Oscillons

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1 Introduction

We start by examining the models described by Amin et al[1] where they discuss flat top oscillon solutions which are spatially localised and are long lived in time. With the intuitive reasoning from Sfakianakis[2]. We begin by understanding oscillons in 1+1D, i.e one spatial and one time dimension.

1.1 Oscillon model in 1+1D

We define the lagrangian density for a scalar field, in analogous to ref[2]

$$\mathcal{L} = \frac{1}{2}((\partial_t \phi)^2 - (\partial_x \phi)^2) - V(\phi) \tag{1}$$

with

$$V = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{\Lambda}{6\epsilon^2}\phi^6$$

the ϵ parameter is a small (constant) number which we will encounter later. Here it just means that the strength of the ϕ^6 term is somehow dependant on the ratio between the Λ parameter and the ϵ parameter. The Λ parameter is proportional to the sixth order coupling strength, which is usually denoted by g. The equation of motion for the field becomes:

$$\frac{d^2\phi}{dt^2} - \frac{d^2\phi}{dx^2} + \phi - \phi^3 + \frac{\Lambda}{\epsilon}\phi^5 = 0 \tag{2}$$

with the change of variables from t to τ given by $t = \epsilon^2\tau$ and from x to ρ given by $x = \epsilon\rho$ and also noting that the oscillons are oscillating only in time and are localized in space, we can write eqn(2) as:

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t}\right)\left(\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \tau}\epsilon^2\right) = \frac{\partial^2\phi}{\partial t^2} + 2\epsilon^2\frac{\partial^2\phi}{\partial\tau\partial t} + \frac{\partial^2\phi}{\partial\tau^2}\epsilon^4 \tag{3}$$

with these the equation of motion becomes:

$$\frac{\partial^2\phi}{\partial t^2} + 2\epsilon^2\frac{\partial^2\phi}{\partial\tau\partial t} + \epsilon^4\frac{\partial^2\phi}{\partial\tau^2} - \epsilon^2\frac{\partial^2}{\partial\rho^2} + \phi - \epsilon^2\phi^3 + \Lambda\epsilon^2\phi^5 \tag{4}$$

to this equation, we add the most general solution in orders of ϵ of the form:

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \dots \tag{5}$$

we get the corresponding equations of motion as:

$$\frac{\partial^2\phi_0}{\partial t^2} + \epsilon\frac{\partial^2\phi_1}{\partial t^2} + \epsilon^2\frac{\partial^2\phi_2}{\partial t^2} + 2\epsilon^2\frac{\partial^2\phi_0}{\partial\tau\partial t} - \epsilon^2\frac{\partial^2\phi_0}{\partial\rho^2} + \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \Lambda\epsilon^2\phi_0^5 + \mathcal{O}(\epsilon^3) = 0 \tag{6}$$

counting in powers of ϵ we have for $\mathcal{O}(1)$:

$$\frac{\partial^2\phi_0}{\partial t^2} + \phi_0 = 0 \tag{7}$$

at $\mathcal{O}(\epsilon)$, we have:

$$\frac{\partial^2\phi_1}{\partial t^2} + \phi_1 = 0 \tag{8}$$

for $\mathcal{O}(\epsilon)$, we then get:

$$\frac{\partial^2\phi_2}{\partial t^2} + 2\frac{\partial^2\phi_0}{\partial t\partial\tau} - \frac{\partial^2\phi_0}{\partial\rho^2} + \phi_2 - \phi_0^3 + \Lambda\phi_0^5 = 0 \tag{9}$$

The general solution for ϕ_0 from eqn(7) is of the form:

$$\phi_0 = \frac{1}{2}(Ae^{-it} + A^*e^{it}) \tag{10}$$

If we substitute this into eqn(9) and taking only the decaying mode solutions, we get:

$$\frac{A_{\rho\rho}}{2} + iA_\tau + \frac{3}{8}|A|^2A = 0 \tag{11}$$

and with the ansatz $A = a(\rho)e^{i\tau/2}$ we get:

$$a_{\rho\rho} - a + \frac{3}{4}a^3 = 0 \tag{12}$$

Looking at the tail end of the above equation, we obtain the required boundary conditions:

$$\frac{d^2a}{d\rho^2} - a = 0 \quad for \rho >> 0, a << 0 \tag{13}$$

the general solution for a then is:

$$a = C_1e^{\rho} + C_2e^{-\rho}$$

ignoring the growing mode, if we take

$$\rho \to \infty, \implies C_2 = 1$$

we get the condition:

$$\frac{a_\rho}{a} = -1 \implies (a_\rho)_{\rho=\infty} = -a$$

if we assume that the trailing end of the envelope is an arbritrary constant κ then:

$$a = \kappa \implies a_\rho = -\kappa$$

evaluating a at $\rho = 0$ then we are left with:

$$\rho = 0, a_\rho = 0$$

We can now use these conditions to break down second order ODE into a first order ODE by:

$$\frac{da}{d\rho} = b, \quad \frac{db}{d\rho} = a - \frac{3}{4}a^3 \tag{14}$$

with the initial conditions:

$$b(0) = 0, \quad b(\infty) = -\kappa$$

2 Computational details

Continuing from the last section, observing the eqn(14) we see that this can be solved numerically by using various computational algorithms. One of the standard ways to solve non-stiff problems is to use the Runge-Kutta[3][4] method for solving ODEs.

2.1 The Runge-Kutta Method

Asumming previous knowledge of the Euler method for solving ODEs, we naturally extend to the Runge-Kutta (RK) methods which, achieve the accuracy of a Taylor series approach without re- quiring the calculation of higher derivatives. Inherently, the RK method exists with many variations exist and can be generalized into:

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h \tag{15}$$

where ϕ is the increment function, which can be interpreted as a representative slope over the interval. The increment function can be written in general as:

$$\phi = a_1k_1 + a_2k_2 + \; + a_nk_n$$

where the as are constants and the ks are

$$
\begin{gathered}
k_1 = f(x_i, y_i) \\
k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h) \\
k_3 = f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \\
. \\
. \\
. \\
k_n = f(x_i + p_{n1} h, y_i + q_{n1,1} k_1 h + q_{n1,2} k_2 h + \; + q_{n1,n1} k_{n1} h)
\end{gathered} \tag{16}
$$

where the ps and qs are constants. The recurrence of the k's in subsequent steps makes the RK method exceptionally efficient for computer calculations. Once the n is fixed, the values of the constants are evaluated by setting eqn(15.) equal to terms in a Taylor series expansion. The local truncation error is $\mathcal{O}(h^3)$ and the global error is $\mathcal{O}(h^2)$. For our purposes, we will be using RK- 4^{th} order, the global truncation error is then $\mathcal{O}(h^4)$.

References

[1] Mustafa A. Amin and David Shirokoff, *Flat-top oscillons in an expanding universe.*, 2010.

[2] Evangelos I. Sfakianakis, *Analysis of Oscillons in the SU (2) Gauged Higgs Model*, 2012

[3] Steven C. Chapra, Raymond P. Canale, *Numerical Methods for Engineers*, Sixth Edition

[4] William H. Press, Saul A. Teukolsky, William T. Vetterling, Brian P. Flannery, *Numerical Recipes in C*, Second Edition