MATH40550, Applied Matrix Theory

Homework 4

Let

$$A := \begin{pmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{10} & \frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{10} & \frac{1}{5} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

1. Use Geršgorin's theorem to find a region that contains all the eigenvalues of A.

2. Find a diagonal matrix D for which $B := DAD^{-1}$ is a symmetric matrix. Does this allow you to reduce the region containing the eigenvalues of A that you found in the previous item?

3. Determine $L := \lim_{n\to\infty} A^n$. Computer software may be used to compute any eigenvalues and eigenvectors needed. Are any eigenvalues and eigenvectors easy to obtain by hand?

4. Compute $\|A^k - L\|_2$ for $k = 1, \ldots, 20$, and determine the minimal k for which $\|A^k - L\|_2 < \frac{1}{10}$. This item should be done with the aid of a computer.

5. Find a rank one matrix R so that $C = A + R$ has the same eigenvalues as A, except 1 is replaced by $\frac{1}{2}$. Compute $\lim_{n\to\infty} C^n$.

6. Consider a more general family of 5×5 tridiagonal matrices:

$$M = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & 0 \\ 0 & 0 & c_3 & a_4 & b_4 \\ 0 & 0 & 0 & c_4 & a_5 \end{pmatrix},$$

   where $a_i, b_i, c_i \in \mathbb{R}$. Prove that M has real eigenvalues, if $b_i c_i > 0$ for $i = 1, \ldots, 4$.