Exercício 3

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EXERCÍCIO 3

Considere os sinais $f(t)$ e $h(t)$ dados por:

$$f(t) = \begin{cases} e^{-\lambda t} & \text{se } t > 0 \\ 0 & \text{se } t \leq 0 \end{cases}$$

$$h(t) = \begin{cases} 1 & \text{se} \quad |t| \leq 1 \\ 0 & \text{se} \quad |t| > 1 \end{cases}$$

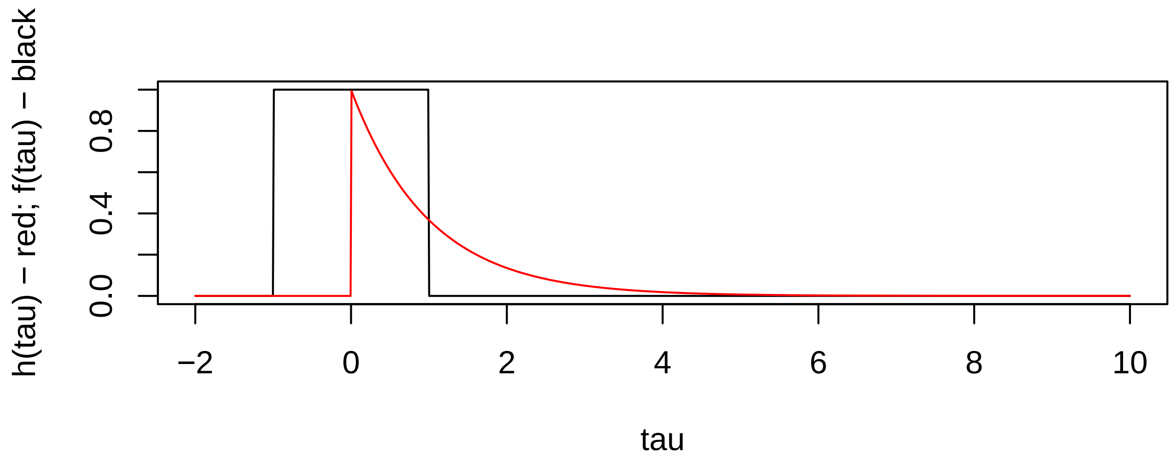
(a) Calcule a integral de convolução

$$g(t) = \int_{\tau=-\infty}^{\infty} f(\tau) \cdot h(t-\tau) d\tau$$

(b) Aplique o Teorema da Convolução para determinar $g(t)$.

```r
f <- function(t, l = 1) (t>0)*exp(-t*l)
h <- function(t) as.numeric(abs(t) < 1)
g <- function(t, tau = seq(-2,10,l=1000)) mean(f(tau) * h(t-tau))

tau <- seq(-2,10,l=1000)
plot(tau, h(tau), type = "l", ylab = "h(tau) - red; f(tau) - black")
lines(tau, f(tau), col = "red")
```



Item (a)

Se $t < -1$

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \qquad (1)\\ &= \int_{-\infty}^{0} 0h(t-\tau)d\tau + \int_{0}^{\infty} f(\tau)0d\tau = 0 \qquad (2)\\ & \qquad (3) \end{aligned}$$

Se $-1 \leq t \leq 1$

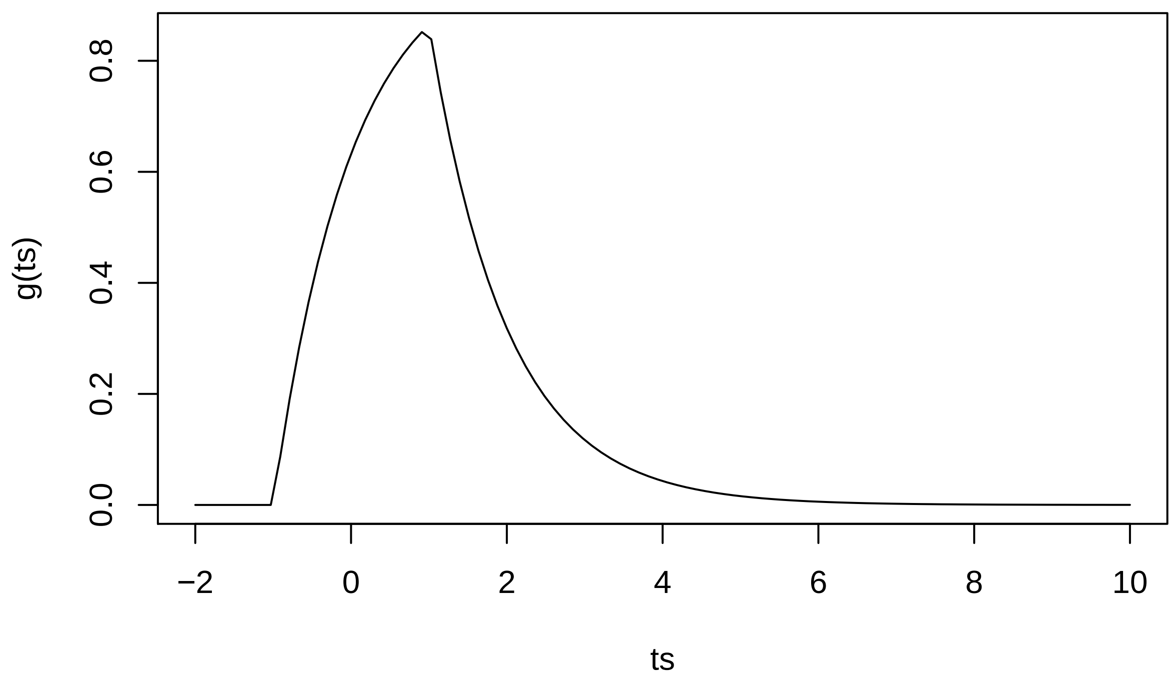
$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \qquad (4)\\ &= \cancel{\int_{t-1}^{0} e^{-\lambda\tau}0d\tau} + \int_{0}^{t+1} e^{-\lambda\tau}1d\tau = \qquad (5)\\ &= \left.\frac{-e^{-\lambda\tau}}{\lambda}\right|_{0}^{t+1} = \frac{1-e^{-\lambda(t+1)}}{\lambda} \qquad (6)\\ & \qquad (7) \end{aligned}$$

Se $t > 1$

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \qquad (8)\\ &= \int_{t-1}^{t+1} e^{-\lambda\tau}d\tau = \qquad (9)\\ &= \left.\frac{-e^{-\lambda\tau}}{\lambda}\right|_{t-1}^{t+1} = \left.\frac{-e^{-\lambda\tau}}{\lambda}\right|_{0}^{t+1} = \frac{e^{-\lambda(t-1)}-e^{-\lambda(t+1)}}{\lambda} \qquad (10)\\ & \qquad (11) \end{aligned}$$

```r
ts  <- seq(-2,10,l=100)
g <- function(t, l = 1) {
  dplyr::case_when(
    t < -1 ~ 0,
    t < 1 ~ (1 - exp(-l*(t+1)))/l,
    t >= 1 ~ (exp(-l*(t-1)) - exp(-l*(t+1)))/l
  )
}

plot(ts, g(ts), type = "l")
```



Item (b)

Seja $F(\omega)$ e $H(\omega)$ as respectivas trasnformadas de Fourier de $f(t)$ e $h(t)$ respectivamente.

$$F(\omega) = \int_0^{\infty} e^{-\lambda\tau} e^{-i2\pi\omega\tau} d\tau = \frac{1}{\lambda + 2i\pi\omega} \quad (12)$$

$$H(\omega) = \int_{-1}^{1} e^{-i2\pi\omega\tau} d\tau = \frac{\sin(2\pi\omega)}{\pi\omega} \quad (13)$$

$$(14)$$

O teorema da convolução garante que

$$\mathcal{F}\{g(t)\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} f(\tau)\cdot h(t-\tau)d\tau\right\} = F(\omega)\cdot H(\omega) = \frac{1}{\lambda + 2i\pi\omega}\frac{\sin(2\pi\omega)}{\pi\omega}$$

Agora aplicamos a transformada de Fourier inversa para retomarmos $g(t)$

$$\begin{aligned} g(t) &= \mathcal{F}^{-1}\{\mathcal{F}\{g(t)\}\} = \mathcal{F}^{-1}\left\{\frac{1}{\lambda + 2i\pi\omega}\frac{\sin(2\pi\omega)}{\pi\omega}\right\} = \quad (15) \\ &= \int_{-\infty}^{\infty} \quad (16) \\ & \quad (17) \end{aligned}$$

```r
## Código pra gerar o gif
animation::saveGIF({
  f <- function(t, l = 1) (t>0)*exp(-t*l)
  h <- function(t) as.numeric(abs(t) < 1)
  g <- function(t, tau = seq(-2,10,l=1000)) sapply(t, function(t) mean(f(tau) * h(t-tau)))

  par(mfrow = c(2,1))
  tau <- seq(-2,10,l=1000)
  ts  <- seq(-2,10,l=100)
  gs <- c()

  for(t in ts) {
    plot(tau, h(t-tau), type = "l")
    lines(tau, f(tau), col = "red")

    gs <- c(gs, g(t))
    plot(ts[1:length(gs)], gs,
         xlim = range(ts), ylim = c(0,0.08),
         type = "l", xlab = "t", ylab = "g(t)")

    Sys.sleep(0.05)
  }
}, interval = 0.05)
```