521324S SSP II — Matlab Simulation Exercise Task 6

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ESTIMATOR CORRELATOR (7 PTS)

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In this lab task, you are asked to reproduce the numerical results for *[Example 5.1, Kay, Vol 2]*, which considers the detection of a Gausssian random signal in white Gaussian noise (WGN). You are subsequently asked to extend the simulation to the general linear model case.

Let $s[n]$ be a zero mean, Gaussian random process with known covariance $\mathbf{C}_\mathbf{s}$. Assume that $w[n]$ is WGN signal with known variance σ^2, and independent of the signal. The detection problem is expressed as

$$\begin{aligned} \mathcal{H}_0 &: x[n] = w[n] \qquad\qquad n = 0, 1, \ldots N-1 \\ \mathcal{H}_1 &: x[n] = s[n] + w[n], \quad n = 0, 1, \ldots N-1 \end{aligned}$$

Task:

1) Assume $\mathbf{C}_\mathbf{s} = \sigma_\mathbf{s}^2\mathbf{I}$. Reproduce the results in Figure 5.1, i.e., plot the probability of detection P_D
   a) With $N = 25$.
   b) With $N = 50$. Then compare to the results in Task 1.a

   Hint:
   - Given $P_{\text{FA}} = Q_{\chi^2_N}(x)$. In Matlab, x can be found by the following commands (inversion of CDF of Chi-squared distribution)

     ```
     >> x=chi2inv(1-P_FA,N);
     ```
   - You can follow the analysis on pages 142–143, or that on pages 145–147 (*Estimator correlator*) in textbook *[Kay, Vol 2]*. Note that the formulations leading to the thresholds γ' and γ'' differ in each case. You can derive γ' by noting that

     $$P_{\text{FA}} = Q_{\chi^2_N}\left(\frac{\gamma'}{\text{var}(T(\mathbf{x}))}\right)$$

     Then, given P_{FA}, we have

     $$\gamma' = \text{var}(T(\mathbf{x}))Q^{-1}_{\chi^2_N}\left(P_{\text{FA}}\right)$$

     where $Q^{-1}_{\chi^2_N}(.)$ is the inverse function of $Q_{\chi^2_N}$.

2) The detection problem with the general linear model (see also Sect. 5.4 of *[Kay, Vol 2]*) can be described

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 &: \mathbf{x} = \mathbf{H}\mathbf{d} + \mathbf{w} = \mathbf{s} + \mathbf{w} \end{aligned}$$

where $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]$ is observation vector; $\mathbf{H}$ is a (known) $N \times M$ channel/filter matrix, the n-th row of $\mathbf{H}$ is tap coefficients of channel/filter at time sample n; and $\mathbf{d}$ is $M \times 1$ data vector, $\mathbf{d} \sim \mathcal{N}(\mathbf{0}, \sigma_{\mathbf{s}}^2 \mathbf{I})$; $\mathbf{w}$ is $N \times 1$ noise vector with $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$.

Show that the covariance of signal $\mathbf{s}$ is

$$\mathbf{C}_{\mathbf{s}} = \sigma_{\mathbf{s}}^2 \mathbf{H}\mathbf{H}^{\mathrm{T}}$$

Then, let $M = 2$, write Matlab code to implement the *estimator-correlator,* and plot the probability of detection P_D.

Hint: To simplify, you can reuse the value γ obtained from P_{FA} as in Task 1.