FACULTY OF INFORMATION TECHNOLOGY AND ELECTRICAL ENGINEERING
DEGREE PROGRAMME IN ELECTRONICS AND COMMUNICATIONS ENGINEERING

# Statistical Signal Processing I

# MATLAB Homework 5


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# 1 DETECTION USING MATCHED FILTER

Signal detection is very critical task in several signal processing applications. The goal of this assignment is apply detection theory to a signal $\mathbf{x}$ in a noisy environment. Let assume that the receiver observes the following signal

$$x[n] = As[n] + w[n], \quad n = 0, \ldots, N-1,$$

where both $A$ and $\mathbf{S} = \left[s[0], s[1], \ldots, s[N-1]\right]^{\mathrm{T}}$ are deterministic variables and **known to the Receiver**. let us consider that $\mathbf{S}$ is a unit-norm Dc level and $\mathbf{w}$ is a white Gaussian noise such that $w[n] \sim \mathcal{N}(0, \sigma^2), n = 0, \ldots, N-1$.

The receiver aims to detect whether the signal $A\mathbf{S}$ is present. This can be formulated using the following two hypotheses

$$H_0 : \mathbf{x} = \mathbf{w}$$

$$H_1 : \mathbf{x} = A\mathbf{s} + \mathbf{w},$$

Figure 1.1. Problem Statement

### 1.1 Derive a detector of your choice and explain clearly when you chose that detector.

Since this involved a known deterministic signal in White Gaussian Noise, I have chosen the matched filter approach to design the detector. The derivation starts from the Neyman-Pearson (NP) criterion for both hypothesis $\mathcal{H}_{\prime}$ and $\mathcal{H}_{\infty}$ shown in Figure 1.1.[1]

$$p(x; \mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[\frac{-1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - As[n])^2\right] \tag{1.1}$$

$$p(x; \mathcal{H}_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[\frac{-1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n])^2\right] \tag{1.2}$$

Then the NP detector decides $\mathcal{H}_1$ if the likelihood ratio shown in equation exceeds a threshold $\gamma$

$$L(x) = \exp\left[\frac{-1}{2\sigma^2} \left(\sum_{n=0}^{N-1} (x[n] - As[n])^2 - \sum_{n=0}^{N-1} x^2[n]\right)\right] > \gamma \tag{1.3}$$

$$\ln L(x) = \frac{-1}{2\sigma^2}\left[\sum_{n=0}^{N-1}(x[n] - As[n])^2 - \sum_{n=0}^{N-1}x^2[n]\right] > \ln\gamma \tag{1.4}$$

$$\frac{1}{\sigma^2}\sum_{n=0}^{N-1}x[n]s[n] - \frac{1}{2\sigma^2}A^2N > \ln\gamma \tag{1.5}$$

$$\sum_{n=0}^{N-1}x[n]s[n] > \frac{\sigma^2}{A}\ln\gamma + \frac{AN}{2} \tag{1.6}$$

$$T(x) = \sum_{n=0}^{N-1}x[n]s[n] > \gamma' \tag{1.7}$$

From equations (1.1) to (1.1), it can be seen that the test statistic $T(x)$ can be derived as shown in Equation (1.1). Under the hypothesis $x[n]$ is Gaussian and $T(x)$ is a linear combination of Gaussian random variables, $T(x)$ is also a Gaussian process [1].

Therefore, the means and variances of the test statistics can be found as shown in the equations for mean calculations 1.8, 1.9 and variance calculations 1.10, 1.11.

$$\mathbb{E}[T;\mathcal{H}_0] = \mathbb{E}\left(\sum_{n=0}^{N-1}w[n]s[n]\right) = 0 \tag{1.8}$$

$$\begin{aligned}\mathbb{E}[T;\mathcal{H}_1] &= \mathbb{E}\left(\sum_{n=0}^{N-1}(As^2[n] + w[n]s[n])\right)\\ &= \mathbb{E}\left(\sum_{n=0}^{N-1}(As^2[n])\right) \text{ ; Since } \mathbf{S} \text{ is unit norm}\\ &= A\end{aligned} \tag{1.9}$$

$$
\begin{aligned}
\mathrm{var}[T;\mathcal{H}_0] &= \mathrm{var}\left(\sum_{n=0}^{N-1} w[n]s[n]\right) \\
&= \sum_{n=0}^{N-1} \mathrm{var}(w[n])s^2[n] \\
&= \sigma^2 \sum_{n=0}^{N-1} s^2[n] \text{ ; Since } \mathbf{S} \text{ is unit norm} \\
&= \sigma^2
\end{aligned} \tag{1.10}
$$

$$
\begin{aligned}
\mathrm{var}[T;\mathcal{H}_1] &= \mathrm{var}\left(\sum_{n=0}^{N-1} As^2[n] + w[n]s[n]\right) \\
&= \sum_{n=0}^{N-1} \mathrm{var}(w[n])s^2[n] \\
&= \sigma^2 \sum_{n=0}^{N-1} s^2[n] \text{ ; Since } \mathbf{S} \text{ is unit norm} \\
&= \sigma^2
\end{aligned} \tag{1.11}
$$

Now that the Test statistic has been determined, derivation of $P_{FA}$ and $P_D$ can be derived as shown in equations 1.12, 1.13 where $Q(x)$ is defined by equation 1.14.

$$
\begin{aligned}
P_{FA} &= Pr[T > \gamma';\mathcal{H}_0] \\
&= Q\left(\frac{\gamma' - 0}{\sqrt{\sigma^2}}\right)
\end{aligned} \tag{1.12}
$$

$$
\begin{aligned}
P_D &= Pr[T > \gamma';\mathcal{H}_1] \\
&= Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2}}\right)
\end{aligned} \tag{1.13}
$$

$$
Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right)\, dt \tag{1.14}
$$

### 1.2 Implement Monte Carlo simulation for your detector

The mean and variances of the simulated values are in line with the theoretical values rending the PDFs of both aligned close to each other as shown in figure 1.2.

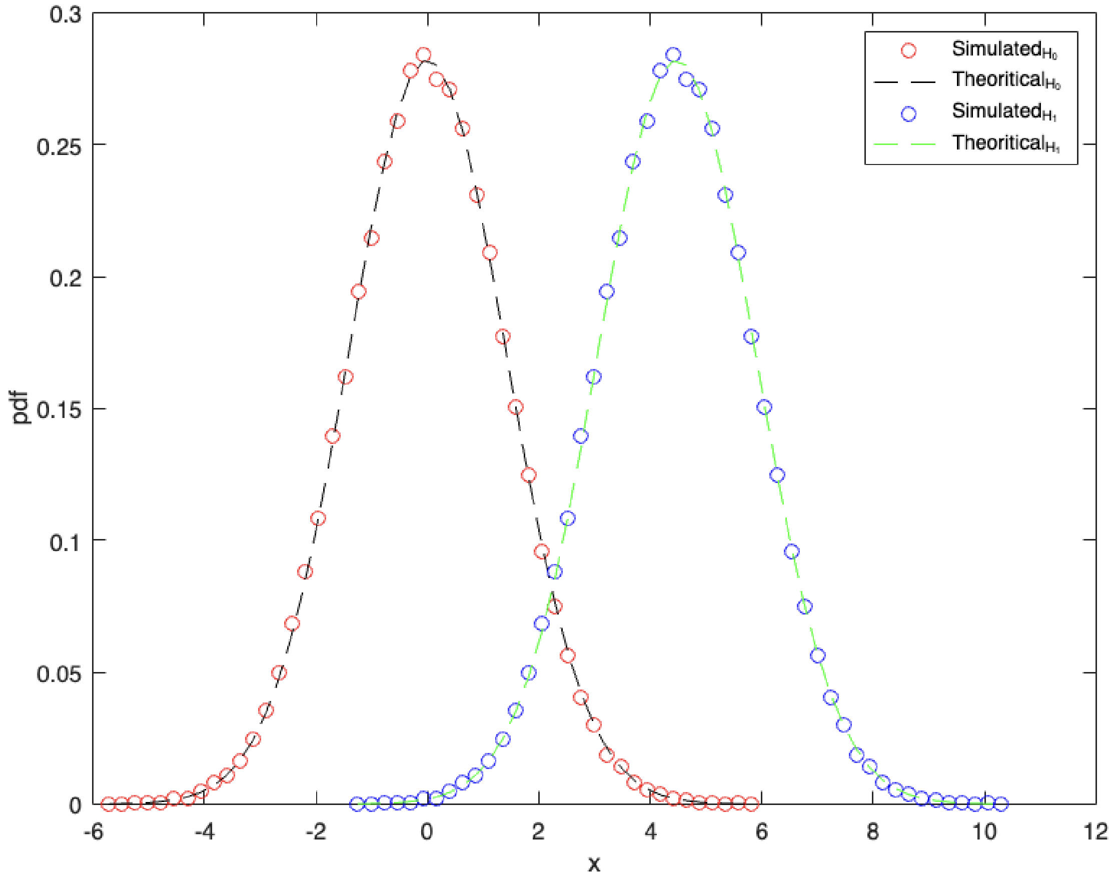



Figure 1.2. PDFs of Theoretical and Simulated Test Statistics of Detector Hypothesis

### 1.3 Plot the ROC curve of your estimator

Figure 1.3 shows the ROC curves generated for this section from Matlab script.

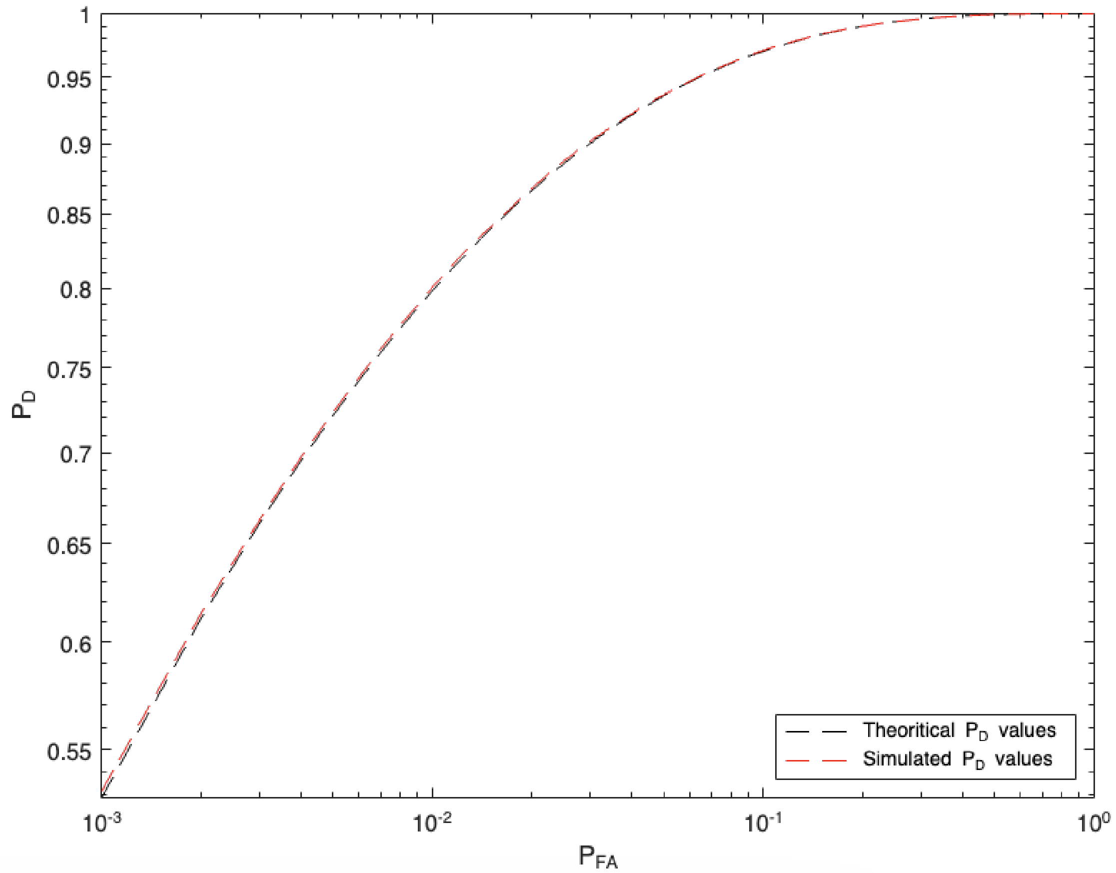



Figure 1.3. ROC curves for theoretical and simulated values

### 1.4 Change the values of $A$ to plot $P_D$ versus $P_{FA}$ for the following signal-to-noise ratio (SNR) values $-5, 0, 5, 10, 15, 20$ dB.

Figure 1.4 shows the behaviour of the ROC curves for different SNR values. It can be seen that as the SNR increases the detection progressively becomes perfect.

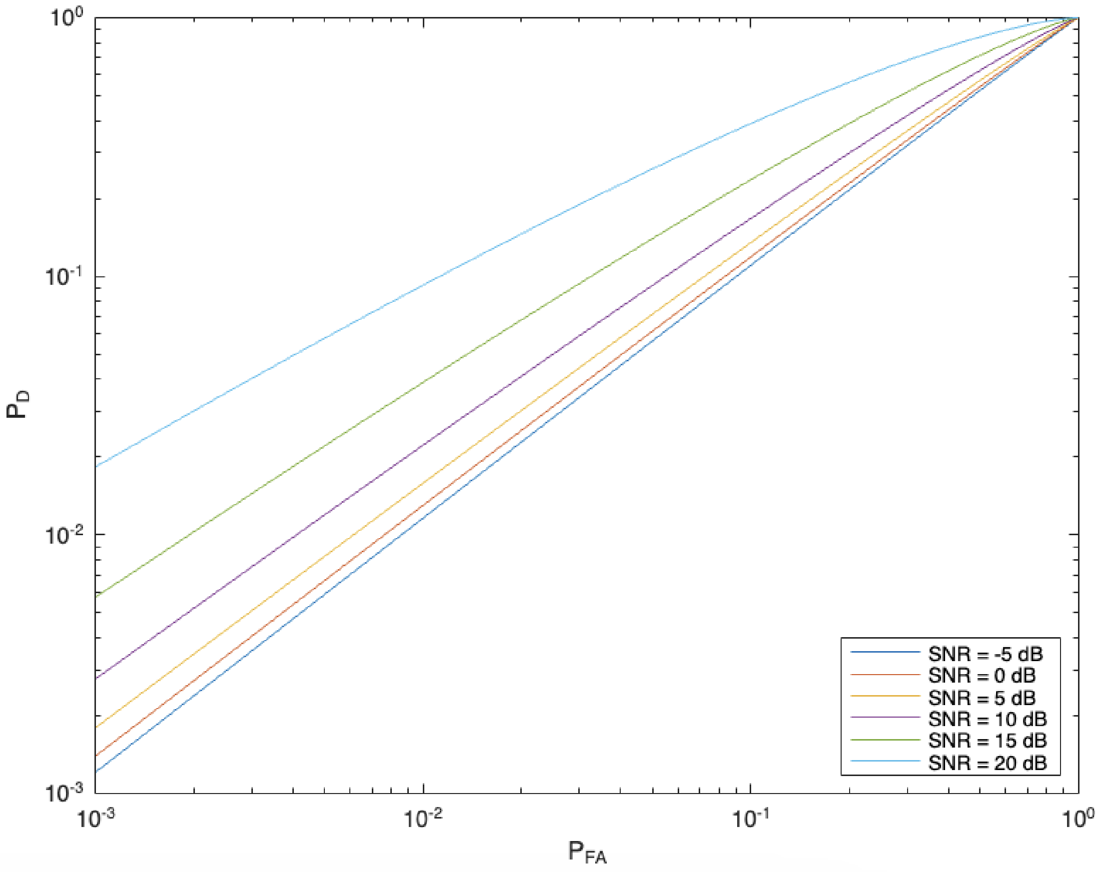



Figure 1.4. $P_D$ Vs $P_{FA}$ for different SNR values

## 1.5 Matlab Code for this Task

### Define parameters

```
N = 100; % number of samples
noiseVar = 2; % Noise Variance of WGN
A = sqrt(20); % DC level component
s = ones(1,N)/sqrt(N); % initialize size for s[n] with DC level unit norm
assert(norm(s) == 1 , 'Norm condition of s[n] is not satisfied');
mcLoops = 1E5;
Matched_H0 = ones(1,mcLoops)*NaN;
Matched_H1 = ones(1,mcLoops)*NaN;
```

### Running Monte Carlo Loops

```
for MC = 1 :mcLoops
    noisePart = randn(1,N)*sqrt(noiseVar);
    signalAndNoise = A*s + noisePart;
    % T(x) = x[n]s[n]
    % T(x) for H0 = \Sigma [w[n]s[n]]
    % T(x) for H1 = \Sigma As[n]s[n] + w[n]s[n]
    Matched_H0(MC) = sum(noisePart.*s);
    Matched_H1(MC) = sum(signalAndNoise.*s);
end
```

### Plotting results

```
[nH0,xHo] = hist(Matched_H0,50);
[nH1,xH1] = hist(Matched_H1,50);

% Theoretical plot of T(x;H0)
mean_H0 = 0; % Eq(1.8) in the report
var_H0 = noiseVar; % Eq(1.10) in the report
theo_pdf_H0 = 1 / (sqrt(2 * pi * var_H0)) * exp(-((xHo - mean_H0).^2) / (2 * var_H0));
bin_widthH0 = xHo(2) - xHo(1);
plot(xHo,nH0/(sum(nH0)*(bin_widthH0)),'ro',xHo,theo_pdf_H0,'k--')
hold on

% Theoretical plot of T(x;H1)
mean_H1 = A; % Eq(1.9) in the report
var_H1 = noiseVar; % Eq(1.11) in the report
theo_pdf_H1 = 1 / (sqrt(2 * pi * var_H1)) * exp(-((xH1 - mean_H1).^2) / (2 * var_H1));
bin_widthH1 = xH1(2) - xH1(1);
plot(xH1,nH1/(sum(nH1)*(bin_widthH1)),'bo',xH1,theo_pdf_H1,'g--')

legend('Simulated_{H_0}','Theoritical_{H_0}','Simulated_{H_1}','Theoritical_{H_1}');
xlabel('x');
ylabel('pdf')
hold off
```

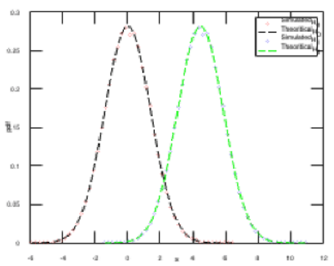


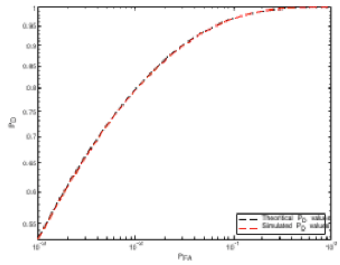
Figure 1.5. Part 1

## ROC Curves

```
PFA = linspace(0, 1, 1000);
gamma = sqrt(noiseVar)*qfuncinv(PFA); % Eq(1.12) in the report
PD_theo = qfunc((gamma-A)/sqrt(noiseVar)); % Eq(1.13) in the report
PD_simu = ones(size(PFA));

for gammaVal = gamma
    PD_simu(gammaVal == gamma) = mean(Matched_H1 > gammaVal);
end

loglog(PFA,PD_theo,'k--',PFA,PD_simu,'r--')
legend('Theoritical P_D values','Simulated P_D values','Location', 'south east');
xlabel('P_{FA}');
ylabel('P_D')
```



## Different SNR Values for PD versus PFA curves

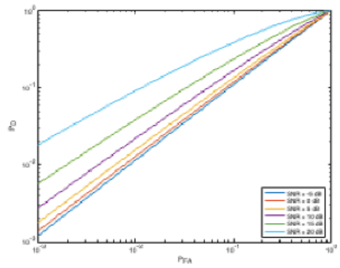
```
SNRValues = [-5, 0, 5, 10, 15, 20];
AValues = ones(size(SNRValues));

for SNR = SNRValues
    AValues(SNR == SNRValues) = calculate_A(SNR, N, noiseVar);
end



for Aind = AValues
    PD_theo = qfunc((gamma-Aind)/sqrt(noiseVar));
    SNRCoreVal  = SNRValues(Aind ==AValues);
    dispName = ['SNR = ',num2str(SNRCoreVal),' dB'];
    loglog(PFA,PD_theo,'DisplayName', dispName)
    hold on

end
hold off
legend('Location', 'south east');
xlabel('P_{FA}');
ylabel('P_D');
```



```
function A = calculate_A(SNR_dB, N, noiseVar)
% Function to convert log to linear scale
    SNR_linear = 10^(SNR_dB/10);
    A = sqrt(SNR_linear * noiseVar / N);
end
```

Figure 1.6. Part 1

# 2 REFERENCES

[1] Kay Steven M. k. (1998) Fundamentals of statistical signal processing. Vol. 2, Detection theory. Prentice-Hall signal processing series, Prentice Hall PTR, Upper Saddle River, N.J., Lisäpainokset: 16th printing 2013.