CS 6375
Binary Classification / Perceptron

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Slides adapted from Nick Rouzzi and Sriraam Natarajan

Reminders

- Homework 1 available soon on eLearning and due in 2 weeks
  - Late homework will not be accepted
- Instructions for getting started with the course, e.g., joining Piazza & taking the preq quiz, are on eLearning

Part I: Recap of Supervised Learning, Linear Separation and Basics of Perceptron

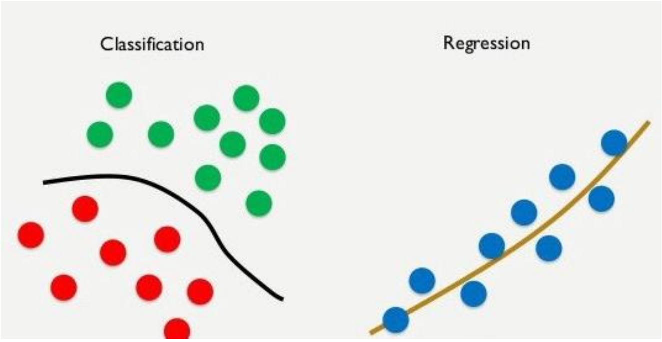
History of Perceptron

- Formally introduced by Rosenblatt in 1958*
- Introduced more like a *General-purpose Machine* rather than a classifier
  - This caused a heated controversy in the 1960’s (NY times articles) etc.
- Soon, the limitations of perceptron's became evident
  - Works only in Linear separable cases
  - Cannot learn a simple XOR function
- However, these were the seeds for *Multi-Layer Perceptron's,* today known as Deep Neural Networks!

*Rosenblatt, Frank (1958), The Perceptron: A Probabilistic Model for Information Storage and Organization in the Brain, Cornell Aeronautical Laboratory, Psychological Review, v65, No. 6, pp. 386–408. doi:10.1037/h0042519

Supervised Learning

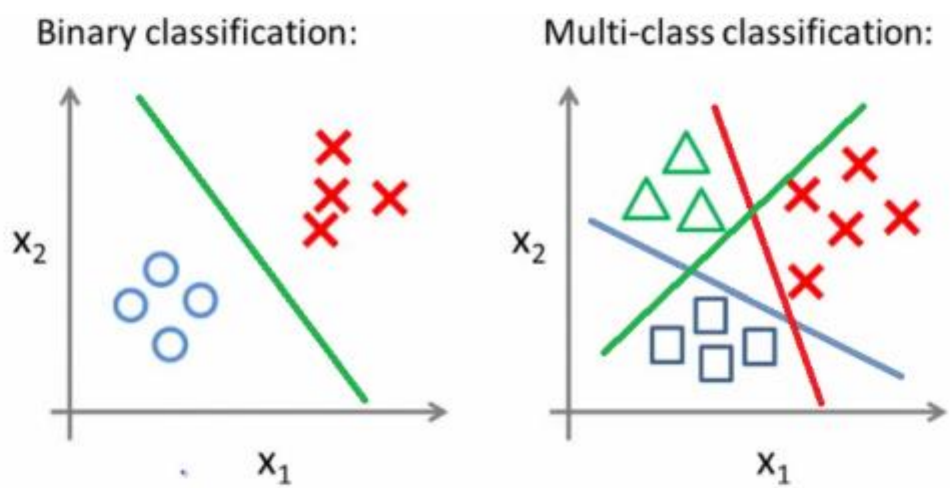
- **Input:** $\left(x^{(1)}, y^{(1)}\right), \dots, \left(x^{(M)}, y^{(M)}\right)$
  - $x^{(m)}$ is the m^{th} data item and $y^{(m)}$ is the m^{th} **label**
- **Goal:** find a function f such that $f\left(x^{(m)}\right)$ is a "good approximation" to $y^{(m)}$
  - Can use it to predict y values for previously unseen x values



Supervised Learning

Classification vs Regression

- Input: pairs of points $(x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^d$
- Regression case: $y^{(m)} \in \mathbb{R}$
- Classification case: $y^{(m)} \in [0, k-1]$ [k-class classification]
- If k = 2, we get Binary classification



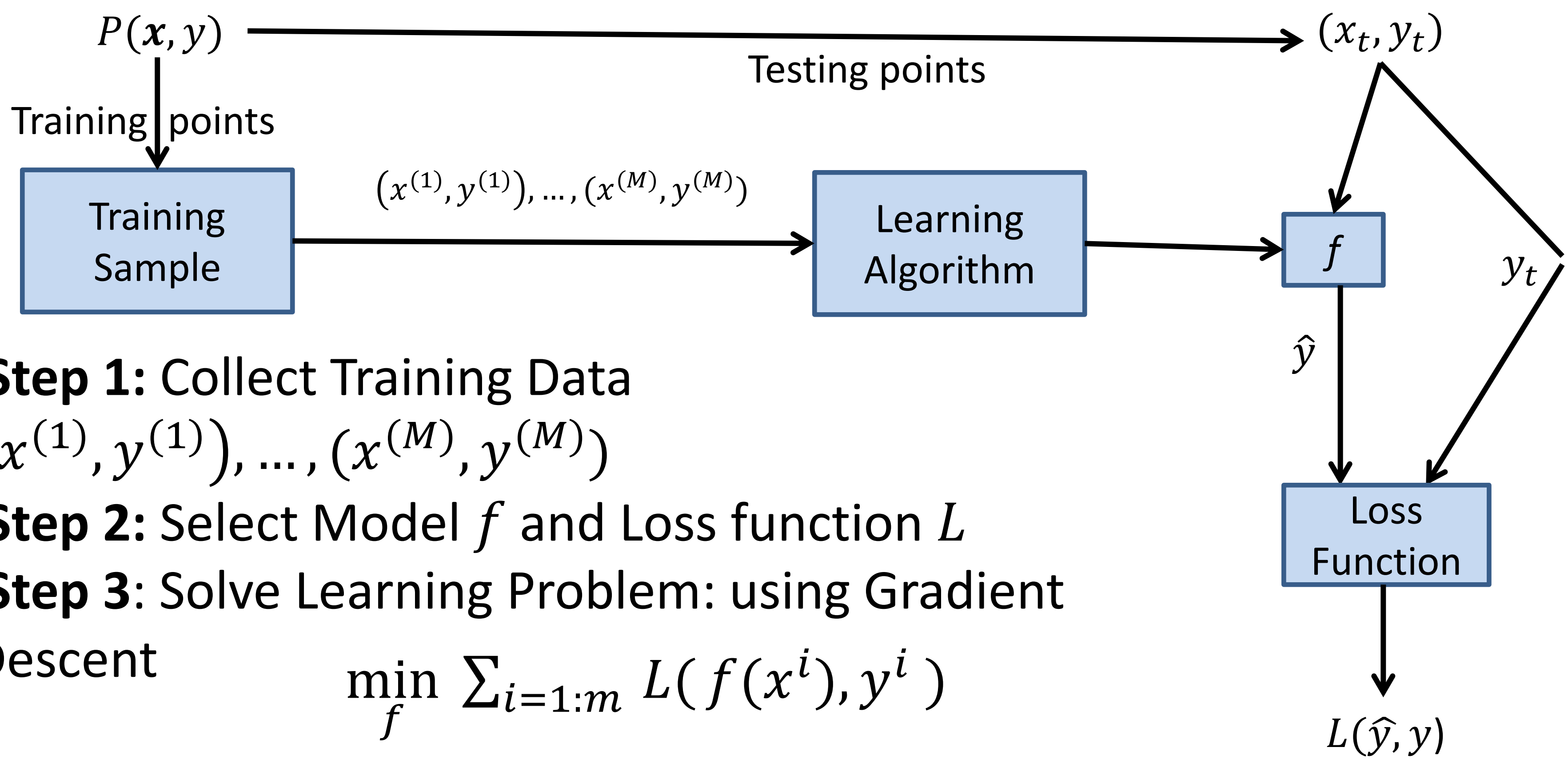
Recap: Hypothesis Space

- Hypothesis space: set of allowable functions $f : X \to Y$

- Goal: find the "best" element of the hypothesis space

  - How do we measure the quality of f?

Recap: Supervised Learning Workflow



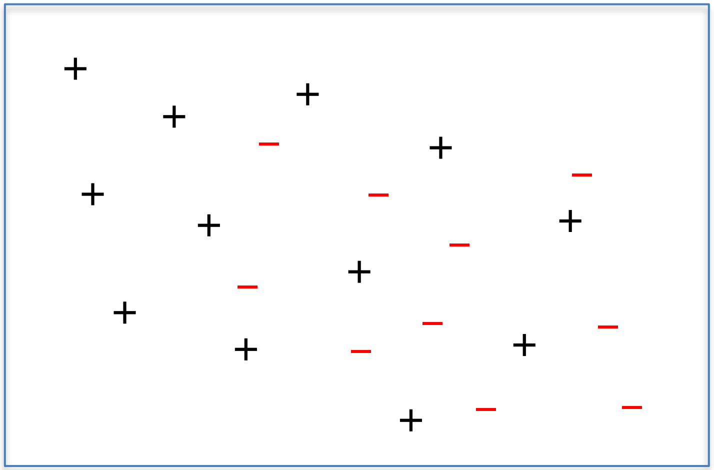
- **Step 1:** Collect Training Data $(x^{(1)}, y^{(1)}), \dots, (x^{(M)}, y^{(M)})$
- **Step 2:** Select Model f and Loss function L
- **Step 3**: Solve Learning Problem: using Gradient Descent $$\min_f \sum_{i=1:m} L(f(x^i), y^i)$$
- **Step 4:** Obtain Predictions $\hat{y}_t = f(x_t)$ on all **Test Data**
- **Step 5:** Evaluation -- Measure the error $Err(\hat{y}_t, y_t)$ averaged over all **Test Data.**

Supervised Learning Workflow Cont…

- **Collect Training Data**
- **Select a hypothesis space** (elements of the space are represented by a collection of parameters)
- **Choose a loss function** (evaluates quality of the hypothesis as a function of its parameters)
- **Minimize loss function using gradient descent** (minimization over the parameters)
- **Evaluate quality of the learned model using test data** – that is, data on which the model was not trained

Binary Classification

- Input $\left(x^{(1)}, y^{(1)}\right), \ldots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
- We can think of the observations as points in $\mathbb{R}^n$ with an associated sign (either +/- corresponding to 0/1)
- An example with $n = 2$



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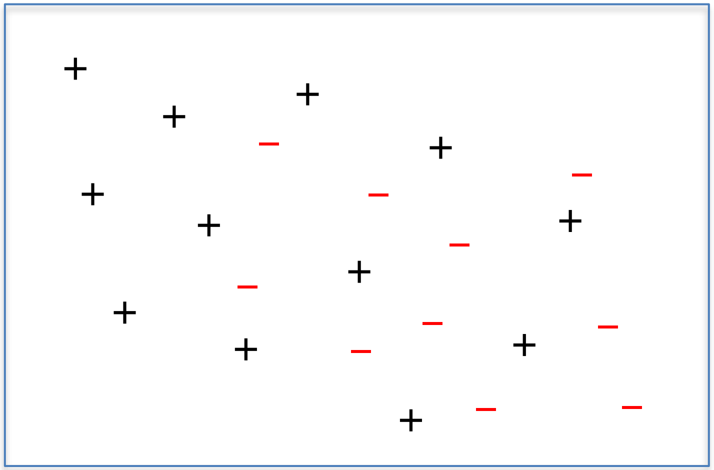

What is a good hypothesis space for this problem?

Binary Classification

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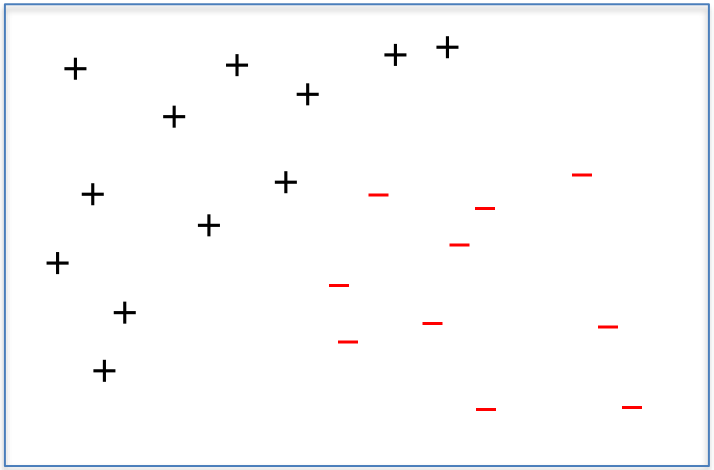

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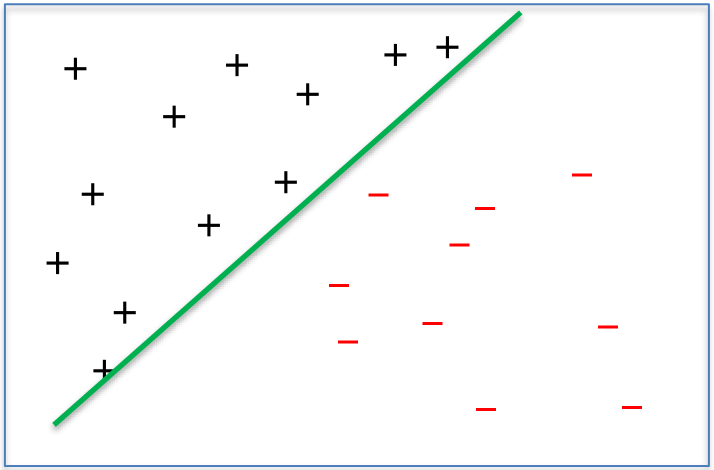


In this case, we say that the observations are linearly separable

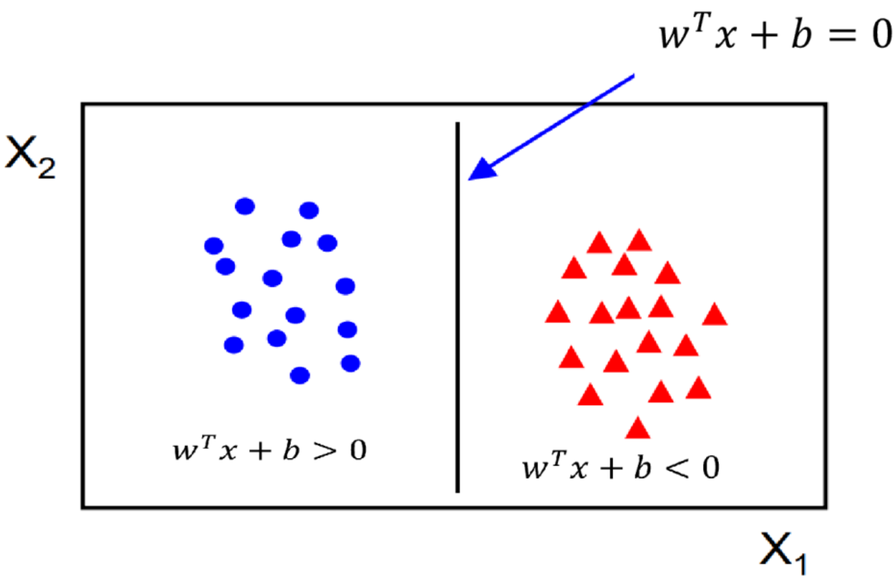
Linear Separators

- In n dimensions, a hyperplane is a solution to the equation

$$w^T x + b = 0$$

  with $w \in \mathbb{R}^n, b \in \mathbb{R}$

- Hyperplanes divide $\mathbb{R}^n$ into two distinct sets of points (called open halfspaces)
  - Half Space 1: $w^T x + b > 0$
  - Half Space 2: $w^T x + b < 0$



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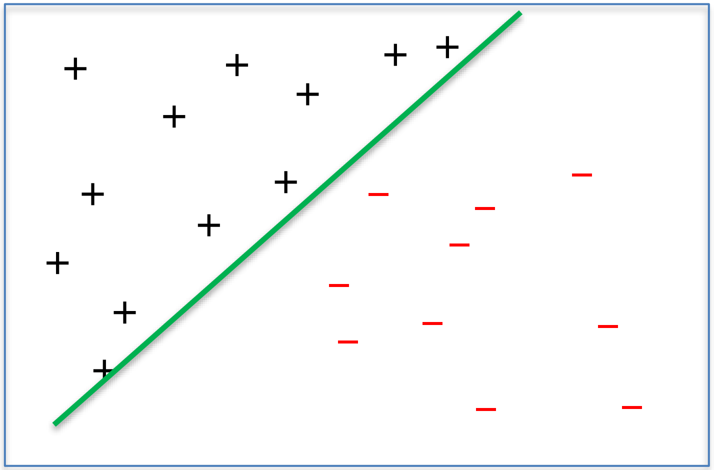

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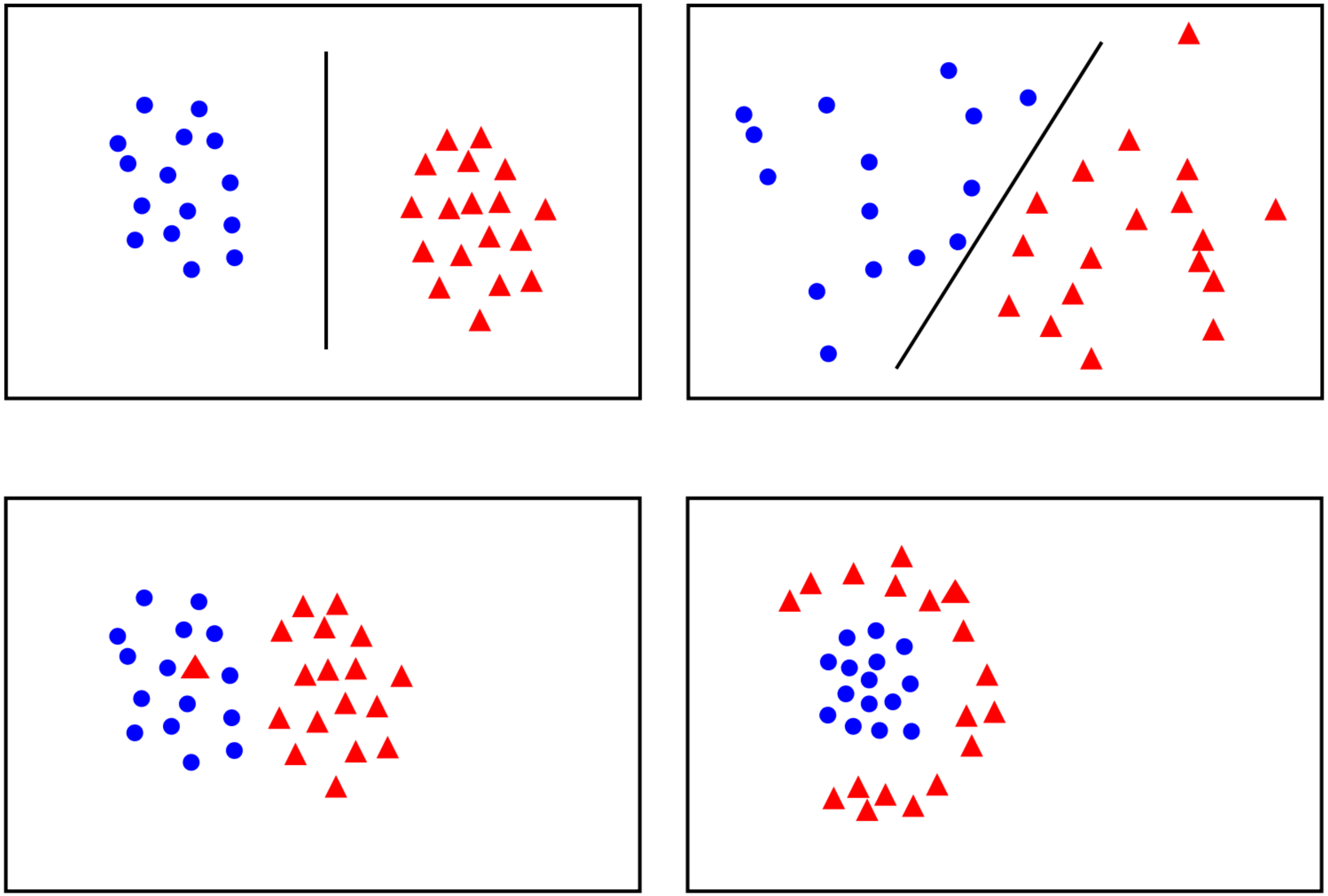
Linear Separable

linearly separable

not linearly separable



The Linearly Separable Case

- Input $\left(x^{(1)}, y^{(1)}\right), \ldots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$

- Hypothesis space: separating hyperplanes

$$f(x) = sign\ (w^T x + b)$$

- How should we choose the loss function?

The 0/1 Loss (Seperable Case)

- Input $(x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
- Hypothesis space: separating hyperplanes

$$f(x) = sign\,(w^T x + b)$$

- How should we choose the loss function?
  - Count the number of misclassifications

$$zero/one\ loss = \frac{1}{2}\sum_m \left|y^{(m)} - sign(w^T x^{(m)} + b)\right|$$

    - Tough to optimize, gradient contains no information

The Perceptron Loss (Seperable Case)

- Input $(x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
- Hypothesis space: separating hyperplanes

$$f(x) = sign\ (w^T x + b)$$

- How should we choose the loss function?
  - Penalize misclassification linearly by the size of the violation

$$perceptron\ loss = \sum_m \max\{0, -y^{(m)}(w^T x^{(m)} + b)\}$$

  - Modified hinge loss (this loss is convex, but not differentiable)

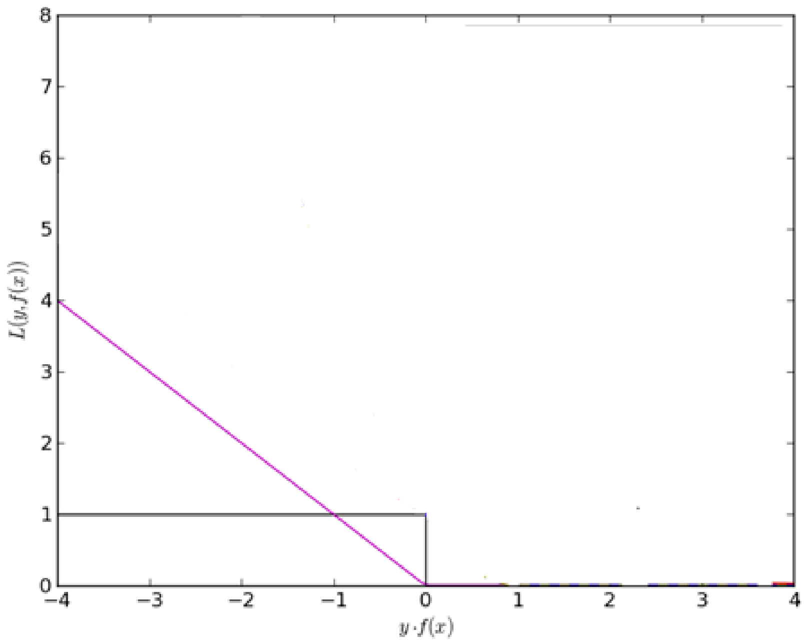
0/1 Loss Vs Perceptron Loss

- Zero/One Loss which counts the number of mis-classifications:

$$zero/one\ loss = \frac{1}{2}\sum_m \left|y^{(m)} - sign(w^T x^{(m)} + b)\right|$$

- Perceptron Loss:

$$perceptron\ loss = \sum_m \max\{0, -y^{(m)}(w^T x^{(m)} + b)\}$$



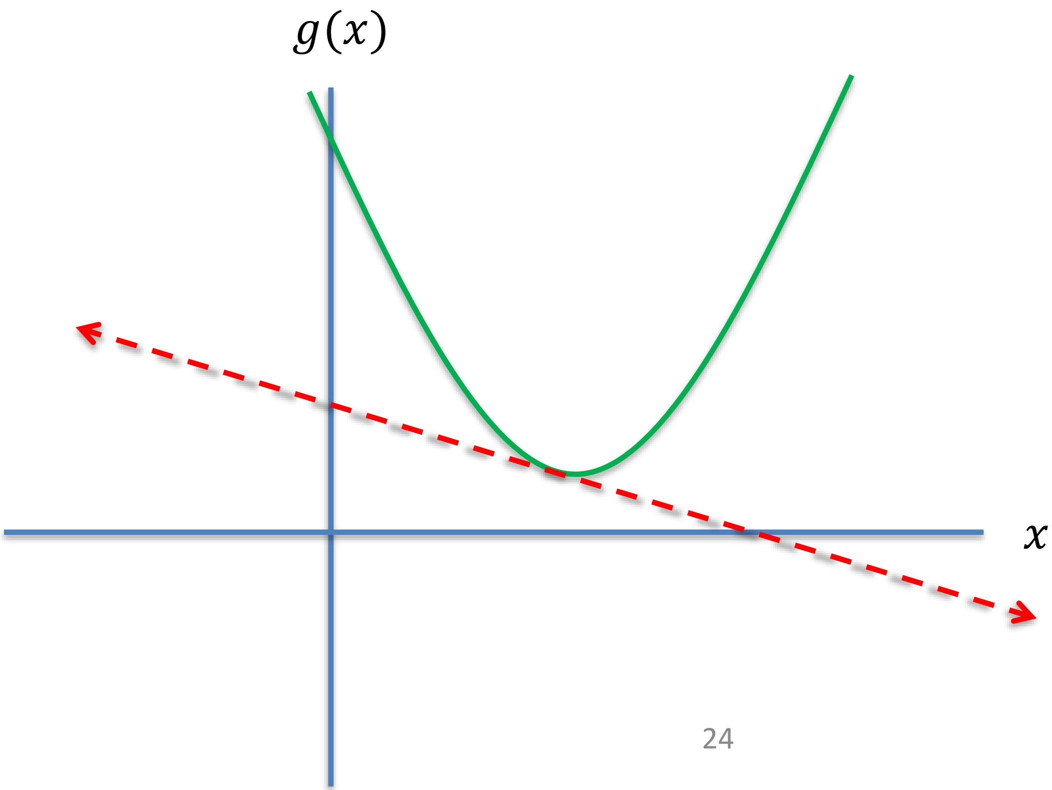
The Perceptron Algorithm

- Try to minimize the perceptron loss using gradient descent
  - The perceptron loss isn't differentiable, how can we apply gradient descent?
  - Need a generalization of what it means to be the gradient of a **convex** function

Part II: (Sub) Gradient Descent and Perceptron

Gradients of Convex Functions

- For a differentiable convex function $g(x)$ its gradients are **linear underestimators**



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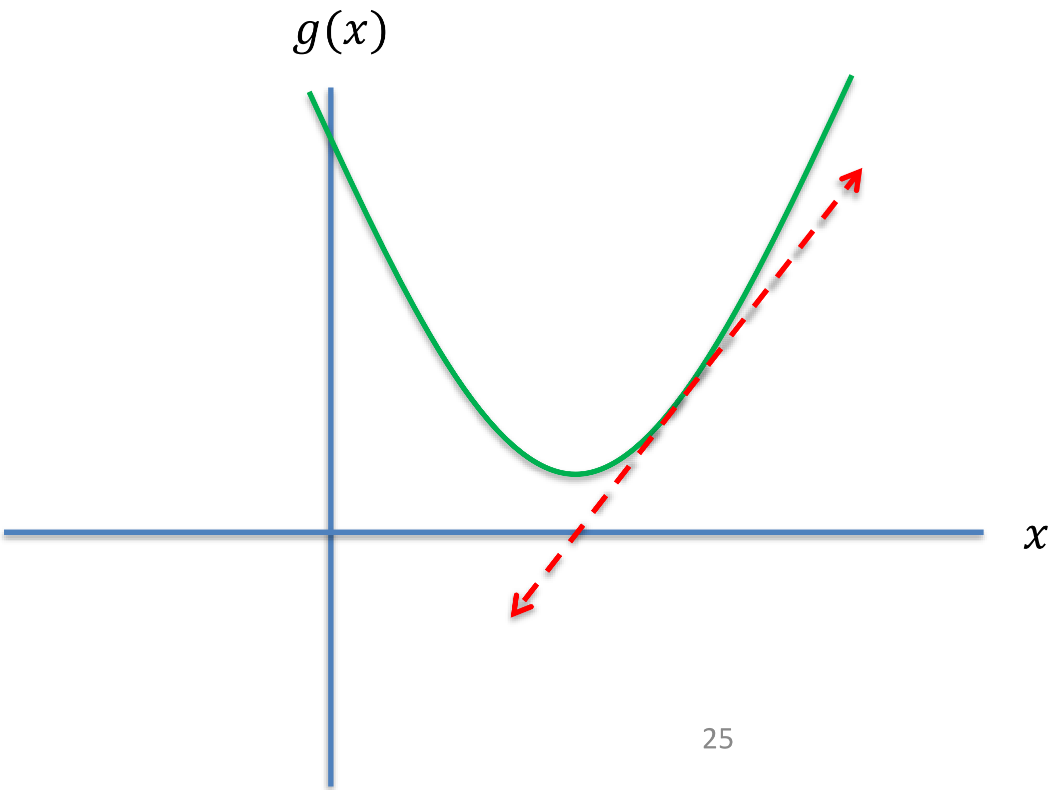
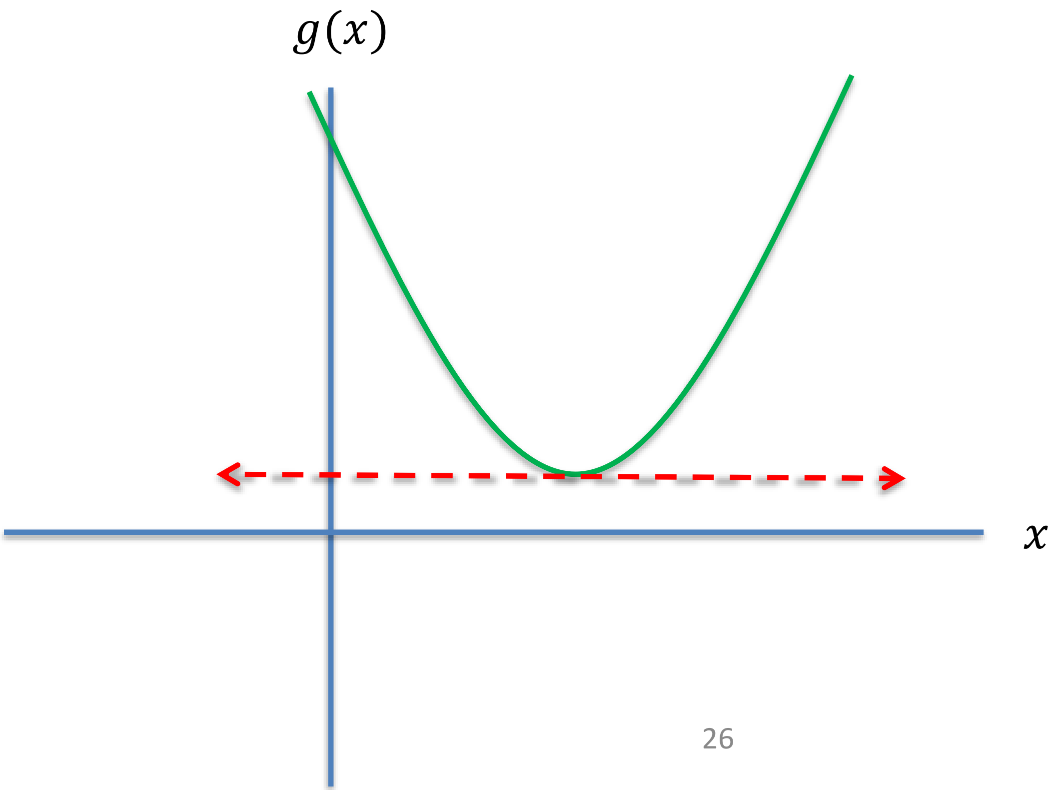


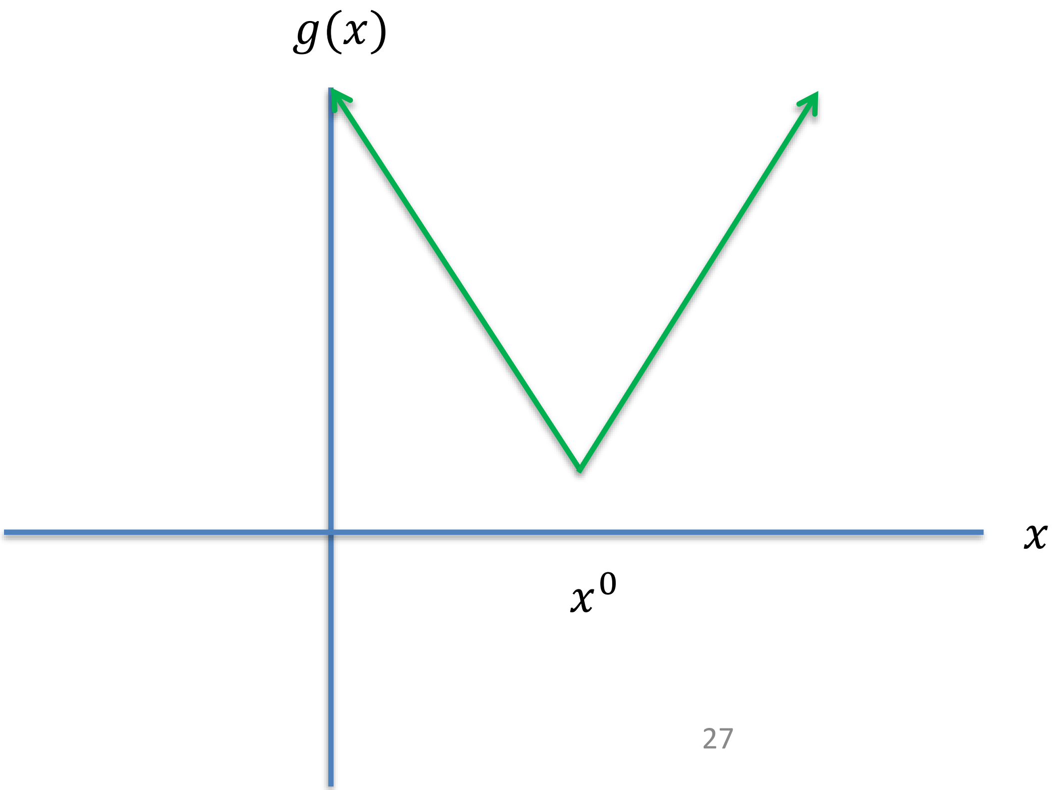
Gradients of Convex Functions

- For a differentiable convex function $g(x)$ its gradients are **linear underestimators**: zero gradient corresponds to a global optimum



Subgradients

- For a convex function $g(x)$, a subgradient at a point x^0 is given by any line, l, such that $l\left(x^0\right) = g(x^0)$ and $l(x) \leq g(x)$ for all x, i.e., it is a linear underestimator



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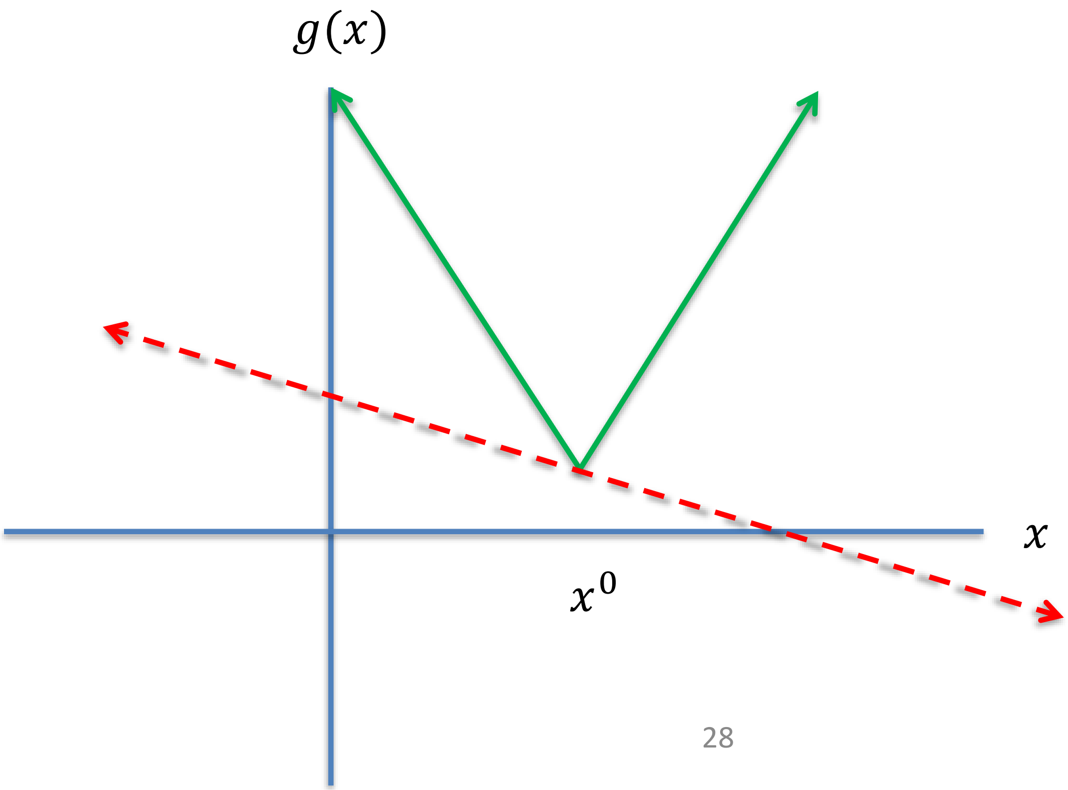

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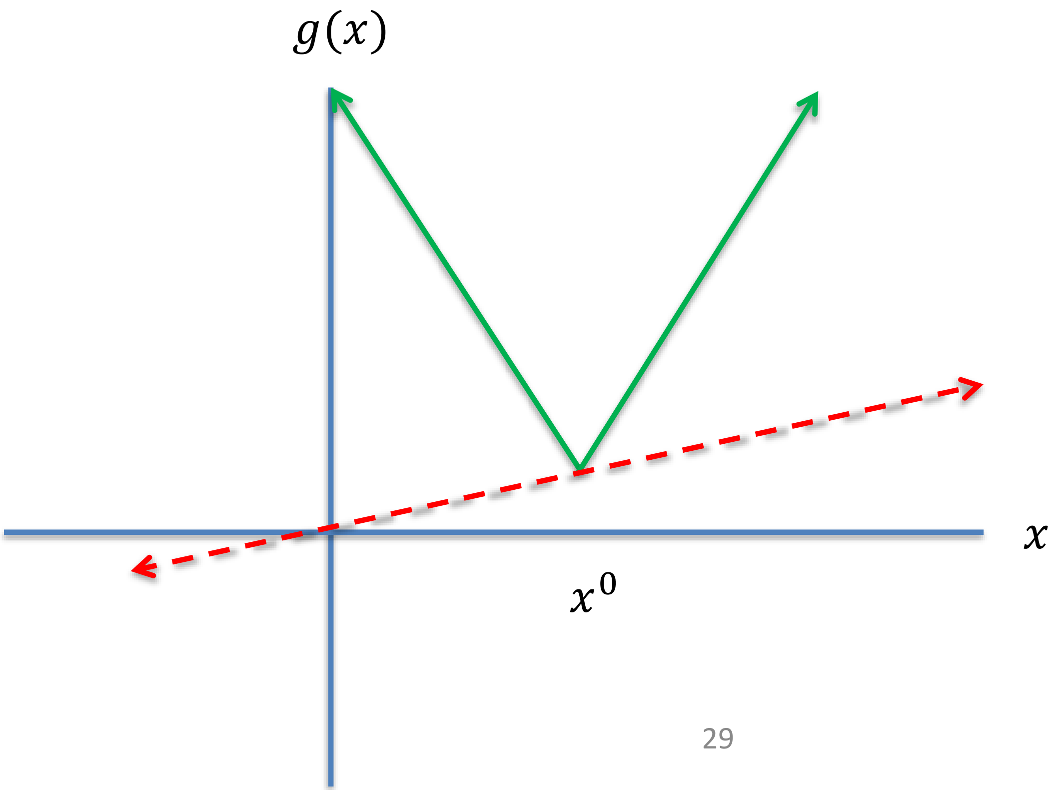

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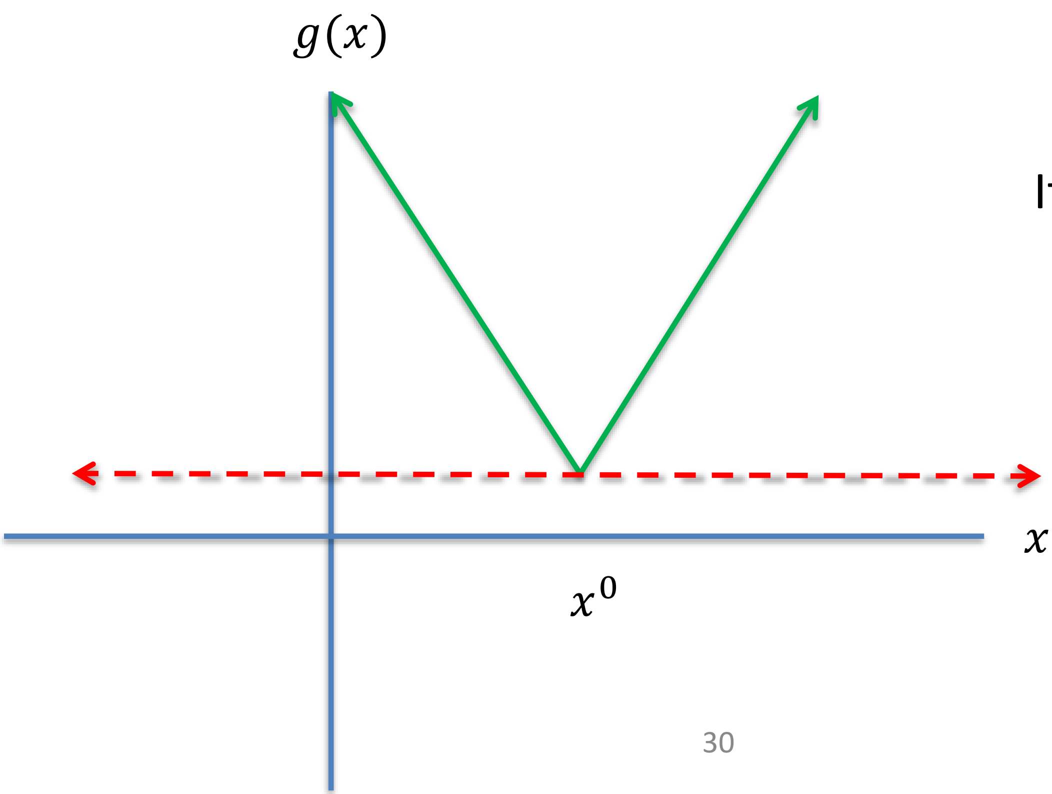

If $\vec{0}$ is a subgradient at x^0, then x^0 is a global minimum

Subgradients

- If a convex function is differentiable at a point x, then it has a unique subgradient at the point x given by the gradient

- If a convex function is not differentiable at a point x, it can have many subgradients

  - E.g., the set of subgradients of the convex function $|x|$ at the point $x = 0$ is given by the set of slopes $[-1,1]$

- Subgradients only guaranteed to exist for convex functions

The Perceptron Algorithm

- Try to minimize the perceptron loss using (sub)gradient descent

The Perceptron Algorithm

- Try to minimize the perceptron loss using (sub)gradient descent

$$\nabla_w(perceptron\ loss) = -\sum_{m=1}^{M}\left(y^{(m)}x^{(m)} \cdot 1_{-y^{(m)}f_{w,b}(x^{(m)})\geq 0}\right)$$

$$\nabla_b(perceptron\ loss) = -\sum_{m=1}^{M}\left(y^{(m)} \cdot 1_{-y^{(m)}f_{w,b}(x^{(m)})\geq 0}\right)$$

The Perceptron Algorithm

- Try to minimize the perceptron loss using (sub)gradient descent

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$$\nabla_b(perceptron\ loss) = -\sum_{m=1}^{M}\left(y^{(m)} \cdot 1_{-y^{(m)}f_{w,b}\left(x^{(m)}\right)\geq 0}\right)$$

Is equal to zero if the m^{th} data point is correctly classified and one otherwise

The Perceptron Algorithm

- Try to minimize the perceptron loss using (sub)gradient descent

$$w^{(t+1)} = w^{(t)} + \gamma_t \sum_{m=1}^{M} \left(y^{(m)} x^{(m)} \cdot 1_{-y^{(m)} f_{w,b}\left(x^{(m)}\right) \geq 0} \right)$$

$$b^{(t+1)} = b^{(t)} + \gamma_t \sum_{m=1}^{M} \left(y^{(m)} \cdot 1_{-y^{(m)} f_{w,b}\left(x^{(m)}\right) \geq 0} \right)$$

- With step size γ_t (also called the learning rate)
- Note that, for convergence of subgradient methods, a diminishing step size, e.g., $\gamma_t = \frac{1}{1+t}$ is required

Stochastic Gradient Descent

- To make the training more practical, stochastic (sub)gradient descent is often used instead of standard gradient descent

- Approximate the gradient of a sum by sampling a few indices (as few as one) uniformly at random and averaging

$$\nabla_x \left[\sum_{m=1}^{M} g_m(x) \right] \approx \frac{1}{K} \sum_{k=1}^{K} \nabla_x g_{m_k}(x)$$

here, each m_k is sampled uniformly at random from $\{1, \dots, M\}$

- Stochastic gradient descent converges to the global optimum under certain assumptions on the step size

Stochastic Gradient Descent

- Setting $K = 1$, we pick a random observation m and perform the following update

if the $\boldsymbol{m^{th}}$ data point is misclassified:

$$w^{(t+1)} = w^{(t)} + \gamma_t y^{(m)} x^{(m)}$$

$$b^{(t+1)} = b^{(t)} + \gamma_t y^{(m)}$$

if the $\boldsymbol{m^{th}}$ data point is correctly classified:

$$w^{(t+1)} = w^{(t)}$$
$$b^{(t+1)} = b^{(t)}$$

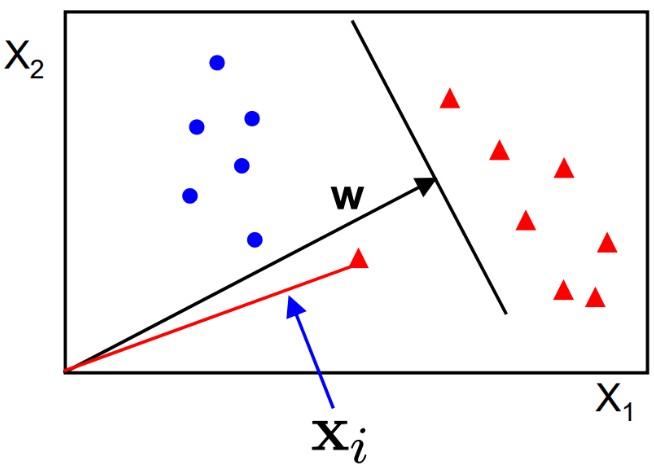
- Sometimes, you will see the perceptron algorithm specified with $\gamma_t = 1$ for all t

Perceptron Example

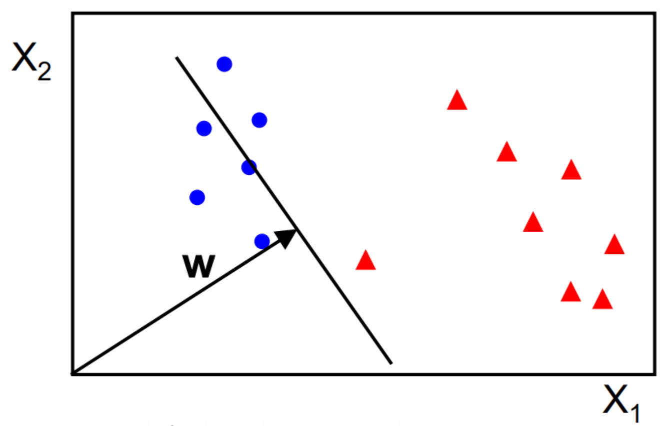
For example in 2D

- Initialize **w** = 0
- Cycle though the data points { $\mathbf{x}_i$, y_i }
  - if $\mathbf{x}_i$ is misclassified then $\mathbf{w} \leftarrow \mathbf{w} + \alpha\,\mathrm{sign}(f(\mathbf{x}_i))\,\mathbf{x}_i$
- Until all the data is correctly classified

before update



after update



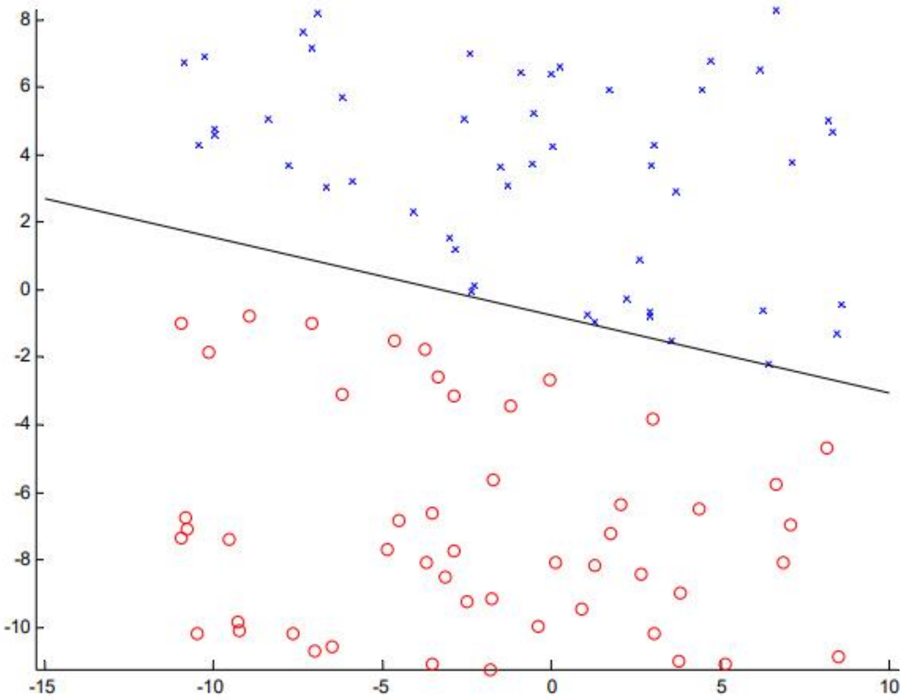
$$\mathbf{w} \leftarrow \mathbf{w} - \alpha\,\mathbf{x}_i$$

NB after convergence $\mathbf{w} = \sum_i^N \alpha_i \mathbf{x}_i$

Part III: More On Perceptron

More on Perceptron

Perceptron example



- if the data is linearly separable, then the algorithm will converge
- convergence can be slow …
- separating line close to training data
- we would prefer a larger margin for generalization

Applications of Perceptron

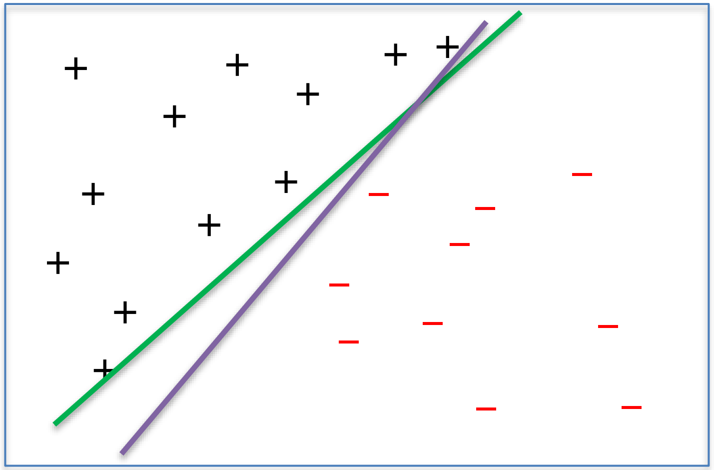
- Spam email classification
  - Represent emails as vectors of counts of certain words (e.g., sir, madam, Nigerian, prince, money, etc.)
  - Apply the perceptron algorithm to the resulting vectors
  - To predict the label of an unseen email
    - Construct its vector representation, x'
    - Check whether or not $w^T x' + b$ is positive or negative

Perceptron Learning Drawbacks

- No convergence guarantees if the observations are not linearly separable

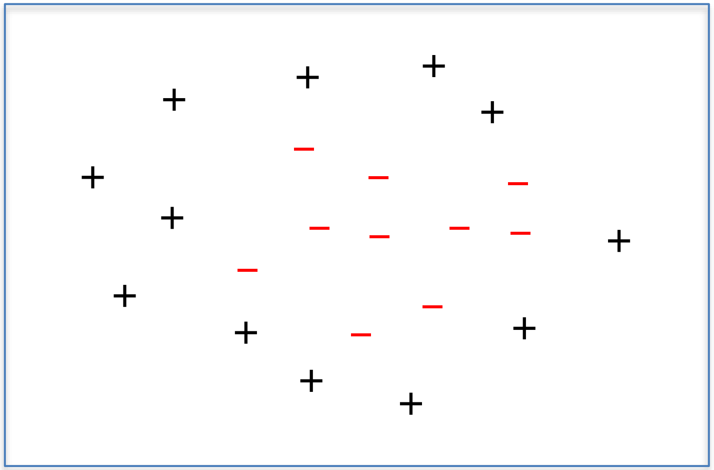
- Can overfit

  - There can be a number of perfect classifiers, but the perceptron algorithm doesn't have any mechanism for choosing between them



What If the Data Isn‘t Separable?

- Input $(x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
- We can think of the observations as points in $\mathbb{R}^n$ with an associated sign (either +/- corresponding to 0/1)
- An example with $n = 2$



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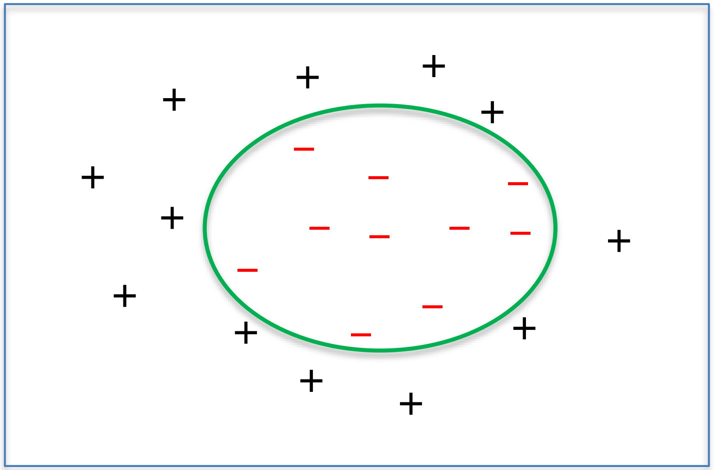
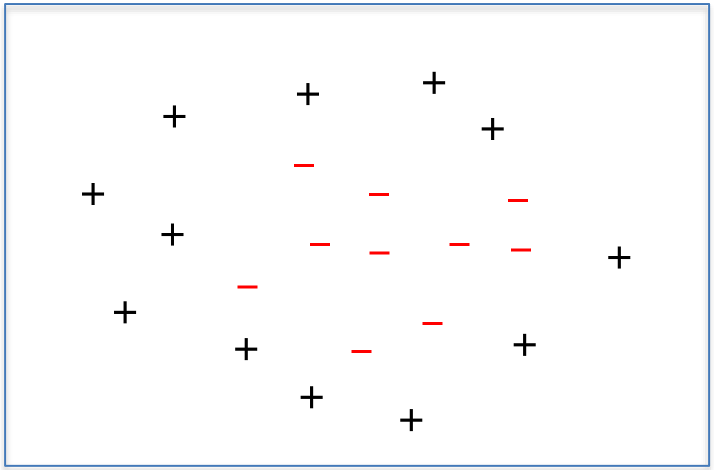


What is a good hypothesis space for this problem?

Adding Features

- Perceptron algorithm only works for linearly separable data



Can add features to make the data linearly separable in a higher dimensional space!

Essentially the same as higher order polynomials for linear regression!

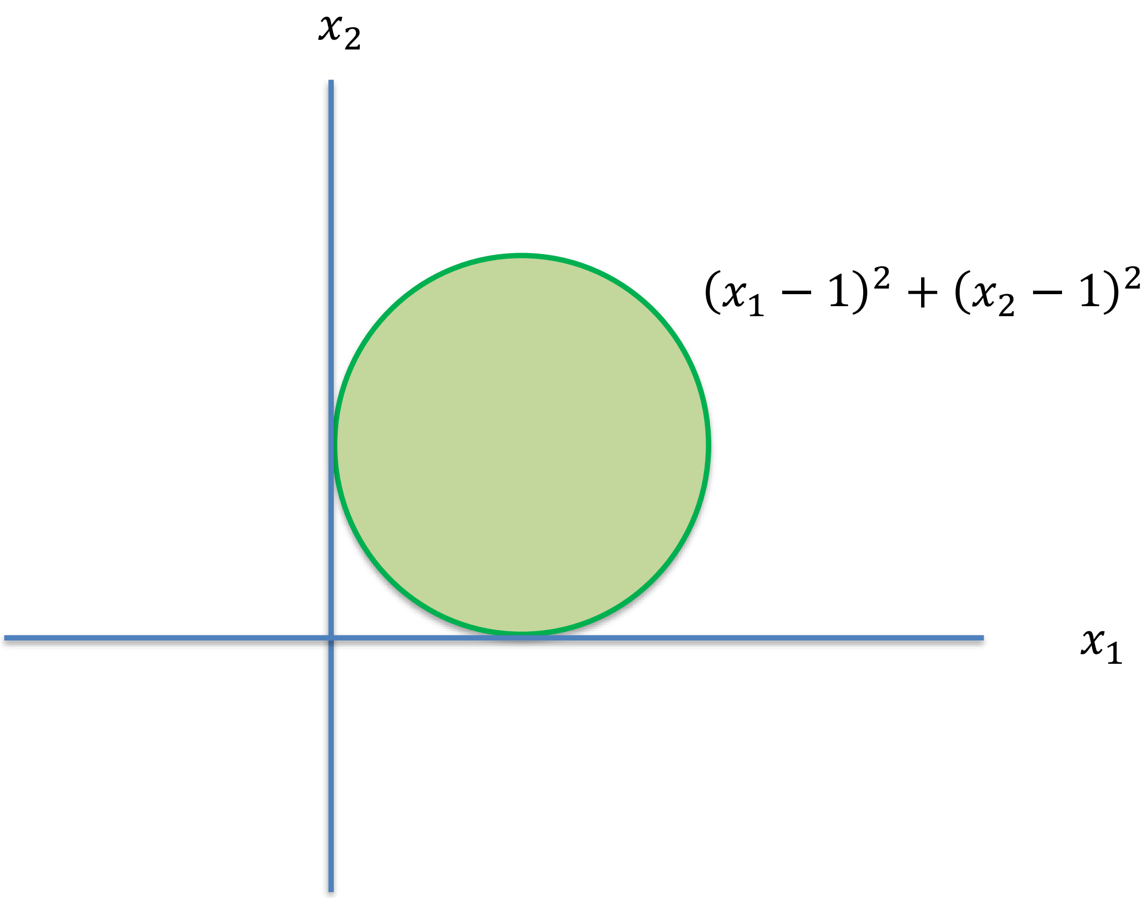
Adding Features

- The idea, choose a feature map $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^k$
  - Given the observations $x^{(1)}, \ldots, x^{(M)}$, construct feature vectors $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$
  - Use $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$ instead of $x^{(1)}, \ldots, x^{(M)}$ in the learning algorithm
  - Goal is to choose ϕ so that $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$ are linearly separable in $\mathbb{R}^k$
  - Learn linear separators of the form $w^T\phi(x)$ (instead of $w^T x$)
- Warning: more expressive features can lead to overfitting!

Adding Features: Examples

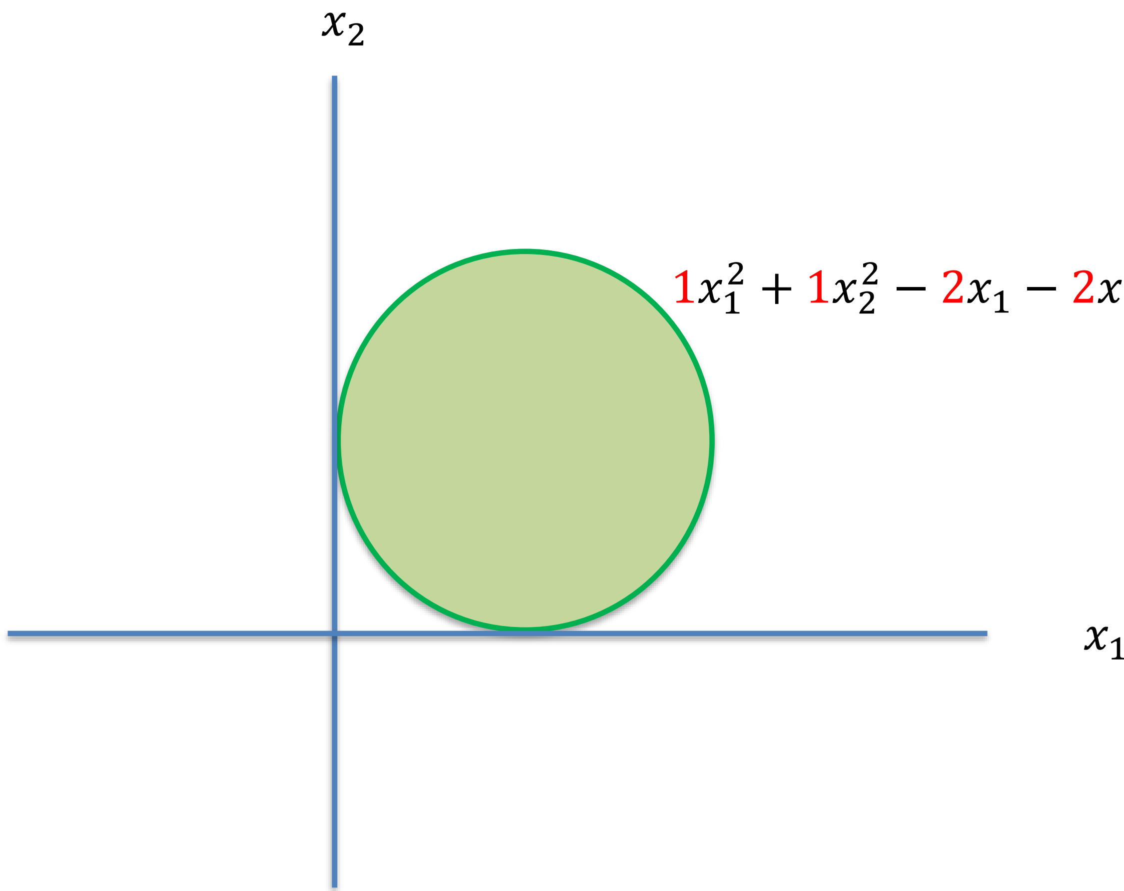
- $\phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
  - This is just the input data, without modification
- $\phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}$
  - This corresponds to a second-degree polynomial separator, or equivalently, elliptical separators in the original space

Adding Features



$$(x_1 - 1)^2 + (x_2 - 1)^2 - 1 \leq 0$$

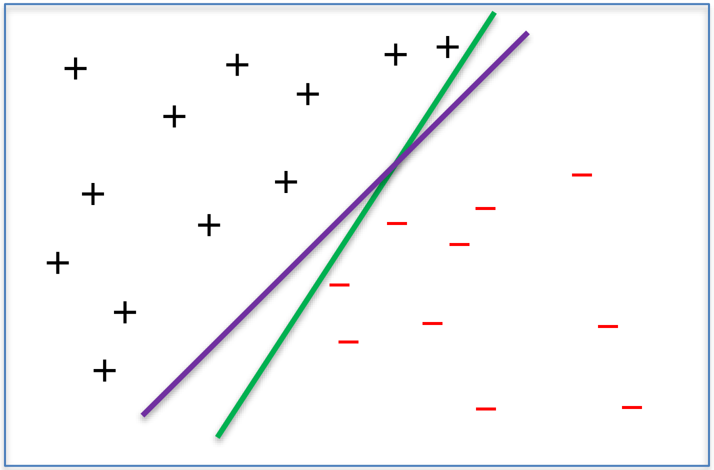
Adding Features



$$1x_1^2 + 1x_2^2 - 2x_1 - 2x_2 - 2 \leq 0$$

Support Vector Machines

- How can we decide between two perfect classifiers?



- What is the practical difference between these two solutions?