These notes correspond to Sections 11.1 and 11.2 in the text.

The Shooting Method for Two-Point Boundary Value Problems

We now consider the *two-point boundary value problem* (BVP)

$$y'' = f(x, y, y'), \quad a < x < b,$$

a second-order ODE, with *boundary conditions*

$$y(a) = \alpha, \quad y(b) = \beta.$$

This problem is guaranteed to have a unique solution if the following conditions hold:

- f, f_y, and $f_{y'}$ are continuous on the domain

$$D = \{(x, y, y') \mid a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}.$$

- $f_y > 0$ on D

- $f_{y'}$ is bounded on D.

There are several approaches to solving this type of problem. The first method that we will examine is called the *shooting method*. It treats the two-point boundary value problem as an *initial value problem* (IVP), in which x plays the role of the time variable, with a being the "initial time" and b being the "final time". Specifically, the shooting method solves the initial value problem

$$y'' = f(x, y, y'), \quad a < x < b,$$

with initial conditions

$$y(a) = \alpha, \quad y'(a) = t,$$

where t must be chosen so that the solution satisfies the remaining boundary condition, $y(b) = \beta$. Since t, being the first derivative of $y(x)$ at $x = a$, is the "initial slope" of the solution, this approach requires selecting the proper slope, or "trajectory", so that the solution will "hit the target" of $y(x) = \beta$ at $x = b$. This viewpoint indicates how the shooting method earned its name. Note that since the ODE associated with the IVP is of second-order, it must normally be rewritten

as a system of first-order equations before it can be solved by standard numerical methods such as Runge-Kutta or multistep methods.

In the case where $y'' = f(x, y, y')$ is a *linear* ODE, selecting the slope t is relatively simple. Let $y_1(x)$ be the solution of the IVP

$$y'' = f(x, y, y'), \quad a < x < b, \quad y(a) = \alpha, \quad y'(a) = 0,$$

and let $y_2(x)$ be the solution of the IVP

$$y'' = f(x, y, y'), \quad a < x < b, \quad y(a) = 0, \quad y'(a) = 1.$$

These can be computed using any analytical or numerical method. Then, the solution of the original BVP has the form

$$y(x) = y_1(x) + t y_2(x)$$

where t is the correct slope, since any linear combination of solutions of the ODE also satisfies the ODE, and the initial values are linearly combined in the same manner as the solutions themselves. To find the proper value of t, we evaluate $y(b)$, which yields

$$y(b) = y_1(b) + t y_2(b) = \beta,$$

and therefore $t = (\beta - y_1(b))/y_2(b)$. It follows that as long as $y_2(b) \neq 0$, then $y(x)$ is the unique solution of the original BVP. This condition is guaranteed to be satisfied due to the previously stated assumptions about $f(x, y, y')$ that guarantee the existence and uniqueness of the solution.

If the ODE is nonlinear, however, then t satisfies a nonlinear equation of the form

$$y(b, t) = 0,$$

where $y(b, t)$ is the value of the solution, at $x = b$, of the IVP specified by the shooting method, with initial sope t. This nonlinear equation can be solved using an iterative method such as the bisection method, fixed-point iteration, Newton's Method, or the Secant Method. The only difference is that each evaluation of the function $y(b, t)$, at a new value of t, is relatively expensive, since it requires the solution of an IVP over the interval $[a, b]$, for which $y'(a) = t$. The value of that solution at $x = b$ is taken to be the value of $y(b, t)$.

If Newton's Method is used, then an additional complication arises, because it requires the derivative of $y(b, t)$, with respect to t, during each iteration. This can be computed using the fact that $z(x, t) = \partial y(x, t)/\partial t$ satisfies the ODE

$$z'' = f_y z + f_{y'} z', \quad a < x < b, \quad z(a, t) = 0, \quad z'(a, t) = 1,$$

which can be obtained by differentiating the original BVP and its boundary conditions with respect to t. Therefore, each iteration of Newton's Method requires *two* IVPs to be solved, but this extra effort can be offset by the rapid convergence of Newton's Method.

Suppose that Euler's method,

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f}(x, \mathbf{y}_i, h),$$

for the IVP $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$, $\mathbf{y}(x_0) = \mathbf{y}_0$, is to be used to solve any IVPs arising from the Shooting Method in conjunction with Newton's Method. Because each IVP, for $y(x, t)$ and $z(x, t)$, is of second order, we must rewrite each one as a first-order system. We first define

$$y^1 = y, \quad y^2 = y', \quad z^1 = z, \quad z^2 = z'.$$

We then have the systems

$$
\begin{aligned}
\frac{\partial y^1}{\partial x} &= y^2, \\
\frac{\partial y^2}{\partial x} &= f(x, y^1, y^2), \\
\frac{\partial z^1}{\partial x} &= z^2, \\
\frac{\partial z^2}{\partial x} &= f_y(x, y^1, y^2)z^1 + f_{y'}(x, y^1, y^2)z^2,
\end{aligned}
$$

with initial conditions

$$y^1(a) = \alpha, \quad y^2(a) = t, \quad z^1(a) = 0, \quad z^2(a) = 1.$$

The algorithm then proceeds as follows:

Choose $t^{(0)}$
Choose h such that $b - a = hN$, where N is the number of steps
for $k = 0, 1, 2, \ldots$ until convergence **do**
    $i = 0$, $y_0^1 = \alpha$, $y_0^2 = t^{(k)}$, $z_0^1 = 0$, $z_0^2 = 1$
    for $i = 0, 1, 2, \ldots, N - 1$ **do**
        $x_i = a + ih$
        $y_{i+1}^1 = y_i^1 + hy_i^2$
        $y_{i+1}^2 = y_i^2 + hf(x_i, y_i^1, y_i^2)$
        $z_{i+1}^1 = z_i^1 + hz_i^2$
        $z_{i+1}^2 = z_i^2 + h[f_y(x_i, y_i^1, y_i^2)z_i^1 + f_{y'}(x_i, y_i^1, y_i^2)z_i^2]$
    end
    $t^{(k+1)} = t^{(k)} - (y_N^1 - \beta)/z_N^1$
end

Changing the implementation to use a different IVP solver, such as a Runge-Kutta method or multistep method, in place of Euler's method only changes the inner loop.

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