United International University

School of Science and Engineering

Final Examination Trimester: Summer-2024

Course Title: Linear Algebra

Course Code: Math 2107 Marks: 50 Time: 2 Hours

Answer all the questions. Answer all parts of a question together.

1. (a) **Determine** whether the given set is a vector space. If not, give at least one axiom that is not satisfied. Unless otherwise stated, assume that vector addition and scalar multiplication are the ordinary operations defined on the set. [5]

   (i) The set of vectors $\{(x, y, z)\epsilon\mathbb{R}^3 \mid x + y + z = 0\}$.

   (ii) The set of all points lies in the first quadrant on $\mathbb{R}^2$.

   (b) **Determine** whether or not each of the following sets W is a subspace of the vector space V. Justify your answers. [5]

   (i) $V = \mathbb{R}^3$ and $W = \{(x, y, z)\epsilon\mathbb{R}^3 \mid x^3 - y^2 = 0\}$.

   (ii) $V = M_{2\times2}(\mathbb{R})$ be the vector space of 2×2 matrices with real coefficients and $W = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon M_{2\times2}(\mathbb{R}) \mid 3a - 2d = 0\right\}$.

2. (a) **Express** the vector $w = (-1,7,2)$ as a linear combination of the vectors $v_1 = (1,3,5), v_2 = (2,-1,3)$ and $v_2 = (-3,2,4)$ in $\mathbb{R}^3$, if possible. Explain the reason. [4]

   (b) Let $V = P_2(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most two and let $S = \{1, 1 + x, 1 + x + x^2\}$. **Prove that** $\text{Span}(S) = V$. [3]

   (c) **Determine** whether the given set of vectors in $\mathbb{R}^n$ is linearly dependent or linearly independent. Explain the reason. [5]

   (i) $v_1 = (5,-2,4), v_2 = (2,-3,5), v_3 = (4,5,-7)$

   (ii) $v_1 = (1,2,3), v_2 = (3,2,9), v_3 = (5,2,-1)$

   (iii) $v_1 = (2,1,3), v_2 = (1,1,0), v_3 = (3,1,1), v_4 = (2,5,-4)$

   (iv) $v_1 = (1,0,0,3), v_2 = (0,1,-2,0), v_3 = (0,-1,1,1)$

   (d) **Determine** whether the following set of vectors form a basis for $\mathbb{R}^3$, or not. [3]

   (i) $v_1 = (1,-2,3),\ v_2 = (0,-3,2)$

   (ii) $v_1 = (1,-2,-1), v_2 = (2,-3,1), v_3 = (5,-8,1)$

3. (a) **Find** bases and the dimensions of the four fundamental subspaces for the matrix [7]

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 2 & 4 & 14 & 4 \\ 2 & 3 & 12 & 3 \end{pmatrix}$$

(b) If A is an 8×7 matrix with nullity 2, then **find** [2]

- (i) rank(A).
- (ii) nullity(A^T).
- (iii) Pivot elements in row echelon form of A.
- (iv) Zero rows in row echelon form of A.

(c) Use the ***Gram–Schmidt process*** to transform the given basis into an orthonormal basis. [6]

$$v_1 = (1, 2, -1), v_2 = (1, -1; 1), v_3 = (-1, 0, 2)$$

4. (a) **Find** the standard matrix of the following operators, and then **find** the image of the corresponding vector **x**. [5]

- (i) Reflection about the line $y = x$ in $\mathbb{R}^2$; $\mathbf{x} = (-2, 3)$
- (ii) Compression in the x-direction with a factor $\frac{1}{3}$ in $\mathbb{R}^2$; $\mathbf{x} = (1, -1)$

(b) **Describe** the matrix operator whose standard matrix is given, and **sketch** its effect on the unit square. [5]

(i) $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, (ii) $A_2 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$