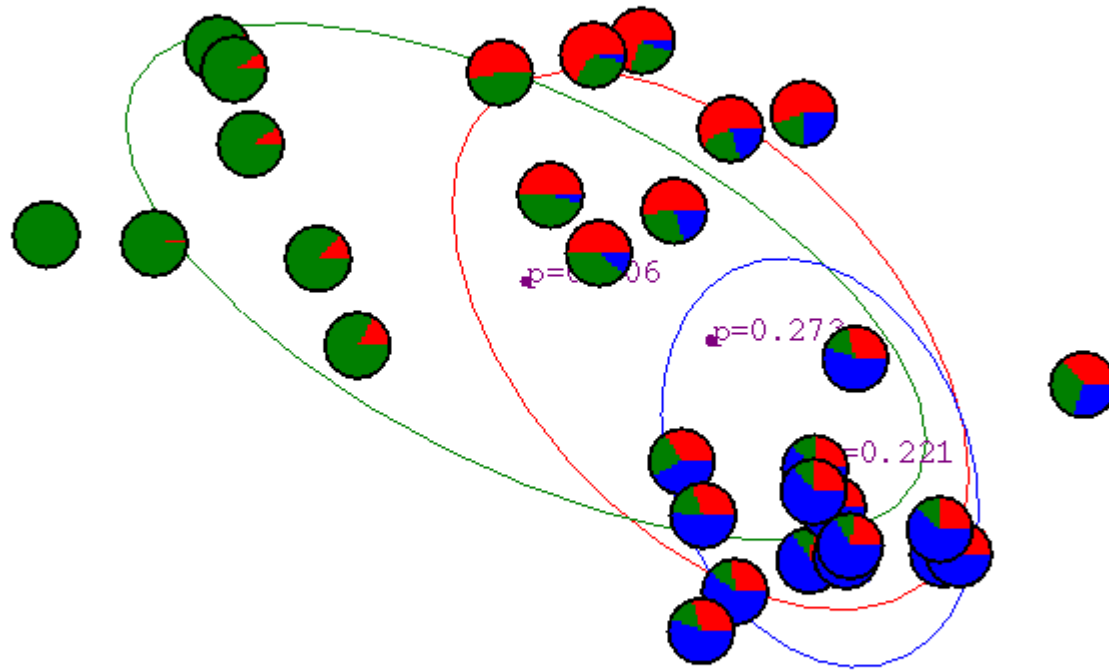
After 2nd iteration

p=0.374

p=0.306

p=0.320

After 3rd iteration

After 4th iteration

p=0.331

p=0.288

After 5th iteration

p=0.322

p=0.285

Properties of EM
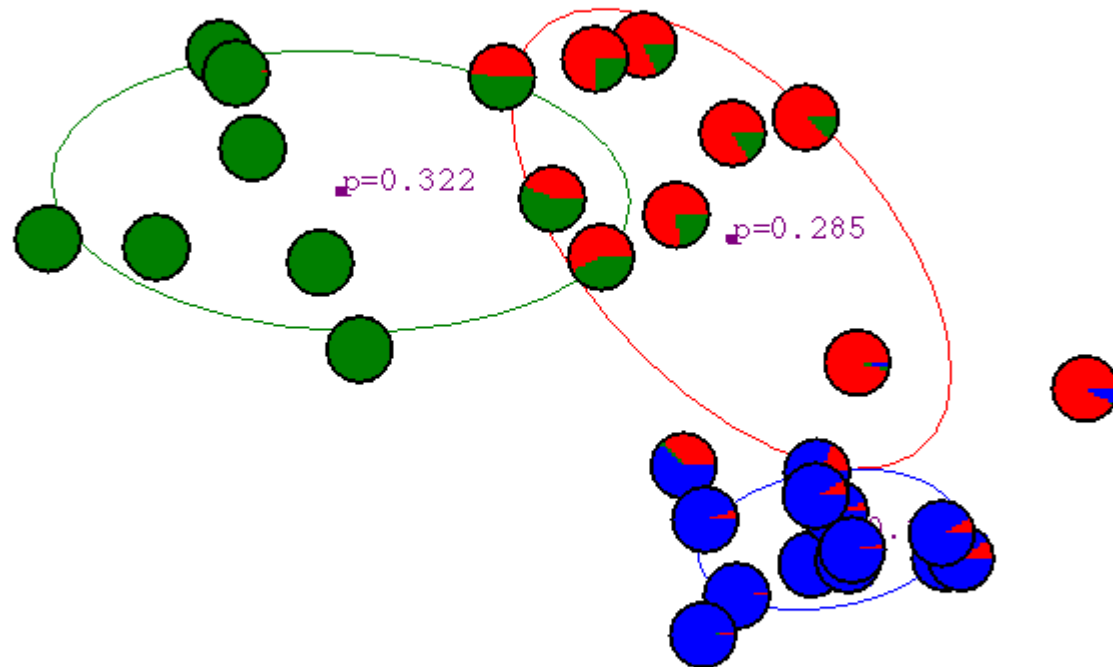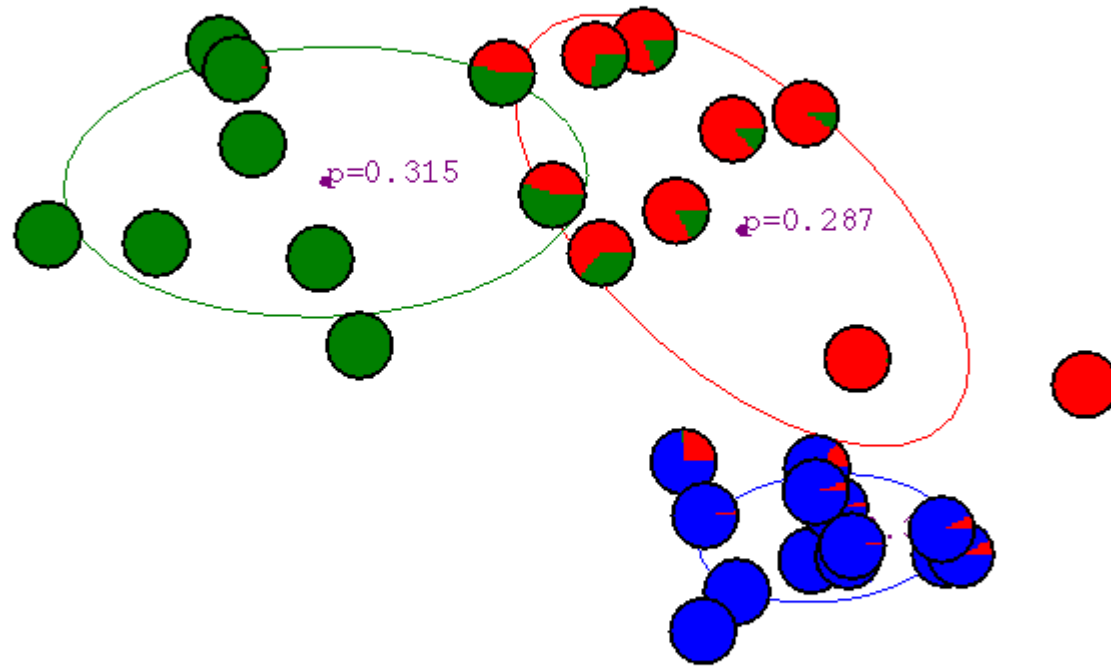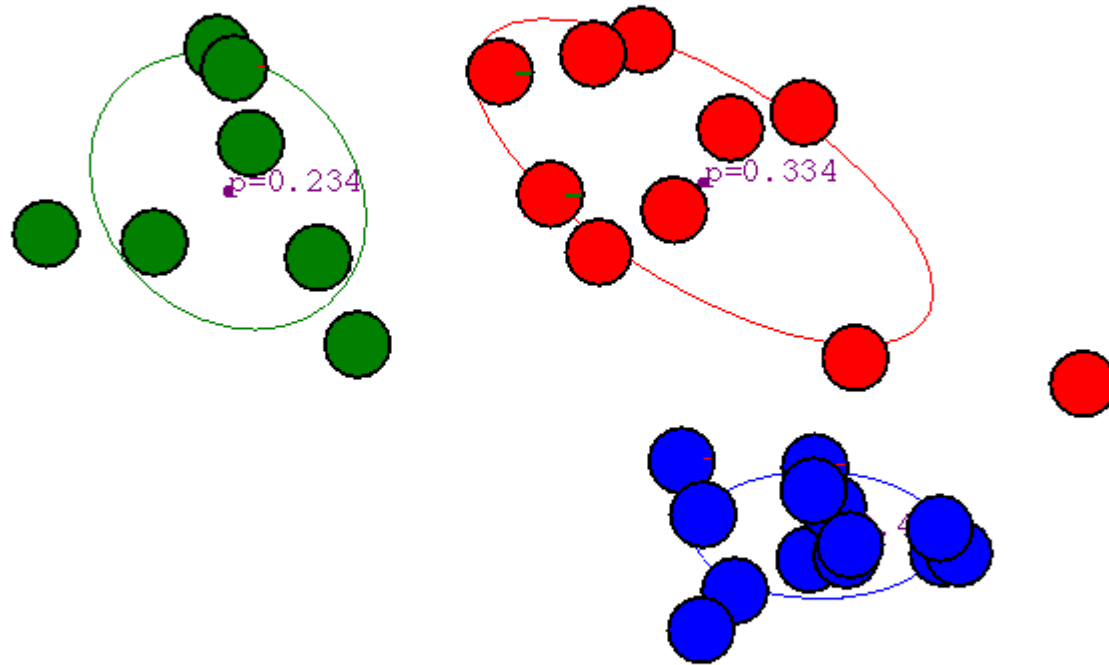
- EM converges to a local optimum

  - This is because each iteration improves the log-likelihood

  - Proof same as k-means (just block coordinate ascent)

    - E-step can never decrease likelihood

    - M-step can never decrease likelihood

- If we make hard assignments instead of soft ones, algorithm is equivalent to k-means!