

Let \mathcal{P} denote the probability of selecting (r_1, \dots, r_n) as samples, or equivalently, (r_1, \dots, r_n) being rows with the top n largest keys, where r_1 has the largest key, r_2 has the second largest key, etc. We wish to show $\mathcal{P} = \prod_{j=1}^n \left(w_j / \sum_{k=j}^N w_k \right)$.

Recall the key for the j -th row (denoted as x_j from now on) is sampled from a probability distribution on $(-\infty, 0)$ with CDF $F_j(x) = e^{w_j \cdot x}$, and therefore the PDF of x_j is $f_j(x) = F_j'(x) = w_j e^{w_j \cdot x}$. Given that $x_1 \geq x_2 \geq \dots \geq x_n$, and also, $x_n \geq x_j$ for $j \in \{n+1, \dots, N\}$, we have

$$\mathcal{P} = \int_{-\infty}^0 f_1(x_1) \int_{-\infty}^{x_1} f_2(x_2) \cdots \int_{-\infty}^{x_n} f_n(x_n) \int_{-\infty}^{x_n} f_{n+1}(x_{n+1}) \cdots \int_{-\infty}^{x_n} f_N(x_N) dx_N \cdots dx_2 dx_1$$

Working through multiple integrals like the ones above with many dots in between would be a bit too hand-wavy, so, in the interest of greater clarity, let's break down \mathcal{P} into $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_n$ instead, with

$$\begin{aligned} \mathcal{P}_0(x_n) &= \int_{-\infty}^{x_n} f_{n+1}(x_{n+1}) \cdots \int_{-\infty}^{x_n} f_N(x_N) dx_N \cdots dx_{n+1} \\ &= \prod_{j=n+1}^N \left(\int_{-\infty}^{x_n} f_j(x_j) dx_j \right) = \prod_{j=n+1}^N \left(F_j(x) \Big|_{-\infty}^{x_n} \right) \\ &= \prod_{j=n+1}^N \left(e^{w_j \cdot x} \Big|_{-\infty}^{x_n} \right) = e^{\left(\sum_{j=n+1}^N w_j \right) \cdot x_n} \end{aligned}$$

(i.e., $\mathcal{P}_0(x_n)$ is the inner-most bunch of integrals beyond $j = n$),

and then define

$$\mathcal{P}_j(x_{n-j}) = \int_{-\infty}^{x_{n-j}} f_{n-j+1}(x_{n-j+1}) \mathcal{P}_{j-1}(x_{n-j+1}) dx_{n-j+1}$$

for $j \in \{1, \dots, n-1\}$, then for $j = 1$,

$$\begin{aligned} \mathcal{P}_1(x_{n-1}) &= \int_{-\infty}^{x_{n-1}} f_n(x_n) \mathcal{P}_0(x_n) dx_n \\ &= \int_{-\infty}^{x_{n-1}} w_n e^{w_n \cdot x_n} \cdot \left[e^{\left(\sum_{k=n+1}^N w_k \right) \cdot x_n} \right] dx_n = w_n \int_{-\infty}^{x_{n-1}} e^{\left(\sum_{k=n}^N w_k \right) \cdot x_n} dx_n \end{aligned}$$

$$= \left(w_n / \sum_{k=n}^N w_k \right) \cdot \left[e^{\left(\sum_{k=n}^N w_k \right) \cdot x_n} \right] \Big|_{-\infty}^{x_{n-1}} = \left(w_n / \sum_{k=n}^N w_k \right) \cdot e^{\left(\sum_{k=n}^N w_k \right) \cdot x_{n-1}}$$

Now all that is remaining is simply an exercise of proof by mathematical induction, where given the induction hypothesis

$$\mathcal{P}_j(x_{n-j}) = \left[\prod_{h=n-j+1}^n \left(w_h / \sum_{k=h}^N w_k \right) \right] \cdot e^{\left(\sum_{j=n-j+1}^N w_j \right) \cdot x_{n-1}}$$

, which is already shown to be true for $j = 1$, show it is true for $j \in \{2, \dots, n-1\}$.