Quantum Field Theory

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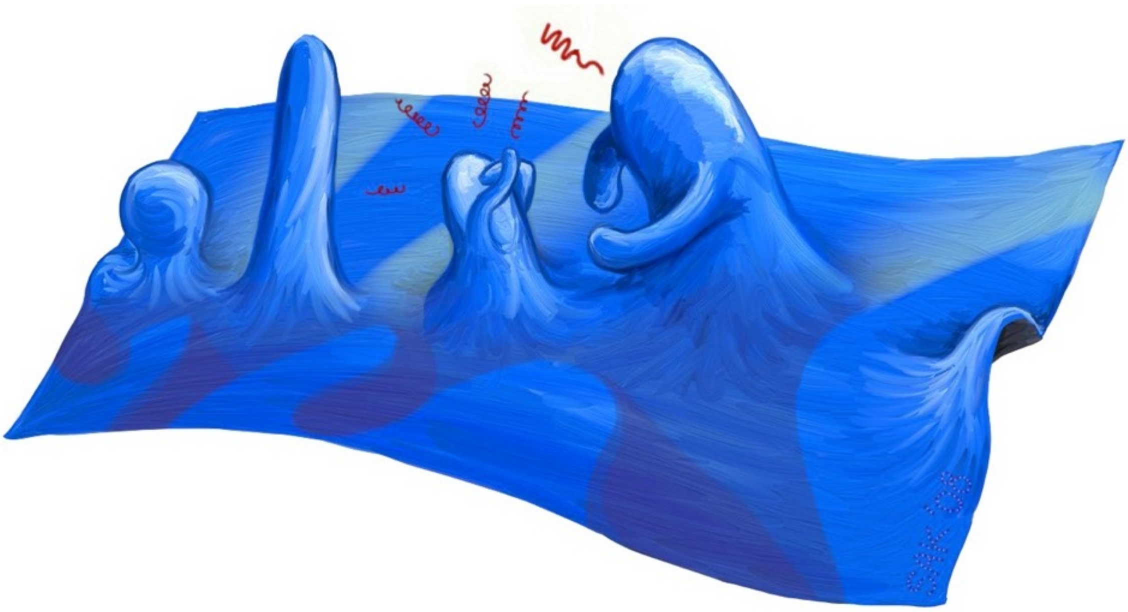

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Preface

Quantum field theory (QFT) is the language in which the fundamental laws of Nature are written. It is the inevitable consequence of trying to unite quantum mechanics with special relativity.

The marriage of quantum mechanics and relativity faces a number of conceptual difficulties. For example, wavepackets in quantum mechanics can spread faster than the speed of light, in apparent violation of the causal speed limit of relativity. Moreover, ordinary quantum mechanics deals with systems in which the number of particles is fixed, but relativity implies that particles can be created at high energies or short distances. To see this, consider a particle of mass m trapped in a box of size L. Due to the uncertainty principle, the momentum of the particle is $\Delta p \geq \hbar/L$. In the relativistic limit, this corresponds to an uncertainty in the energy of

$$\Delta E \geq \frac{\hbar c}{L}\,.$$

If $\Delta E \geq 2mc^2$, then there is enough energy available to create a particle-antiparticle pair. These are virtual particles that only live for a very short amount of time. Nevertheless, on scales smaller than the Compton wavelength $\lambda_{\mathrm{C}} \equiv \hbar/mc$ a single-particle description loses its meaning since the probability of creating particle-antiparticle pairs is large. Attempts to write down a one-particle Schrödinger equation for the system are doomed to fail. Indeed, naive attempts lead to negative probabilities, infinite towers of negative energy eigenstates or a breakdown of causality. As we will see, these issues are resolved if the fundamental building blocks of Nature are quantum fields, and particles are just bundles of energy and momentum of the fields.

Historically, QFT was first developed (by Born, Heisenberg and Jordan) as a quantum theory of the electromagnetic field. We know that ripples in the electromagnetic field propagate as electromagnetic waves. In quantum mechanics, these ripples get tied up into particles called photons. Shortly afterwards, Dirac derived a relativistic quantum theory of matter particles like electrons. For a while, a distinction was made between the quantum theory of particles and that of fields: the electron was considered a particle and the electromagnetic field a field, although it also has particle-like excitations. However, it was soon suggested (by Jordan, Wigner, Heisenberg, Pauli, Weisskopf, Oppenheimer and others) that quantum field theory applies to everything, not just to electromagnetism. In the modern view of physics, everything is a field. There is an electron field, a photon field, a quark field, a graviton field, a pion field, a Higgs field, etc. These fields are the primary objects. Particles are derived quantities arising as "excitations" of fields. In this view of the world the electron is an excitation of the electron field, the photon is an excitation of the electromagnetic field, and so on. The quantum dynamics of the fields determines how particles are created and destroyed, and how they interact.

Quantum field theory has a number of important consequences, both conceptual and experimental:

- *Indistinguishable particles.* QFT explains the deep fact that "all electrons are the same". The particles associated with a field are indistinguishable since they are just excitations of the same underlying field.

- *Quantum statistics.* Particles with even spin are bosons, those with odd spin are fermions. According to the spin-statistics theorem, the physical state of a system is even (odd) under the exchange of identical bosons (fermions). In QFT, this fact will emerge as a natural consequence of the formalism.

- *Antiparticles.* For every particle in Nature there exists a corresponding antiparticle. In QFT, these antiparticles must be added in order for the theory to respect causality.

The above features are derived from *free* (i.e. non-interacting) quantum field theory. Interactions in QFT lead to a few additional features:

- *Particle creation and destruction.* In QFT, particles can be created, annihilated, and change their identity. The probabilities for these processes to occur are derived from the dynamics of the corresponding quantum fields and especially depend on the nonlinear interactions between the fields.

- *Forces from particle exchange.* Classical waves with nonlinear interactions can scatter from each other. In QFT, this scattering is mediated by the exchange of intermediate particles often called *virtual* particles.

As these examples show, fundamental features of the physical world are explained by quantum field theory. At a more practical level, it also just works. Some of the most precise predictions in the history of physics were made using quantum field theory. For example, QFT predicts an anomalous magnetic dipole moment for the muon. This prediction agrees with experiment at an incredible level of precision:

$$\begin{aligned}(g_\mu - 2)_{\text{theor.}} &= 0.00233183478(308)\,,\\ (g_\mu - 2)_{\text{exp.}} &= 0.00233184600(168)\,.\end{aligned}$$

This is just one of many predictions of QFT that have been confirmed by experiment.

QFT is the theoretical foundation of many areas of physics. A few examples are:

- *Particle physics.* The Standard Model (SM) of particle physics, which accounts for all observed phenomena on length scales larger than 10^{-18} meters, is a quantum field theory. In this course, we will lay the conceptual foundations for the physics of the SM.

- *Condensed matter physics.* Although in this course we will deal exclusively with relativistic field theories, QFT has also found important applications to non-relativistic theories such as those found in condensed matter systems. Sound waves in metals and crystals, Fermi liquids of weakly interacting electrons, fluids and superfluids, the behavior of systems near phase transitions, etc. are all described by QFTs.

- *Cosmology.* There is growing evidence that the early universe went through a period of inflationary expansion. During inflation, the rapid expansion of the spacetime amplified the fluctuations in certain quantum fields, such as the inflaton field and the metric tensor. These fluctuations are believed to be the seeds for the large-scale structure of the universe. Computing this effect is a beautiful application of QFT in curved spacetime.

- *Black holes.* The quantization of fields near the horizon of black holes (BHs) leads to interesting new effects. Hawking famously showed, using QFT in the curved BH background, that BHs aren't truly black, but radiate quantum mechanically.

- *General relativity.* At long distances, even general relativity (GR) can be described as an effective QFT, namely that of a massless spin-2 particle (the graviton) interacting with the degrees of freedom of the SM.
- *String theory.* At long distances, string theory also reduces to an effective QFT, namely supergravity. At short distances, the extended nature of the string becomes important and a point particle description breaks down.
- *Mathematics.* QFT has also proven to be a source of inspiration for mathematicians. For example, certain QFTs lead to the definition of novel topological invariants including various knot invariants and invariants for higher-dimensional manifolds.

Outline of the Course

In Chapter 1, we review key elements of classical field theory. We present both the Lagrangian and Hamiltonian formalism. We show how continuous symmetries lead to conserved quantities via Noether's theorem.

In Chapter 2, we begin our study of quantum field theory with the quantization of a free scalar field. We show that the quantum excitations of the field can be interpreted as particles and that the quantization of a complex scalar field leads to the concept of antiparticles. We introduce the concept of the Feynman propagator.

In Chapter 3, we study interactions. We introduce the concept of the S-matrix and derive its Dyson expansion in perturbation theory. We show how the individual terms in the Dyson expansion are evaluated using Wick's theory. We introduce Feynman diagrams as an efficient way to organize these computations. We discuss the relation between the S-matrix and observables such as decay rates and cross sections. Using the example of scalar Yukawa theory, we show how the forces between particles arise from virtual particle exchange. We derive the Yukawa potential from the non-relativistic limit of the scattering amplitude.

In Chapter 4, we discuss how to embed particles with spin into quantum fields. We derive the unique Lagrangians that propagate particles of spin 1 (e.g. photons) and spin $\frac{1}{2}$ (e.g. electrons).

In Chapter 5, we quantize the Dirac Lagrangian. We elucidate the origin of the spin-statistics theorem. We derive the Feynman propagator for fermions. We repeat our treatment of Yukawa theory for the scattering of spin-$\frac{1}{2}$ particles. We find that the force mediated by spin-0 particles is universally attractive.

In Chapter 6, we quantize the Maxwell Lagrangian. The gauge redundancy of massless vector fields makes this a subtle enterprise. We perform the quantization in Coulomb gauge and in Lorentz gauge, highlighting the distinct challenges in both cases. We derive the gauge field propagator in both gauges.

In Chapter 7, the course culminates in a brief discussion of Quantum Electrodynamics (QED). We first show that the interactions between fermions and photons is uniquely fixed by gauge invariance. We apply the resulting Feynman rules to a few examples. Finally, show that the force mediated by spin-1 particles can be repulsive or attractive depending on the charges of the scattering particles.

A number of appendices contain relevant reference material. In Appendix A, we review aspects of Fourier analysis, complex analysis and group theory that play an essential role in QFT. In Appendix B, we recall basic features of special relativity and quantum mechanics. In Appendix C, we present solutions to the exercises that appear at the end of each chapter. Finally, Appendix D provides a guide to the vast literature on QFT.

To improve the readability of the text, the details of longer calculations are sometimes put into boxes. These may be omitted in a first reading. Sections marked with a superscript $*$ will be non-examinable. It is still advised to skimmed them, since qualitative features of theses sections do reappear in other parts of the notes.

Scales and Units

Since we will be working mostly with relativistic field theories, it will make sense to use so-called *natural units*, where the speed of light and Planck's constant are set to unity: $c = \hbar \equiv 1$. This allows us to express all dimensionful quantities in terms of a single scale which we choose to be mass or, equivalently, energy (since the famous $E = mc^2$ has become $E = m$). In particle physics, the standard unit of energy is the electron volt, eV, or more often GeV $= 10^9$ eV and TeV $= 10^{12}$ eV. To convert the unit of energy back to units of length or time, we have to insert the relevant powers of c and $\hbar$. To do this, we just have to remember any equation that relates a length scale or time scale to a mass scale or energy scale. For example, above we encountered the Compton wavelength $\lambda_{\rm C} = \hbar/(mc)$. Using $\frac{1}{2}\hbar c \approx 10^{-7}\text{eV}\cdot\text{m}$, one then finds that the length scale corresponding to the electron with mass $m_e \approx 511\,\text{keV}$ is $\lambda_e \approx 2 \times 10^{-12}\,\text{m}$. Other length scales follow from the proportionality, $L \propto m^{-1}$. The following table lists some relevant mass and length scales of elementary particle physics and cosmology:

Quantity	Mass	Length
Observable universe	10^{-33} eV	10^{27} m
Cosmological constant (Λ)	10^{-3} eV	10^{-3} m
Neutrinos (ν)	1 eV	10^{-6} m
Electron (e)	511 keV	10^{-12} m
Muon (μ)	106 MeV	10^{-14} m
Charm quark (c)	1.3 GeV	10^{-15} m
Tau (τ)	1.8 GeV	10^{-15} m
Bottom quark (b)	4.6 GeV	10^{-16} m
Higgs boson (h)	125 GeV	10^{-17} m
Top quark (t)	175 GeV	10^{-17} m
Electroweak scale (v)	250 GeV	10^{-17} m
LHC energy	14 TeV	10^{-18} m
GUT scale	10^{15} GeV	10^{-30} m
Planck scale ($M_{\rm pl}$)	10^{18} GeV	10^{-34} m

While there are still important puzzles on the largest and smallest scales, between 10^{-18} m and 10^{-6} m the Standard Model of particle physics describes all observed phenomena. This remarkable success is based on our understanding of quantum field theory. In this course, I hope to give you a first glimpse at this.

Acknowledgements

This course is far from original and I have benefited greatly from many excellent resources on QFT (see Appendix D). A special thank you to David Tong for writing the brilliant set of QFT lectures that forms the backbone of these lectures. Thanks to Alejandra Castro for sharing her experience in teaching a previous version of this course. Thanks to Mustafa Amin for discussions on the pedagogy of teaching a course on QFT. Finally, I am grateful to the TAs of this course (Horng Sheng Chia, Victor Godet and Lorenzo Zoppi) for their careful reading of the lecture notes and for their great help in writing the solutions to many of the problems.

1 Classical Field Theory

In classical physics, the primary reason for introducing *fields* is to construct laws of Nature that are *local*. The old laws of Coulomb and Newton involve "action at a distance". This means that the force felt by an electron (or planet) changes instantaneously if a distant proton (or star) moves. This situation is philosophically unsatisfactory. More importantly, it is also experimentally wrong. The field theories of Maxwell and Einstein remedy the situation, with all interactions mediated in a local fashion by the field.

In this chapter, we will give a concise introduction to classical field theory. We will only present the absolute minimum required for our study of the quantum theory of fields. Additional elements of field theory will be introduced as we go along.

1.1 Dynamics

A field is a quantity defined at every spacetime point

$$\phi_a(\mathbf{x}, t) \,, \tag{1.1.1}$$

where a is a discrete label that charaterizes the type of field. In this chapter and the next, we will mostly deal with real scalar fields ϕ, i.e. fields that that a single value at every point in spacetime. We will also consider complex scalar fields $\psi \equiv (\phi_1 + i\phi_2)/\sqrt{2}$ which have two real components. The excitations of complex fields will be particle/antiparticle pairs. Spinor fields ψ_α have two or four components. Excitations of spinors will be fermionic particles. The vector field A_μ has four components although not all of them will be independent. The excitations of A_μ are photons. The gravitational field is described by a tensor field $g_{\mu\nu}$ with 10 components. Only two of them are physical and their excitations correspond to the two physical degrees of freedom of the graviton.

Note that, in QFT, the position $\mathbf{x}$ in $\phi_a(\mathbf{x}, t)$ is considered to be a label (on the same footing as the label a) and *not* a dynamical variable. Since $\mathbf{x}$ is continuous, QFT deals with an infinite number of degrees of freedom — at least one for every point in space. This is in contrast to QM which describes systems with a finite number of degrees of freedom, typically defined by generalized coordinates $q_a(t)$, where $a = 1, \dots, N$. In principle, we can get from QM to QFT by taking $N \to \infty$ (see §2.1). In practice, the limit is subtle and encodes a lot of the richness of quantum field theory.

1.1.1 Lagrangian Formalism

The dynamics of the field $\phi_a(t, \mathbf{x})$ can be derived from a Lagrangian. In this course, we will study Lagrangians that are functionals of the field ϕ_a and their derivative $\partial_\mu \phi_a$, i.e.

$$L = \int \mathrm{d}^3x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a) \, , \tag{1.1.2}$$

where $\mathcal{L}$ is the *Lagrangian density.* We will follow standard practice and refer to $\mathcal{L}$ simply as the Lagrangian. On the interval $t \in [t_1, t_2]$, the *action* of the field then is

$$S = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x \, \mathcal{L} = \int \mathrm{d}^4x \, \mathcal{L} \, . \tag{1.1.3}$$

The evolution of the field configuration $\phi_a(t, \mathbf{x})$ between t_1 and t_2 follows from the *principle of least action.* An infinitesimal change of the field, $\delta\phi_a(t, \mathbf{x})$, implies the following variation of the action

$$\delta S = \int \mathrm{d}^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right\} \tag{1.1.4}$$

$$= \int \mathrm{d}^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) \right\} . \tag{1.1.5}$$

The last term in (1.1.5) is a total derivative and vanishes for any variation $\delta\phi_a$ that decays at spatial infinity and which obeys $\delta\phi_a(t_1, \mathbf{x}) = \delta\phi_a(t_2, \mathbf{x}) = 0$. Setting $\delta S = 0$ then leads to the *Euler-Lagrange equation*

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0} \, . \tag{1.1.6}$$

We will now apply this result to number of examples.

Scalar field theory

The Lagrangian of a real scalar field $\phi(x)$ is

$$\mathcal{L} \; = \; \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \, , \tag{1.1.7}$$

where $\eta^{\mu\nu}$ is the Minkowski metric

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{1.1.8}$$

Writing out the spacetime derivatives, we have

$$\mathcal{L} \; = \; \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 \, . \tag{1.1.9}$$

We see that the Lagrangian takes the familiar form of kinetic energy minus potential energy, $L = T - V$, if we identify

$$T = \int \mathrm{d}^3x \, \frac{1}{2} \dot{\phi}^2 \, , \tag{1.1.10}$$

$$V = \int \mathrm{d}^3x \, \left[\frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right] . \tag{1.1.11}$$

The first term in (1.1.11) is called the gradient energy, while "potential energy", or just "potential", is usually reserved for the last term. Finally, it is standard jargon in QFT to refer to $\frac{1}{2}(\partial_\mu\phi)^2$ as the kinetic term and $\frac{1}{2}m^2\phi^2$ as the potential term (or mass term).

Substituting

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi\,, \tag{1.1.12}$$

into the Euler-Lagrange equation (1.1.6), we obtain the so-called *Klein-Gordon equation*

$$\boxed{\partial_\mu\partial^\mu\phi + m^2\phi = 0}\,, \tag{1.1.13}$$

where the combination $\partial_\mu\partial^\mu \equiv \Box$ is the d'Alembertian operator.

The Lagrangian of a complex scalar field $\psi(x) \equiv (\phi_1 + i\phi_2)/\sqrt{2}$ is

$$\mathcal{L} \,=\, \eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi^* - m^2\psi\psi^*\,, \tag{1.1.14}$$

where ψ^* is the complex conjugate of ψ. Written in terms of the real and imaginary parts of the field, ϕ_1 and ϕ_2, this is just two copies of the Lagrangian (1.1.7). The equations of motion can be derived by treating ψ and ψ^* as independent. Using

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)} = \partial^\mu\psi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\psi^*} = -m^2\psi\,, \tag{1.1.15}$$

we then find

$$\partial_\mu\partial^\mu\psi + m^2\psi = 0\,, \tag{1.1.16}$$

and similarly for ψ^*.

Electrodynamics

Another important field theory is electrodynamics. Recall that in relativity, the electric scalar potential ϕ and the magnetic vector potential $\mathbf{A}$ combine into the four-vector potential $A_\mu = (\phi, \mathbf{A})$. The Lagrangian for A_μ is

$$\mathcal{L} \,=\, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu\,, \tag{1.1.17}$$

where the field-strength tensor is $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and the vector current is $J_\mu = (\rho, \mathbf{j})$. The Maxwell Lagrangian can also be written as

$$\mathcal{L} \,=\, -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2 - A_\mu J^\mu\,. \tag{1.1.18}$$

Using

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + (\partial_\rho A^\rho)\eta^{\mu\nu} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial A_\nu} = -J^\nu\,, \tag{1.1.19}$$

the Euler-Lagrange equation then leads to

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\nu\,, \tag{1.1.20}$$

or, more compactly,

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \,. \tag{1.1.21}$$

This is the inhomogeneous *Maxwell equation.* Indeed, in terms of $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk}B^k$, equation (1.1.21) becomes

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho \,, \tag{1.1.22}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \,, \tag{1.1.23}$$

which are the familiar Maxwell equations for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$.

1.1.2 Hamiltonian Formalism

The canonical quantization of fields in Chapter 2 will be based on the Hamiltonian formulation of classical dynamics. Let us define the *momentum density* conjugate to the field $\phi_a(x)$ as

$$\pi^a(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \,. \tag{1.1.24}$$

The *Hamilonian density* is then given by

$$\mathcal{H} \equiv \pi^a \dot{\phi}_a - \mathcal{L} \,, \tag{1.1.25}$$

where, as in classical mechanics, the field $\dot{\phi}_a$ should be eliminated in favor of π^a everywhere in $\mathcal{H}$. The Hamiltonian is $H = \int \mathrm{d}^3x\, \mathcal{H}$. Writing the Lagrangian as $\mathcal{L} = \pi^a\dot{\phi}_a - \mathcal{H}$, the variation of the action reads

$$\delta S = \int \mathrm{d}^4x \left\{ \delta\pi^a \dot{\phi}_a + \pi^a \delta\dot{\phi}_a - \delta\pi^a \frac{\partial \mathcal{H}}{\partial \pi^a} - \delta\phi_a \frac{\partial \mathcal{H}}{\partial \phi_a} - \delta(\partial^i \phi_a) \frac{\partial \mathcal{H}}{\partial(\partial_i \phi_a)} \right\} \tag{1.1.26}$$

$$= \int \mathrm{d}^4x \left\{ \delta\pi^a \left[\dot{\phi}_a - \frac{\partial \mathcal{H}}{\partial \pi^a} \right] + \delta\phi_a \left[-\dot{\pi}^a - \frac{\partial \mathcal{H}}{\partial \phi_a} + \partial_i \left(\frac{\partial \mathcal{H}}{\partial(\partial_i \phi_a)} \right) \right] \right\} . \tag{1.1.27}$$

For arbitrary variations $\delta\phi_a$ and $\delta\pi^a$, the condition $\delta S = 0$ can only be satisfied if

$$\dot{\phi}_a = \frac{\partial \mathcal{H}}{\partial \pi^a} \quad \text{and} \quad \dot{\pi}^a = -\frac{\partial \mathcal{H}}{\partial \phi_a} + \partial_i \left(\frac{\partial \mathcal{H}}{\partial(\partial_i \phi_a)} \right) . \tag{1.1.28}$$

These are *Hamilton's equations.*

As a simple example, consider the Klein-Gordon Lagrangian (1.1.7):

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \,, \tag{1.1.29}$$

for a general potential $V(\phi)$. The conjugate momentum density is $\pi = \dot{\phi}$ and the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \,. \tag{1.1.30}$$

We see that the Hamilonian is a sum of the kinetic energy $\frac{1}{2}\pi^2$, the gradient energy $\frac{1}{2}(\nabla\phi)^2$ and the potential energy $V(\phi)$. Using

$$\dot{\pi} = -\frac{\partial \mathcal{H}}{\partial \phi} + \partial_i \left(\frac{\partial \mathcal{H}}{\partial(\partial_i \phi)} \right) , \tag{1.1.31}$$

we recover the Klein-Gordon equation

$$\ddot{\phi} - \nabla^2\phi = -\frac{dV}{d\phi}\,. \tag{1.1.32}$$

A drawback of the Hamiltonian formalism is that manifest Lorentz invariance is lost. Of course, the physics is Lorentz invariant despite the appearance of some intermediate steps in the calculations.

1.2 Symmetries

Symmetries play an important role in field theory. As in classical mechanics, the reason is Noether's theorem. We will first derive the field theory version of Noether's theorem and then discuss some of its consequences.

1.2.1 Noether's Theorem

Every continuous symmetry of the Lagrangian gives rise to a conserved current, a vector J^μ that satisfies $\partial_\mu J^\mu = 0$ after imposing the equation of motion (1.1.6).

The proof of Noether's theorem is instructive and explains why it is such a powerful result. Consider an infinitesimal transformation $\delta\phi_a = X_a(\phi)$. This is a symmetry of the theory, if it changes the Lagrangian by at most a total derivative

$$\delta\mathcal{L} = \partial_\mu F^\mu\,, \tag{1.2.33}$$

for some $F^\mu(\phi)$. The change of the Lagrangian for an *arbitrary* transformation $\delta\phi_a$ can be read of from (1.1.5):

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right). \tag{1.2.34}$$

The term in square bracket vanishes if the Euler-Lagrange equation is satisfied. We are then left only with the total derivative term

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right). \tag{1.2.35}$$

For a symmetry transformation $\delta\phi = X(\phi)$, this has to equal (1.2.33), so that we get

$$\partial_\mu J^\mu = 0\,, \quad \text{for} \quad \boxed{J^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}X_a(\phi) - F^\mu(\phi)}\,. \tag{1.2.36}$$

If the symmetry involves more than one field, the first term in (1.2.36) will involve a sum over the fields (or field components).

Associated with the Noether current is a *conserved charge* $Q \equiv \int \mathrm{d}^3x\, J^0$.

To see this, we simply take the time derivative of the charge

$$\frac{dQ}{dt} = \int \mathrm{d}^3x\,\frac{dJ^0}{dt} = -\int \mathrm{d}^3x\,\boldsymbol{\nabla}\cdot\boldsymbol{J}\,, \tag{1.2.37}$$

which is zero if $\boldsymbol{J}$ falls off sufficiently fast at $|\mathbf{x}| \to \infty$.

1.2.2 Spacetime Symmetries

In this course, we will study *relativistic* field theories, i.e. theories that are invariant under spacetime translations and Lorentz transformations. In this section, we discuss the Noether charges associated with these symmetries.

Translations

In classical mechanics, translation invariance implies conservation of momentum and time independence implies conservation of energy. We will now derive the analogous statements in field theory.

Let us write an infinitesimal spacetime translation as

$$x^\mu \to (x')^\mu = x^\mu + \epsilon^\mu \,. \tag{1.2.38}$$

The corresponding transformation of a scalar field is

$$\phi'(x') = \phi(x) \,. \tag{1.2.39}$$

In other words, the new field evaluated at the new position equals the old field at the old position.[1] We define the field variation as

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) \,, \tag{1.2.40}$$

i.e. as the difference between the transformed field and the original field *at the same point*. Using (1.2.39) and (1.2.38), we get

$$\begin{aligned} \delta\phi(x) &= \phi'(x) - \phi(x) \\ &= \phi(x-\epsilon) - \phi(x) = -\epsilon^\nu \partial_\nu \phi \,, \end{aligned} \tag{1.2.41}$$

where the minus sign in the transformation arises because we are making an active, rather than passive, transformation. If the Lagrangian doesn't depend explicitly on the spacetime coordinates (but only implicitly through its dependence on ϕ), then it transforms in the same way as the scalar field

$$\delta\mathcal{L} = -\epsilon^\nu \partial_\nu \mathcal{L} \,. \tag{1.2.42}$$

This takes the form of (1.2.33), with $F^\mu = -\epsilon^\mu \mathcal{L}$. Together with $X = -\epsilon^\nu \partial_\nu \phi$ from (1.2.41), this can be substituted into (1.2.36) to give

$$J^\mu = -\epsilon^\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right] \equiv -\epsilon^\nu \, T^\mu{}_\nu \,. \tag{1.2.43}$$

Stripping off the arbitrary constant parameters ϵ^ν, this leads to four conserved currents, $T^\mu{}_\nu = (J^\mu)_\nu$, one for each value of $\nu = 0,1,2,3$:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \,. \tag{1.2.44}$$

[1] Alternatively, one might define $\phi'(x) = \phi(x')$ resulting in the opposite transformation rules. This is called a *passive* transformation, while (1.2.39) is an *active* transformation.

This quantity is called the *energy-momentum tensor*. Current conservation implies $\partial_\mu T^\mu{}_\nu = 0$. The four conserved charges are

$$E = \int \mathrm{d}^3x\, T^{00} = \int \mathrm{d}^3x \left(\pi\dot{\phi} - \mathcal{L}\right), \tag{1.2.45}$$

$$P^i = \int \mathrm{d}^3x\, T^{0i} = -\int \mathrm{d}^3x\ \pi\partial_i\phi\,. \tag{1.2.46}$$

This describes the total energy and momentum of the field configuration. Notice that the integrand in (1.2.45) is the Hamiltonian density $\mathcal{H}$ as defined in (1.1.25).

Comment.—Let us remark that the definition (1.2.44) is not guaranteed to lead to a symmetric energy-momentum tensor, i.e. we may find $T^{\mu\nu} \neq T^{\nu\mu}$. However, it is always possible to define

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} + \partial_\rho \Gamma^{\rho\mu\nu}\,, \tag{1.2.47}$$

with $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu}$, so that $\Theta^{\mu\nu} = \Theta^{\nu\mu}$. Since $\partial_\mu\partial_\rho\Gamma^{\rho\mu\nu} = 0$, the new energy-momentum tensor $\Theta^{\mu\nu}$ is also a conserved current.

Example.—Applying Noether's definition of the energy-momentum tensor to the Maxwell Lagrangian (1.1.21), one finds

$$T^{\mu\nu} = (\partial^\nu A^\mu)(\partial_\rho A^\rho) - (\partial^\mu A_\rho)(\partial^\nu A^\rho) + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,. \tag{1.2.48}$$

The first term in this expression is clearly not symmetric under the interchange of μ and ν, so that $T^{\mu\nu} \neq T^{\nu\mu}$. A symmetric version of the energy-momentum tensor is

$$\Theta^{\mu\nu} = T^{\mu\nu} - \partial_\rho(F^{\rho\mu}A^\nu)\,. \tag{1.2.49}$$

This is of the form of (1.2.47), with $\Gamma^{\rho\mu\nu} = -F^{\rho\mu}A^\nu$.

Lorentz transformations

In classical mechanics, rotational invariance implies the conservation of angular momentum. What does this correspond to in field theory? Relativistic theories, moreover, are invariant under Lorentz boosts. What conserved quantity do they correspond to?

Consider the Lorentz transformation

$$x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu\, x^\nu\,, \tag{1.2.50}$$

where $\Lambda^\mu{}_\nu$ satisfies $\eta^{\mu\nu} = \Lambda^\mu{}_\rho\, \eta^{\rho\sigma}\Lambda^\nu{}_\sigma$, so that the line element $\mathrm{d}s^2 \equiv \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is invariant. A theory is Lorentz invariant if its Lagrangian is a *Lorentz scalar*.[2] In practice, this simply means that all terms in the Lagrangian have appropriately contracted Lorentz indices.

[2] Although the change of the integration variable from d^4x to d^4x' introduces a Jacobian factor $\det(\Lambda)$, this is unity for a Lorentz transformation.

Examples.—To avoid clutter, we will sometimes use matrix notation and drop the indices, e.g. $x \to x' = \Lambda x$. Under the Lorentz transformation (1.2.50), a scalar field then transforms as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)\,, \tag{1.2.51}$$

where Λ^{-1} appears because we are making an active transformation. The derivative of the scalar field transforms as a vector, namely

$$(\partial_\mu\phi)(x) \to (\Lambda^{-1})^\nu{}_\mu(\partial_\nu\phi)(\Lambda^{-1}x)\,. \tag{1.2.52}$$

With this it is easy to show that the Klein-Gordon Lagrangian (1.1.7) is indeed Lorentz invariant. Similarly, a vector field A^μ transforms as

$$A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)\,. \tag{1.2.53}$$

It is then a simple exercise to show that the Maxwell Lagrangian (1.1.21) is Lorentz invariant.

To determine the Noether currents associated with the Lorentz symmetries, we write

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu\,, \tag{1.2.54}$$

where $\omega^\mu{}_\nu$ is an infinitesimal parameter. The requirement $\eta^{\mu\nu} = \Lambda^\mu{}_\rho\,\eta^{\rho\sigma}\Lambda^\nu{}_\sigma$ implies

$$\begin{aligned}\eta^{\mu\nu} &= (\delta^\mu_\rho + \omega^\mu{}_\rho)\,\eta^{\rho\sigma}(\delta^\nu_\sigma + \omega^\nu{}_\sigma) \\ &= \eta^{\mu\nu} + \omega^{\mu\nu} + \omega^{\nu\mu} + \mathcal{O}(\omega^2)\,,\end{aligned} \tag{1.2.55}$$

i.e. the transformation parameter must be anti-symmetric

$$\omega^{\mu\nu} = -\omega^{\nu\mu}\,. \tag{1.2.56}$$

Note that the counting works: an anti-symmetric 4×4 matrix has six independent components, corresponding to three rotations and three boosts. A real scalar field then transforms as

$$\begin{aligned}\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) &= \phi(x - \omega x) \\ &= \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu\phi(x)\,,\end{aligned} \tag{1.2.57}$$

where Λ^{-1} and $-\omega x$ appear because we are considering an active transformation. The variation of the field is

$$\delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi\,. \tag{1.2.58}$$

Similarly, the variation of the Lagrangian is

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu \mathcal{L})\,, \tag{1.2.59}$$

where we used $\omega^\mu{}_\mu = 0$. Since the Lagrangian only changes by a total derivative, we can apply Noether's theorem. Using $X = -\omega^\mu{}_\nu x^\nu \partial_\mu\phi$ and $F^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$, the Noether current (1.2.36) becomes

$$\begin{aligned}J^\mu = -\omega^\rho{}_\sigma \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi - \delta^\mu_\rho\mathcal{L}\right] x^\sigma &= -\omega^\rho{}_\sigma\, T^\mu{}_\rho\, x^\sigma \\ &= -\frac{1}{2}\omega_{\rho\sigma}\Big[T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\rho\Big] \equiv -\frac{1}{2}\omega_{\rho\sigma}\,(\mathcal{J}^\mu)^{\rho\sigma}\,.\end{aligned} \tag{1.2.60}$$

Stripping off the arbitrary parameters $\omega_{\rho\sigma}$, we obtain six different currents, one for each value of $\rho \neq \sigma$:

$$(\mathcal{J}^\mu)^{\rho\sigma} \equiv x^\rho \, T^{\mu\sigma} - x^\sigma \, T^{\mu\rho} \,, \tag{1.2.61}$$

with $\partial_\mu (\mathcal{J}^\mu)^{\rho\sigma} = 0$. The associated conserved charges are $Q^{\rho\sigma} = \int \mathrm{d}^3x \, (\mathcal{J}^0)^{\rho\sigma}$. For $\rho, \sigma \neq 0$ the transformation is a rotation and the conserved charge is the total *angular momentum* of the field

$$Q^{ij} = \int \mathrm{d}^3x \, (x^i T^{0j} - x^j T^{0i}) \,. \tag{1.2.62}$$

For $\rho = 0$ and $\sigma = i$, the transformation is a boost and the conserved charges are

$$Q^{0i} = \int \mathrm{d}^3x \, (x^0 T^{0i} - x^i T^{00}) \,. \tag{1.2.63}$$

The conservation of Q^{0i} implies

$$\begin{aligned} 0 = \frac{dQ^{0i}}{dt} &= \int \mathrm{d}^3x \, T^{0i} + t \int \mathrm{d}^3x \, \frac{dT^{0i}}{dt} - \frac{d}{dt} \int \mathrm{d}^3x \, x^i T^{00} \\ &= P^i + t \frac{dP^i}{dt} - \frac{d}{dt} \int \mathrm{d}^3x \, x^i T^{00} \,. \end{aligned} \tag{1.2.64}$$

Since the momentum P_i is also conserved, we conclude that

$$\frac{d}{dt} \int \mathrm{d}^3x \, x^i T^{00} = const., \tag{1.2.65}$$

i.e. the center of energy of the field travels with constant velocity. This is the field theory equivalent of Newton's first law.

1.2.3 Internal Symmetries

Our examples so far have involved transformations of the spacetime coordinates. But there are other types of transformations. An internal symmetry only involves a transformation of the fields and acts the same at every point of spacetime.

- For example, the Klein-Gordon Lagrangian (1.1.7) in the massless limit,

  $$\mathcal{L} \, = \, \frac{1}{2} (\partial_\mu \phi)^2 \,, \tag{1.2.66}$$

  is invariant under a constant shift of the field value

  $$\phi \to \phi + \alpha \,. \tag{1.2.67}$$

  Substituting $F^\mu = 0$ and $X = \alpha$ into (1.2.36), we find

  $$J^\mu = \partial^\mu \phi \,, \tag{1.2.68}$$

  where the overall normalization has been chosen arbitrarily.

- As a slightly less trivial example, consider a complex scalar field whose Lagrangian is

  $$\mathcal{L} \, = \, \partial_\mu \psi \partial^\mu \psi^* - V(\psi \psi^*) \,, \tag{1.2.69}$$

where $V(\psi\psi^*)$ is a general potential. This Lagrangian is invariant under the phase rotation $\psi \to e^{i\alpha}\psi$. The infinitesimal variations of the fields ψ and ψ^* are

$$\delta\psi = i\alpha\psi\,, \quad \delta\psi^* = -i\alpha\psi^*\,. \tag{1.2.70}$$

Using $F^\mu = 0$ and $X = -i\alpha\psi^*$, $X_* = i\alpha\psi$, we get

$$J^\mu = i(\partial^\mu\psi^*)\psi - i\psi^*(\partial^\mu\psi)\,. \tag{1.2.71}$$

We will see that the conserved charges corresponding to these currents can be interpreted as conserved particle numbers.

1.3 Problems

1. The evolution of a complex field $\psi(x)$ is governed by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi \partial^\mu \psi^* - m^2 \psi \psi^* - \frac{\lambda}{2}(\psi\psi^*)^2 \,.$$

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta\psi = i\alpha\psi \,, \quad \delta\psi^* = -i\alpha\psi^* \,,$$

where α is a constant. Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ψ and ψ^*.

2. Consider the Lagrangian density for a triplet of real scalar fields,

$$\mathcal{L} = \sum_{a=1}^{3} \left[\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a \right] .$$

Show that the theory is invariant under the infinitesimal $SO(3)$ transformation

$$\phi_a \to \phi_a + \theta \epsilon_{abc} n_b \phi_c \,,$$

where θ is a constant and n_b is a unit vector. Compute the Noether current J^μ. Deduce that the three quantities

$$Q_a = \int \mathrm{d}^3 x \; \epsilon_{abc} \dot{\phi}_b \phi_c$$

are conserved. Verify this directly using the field equations satisfied by ϕ_a.

3. A class of interesting theories is invariant under the following scaling transformation

$$x^\mu \to (x')^\mu = \lambda x^\mu \,,$$
$$\phi(x) \to \phi'(x) = \lambda^{-D} \phi(\lambda^{-1} x) \,,$$

where D is called the scaling dimension of the field. Consider the action for a real scalar field given by

$$S = \int \mathrm{d}^4 x \left[\frac{1}{2} \partial_\mu \phi \partial^\nu \phi - \frac{1}{2} m^2 \phi^2 - g\phi^p \right] .$$

Find the scaling dimension D such that the derivative term remains invariant. For what values of m and p is the scaling transformation a symmetry of the theory? How do these conclusions change for a scalar field living in an $(n+1)$-dimensional spacetime instead of a $(3+1)$-dimensional spacetime? In $(3+1)$ dimensions, use Noethers theorem to construct the conserved current associated with scaling invariance.

4. The Lagrangian density for a massive vector field B_μ is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 B_\mu B^\mu \,,$$

where $F_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$. Derive the equations of motion and show that for $m \neq 0$ they imply

$$\partial_\mu B^\mu = 0 \,.$$

Further show that B_0 can be eliminated completely in terms of the other fields by

$$(\partial_i \partial^i + m^2) B_0 = \partial^i \dot{B}_i \,.$$

Construct the momenta Π_i conjugate to B_i, $i = 1, 2, 3$ and show that the momentum conjugate to B_0 vanishes. Construct the Hamiltonian density $\mathcal{H}$ in terms of B_0, B_i and Π_i.

5. A system that is invariant under both spacetime translations and Lorentz transformations is said to be *Poincaré invariant*. Show that this requires the energy-momentum tensor of the system to be symmetric.

2 Quantization of Free Fields

This chapter begins our exploration of the quantum theory of fields. We will start with a treatment of *free fields*, that is fields with linear equations of motion and therefore no interactions. A special feature of linear equations of motions is that distinct Fourier modes evolve independently. Moreover, each Fourier mode satisfies the equation of motion of a simple harmonic oscillator and can therefore be quantized in the usual way (see §B.2).

2.1 A Lattice Model*

The aim of this chapter is to quantize the Klein-Gordon theory of a free scalar field. (Multi-component spinor fields and vector fields will be considered in later chapters.) However, before we get to this, we will study a simple lattice model. This example will provide a useful bridge between the familiar territory of quantum mechanics and the new realm of quantum field theory.

Balls and springs

Consider N balls of equal masses μ arranged in a line and connected by springs. This is a simple model of a one-dimensional crystal, so let's call the balls "atoms". In equilibrium, the atoms are separated by a distance ℓ. We denote the position of each atom by

$$x_a(t) = a\ell + \phi_a(t)\,, \quad a = 1, \dots, N, \tag{2.1.1}$$

where ϕ_a describes the deviation of the a-th atom from its equilibrium position $a\ell$. The Lagrangian of the system should then take the following form

$$L = \sum_a \left[\frac{1}{2}\mu\dot{\phi}_a^2 - \frac{g_2}{2}\left(\frac{\phi_a - \phi_{a+1}}{\ell}\right)^2 - \frac{g_3}{3!}\left(\frac{\phi_a - \phi_{a+1}}{\ell}\right)^3 - \cdots\right], \tag{2.1.2}$$

where we have Taylor expanded the potential for small deviations from equilibrium, $\phi_a \ll \ell$. We have found it convenient to keep the lattice spacing ℓ explicit. The parameters g_n then have units of energy. Assuming all the g_n's to be of the same order, the quadratic term in the potential dominates and we can write the Lagrangian as

$$L \approx \sum_a \frac{\mu}{2}\left[\dot{\phi}_a^2 - c_s^2\left(\frac{\phi_a - \phi_{a+1}}{\ell}\right)^2\right], \tag{2.1.3}$$

where we have defined $c_s^2 \equiv g_2/\mu$. This is just a theory of coupled harmonic oscillators.

The Euler-Lagrange equations corresponding to (2.1.3) are

$$\ddot{\phi}_a + c_s^2\left(\frac{2\phi_a - \phi_{a+1} - \phi_{a-1}}{\ell^2}\right) = 0\,. \tag{2.1.4}$$

To solve this set of equations, we write ϕ_a as a discrete Fourier series

$$\phi_a(t) = \frac{1}{\sqrt{N}} \sum_k e^{ikx_a}\, \phi_k(t)\,. \tag{2.1.5}$$

In writing this we have assumed periodic boundary conditions. The sum over k runs over N discrete values in the range $\Delta k = 2\pi/\ell$. We choose this range to be $-\pi/\ell < k \leq \pi/\ell$. Substituting (2.1.5) into (2.1.4), we find that each Fourier mode satisfies (see box below)

$$\ddot{\phi}_k + 4\,\frac{c_s^2}{\ell^2}\, \sin^2(k\ell/2)\, \phi_k = 0\,. \tag{2.1.6}$$

We see that each Fourier mode $\phi_k(t)$ evolves independently.

Derivation.—Substituting (2.1.5) in (2.1.4) gives

$$\begin{aligned}
\frac{1}{\sqrt{N}} \sum_k e^{ikx_a} \ddot{\phi}_k(t) &= -\frac{c_s^2}{\ell^2}\frac{1}{\sqrt{N}}\left[2\sum_k e^{ikx_a}\phi_k - \sum_k e^{ikx_{a+1}}\phi_k - \sum_k e^{ikx_{a-1}}\phi_k\right] \\
&= -\frac{c_s^2}{\ell^2}\frac{1}{\sqrt{N}}\left[2\sum_k e^{ik\,a\ell}\phi_k - \sum_k e^{ik(a+1)\ell}\phi_k - \sum_k e^{ik(a-1)\ell}\phi_k\right] \\
&= -\frac{c_s^2}{\ell^2}\frac{1}{\sqrt{N}}\sum_k e^{ikx_a}\left[2 - e^{ik\ell} - e^{-ik\ell}\right]\phi_k \\
&= -\frac{c_s^2}{\ell^2}\frac{1}{\sqrt{N}}\sum_k e^{ikx_a}\left[2 - 2\cos(k\ell)\right]\phi_k\,.
\end{aligned} \tag{2.1.7}$$

Taking the inverse Fourier transform, we find that the Fourier modes must satisfy

$$\ddot{\phi}_k = -2\frac{c_s^2}{\ell^2}\left[1 - \cos(k\ell)\right]\phi_k\,, \tag{2.1.8}$$

which confirms the result (2.1.6).

By going to Fourier space, we have decoupled, or diagonalized, the equations of motion. The problem has been reduced to N independent harmonic oscillators, with frequencies

$$\omega_k = 2\frac{c_s}{\ell}|\sin(k\ell/2)|\,. \tag{2.1.9}$$

For wavelengths much larger than the lattice spacing, $k\ell \ll 1$, the dispersion relation becomes $\omega_k \to c_s|k|$ and the equation of motion takes the form of a wave equation

$$\left(\partial_t^2 - c_s^2 k^2\right)\phi_k = 0\,. \tag{2.1.10}$$

The solution of this equation are sound waves travelling at speed c_s.

Diagonalizing the equations of motion also diagonalizes the Lagrangian and Hamiltonian. For example, substituting the Fourier series (2.1.5) into (2.1.3), we get

$$L = \frac{1}{2}\sum_k \left[\dot{\phi}_k\dot{\phi}_{-k} - \omega_k^2\, \phi_k\phi_{-k}\right], \tag{2.1.11}$$

where we have rescaled the coordinates to set $\mu \equiv 1$. You might be surprised to see $\phi_k\phi_{-k}$, instead of ϕ_k^2, but note that $\phi_k^\dagger = \phi_{-k}$, so that $\phi_a^\dagger = \phi_a$. The momentum conjugate to ϕ_a is $\pi_a \equiv \dot{\phi}_a$ and the Hamiltonian becomes

$$H = \frac{1}{2}\sum_k \left[\pi_k\pi_{-k} + \omega_k^2\,\phi_k\phi_{-k}\right], \tag{2.1.12}$$

which is just the sum of Hamiltonians for each independent oscillator.

Derivation.—As an example, consider the Fourier expansion of the kinetic term $\sum_a \frac{1}{2}\dot{\phi}_a^2$. Substituting (2.1.5), we find

$$\begin{aligned}\sum_a \frac{1}{2}\dot{\phi}_a^2 &= \sum_a \frac{1}{2N}\sum_{k,k'} e^{i(k+k')x_a}\,\dot{\phi}_k\dot{\phi}_{k'} \\ &= \frac{1}{2N}\sum_{k,k'} N\delta_{k,-k'}\,\dot{\phi}_k\dot{\phi}_{k'} \\ &= \frac{1}{2}\sum_k \dot{\phi}_k\dot{\phi}_{-k}\,. \end{aligned}\tag{2.1.13}$$

A similar analysis of the second term in the Lagrangian leads to

$$-\sum_a \frac{c_s^2}{2\ell^2}(\phi_a - \phi_{a+1})^2 = -\frac{1}{2}\sum_k \omega_k^2\,\phi_k\phi_{-k}\,, \tag{2.1.14}$$

so that the Lagrangian takes the form (2.1.11).

Quantization of phonons

To quantize the collection of sound waves, we promote the oscillator amplitude and its conjugate momentum to operators, $\phi_a, \pi_b \to \hat{\phi}_a, \hat{\pi}_b$, and impose the following commutation relations

$$\begin{aligned}\left[\hat{\phi}_a, \hat{\pi}^b\right] &= i\hbar\delta_a^b \equiv i\delta_a^b\,, \\ \left[\hat{\phi}_a, \hat{\phi}_b\right] &= \left[\hat{\pi}^a, \hat{\pi}^b\right] = 0\,.\end{aligned}\tag{2.1.15}$$

To find the spectrum of energy eigenstates (normal modes), we write the Fourier modes of the operators as (see §B.2)

$$\hat{\phi}_k \equiv \frac{1}{\sqrt{2\omega_k}}\left(\hat{a}_k + \hat{a}_{-k}^\dagger\right), \quad \hat{\pi}_k \equiv -i\sqrt{\frac{\omega_k}{2}}\left(\hat{a}_k - \hat{a}_{-k}^\dagger\right), \tag{2.1.16}$$

where the ladder operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ satisfy

$$\begin{aligned}\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] &= \delta_{kk'}\,, \\ \left[\hat{a}_k, \hat{a}_{k'}\right] &= \left[\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\right] = 0\,.\end{aligned}\tag{2.1.17}$$

The lowering operators $\hat{a}_k$ annihilate the vacuum state, i.e. $\hat{a}_k|0\rangle = 0$. Acting on the vacuum with the raising operators $\hat{a}_k^\dagger$ creates excitations with momentum k called *phonons*. The Hilbert space of the theory is generated by states like

$$|n_{k_1}, n_{k_2}, \ldots\rangle \equiv \frac{1}{\sqrt{n_{k_1}!\,n_{k_2}!\cdots}}\,(\hat{a}_{k_1}^\dagger)^{n_{k_1}}(\hat{a}_{k_2}^\dagger)^{n_{k_2}}\cdots|0\rangle\,, \tag{2.1.18}$$

which describes n_i phonon excitations with momentum k_i.

Substituting (2.1.16) into (2.1.12), we obtain

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right), \tag{2.1.19}$$

where we have restored Planck's constant. We see that the total energy is just the sum of the energies carried by each k mode. Even the vacuum has a finite zero-point energy, which now receives a sum over all k modes.

Continuum limit

Let us consider the continuum limit of the theory. This means sending the lattice spacing $\ell \to 0$, while keeping other parameters fixed. Of course, since ℓ is dimensionful, this limit only makes sense in comparison to another length scale. What we really mean by the continuum limit is that ℓ/L goes to zero, where L is some length scale at which we are conducting an experiment. Since we have modes with inverse wavelength k, this means $k\ell \to 0$. This limit is achieved by taking $N \to \infty$, in which case the wavenumber k becomes a continuous parameter.

The dispersion relation (2.1.9) then becomes

$$\omega_k = 2\frac{c_s}{\ell}|\sin(k\ell/2)| \xrightarrow{k\ell\to 0} c_s|k|\,. \tag{2.1.20}$$

and the Lagrangian (2.1.11) reduces to

$$L \;\to\; \int \mathrm{d}k\, \frac{1}{2}\left[\dot{\phi}_k\dot{\phi}_{-k} - c_s^2 k^2 \phi_k\phi_{-k}\right] \tag{2.1.21}$$

$$= \int \mathrm{d}x\, \frac{1}{2}\left[(\partial_t\phi)^2 - c_s^2(\partial_x\phi)^2\right]. \tag{2.1.22}$$

The final form of the Lagrangian could also have been obtained directly from (2.1.3) by noting that $(\phi_a - \phi_{a-1})/\ell \to \partial_x\phi$ in the continuum limit.

The Hamiltonian (2.1.12) becomes

$$\hat{H} = \int_{-\infty}^{\infty} \mathrm{d}k\, \omega_k \left(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}\right) = \int_{-\infty}^{\infty} \mathrm{d}k\, c_s|k| \left(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}\right). \tag{2.1.23}$$

Notice that the zero-point contribution to the integral is infinite. This troublesome feature of the continuum limit will occupy us more below.

Had we extended our balls and springs example to $3+1$ spacetime dimension, we would have found

$$L = \int \mathrm{d}^3x\, \frac{1}{2}\left(\dot{\phi}^2 - c_s^2(\partial_i\phi)^2\right), \tag{2.1.24}$$

instead of (2.1.22). If we identify the speed of sound c_s with the speed of light c, then this is the same as the Lagrangian for a massless scalar field

$$L = \int \mathrm{d}^3x\, \frac{1}{2}\left(\dot{\phi}^2 - c^2(\partial_i\phi)^2\right) = \int \mathrm{d}^3x\, \frac{1}{2}(\partial_\mu\phi)^2\,. \tag{2.1.25}$$

We therefore expect that the quantization of the Klein-Gordon theory should also lead to phonon-like excitations.

2.2 Klein-Gordon Equation

The unique Poincaré invariant action for a scalar field ϕ that is quadratic in ϕ and only has two derivatives is the Klein-Gordon action, with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\,. \tag{2.2.26}$$

We would have obtained this Lagrangian from the continuum limit of the 'balls and springs' model if we had added a potential term of the form $\sum_a \lambda\phi_a^2$. This would correspond to extra springs that pull the balls back to their equilibrium positions independent of the positions of their nearest neighbours.

The Euler-Lagrange equation corresponding to (2.2.26) is the Klein-Gordon equation,

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \mathbf{x}^2} + m^2\right]\phi(t,\mathbf{x}) = 0\,. \tag{2.2.27}$$

The gradient term couples the field values at nearby spatial locations. This is what allows for propagating wave solutions to the equation. As in the discrete lattice model, we decouple the equations by going to Fourier space. Substituting a Fourier expansion of the field,

$$\phi(t,\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}}\phi(t,\mathbf{p})\,, \tag{2.2.28}$$

the Klein-Gordon equation becomes

$$\left[\frac{\partial^2}{\partial t^2} + (\mathbf{p}^2 + m^2)\right]\phi(t,\mathbf{p}) = 0\,. \tag{2.2.29}$$

We see that each Fourier mode $\phi(t,\mathbf{p})$ satisfies the equation of a harmonic oscillator with frequency $\omega_\mathbf{p} \equiv \sqrt{\mathbf{p}^2+m^2}$. As advertised, the different momentum modes evolve independently. The field $\phi(t,\mathbf{x})$ can therefore be treated as a sum over independent harmonic oscillators, whose quantization we will consider next.

2.3 Canonical Quantization

We quantize the system by promoting the fields ϕ, π to operators $\hat\phi, \hat\pi$ and imposing the canonical commutation relations

$$\begin{aligned} \big[\hat\phi(\mathbf{x}), \hat\pi(\mathbf{y})\big] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \\ \big[\hat\phi(\mathbf{x}), \hat\phi(\mathbf{y})\big] = \big[\hat\pi(\mathbf{x}), \hat\pi(\mathbf{y})\big] &= 0\,. \end{aligned} \tag{2.3.30}$$

This is the analog of (2.1.15) with the Kronecker delta replaced by a Dirac delta function. For now, we are working in the *Schrödinger picture* where the field operators are time-independent and the states evolve according to the Schrödinger equation. In analogy to (2.1.16), we write

$$\hat\phi(\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(\hat a_\mathbf{p} + \hat a^\dagger_{-\mathbf{p}}\right)e^{i\mathbf{p}\cdot\mathbf{x}}\,, \tag{2.3.31}$$

$$\hat\pi(\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_\mathbf{p}}{2}}\left(\hat a_\mathbf{p} - \hat a^\dagger_{-\mathbf{p}}\right)e^{i\mathbf{p}\cdot\mathbf{x}}\,, \tag{2.3.32}$$

where it must be $\hat a^\dagger_{-\mathbf{p}}$ rather than $\hat a^\dagger_\mathbf{p}$, so that $\hat\phi(\mathbf{x})$ and $\hat\pi(\mathbf{x})$ are Hermitian, i.e. $\hat\phi^\dagger(\mathbf{x}) = \hat\phi(\mathbf{x})$ and $\hat\pi^\dagger(\mathbf{x}) = \hat\pi(\mathbf{x})$.

Exercise.—Show that the mode expansion (2.3.31) can also be written as

$$\begin{aligned}\hat{\phi}(\mathbf{x}) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(\hat{a}_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{p}}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(\hat{a}_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{x}} + h.c.\right) ,\end{aligned} \tag{2.3.33}$$

and similarly for $\hat{\pi}(\mathbf{x})$.

Substituting the mode expansions into (2.3.30), we get

$$\begin{aligned}\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,, \\ \left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\right] = \left[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] &= 0\,.\end{aligned} \tag{2.3.34}$$

The Hamiltonian becomes

$$\hat{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)\right). \tag{2.3.35}$$

Derivation.—The Hamiltonian of the free Klein-Gordon theory is

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3x\,\left[\hat{\pi}^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\right]. \tag{2.3.36}$$

Substituting (2.3.31) and (2.3.32), we find

$$\begin{aligned}\hat{H} = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{(2\pi)^6}\, e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \Bigg\{ &-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4}\left(\hat{a}_{\mathbf{p}} - \hat{a}^\dagger_{-\mathbf{p}}\right)\left(\hat{a}_{\mathbf{p}'} - \hat{a}^\dagger_{-\mathbf{p}'}\right) \\ &+ \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}\left(\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\right)\left(\hat{a}_{\mathbf{p}'} + \hat{a}^\dagger_{-\mathbf{p}'}\right)\Bigg\}.\end{aligned} \tag{2.3.37}$$

We first perform the integral over x. This leads to $(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{p}')$, which allows us to integrate over p' and hence obtain

$$\hat{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{4}\left[\left(\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\right)\left(\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\right) - \left(\hat{a}_{\mathbf{p}} - \hat{a}^\dagger_{-\mathbf{p}}\right)\left(\hat{a}_{-\mathbf{p}} - \hat{a}^\dagger_{\mathbf{p}}\right)\right], \tag{2.3.38}$$

where we have used that $\mathbf{p}\cdot\mathbf{p} + m^2 = \omega^2_{\mathbf{p}}$. This rearranges into

$$\begin{aligned}\hat{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{2}\left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\hat{a}_{-\mathbf{p}}\right) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{1}{2}\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{p}}\right]\right) \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)\right),\end{aligned} \tag{2.3.39}$$

which confirms (2.3.35).

We note that the result (2.3.35) contains two types of divergences: First, the delta function is evaluated at zero where it is infinite. Second, the integral over $\omega_{\mathbf{p}}$ diverges at large p. To understand what is going on, let us look at the vacuum contribution.

Vacuum energy

The vacuum is defined as the state annihilated by all $\hat{a}_{\mathbf{p}}$, i.e.

$$\hat{a}_{\mathbf{p}}|0\rangle = 0 \quad \forall\, \mathbf{p}\,. \tag{2.3.40}$$

The ground state energy then is

$$E_0 \equiv \langle 0|\hat{H}|0\rangle = \int \mathrm{d}^3p\, \frac{\omega_{\mathbf{p}}}{2}\, \delta^{(3)}(0) = \infty\,. \tag{2.3.41}$$

The delta-function divergence arises because we are integrating over an infinite space. To see this, put the theory in a box of side length L and volume $V = L^3$. Taking the limit $L \to \infty$, we get

$$(2\pi)^3\delta^{(3)}(0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} \mathrm{d}^3x\; e^{i\mathbf{p}\cdot\mathbf{x}}\Big|_{\mathbf{p}=0} = \lim_{L\to\infty} \int_{-L/2}^{L/2} \mathrm{d}^3x = V\,. \tag{2.3.42}$$

We can remove the *infrared divergence* associated with $\delta^{(3)}(0)$ by computing the energy density, rather than the energy, i.e.

$$\rho_0 \equiv \frac{E_0}{V} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{2}\,, \tag{2.3.43}$$

which is the sum of the ground state energies of each harmonic oscillator. The integral in (2.3.43) is still infinite because it diverges at high momenta. This so-called *ultraviolet divergence* arises because we are extending the theory to infinite energies. This extrapolation is clearly naive. In reality, we expect the theory to break down at a finite energy, so the integral in (2.3.43) should contain a finite momentum cutoff.

A practical way to deal with the UV divergence is to appeal to the fact that physics is only sensitive to energy differences.[1] We therefore simply subtract the divergent constant and write the Hamiltonian as

$$:\hat{H}: \;= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \omega_{\mathbf{p}}\, \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\,, \tag{2.3.44}$$

so that $\langle 0|:\hat{H}:|0\rangle = 0$. A convenient way to automatically subtract the divergent vacuum contribution is to *normal order* an operator. This corresponds to moving all annihilation operators $\hat{a}_{\mathbf{p}}$ to the right in any product of ladder operators, e.g.

$$:\hat{a}_{\mathbf{p}}\,\hat{a}^\dagger_{\mathbf{q}}\,\hat{a}_{\mathbf{r}}: \;=\; \hat{a}^\dagger_{\mathbf{q}}\,\hat{a}_{\mathbf{p}}\,\hat{a}_{\mathbf{r}}\,. \tag{2.3.45}$$

From now on, we will assume that all operators are normal ordered.

[1] This is not true for gravity. The absolute energy appears in the Einstein equation and affects the expansion of the universe. The UV divergence in (2.3.43) is the *cosmological constant problem.*

Particles

Consider the state[2]

$$|\mathbf{p}\rangle \equiv \sqrt{2\omega_{\mathbf{p}}}\,\hat{a}^\dagger_{\mathbf{p}}|0\rangle\,. \tag{2.3.46}$$

Acting with the Hamiltonian operator on this state, we find

$$\begin{aligned}\hat{H}|\mathbf{p}\rangle = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\,\omega_{\mathbf{q}}\,\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}\,(\sqrt{2\omega_{\mathbf{p}}}\hat{a}^\dagger_{\mathbf{p}})\,|0\rangle &= \sqrt{2\omega_{\mathbf{p}}}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\,\omega_{\mathbf{q}}\,\hat{a}^\dagger_{\mathbf{q}}\big[\hat{a}_{\mathbf{q}},\hat{a}^\dagger_{\mathbf{p}}\big]\,|0\rangle \\ &= \sqrt{2\omega_{\mathbf{p}}}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\,\omega_{\mathbf{q}}\,\hat{a}^\dagger_{\mathbf{q}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,|0\rangle \\ &= \omega_{\mathbf{p}}|\mathbf{p}\rangle\,, \end{aligned} \tag{2.3.47}$$

i.e. the state is an eigenstate of the Hamiltonian with energy $E_{\mathbf{p}} = \omega_{\mathbf{p}}$. Similar exercises show that it is eigenstate of the momentum operator with momentum $\mathbf{p}$ and has zero intrinsic angular momentum.

Exercise.—Show that the momentum operator is

$$\hat{\mathbf{P}} = -\int \mathrm{d}^3x\;\hat{\pi}\boldsymbol{\nabla}\hat{\phi} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathbf{p}\,\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\,, \tag{2.3.48}$$

and hence $\hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$.

Exercise.—In Chapter 1, we derived the classical angular momentum of a field

$$J_i = \epsilon_{ijk}\int \mathrm{d}^3x\,\left(x^jT^{0k} - x^kT^{0j}\right). \tag{2.3.49}$$

After normal ordering, the corresponding quantum operator $\hat{J}_i$ can be written as

$$\hat{J}_i = -i\epsilon_{ijk}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\hat{a}^\dagger_{\mathbf{p}}\left(p^j\frac{\partial}{\partial p_k} - p^k\frac{\partial}{\partial p_j}\right)\hat{a}_{\mathbf{p}}\,. \tag{2.3.50}$$

Show that $\hat{J}_i|\mathbf{p}=0\rangle = 0$, i.e. a stationary one-particle state $|\mathbf{p}=0\rangle$ has zero angular momentum.

We conclude that the quantization of a scalar field gives rise to *spin-0 particles.* We will encounter fields giving rise to spin-$\frac{1}{2}$ and spin-1 particles later in these lectures.

Multiparticle states are defined as

$$|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle \propto \hat{a}^\dagger_{\mathbf{p}_1}\cdots\hat{a}^\dagger_{\mathbf{p}_n}|0\rangle\,. \tag{2.3.51}$$

These states satisfy *Bose statistics*, i.e they are symmetric under the exchange of any two particles, e.g. $|\mathbf{p}_1,\mathbf{p}_2\rangle = |\mathbf{p}_2,\mathbf{p}_1\rangle$. This is a direct consequence of that fact that the creation operators $\hat{a}^\dagger$ commute among themselves. We note that unlike in quantum mechanics, the spin statistics is an output, not an input.

[2]The factor of $\sqrt{2\omega_{\mathbf{p}}}$ is included so that the inner product of two states is Lorentz invariant,

$$\langle\mathbf{p}_1|\mathbf{p}_2\rangle = 2\omega_{\mathbf{p}_1}(2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2)\,.$$

The proof is somewhat involved and will be given below.

Antiparticles

In Chapter 1, we showed that the Lagrangian of a complex scalar field $\psi(x)$ is

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi \,. \tag{2.3.52}$$

We treat ψ and ψ^* as independent fields, whose conjugate momenta are $\pi = \dot{\psi}^*$ and $\pi^* = \dot{\psi}$. We write the complex field operators as

$$\hat{\psi}(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(\hat{b}_\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{c}^\dagger_\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \tag{2.3.53}$$

$$\hat{\pi}(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_\mathbf{p}}{2}} \left(\hat{b}^\dagger_\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{c}_\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \tag{2.3.54}$$

where $\hat{b}_\mathbf{p} \neq \hat{c}_\mathbf{p}$ because the operator $\hat{\psi}$ is not Hermitian. The commutation relations are

$$\begin{aligned} \left[\hat{\psi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \\ \left[\hat{\psi}(\mathbf{x}), \hat{\pi}^\dagger(\mathbf{y})\right] &= 0\,, \end{aligned} \tag{2.3.55}$$

as well as others related by complex conjugation. This implies

$$\begin{aligned} \left[\hat{b}_\mathbf{p}, \hat{b}^\dagger_\mathbf{q}\right] &= (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})\,, \\ \left[\hat{c}_\mathbf{p}, \hat{c}^\dagger_\mathbf{q}\right] &= (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})\,, \end{aligned} \tag{2.3.56}$$

and $\left[\hat{b}_\mathbf{p}, \hat{b}_\mathbf{q}\right] = \left[\hat{c}_\mathbf{p}, \hat{c}_\mathbf{q}\right] = \left[\hat{b}_\mathbf{p}, \hat{c}_\mathbf{q}\right] = \left[\hat{b}_\mathbf{p}, \hat{c}^\dagger_\mathbf{q}\right] = 0$. We see that quantizing a complex scalar field gives rise to two creation operators $\hat{b}^\dagger_\mathbf{p}$ and $\hat{c}^\dagger_\mathbf{p}$. We call the states created by $\hat{c}^\dagger_\mathbf{p}$ *antiparticles.*

Indeed, computing the Hamiltonian and the momentum, we find

$$\hat{H} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \,\omega_\mathbf{p} \left(\hat{b}^\dagger_\mathbf{p} \hat{b}_\mathbf{p} + \hat{c}^\dagger_\mathbf{p} \hat{c}_\mathbf{p} \right), \tag{2.3.57}$$

$$\hat{P}^i = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \,p^i \left(\hat{b}^\dagger_\mathbf{p} \hat{b}_\mathbf{p} + \hat{c}^\dagger_\mathbf{p} \hat{c}_\mathbf{p} \right). \tag{2.3.58}$$

We see that the quanta of the complex scalar field are given by two different particle species with the same mass, created by $\hat{b}^\dagger_\mathbf{p}$ and $\hat{c}^\dagger_\mathbf{p}$.

In Chapter 1, we have seen that the theory defined by the Lagrangian (2.3.52) has the classically conserved charge

$$Q = i \int \mathrm{d}^3 x \left(\dot{\psi}^* \psi - \psi^* \dot{\psi} \right) = i \int \mathrm{d}^3 x \,(\pi\psi - \psi^* \pi^*)\,. \tag{2.3.59}$$

The corresponding normal-ordered operator is

$$\hat{Q} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\hat{c}^\dagger_\mathbf{p} \hat{c}_\mathbf{p} - \hat{b}^\dagger_\mathbf{p} \hat{b}_\mathbf{p} \right) = \hat{N}_c - \hat{N}_b \,. \tag{2.3.60}$$

We see that $\hat{Q}$ counts the difference in the number of antiparticles (created by $\hat{c}^\dagger_\mathbf{p}$) and the number of particles (created by $\hat{b}^\dagger_\mathbf{p}$).

Heisenberg picture

So far, we have worked in the Schrödinger picture where the operators are time-independent and the states evolve according to the Schrödinger equation. We now switch to the Heisenberg picture where the states are time-independent and the operators evolve in time. This will make the relativistic properties of the quantum theory more manifest.

The relation between operators in the Heisenberg and Schrödinger pictures is

$$\hat{\mathcal{O}}_{\rm H}(t,\mathbf{x}) = e^{i\hat{H}t}\,\hat{\mathcal{O}}_{\rm S}(\mathbf{x})\,e^{-i\hat{H}t}\,, \tag{2.3.61}$$

so that

$$\frac{d\hat{\mathcal{O}}_{\rm H}}{dt} = i[\hat{H},\hat{\mathcal{O}}_{\rm H}]\,. \tag{2.3.62}$$

In the following, we will write $\hat{\phi}_{\rm H}(t,\mathbf{x}) \equiv \phi(t,\mathbf{x}) = \phi(x)$ and $\hat{\phi}_{\rm S}(\mathbf{x}) \equiv \phi(\mathbf{x})$. Whether an operator is in the Schrödinger or Heisenberg picture will be indicated by the argument, $(\mathbf{x})$ vs. $(t,\mathbf{x}) \equiv (x)$, or will be clear from the context. To avoid clutter, we have also dropped the hats on the field operators. No more hats from now on.

The commutation relations in (2.3.30) are now to be understood as *equal time* commutation relations

$$\begin{aligned} \big[\phi(t,\mathbf{x}),\pi(t,\mathbf{y})\big] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ \big[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})\big] = \big[\pi(t,\mathbf{x}),\pi(t,\mathbf{y})\big] &= 0\,. \end{aligned} \tag{2.3.63}$$

Using (2.3.62), we can show that the field operator $\phi(x)$ obeys the Klein-Gordon equation

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0\,. \tag{2.3.64}$$

Derivation.—The evolution equations for the field operators $\phi(x)$ and $\pi(x)$ are

$$\begin{aligned} \dot{\phi} = i[H,\phi] &= \frac{i}{2}\int {\rm d}^3y[\pi^2(y) + (\nabla\phi(y))^2 + m^2\phi^2(y),\phi(x)]\\ &= i\int {\rm d}^3y\,\pi(y)(-i)\delta^{(3)}(\mathbf{y}-\mathbf{x})\\ &= \pi(x) \end{aligned} \tag{2.3.65}$$

$$\begin{aligned} \dot{\pi} = i[H,\pi] &= \frac{i}{2}\int {\rm d}^3y[\pi^2(y) + (\nabla\phi(y))^2 + m^2\phi^2(y),\pi(x)]\\ &= \frac{i}{2}\int {\rm d}^3y\,\Big(\nabla_y[\phi(y),\pi(x)]\nabla\phi(y) + \nabla\phi(y)\nabla_y[\phi(y),\pi(x)] + 2im^2\phi(y)\delta^{(3)}(\mathbf{y}-\mathbf{x})\Big)\\ &= -\int {\rm d}^3y\,\Big(\nabla_y\delta^{(3)}(\mathbf{y}-\mathbf{x})\nabla\phi(y)\Big) - m^2\phi(x)\\ &= \nabla^2\phi - m^2\phi\,. \end{aligned} \tag{2.3.66}$$

Combining (2.3.65) and (2.3.66), we find that the field operator obeys the Klein-Gordon equation (2.3.64).

The mode expansion of the Heisenberg operator $\phi(t,\mathbf{x})$ follows directly from that of Schrödinger operator $\phi(\mathbf{x})$. Using $[H,a_{\mathbf{p}}] = -\omega_{\mathbf{p}}a_{\mathbf{p}}$ and $[H,a_{\mathbf{p}}^\dagger] = +\omega_{\mathbf{p}}a_{\mathbf{p}}^\dagger$, we have

$$e^{iHt}a_{\mathbf{p}}e^{-iHt} = e^{-i\omega_{\mathbf{p}}t}a_{\mathbf{p}}\,, \tag{2.3.67}$$

$$e^{iHt}a_{\mathbf{p}}^\dagger e^{-iHt} = e^{+i\omega_{\mathbf{p}}t}a_{\mathbf{p}}^\dagger\,. \tag{2.3.68}$$

Applying (2.3.61) to (2.3.31), we then find

$$\phi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{+ip\cdot x} \right) , \tag{2.3.69}$$

where $p \cdot x \equiv E_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x}$.

Lorentz invariance.—Let us show that the integral in (2.3.69) is Lorentz invariant. Of course, $\mathrm{d}^3 p$ on its own is not Lorentz invariant. We therefore start from $\mathrm{d}^4 p$, where $p = (p^0, \mathbf{p})$, which clearly is Lorentz invariant. The integral has to be over physical states satisfying the mass shell condition $p^2 = m^2$ and having positive energy. We enforce this by writing

$$\mathrm{d}^4 p \, \delta(p^2 - m^2)\theta(p^0) . \tag{2.3.70}$$

Using the identity

$$\delta[f(x)] = \sum_{\{x | f(x) = 0\}} \frac{1}{|f'(x)|} \delta(x) , \tag{2.3.71}$$

we have

$$\begin{aligned} \delta(p^2 - m^2)\theta(p^0) &= \frac{1}{2E_{\mathbf{p}}} \left[\delta(p^0 - E_{\mathbf{p}})\theta(p^0) + \delta(p^0 + E_{\mathbf{p}})\theta(p^0) \right] \\ &= \frac{1}{2E_{\mathbf{p}}} \delta(p^0 - E_{\mathbf{p}})\theta(p^0) \end{aligned} \tag{2.3.72}$$

As a Lorentz-invariant measure we therefore use

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \bigg|_{p^0 = E_{\mathbf{p}} > 0} . \tag{2.3.73}$$

As an aside, we note that this explains why we required the extra factor of $2\omega_{\mathbf{p}}$ in (**??**):

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \, 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = 1 . \tag{2.3.74}$$

The mode expansion (2.3.75) can then be written as

$$\phi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left((\sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}) \, e^{-ip\cdot x} + (\sqrt{2E_{\mathbf{p}}} a^\dagger_{\mathbf{p}}) \, e^{+ip\cdot x} \right) , \tag{2.3.75}$$

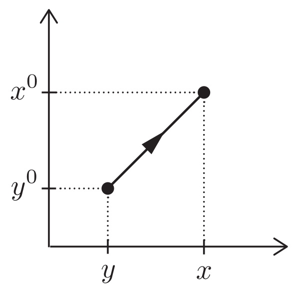
i.e. in terms of the Lorentz-invariant integral measure (2.3.73) and the Lorentz-invariant creation operators $\sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}$.

2.4 Propagators

So far, we have quantized classical field theories and created particles out of the vacuum. What happens next? In a free field theory, not much! Free particles can propagate in space and time, but do not interact. In this section, we will introduce the concept of the *propagator* for free scalar fields. The propagator will also play an important role in weakly interacting theories (Ch. 3) where the evolution between the interactions is described by freely propagating particles.

Klein-Gordon correlator

Consider creating a particle at $y^\mu \equiv (y^0, \mathbf{y})$, letting it evolve for a time $t = x^0 - y^0$ and measuring it at $x^\mu \equiv (x^0, \mathbf{x})$:



The amplitude for this process is described by the *Klein-Gordon correlator*

$$D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\, e^{-ip\cdot(x-y)}\,, \tag{2.4.76}$$

where the second equality follows from substituting the mode expansion (2.3.53).

Derivation.—Substituting (2.3.53) into the definition of the Klein-Gordon correlator, we get

$$\begin{aligned} D(x-y) &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}}\langle 0|a_\mathbf{p}a^\dagger_\mathbf{q}|0\rangle e^{-ipx+iqy} \\ &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}}\langle 0|[a_\mathbf{p},a^\dagger_\mathbf{q}]|0\rangle e^{-ipx+iqy} \\ &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\, e^{-ipx+iqy} \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\, e^{-ip\cdot(x-y)}\,, \end{aligned} \tag{2.4.77}$$

which is the result shown in (2.4.76).

Let us evaluate (2.4.76) for purely spacelike separations, i.e. $x-y=(0,\mathbf{r})$. We get

$$\begin{aligned} D(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{r}} &= \frac{2\pi}{(2\pi)^3}\int_0^\infty \mathrm{d}p\,\frac{p^2}{2E_\mathbf{p}}\frac{e^{ipr}-e^{-ipr}}{ipr} \\ &= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^\infty \mathrm{d}p\,\frac{p\,e^{ipr}}{\sqrt{p^2+m^2}} \\ &= \frac{1}{4\pi^2 r}\int_m^\infty \mathrm{d}\rho\,\frac{\rho\,e^{-\rho r}}{\sqrt{\rho^2-m^2}} \\ &= \frac{1}{4\pi^2 r} m K_1(mr)\,, \end{aligned} \tag{2.4.78}$$

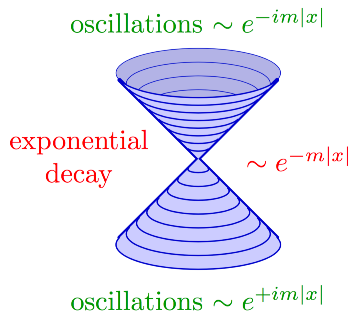
where $K_1(x)$ is a modified Bessel function, which scales as $\sqrt{\pi/(2x)}\,e^{-x}$ in the limit $x\to\infty$. Taking $r\to\infty$, we therefore find

$$D(x-y) \sim e^{-mr}\,, \tag{2.4.79}$$

which is small, but non-zero.

> *Exercise.*—Show that for purely timelike separations, i.e. $x - y = (t, 0)$, the correlator behaves as $D(x-y) \sim e^{-imt}$, in the limit $t \to \pm\infty$.

The behavior of the correlator is summarized in the following figure:



The leakage outside the lightcone should worry us. We will return to this issue in §2.5.

Feynman propagator

The most important quantity in interacting QFTs (see Ch. 3) is the *Feynman propagator*

$$\Delta_F(x-y) \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \,, \tag{2.4.80}$$

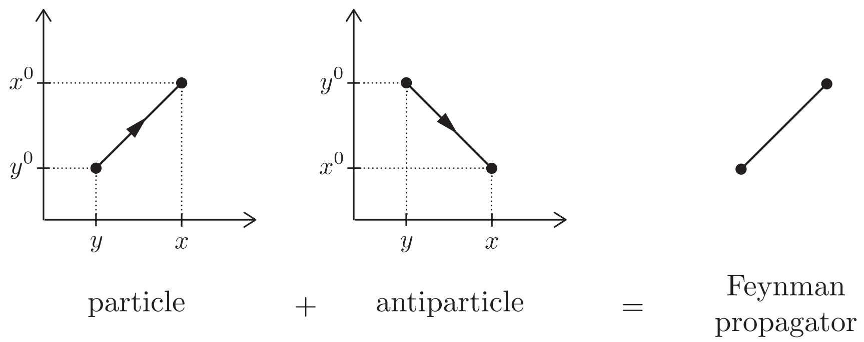
where T denotes *time ordering*, i.e. moving all operators evaluated at later times to the left. For scalar fields, time ordering is defined as

$$T\{\phi(x)\phi(y)\} = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & x^0 < y^0 \end{cases} \tag{2.4.81}$$

and Feynman propagator becomes

$$\Delta_F(x-y) = \theta(x^0 - y^0)\, D(x-y) + \theta(y^0 - x^0)\, D(y-x)\,. \tag{2.4.82}$$

We see that the propagator is made up of two parts. The first term describes a particle created at y and destroyed at x, while the second term captures an antiparticle created at x and annihilated at y:



Comment.—To see that this is really the sum of the propagation of a particle and the corresponding antiparticle, consider the Feynman propagator for a complex field

$$\begin{aligned}\Delta_F(x-y) &\equiv \langle 0|T\{\psi(x)\psi^\dagger(y)\}|0\rangle \\ &= \theta(x^0-y^0)\,\langle 0|\psi(x)\psi^\dagger(y)|0\rangle + \theta(y^0-x^0)\,\langle 0|\psi^\dagger(y)\psi(x)|0\rangle\,. \end{aligned} \tag{2.4.83}$$

This makes it manifest that the first term describes a particle created at y and destroyed at x, while the second term corresponds to an antiparticle created at x and annihilated at y.

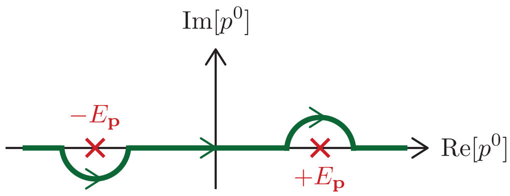
Substituting the explicit expression of the Klein-Gordon correlator, we find

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)\,e^{-ip\cdot(x-y)} + \theta(y^0-x^0)\,e^{ip\cdot(x-y)}\right]. \tag{2.4.84}$$

This representation of the Feynman propagator is not very useful. However, a bit of complex analysis (see Appendix A) lets us write it in terms of a four-momentum integral

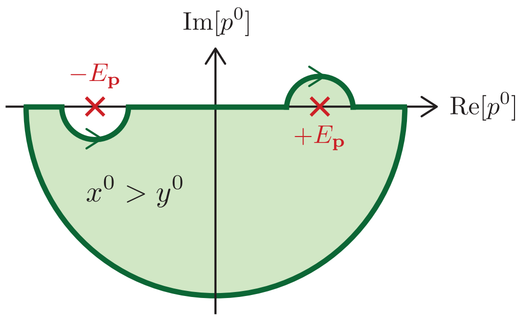
$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i}{p^2-m^2}\,. \tag{2.4.85}$$

This is the first time that we are integrating over the four-momentum $p^\mu = (p^0, \mathbf{p})$. Up until now, we integrated only over three-momentum, with p^0 fixed by the mass-shell condition to be $p^0 = E_{\mathbf{p}}$. In the integral in (2.4.85), we do not have this restriction on p^0. Note further that the integrand has poles at $p^0 = \pm E_{\mathbf{p}}$. To make the integral well-defined, we need to specify an integration contour that avoids these singularities in the p^0-integral. The contour that leads to (2.4.84) is



(2.4.86)

To see this, consider first the case $x^0 > y^0$. We must then close the contour in the lower half-plane:

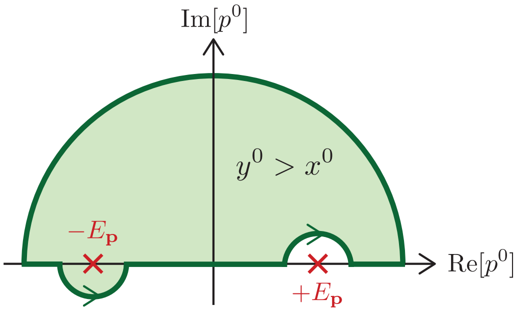


(2.4.87)

This encloses the pole at $p^0 = +E_{\mathbf{p}}$. The residue is $-2\pi i/(2E_{\mathbf{p}})$, where the minus sign arises because the orientation of the contour is clockwise. We therefore obtain

$$\begin{aligned}\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^4}\frac{-2\pi i}{2E_{\mathbf{p}}} i e^{-iE_{\mathbf{p}}(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)} \\ &= D(x-y)\,. \end{aligned} \tag{2.4.88}$$

For $y^0 > x^0$, we must close the contour in the upper half-plane



(2.4.89)

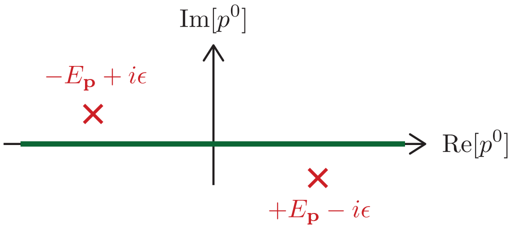
This encloses the pole at $p^0 = -E_{\mathbf{p}}$, with residue $2\pi i/(-2E_{\mathbf{p}})$. We then find

$$\begin{aligned}\Delta_F(x-y) &= \int \frac{\mathrm{d}^3p}{(2\pi)^4}\frac{2\pi i}{(-2E_{\mathbf{p}})} i e^{iE_{\mathbf{p}}(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{-ip(y-x)} \\ &= D(y-x)\,. \end{aligned} \tag{2.4.90}$$

This confirms that (2.4.85) is equivalent to (2.4.84) if we choose the contour (2.4.86). Instead of specifying the contour, we can modify the integrand in (2.4.85) as follows

$$\boxed{\Delta_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}}\,, \tag{2.4.91}$$

where $\epsilon > 0$ is an infinitesimal constant. The $i\epsilon$ in (2.4.91) is a trick to enforce the contour in (2.4.86) automatically. It achieves that the two poles at $p^0 = \pm E_{\mathbf{p}}$ are shifted up and down into the complex plane:



(2.4.92)

This leads to the same result as the contour in (2.4.86).

2.5 Causality

In (2.4.79), we saw that the Klein-Gordon correlator leaks outside of the lightcone. To see if this is a violation of causality, we have to ask the following question: Can a measurement at the spacetime point x physically influence that at a spacelike separated spacetime point y? To this end, we consider the commutator $[\phi(x), \phi(y)]$: if this commutator vanishes, one measurement cannot affect the other. We know that the commutator vanishes for $x^0 = y^0$, but need to check if it also vanishes at unequal times. Inserting the commutator in between two vacuum states, we have

$$\Delta(x-y) \equiv \langle 0|\big[\phi(x),\phi(y)\big]|0\rangle = D(x-y) - D(y-x) \tag{2.5.93}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right). \tag{2.5.94}$$

For spacelike separated points, $(x-y)^2 < 0$, we can perform a Lorentz transformation on the second term that takes $(x-y) \to -(x-y)$. The two terms are then equal and cancel to give zero. Causality is preserved. For timelike separated points, $(x-y)^2 > 0$, there is not Lorentz transformation that takes $(x-y) \to -(x-y)$. In that case, the amplitude is non-zero and scales as $e^{-imt} - e^{imt}$ for $\mathbf{x}-\mathbf{y} = 0$.

Note that in the calculation we observed a cancellation. The particle created at x and annihilated at y cancelled against the particle created at y and annihilated at x. For a complex field this cancellation is between particles and antiparticles. Causality in quantum field theory requires the existence of antiparticles!

2.6 Sources*

Consider adding an external source to the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = \rho(x)\,. \tag{2.6.95}$$

We assume that this source is non-zero only for a finite interval of time, $t_0 < t < t_1$. Before and after, the field is free. The field configuration for $t < t_0$ is $\phi_{\rm i}(x)$, while for $t > t_1$ we write

$$\phi_{\rm f}(x) = \phi_{\rm i}(x) + \Delta\phi(x)\,. \tag{2.6.96}$$

The change in the field configuration is

$$\Delta\phi(x) = \int \mathrm{d}^4 y\, G_R(x-y)\rho(y)\,, \tag{2.6.97}$$

where G_R is called the *Green's function* (or *retarded propagator*). It satisfies

$$(\partial_\mu \partial^\mu + m^2) G_R(x) = \delta^{(4)}(x)\,, \tag{2.6.98}$$

$$G_R(x) = 0 \quad \text{for} \quad x^0 < 0\,. \tag{2.6.99}$$

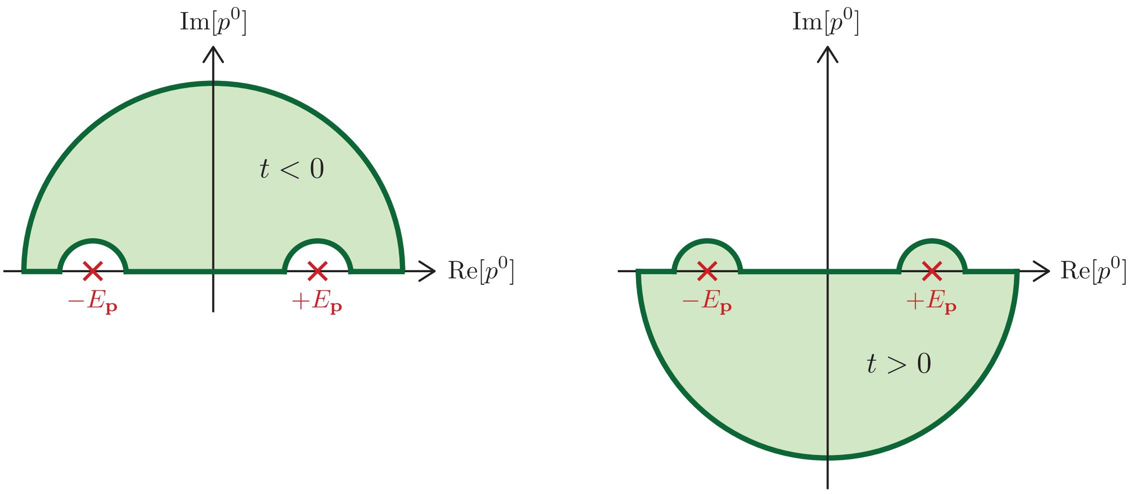
Equation (2.6.98) is solved conveniently in momentum space

$$(p^2 + m^2)G(p) = 1 \quad \Rightarrow \quad G(p) = \frac{1}{p^2+m^2}\,. \tag{2.6.100}$$

To incorporate the constraint (2.6.99), we write

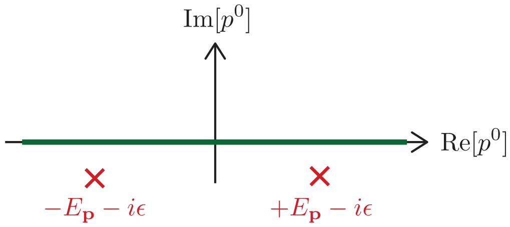
$$\begin{aligned} G(x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, e^{ip\cdot x} \frac{1}{p^2+m^2} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}} \int \frac{\mathrm{d}p^0}{2\pi}\, e^{-ip^0 t} \frac{-1}{(p^0)^2 - E_{\mathbf{p}}^2} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}} \int \frac{\mathrm{d}p^0}{2\pi} \frac{-1}{2E_{\mathbf{p}}} \left(\frac{e^{-ip^0 t}}{p^0 - E_{\mathbf{p}}} - \frac{e^{-ip^0 t}}{p^0 + E_{\mathbf{p}}} \right). \end{aligned} \tag{2.6.101}$$

To evaluate this integral, we need to flex our complex analysis muscles (see Appendix A). The integral over p^0 runs from $-\infty$ to $+\infty$. To apply the residue theorem, we need to close the contour. We will use a large semi-circle. For $t < 0$, the contour needs to be closed in the upper half, for $t > 0$ in the lower half. The integrand has two poles on the real axis, $p^0 = \pm E_{\mathbf{p}}$. We need to decide how they contribute to residues. For this, we need to remember that we want $G_R(x) = 0$ for $t < 0$. This is achieved by the following contours:



Alternatively, we can shift the poles slightly into the lower half-plane:

$$G_R(p) = \frac{1}{p^2 + m^2 - ip^0\epsilon} \qquad \leftrightarrow \tag{2.6.102}$$



Since there are now no poles in the upper half-plane, we have $G_R(x) = 0$ for $t < 0$, as required. For $t > 0$, however, both poles contribute a residue, and we obtain

$$\begin{aligned} G_R(x) &= i\theta(t) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{+ip\cdot x} - e^{-ip\cdot x}\right) \\ &= \theta(t) \left(D(x) - D(-x)\right) \\ &= \theta(t)\Delta(x) \,, \end{aligned} \tag{2.6.103}$$

where $\Delta(x)$ was defined in (2.5.93). Substituting this into (2.6.97), we get

$$\Delta\phi(x) = \int \mathrm{d}^4y \, \Delta(x-y) \, \rho(y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(ie^{ip\cdot x}\rho(p) + h.c.\right) . \tag{2.6.104}$$

Let us compare this to the mode expansion of the homogeneous Klein-Gordon equation

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} \, e^{ip\cdot x} + h.c.\right) . \tag{2.6.105}$$

The relation between the annihilation operators before and after, $a_{\mathrm{i},\mathbf{p}}$ and $a_{\mathrm{f},\mathbf{p}}$, therefore is

$$a_{\mathrm{f},\mathbf{p}} = a_{\mathrm{i},\mathbf{p}} + i\rho(E_{\mathbf{p}}, \mathbf{p})/\sqrt{2E_{\mathbf{p}}} \, . \tag{2.6.106}$$

The associated change in the particle number, $\Delta N \equiv \langle 0|N_{\mathrm{f}}|0\rangle - \langle 0|N_{\mathrm{i}}|0\rangle$, is

$$\Delta N = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \, |\rho(p)|^2 \, . \tag{2.6.107}$$

Note that ΔN is not manifestly integer for an external field. However, if ρ is itself a quantum operator related to another field, the combination $|\rho(p)|^2$ should again lead to an integer N.

2.7 Problems

1. Consider a string of length ℓ, mass per unit length σ and tension τ. The string is fixed at the end points. Let y be the transverse dispacement of the string. The Lagrangian governing the evolution of $y(x,t)$ is

$$L = \int_0^\ell \mathrm{d}x \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{\tau}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right],$$

where x is the position along the string.

i) Writing

$$y(x,t) = \sqrt{\frac{2}{\ell}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right) q_n(t)\,,$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{\tau}{2} \left(\frac{n\pi}{\ell}\right)^2 q_n^2 \right].$$

Derive the equations of motion. Hence, show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies $\omega_n \equiv \sqrt{\tau/\sigma}\, n\pi/\ell$.

ii) Show that the Hamiltonian of the system is

$$H = \frac{1}{2} \sum_{n=1}^{\infty} \left(p_n^2 + \omega_n^2 q_n^2 \right).$$

After quantization, q_n, p_n become operators $\hat{q}_n, \hat{p}_n$ satisfying

$$\begin{aligned} \left[\hat{q}_n, \hat{p}_m\right] &= i\delta_{nm}\,, \\ \left[\hat{q}_n, \hat{q}_m\right] = \left[\hat{p}_n, \hat{p}_m\right] &= 0\,. \end{aligned}$$

Introduce the ladder operators

$$\hat{a}_n \equiv \sqrt{\frac{\omega_n}{2}} \hat{q}_n + \frac{i}{\sqrt{2\omega_n}} \hat{p}_n\,, \quad \hat{a}_n^\dagger \equiv \sqrt{\frac{\omega_n}{2}} \hat{q}_n - \frac{i}{\sqrt{2\omega_n}} \hat{p}_n\,.$$

Show that they satisfy

$$\begin{aligned} \left[\hat{a}_n, \hat{a}_m^\dagger\right] &= \delta_{nm}\,, \\ \left[\hat{a}_n, \hat{a}_m\right] = \left[\hat{a}_n^\dagger, \hat{a}_m^\dagger\right] &= 0\,. \end{aligned}$$

Show that the Hamiltonian can be written as

$$\hat{H} = \sum_{n=1}^{\infty} \frac{\omega_n}{2} \left(\hat{a}_n \hat{a}_n^\dagger + \hat{a}_n^\dagger \hat{a}_n \right).$$

Given the existence of a ground state $|0\rangle$ such that $\hat{a}_n|0\rangle = 0$, explain how, after removing the vacuum energy, the Hamiltonian can be expressed as

$$\hat{H} = \sum_{n=1}^{\infty} \omega_n \hat{a}_n^\dagger \hat{a}_n\,.$$

Show further that $[\hat{H}, \hat{a}_n^\dagger] = \omega_n \hat{a}_n^\dagger$ and hence calculate the energy of the state

$$|l_1, l_2, \ldots, l_N\rangle = (\hat{a}_1^\dagger)^{l_1} (\hat{a}_2^\dagger)^{l_2} \ldots (\hat{a}_N^\dagger)^{l_N} |0\rangle\,.$$

2. Consider a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \,.$$

The mode expansion of the field operators in the Schrödinger picture is

$$\hat{\phi}(\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(\hat{a}_\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\right),$$
$$\hat{\pi}(\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_\mathbf{p}}{2}}\left(\hat{a}_\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\right).$$

i) Show that the commutation relations

$$\left[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,$$
$$\left[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})\right] = \left[\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] = 0\,.$$

imply that

$$\left[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{q}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,,$$
$$\left[\hat{a}_\mathbf{p}, \hat{a}_\mathbf{q}\right] = \left[\hat{a}^\dagger_\mathbf{p}, \hat{a}^\dagger_\mathbf{q}\right] = 0\,.$$

ii) Show that, after normal ordering, the conserved four-momentum $P^\mu = \int \mathrm{d}^3x\, T^{0\mu}$ takes the operator form

$$\hat{P}^\mu = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, p^\mu\, \hat{a}^\dagger_\mathbf{p}\hat{a}_\mathbf{p}\,,$$

where $p^0 = E_\mathbf{p}$.

iii) The classical angular momentum of a field is

$$J_i = \epsilon_{ijk}\int \mathrm{d}^3x\left(x^jT^{0k} - x^kT^{0j}\right).$$

Write down the explicit form of this expression for a free real scalar field. Show that, after normal ordering, the quantum operator $\hat{J}_i$ can be written as

$$\hat{J}_i = -i\epsilon_{ijk}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\hat{a}^\dagger_\mathbf{p}\left(p^j\frac{\partial}{p_k} - p^k\frac{\partial}{\partial p_j}\right)\hat{a}_\mathbf{p}\,.$$

Hence, confirm that the quanta of the scalar field have spin zero (i.e. a stationary one-particle state $|\mathbf{p}=0\rangle$ has zero angular momentum).

3. Consider the Lagrangian for a complex scalar field ψ given by

$$\mathcal{L} = +i\psi^*\partial_t\psi - \frac{1}{2m}\nabla\psi^*\cdot\nabla\psi\,.$$

i) Determine the equation of motion, the energy-momentum tensor and the conserved current arising from the symmetry $\psi \to e^{i\alpha}\psi$. Show that the momentum conjugate to ψ is $i\psi^*$ and compute the classical Hamiltonian.

ii) We now wish to quantize this theory. We will work in the Schrödinger picture. Explain why the correct commutation relations are

$$\left[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y})\,,$$
$$\left[\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})\right] = \left[\hat{\psi}^\dagger(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})\right] = 0\,.$$

Expand the operators as

$$\hat{\psi}(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, \hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\,,$$
$$\hat{\psi}^\dagger(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, \hat{a}^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\,.$$

Determine the commutation relations obeyed by $\hat{a}_{\mathbf{p}}$ and $\hat{a}^\dagger_{\mathbf{p}}$. Why do we have only a single set of creation and annihilation operators $\hat{a}_{\mathbf{p}}$, $\hat{a}^\dagger_{\mathbf{p}}$ even though ψ is complex? What is the physical significance of this fact? Show that one-particle states have the energy appropriate to a free non-relativistic particle of mass m.

3 Interacting Fields

Interactions are what make quantum field theories interesting. However, the field equations of interacting theories are nonlinear and hard to solve even classically. The quantum theory of interacting fields is even more challenging since quantum fluctuations will create, destroy and mix particles. Although in general it is next to impossible to find exact solutions to interacting QFTs, if the interactions are weak enough we can obtain very accurate approximate solutions. This perturbative treatment of interacting quantum fields is the subject of this chapter.

3.1 Classification of Interactions

The dynamics of a free real scalar field is determined by the Klein-Gordon Lagrangian (1.1.7). To this we now add the following perturbations

$$\Delta\mathcal{L} = -\sum_{n\geq 3} \frac{\lambda_n}{n!}\phi^n\,, \tag{3.1.1}$$

where the *coupling constants* λ_n determine the strength of the interactions. Naively, one may worry that we have added an infinite number of interactions ϕ^n, for $n \geq 3$. However, at low energies, only a finite number of these terms is relevant. This follows from simple dimensional analysis. Recall that we are working in units where $\hbar \equiv 1$, so the action is dimensionless. We denote this by $[S] = 0$. The mass dimension of the Lagrangian density then is $[\mathcal{L}] = 4$, since $S = \int \mathrm{d}^4x\,\mathcal{L}$ and $[\mathrm{d}^4x] = -4$. The dimension of the field ϕ can then be read of from the kinetic term $(\partial_\mu\phi)^2$: using $[\partial_\mu] = 1$, we find $[\phi] = 1$. This implies sensible dimensions for the mass of the field, namely $[m] = 1$. Finally, the dimensions of the coupling constants are

$$[\lambda_n] = 4 - n\,, \tag{3.1.2}$$

which can be positive or negative.

We wish to estimate how much each of the different contributions in (3.1.1) contribute to an observable evaluated at an energy scale E. The dimensionless strength of the interactions in (3.1.1) is λ_n/E^{4-n}. Depending on the value of n, we have three different categories of interactions:

- $n = 3$: The dimensionless strength of the interaction is λ_3/E. This means that $\lambda_3\phi^3$ is a small perturbation at high energies, $E \gg \lambda_3$, but becomes important at low energies, $E \ll \lambda_3$. Interactions with this type of behaviour are called *relevant*.
- $n = 4$: The dimensionless interaction strength is λ_4. This means that $\lambda_4\phi^4$ is a small perturbation if $\lambda_4 \ll 1$. Interactions like this one, which are equally important at all energies, are called *marginal*.

- $n \geq 5$: The dimensionless interaction strength is $\lambda_n E^{n-4}$, which is small at low energies and becomes large at high energies. Such interactions are called *irrelevant*, since that is what they are at low energies.

At sufficiently low energies, we mostly care about relevant and marginal interactions. This also explains why we didn't include interactions with derivatives, such as $\phi(\partial\phi)^2$ or $(\partial\phi)^4$. Instead of considering the infinite number of interaction terms in (3.1.1), only a handful are actually needed at low energies, namely ϕ^3 and ϕ^4 in the case of a real scalar field.

Suppose that one day you figure it all out. You discover the theory that describes the world at very high energies, say at the GUT scale or the Planck scale. Whatever the fundamental scale of that ultimate theory is, let's call it Λ. Being practically minded, you wish to describe physics at much lower energies $E \ll \Lambda$. For simplicity, we assume that the laws of physics at those energies are described by a real scalar field. This scalar field will have some complicated interaction terms like in (3.1.1). In principle, the precise form of the interactions can be derived from your ultimate theory. In practice, you may be able to save some energy by developing an accurate low-energy approximation. This approximation is called *effective field theory* (EFT). In fact, this EFT can be written down even without knowing the ultimate theory. Moreover, the accuracy of the EFT can be improve to any desired level. To see this, we write the couplings in (3.1.1) as

$$\lambda_n \equiv \frac{g_n}{\Lambda^{n-4}}\,, \tag{3.1.3}$$

where the parameters g_n are dimensionless numbers whose precise values are, in principle, computable from your ultimate theory. Typically, one expects the couplings g_n to be of order one. At small energies, $E \ll \Lambda$, the contribution from the terms ϕ^n, with $n \geq 4$, will be suppressed by powers of $(E/\Lambda)^{n-4}$. This is usually a suppression by many orders of magnitude—e.g. for the energies explored by the LHC we have $E/M_{\rm pl} \sim 10^{-16}$. This shows that the first few terms in (3.1.1) give an extremely good approximation to the predictions of the full theory. It also means that if we only have access to low-energy experiments (which we do!), it is going to be very difficult to figure out the high-energy theory (which it is!), because its effects are highly diluted except for the relevant and marginal interactions.

In this chapter, we will study two examples of interacting quantum field theories:

- The first is the so-called ϕ^4 *theory*:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,. \tag{3.1.4}$$

  Notice that by suppressing the ϕ^3 term, the theory has gained a $\phi \to -\phi$ symmetry.

- Some aspects of particle scattering are better illustrated by the *scalar Yukawa theory*:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - M^2\psi^*\psi - g\psi^*\psi\phi\,. \tag{3.1.5}$$

  This couples a real scalar ϕ to a complex scalar ψ. In the limit $m \to 0$, this theory provides a cartoon version of *quantum electrodynamics (QED)*, with the excitations of ϕ playing the role of *photons* and the excitations of ψ corresponding to *electrons* and *positrons*. Of course, in reality photons are spin-1 particles and electrons/positrons carry spin 1/2, so they are not represented by scalar fields. However, many of the qualitative features of QED are nevertheless captured by the model in (3.1.5). QED in all of its glory will be studied in Chapter 6.

3.2 Interaction Picture

Recall that we presented the quantization of free field theories in two distinct representations: the Schrödinger picture, where the operators are time-independent and the states are evolving, and the Heisenberg picture, where the states are time-independent and the operators are not. To describe interacting field theories a third representation will be useful, the so-called *interaction picture*, where both states and operators have some time dependence.

Let us split the Hamiltonian into a part that can be solved exactly (usually the free theory) and a small perturbation

$$H = H_0 + H_{\text{int}} \,. \tag{3.2.6}$$

We then demand that operators in the interaction picture $\mathcal{O}_{\text{I}}$ evolve according to the free part of the Hamiltonian:

$$\mathcal{O}_{\text{I}}(t) = e^{iH_0 t}\, \mathcal{O}_{\text{S}}\, e^{-iH_0 t} \,. \tag{3.2.7}$$

This is like in the Heisenberg picture, except that the evolution is determine by the free Hamiltonian H_0, rather than the full Hamiltonian H. Like in the Heisenberg picture, we have an equation of motion for the operator

$$i\frac{d\mathcal{O}_{\text{I}}}{dt} = [\mathcal{O}_{\text{I}}(t), H_0] \,. \tag{3.2.8}$$

Unlike in the Heisenberg picture, the states in the interaction picture aren't stationary. This evolution is determined by the interacting part of the Hamiltonian H_{int}. To find the correct evolution equation, we compare the matrix element $\langle\phi_{\text{S}}(t)|\mathcal{O}_{\text{S}}|\psi_{\text{S}}(t)\rangle$ in the Schrödinger picture to that in the interaction picture $\langle\phi_{\text{I}}(t)|\mathcal{O}_{\text{I}}|\psi_{\text{I}}(t)\rangle$. This implies

$$\langle\phi_{\text{S}}(t)|\mathcal{O}_{\text{S}}|\psi_{\text{S}}(t)\rangle = \langle\phi_{\text{I}}(t)|e^{iH_0 t}\mathcal{O}_{\text{S}}e^{-iH_0 t}|\psi_{\text{I}}(t)\rangle \,. \tag{3.2.9}$$

For this equality to hold, we require

$$|\psi_{\text{I}}(t)\rangle = e^{iH_0 t}|\psi(t)\rangle \,. \tag{3.2.10}$$

Differentiating this relation with respect to time, we get

$$\begin{aligned} i\frac{d}{dt}|\psi_{\text{I}}(t)\rangle &= e^{iH_0 t}\left(-H_0 + i\frac{d}{dt}\right)|\psi_{\text{S}}(t)\rangle \\ &= e^{iH_0 t}\left(-H_0 + H\right)|\psi_{\text{S}}(t)\rangle \\ &= e^{iH_0 t}(H_{\text{int}})e^{-iH_0 t}\,|\psi_{\text{I}}(t)\rangle \,. \end{aligned} \tag{3.2.11}$$

Defining $H_{\text{I}} \equiv e^{iH_0 t}(H_{\text{int}})e^{-iH_0 t}$, this becomes

$$\boxed{i\frac{d}{dt}|\psi_{\text{I}}(t)\rangle = H_{\text{I}}(t)|\psi_{\text{I}}(t)\rangle} \,. \tag{3.2.12}$$

This is like the evolution equation for a state in the Schrödinger picture, except that it involves H_{I}, rather than H.

3.3 Dyson's Formula

Our task then is to solve (3.2.12). Formally, we can write the solution as

$$|\psi_{\rm I}(t)\rangle = U(t,t_0)|\psi_{\rm I}(t_0)\rangle\,, \tag{3.3.13}$$

where $U(t,t_0)$ is a unitary time-evolution operator. The relation between U and $H_{\rm I}$ is to be determined. Substituting (3.3.13) into (3.2.12), we get

$$i\frac{d}{dt}U(t,t_0) = H_{\rm I}(t)\,U(t,t_0)\,. \tag{3.3.14}$$

We have to resist the temptation to write the solution to this as

$$U(t,t_0) \overset{?}{=} \exp\left(-i\int_{t_0}^{t} {\rm d}t'\, H_{\rm I}(t')\right). \tag{3.3.15}$$

This naive solution ignores the fact that $H_{\rm I}$ is an operator and, in particular, that it doesn't commute with itself at unequal times, i.e. $[H_{\rm I}(t), H_{\rm I}(t')] \neq 0$. To understand the problem, try substituting (3.3.15) into (3.3.14).

As we will show in the insert below, the correct solution is in terms of a *time-ordered exponential*

$$\boxed{U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} {\rm d}t'\, H_{\rm I}(t')\right)}\,. \tag{3.3.16}$$

This famous expression is called *Dyson's formula* (because it was first derived by Dirac).

Derivation.—Let us solve the Schrödinger equation (3.3.14) iteratively, starting with the assumption that $H_{\rm I}$ is a small perturbation. At zeroth order, we have $U(t,t_0)\approx 1$. We use this in the right-hand-side of (3.3.14), i.e. $i\partial_t U(t,t_0) \approx H_{\rm I}(t)$, and integrate, to get the first-order correction

$$U(t,t_0) \approx 1 - i\int_{t_0}^{t} {\rm d}t_1\, H_{\rm I}(t_1)\,. \tag{3.3.17}$$

Substituting the first-order solution into the right-hand side of the Schrödinger equation and integrating yields the second-order correction

$$\begin{aligned} U(t,t_0) \approx 1 &- i\int_{t_0}^{t} {\rm d}t_1\, H_{\rm I}(t_1) \\ &- \int_{t_0}^{t}{\rm d}t_1 \int_{t_0}^{t_1} {\rm d}t_2\, H_{\rm I}(t_1) H_{\rm I}(t_2)\,. \end{aligned} \tag{3.3.18}$$

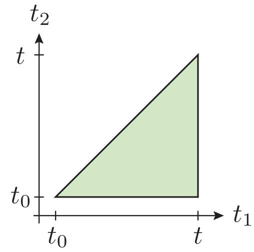
Similarly, the third-order solution is

$$\begin{aligned} U(t,t_0) \approx 1 &- i\int_{t_0}^{t} {\rm d}t_1\, H_{\rm I}(t_1) \\ &- \int_{t_0}^{t}{\rm d}t_1 \int_{t_0}^{t_1} {\rm d}t_2\, H_{\rm I}(t_1) H_{\rm I}(t_2) \\ &+ i\int_{t_0}^{t}{\rm d}t_1 \int_{t_0}^{t_1} {\rm d}t_2 \int_{t_0}^{t_2} {\rm d}t_3\, H_{\rm I}(t_1) H_{\rm I}(t_2) H_{\rm I}(t_3)\,. \end{aligned} \tag{3.3.19}$$

This procedure could be continued indefinitely.

The multiple integrals with a nested sequence of boundaries are a bit inconvenient. Luckily, we can write this in a nicer way. As an example, consider the quadratic term

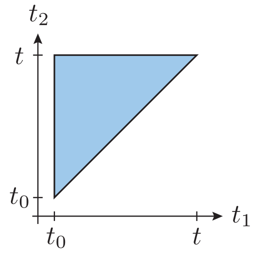
$$\mathcal{I}_2 \equiv -\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2\, H_\mathrm{I}(t_1) H_\mathrm{I}(t_2) \quad = \tag{3.3.20}$$



,

where the diagram illustrates the integration region. The integral can also be written as

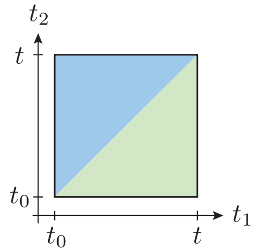
$$\mathcal{I}_2 = -\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_1}^{t} \mathrm{d}t_2\, H_\mathrm{I}(t_2) H_\mathrm{I}(t_1) \quad = \tag{3.3.21}$$



.

Crucially, in both versions of the integral, the factors of H_I in the integrands are *time ordered*. We can thus write the integral as the average of the two equivalent representations where the two triangular integration regions combine to a square

$$\mathcal{I}_2 = -\frac{1}{2} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2\, T\{H_\mathrm{I}(t_1) H_\mathrm{I}(t_2)\} \quad = \tag{3.3.22}$$



.

This can also be written as

$$\mathcal{I}_2 = \frac{1}{2} T \left(-i \int_{t_0}^{t} \mathrm{d}t'\, H_\mathrm{I}(t') \right)^2 . \tag{3.3.23}$$

All terms in the perturbative expansion of $U(t, t_0)$ are naturally time ordered, so the above procedure generalizes to higher orders. For the n-th order term, we get

$$\frac{1}{n!} T \left(-i \int_{t_0}^{t} \mathrm{d}t'\, H_\mathrm{I}(t') \right)^n . \tag{3.3.24}$$

The factor $1/n!$ arises because we are averaging over $n!$ equivalent integration regions in an n-dimensional hypercube. Summing all terms, we finally obtain the time evolution operator in terms of a time-ordered exponential

$$U(t, t_0) = T \exp \left(-i \int_{t_0}^{t} \mathrm{d}t'\, H_I(t') \right) , \tag{3.3.25}$$

which is the claimed solution (3.3.16).

Check.—It is easy to confirm by direct substitution that (3.3.16) indeed solves (3.3.14). First, note that everything within a time-ordered product commutes. Taking the derivative of (3.3.16) with respect to t, we find

$$i\frac{d}{dt}U(t,t_0) = i\frac{d}{dt}T\exp\left(-i\int_{t_0}^{t}\mathrm{d}t'\, H_I(t')\right) = T\left[H_\mathrm{I}(t)\exp\left(-i\int_{t_0}^{t}\mathrm{d}t'\, H_I(t')\right)\right]. \tag{3.3.26}$$

Next, we note that t is the latest time in the problem. The time ordering therefore puts $H_\mathrm{I}(t)$ to the left and we can pull it out of the time-ordered product

$$i\frac{d}{dt}U(t,t_0) = H_\mathrm{I}(t)\,T\exp\left(-i\int_{t_0}^{t}\mathrm{d}t'\, H_I(t')\right) = H_\mathrm{I}(t)\,U(t,t_0)\,. \tag{3.3.27}$$

This confirms that (3.3.16) is indeed a solution of (3.3.14).

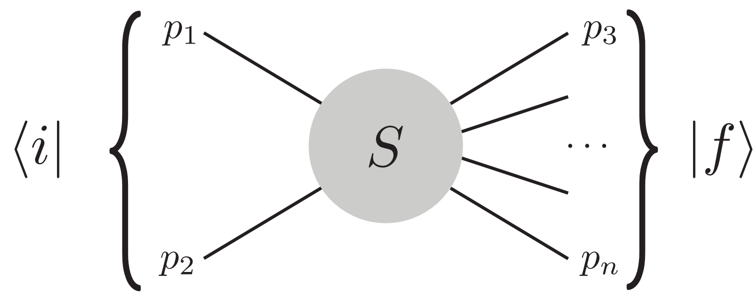
Although Dyson's formula for the evolution operator is beautifully compact, in practice, we only use its perturbative expansion

$$U(t,t_0) = 1 - i\int_{t_0}^{t}\mathrm{d}t'\, H_\mathrm{I}(t') - \int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\, H_\mathrm{I}(t')H_\mathrm{I}(t'') + \cdots\,. \tag{3.3.28}$$

To describe all of the processes covered in this course, we will never have to go beyond second order in the Dyson series.

3.4 Scattering Matrix

Particle colliders like the LHC perform scattering experiments:



Two bunches of particles are accelerated to very high velocities and made to collide. Whenever two particles from the bunches come very close, they produce some complicated interacting quantum state. This state evolves into several particles moving away in various directions. The outgoing particles of each scattering event are measured and recorded. With some luck this produces final states involving interesting new physics.

We will assume that the initial and final states are eigenstates of the free theory, i.e. $|i\rangle$ at $t\to-\infty$ and $|f\rangle$ at $t\to+\infty$ are taken to be eigenstates of H_0, not of the full H. Naively, this seems to be a harmless and reasonable assumption: At $t\to\pm\infty$, the particles are far separated and don't feel the effects of the mutual interactions. However, due to the uncertainty principle and the related existence of virtual particles, even a single particle, far from the interaction region, is never truly alone in QFT. In this course, we will close our eyes to this subtlety.

The amplitude for the state $|i\rangle$ to evolve into the state $|f\rangle$ is

$$\boxed{\langle f|S|i\rangle \equiv \lim_{t_\pm \to \pm\infty} \langle f|U(t_+, t_-)|i\rangle}\,, \tag{3.4.29}$$

where the unitary operator S, called the *S-matrix*, is defined as

$$S \equiv T \exp\left(-i \int_{-\infty}^{+\infty} \mathrm{d}t\, H_{\rm I}\right). \tag{3.4.30}$$

Conservation of energy and momentum means that the sums of the initial and final four-momenta, p_i and p_f, are equal. In the S-matrix elements (3.4.29) this is reflected by an overall delta function, i.e. we will always find

$$\langle f|S|i\rangle \equiv (2\pi)^4 \delta^{(4)}(p_f - p_i)\, i\mathcal{A}\,, \tag{3.4.31}$$

where $\mathcal{A}$ is called the *scattering amplitude* and the factor of i is conventional. In §3.10, we will see how to convert scattering amplitudes to physical observables, such as cross sections and decay rates. For now, we will consider $\mathcal{A}$ to be the main output of a scattering calculation.

3.5 Wick's Theorem

To compute the S-matrix elements, Dyson's formula will ask us to evaluate quantities like

$$\langle f|\, T\{\mathcal{H}_{\rm I}(x_1)\ldots\mathcal{H}_{\rm I}(x_n)\}|i\rangle\,. \tag{3.5.32}$$

This can quickly become very complicated, so in order to stay sane, we will first write terms like (3.5.32) in a simpler form.

Say we wish to calculate $2 \to 2$ scattering in ϕ^4 theory. In that case (3.5.32) becomes

$$\langle \mathbf{p}_3, \mathbf{p}_4|\, T\{\mathcal{H}_{\rm I}(x_1)\ldots\mathcal{H}_{\rm I}(x_n)\}|\mathbf{p}_1, \mathbf{p}_2\rangle \propto \langle 0|a_{\mathbf{p}_3} a_{\mathbf{p}_4}\, T\{\mathcal{H}_{\rm I}(x_1)\ldots\mathcal{H}_{\rm I}(x_n)\}\, a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2}|0\rangle \tag{3.5.33}$$

$$= \langle 0|\, T\{a_{\mathbf{p}_3} a_{\mathbf{p}_4}\, \mathcal{H}_{\rm I}(x_1)\ldots\mathcal{H}_{\rm I}(x_n)\, a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2}\}|0\rangle\,. \tag{3.5.34}$$

The interaction Hamiltonian contains products of the fields $\phi(x)$ made up of creation and annihilation operators. The strategy for evaluating terms like (3.5.33) is to first relate the time-ordered product in (3.5.32) to a normal-ordered product, i.e. we will try move all annihilation operators a to the right and all creation operators $a^\dagger$ to the left, picking up delta functions along the way. Wick's theorem provides a systematic way for doing that.

Let us recall the definition of normal ordering: We first split the field ϕ into pure creation operators ϕ^+ and pure annihilation operators ϕ^-, i.e. $\phi(x) = \phi^+(x) + \phi^-(x)$, where

$$\phi^+(x) \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}}\, a^\dagger_{\mathbf{p}}\, e^{+ip\cdot x}\,, \tag{3.5.35}$$

$$\phi^-(x) \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}}\, a_{\mathbf{p}}\, e^{-ip\cdot x}\,. \tag{3.5.36}$$

The normal ordering of a string of fields, $\phi(x_1)\phi(x_2)\ldots\phi(x_n)$, is defined such that all factors of ϕ^+ are to the left of all factors of ϕ^-. For example, consider the product of two fields:

$$\begin{aligned}\phi_1\phi_2 &= \left(\phi_1^+ + \phi_1^-\right)\left(\phi_2^+ + \phi_2^-\right) \\ &= \phi_1^+\phi_2^+ + \phi_1^-\phi_2^- + \phi_1^+\phi_2^- + \phi_1^-\phi_2^+\,,\end{aligned} \tag{3.5.37}$$

where I have introduced the shorthand $\phi_n \equiv \phi(x_n)$. The order of the first two terms on the right-hand-side is irrelevant, the third term is in normal order, but the fourth term is not. Interchanging the factors in the last term, $\phi_1^- \phi_2^+ \to \phi_2^+ \phi_1^-$, we arrive at the normal-ordered form of the product

$$N\{\phi_1\phi_2\} = \phi_1^+\phi_2^+ + \phi_1^-\phi_2^- + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- \,. \tag{3.5.38}$$

The time-ordering of the same product is

$$\begin{aligned} T\{\phi_1\phi_2\} &= \phi_1^+\phi_2^+ + \phi_1^-\phi_2^- \\ &\quad + \theta(t_1 - t_2)\left[\phi_1^+\phi_2^- + \phi_1^-\phi_2^+\right] \\ &\quad + \theta(t_2 - t_1)\left[\phi_2^+\phi_1^- + \phi_2^-\phi_1^+\right] . \end{aligned} \tag{3.5.39}$$

Subtracting (3.5.38) from (3.5.39), we get

$$\begin{aligned} T\{\phi_1\phi_2\} - N\{\phi_1\phi_2\} &= \theta(t_1 - t_2)[\phi_1^-, \phi_2^+] + \theta(t_2 - t_1)[\phi_2^-, \phi_1^+] \\ &= \theta(t_1 - t_2)D(x_1 - x_2) + \theta(t_2 - t_1)D(x_2 - x_1) \\ &= \Delta_F(x_1 - x_2)\,. \end{aligned} \tag{3.5.40}$$

We see that subtracting the normal-ordered product from the time-ordered product has given us the *Feynman propagator*.

Wick's theorem is a generalisation of this result to an arbitrary number of fields. To formulate the theorem, it is helpful to first introduce the concept of a *Wick contraction*. Such a contraction between two, not necessarily adjacent, fields $\phi_k \equiv \phi(x_k)$ and $\phi_l \equiv \phi(x_l)$ replaces the relevant two field operators by their Feynman propagator $\Delta_F(x_k - x_l)$:

$$\Big[\dots \phi_{k-1}\wick{\c\phi_k \phi_{k+1} \dots \phi_{l-1} \c\phi_l}\phi_{l+1}\dots\Big] = \Delta_F(x_k - x_l)\Big[\dots \phi_{k-1}\phi_{k+1}\dots\phi_{l-1}\phi_{l+1}\dots\Big] . \tag{3.5.41}$$

Wick's theorem then states that

$$\boxed{T\{\phi_1\phi_2\dots\phi_n\} = N\{\phi_1\phi_2\dots\phi_n + \text{all possible contractions}\}}\ . \tag{3.5.42}$$

This is best illustrated by examples:

$$T\{\phi_1\phi_2\} = N\{\phi_1\phi_2\} + \wick{\c\phi_1\c\phi_2} \tag{3.5.43}$$

$$T\{\phi_1\phi_2\phi_3\} = N\{\phi_1\phi_2\phi_3\} + \wick{\c\phi_1\c\phi_2\phi_3} + \wick{\c\phi_1\phi_2\c\phi_3} + \wick{\phi_1\c\phi_2\c\phi_3} \tag{3.5.44}$$

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} = N\{&\phi_1\phi_2\phi_3\phi_4 + \wick{\c\phi_1\c\phi_2\phi_3\phi_4} + \wick{\c\phi_1\phi_2\c\phi_3\phi_4} + \wick{\c\phi_1\phi_2\phi_3\c\phi_4} \\ &+ \wick{\phi_1\c\phi_2\c\phi_3\phi_4} + \wick{\phi_1\c\phi_2\phi_3\c\phi_4} + \wick{\phi_1\phi_2\c\phi_3\c\phi_4}\,\} \\ &+ \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} \end{aligned} \tag{3.5.45}$$

Since the contractions are c-numbers they can be pulled out of the normal-ordering operation, e.g. $N\{\wick{\c\phi_1\phi_2\c\phi_3}\phi_4\} = \wick{\c\phi_1\c\phi_3}N\{\phi_2\phi_4\}$.

Proof.—Wick's theorem is proved by induction:
- Assume that the statement holds for $n-1$ fields.
- Arrange n fields in proper time order $\phi_n \ldots \phi_1$ with $t_n > \ldots > t_1$.
- Consider $T\{\phi_n \ldots \phi_1\} = (\phi_n^+ + \phi_n^-)\, T\{\phi_{n-1} \ldots \phi_1\}$.
- Replace $T\{\phi_{n-1} \ldots \phi_1\}$ by contracted normal-ordered products.
- Since ϕ_n^+ is already in normal order, it can be pulled into $N\{\ldots\}$.
- Commute ϕ_n^- past all the remaining fields in the normal ordering.
- For every uncontracted field ϕ_k in $N\{\ldots\}$, pick up a factor of $-i\Delta_F(x_n - x_k)$.

Why is this re-writing of the time-ordered product of field operators useful? Recall that $\langle 0|N\{\text{anything}\}|0\rangle = 0$. This means that anything that is not fully contracted will not contribute to the vacuum expectation value of the time-ordered product. For example, in (3.5.45) only the last three terms survive

$$
\begin{aligned}
\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle &= \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} \\
&= \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\
&\quad + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)\,.
\end{aligned} \tag{3.5.46}
$$

Since we have an explicit form for the Feynman propagator, cf. (2.4.86), Wick's theorem gives a direct way to compute time-ordered correlators and S-matrix elements.

3.6 Scattering in ϕ^4 Theory

The last section was pretty abstract, so let's apply this to a concrete example, namely $2 \to 2$ scattering in ϕ^4 theory. Along the way, we will introduce an elegant way to describe the terms in the perturbative expansion of the S-matrix by diagrams, and will develop simple rules that associate amplitudes to each diagram.

The relativistically normalized initial and final states are

$$
\begin{aligned}
|i\rangle &= \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}\, a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2}|0\rangle \equiv |\mathbf{p}_1, \mathbf{p}_2\rangle\,, \\
|f\rangle &= \sqrt{4E_{\mathbf{p}_3}E_{\mathbf{p}_4}}\, a^\dagger_{\mathbf{p}_3} a^\dagger_{\mathbf{p}_4}|0\rangle \equiv |\mathbf{p}_3, \mathbf{p}_4\rangle\,,
\end{aligned} \tag{3.6.47}
$$

and the interaction Hamiltonian (density) is

$$
\mathcal{H}_\mathrm{I} = \frac{\lambda}{4!}\phi^4\,. \tag{3.6.48}
$$

We wish to evaluate the S-matrix order-by-order in an expansion in small λ:

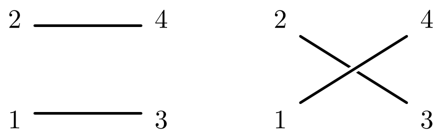
$$
\begin{aligned}
\langle f|S|i\rangle &= \langle \mathbf{p}_3, \mathbf{p}_4|\; T\exp\left(-i\int \mathrm{d}^4x\, \mathcal{H}_\mathrm{I}\right)|\mathbf{p}_1, \mathbf{p}_2\rangle \\
&= \langle \mathbf{p}_3, \mathbf{p}_4|\,\mathbf{p}_1, \mathbf{p}_2\rangle - i\,\frac{\lambda}{4!}\int \mathrm{d}^4x\;\langle \mathbf{p}_3, \mathbf{p}_4|\,T\{\phi^4(x)\}\,|\mathbf{p}_1, \mathbf{p}_2\rangle \\
&\qquad - \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y\,\langle \mathbf{p}_3, \mathbf{p}_4|\,T\{\phi^4(x)\phi^4(y)\}\,|\mathbf{p}_1, \mathbf{p}_2\rangle + \cdots\,, \\
&\equiv \sum_{n=0}^{\infty} \mathcal{S}_n\,,
\end{aligned} \tag{3.6.49}
$$

where $\mathcal{S}_n$ denotes the sum of all terms of order λ^n. Let us look at the individual terms in this expansion in turn:

- At order λ^0, we have

$$\begin{aligned}\mathcal{S}_0 &\equiv \langle \mathbf{p}_3, \mathbf{p}_4 |\, \mathbf{p}_1, \mathbf{p}_2\rangle \\ &= \sqrt{4E_{\mathbf{p}_3}E_{\mathbf{p}_4}4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}\,\langle 0|a_{\mathbf{p}_3}a_{\mathbf{p}_4}a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}|0\rangle \\ &= 4E_{\mathbf{p}_1}E_{\mathbf{p}_2}(2\pi)^6\left(\delta^{(3)}(\mathbf{p}_3-\mathbf{p}_1)\delta^{(3)}(\mathbf{p}_4-\mathbf{p}_2)+\delta^{(3)}(\mathbf{p}_3-\mathbf{p}_2)\delta^{(3)}(\mathbf{p}_4-\mathbf{p}_1)\right). \qquad (3.6.50)\end{aligned}$$

The delta functions force the final state to be identical to the initial state. This trivial part of the S-matrix corresponds to the free propagation of the particles without the interaction taking place. We can represent the two contributions in (3.6.50) by the following spacetime pictures:



We can think of these diagrams as representing particles travelling through spacetime with time running in the horizontal direction.

- At order λ, we need to evaluate

$$\langle \mathbf{p}_3, \mathbf{p}_4 |\, T\{\phi_x\phi_x\phi_x\phi_x\}\,|\mathbf{p}_1, \mathbf{p}_2\rangle\,. \qquad (3.6.51)$$

Using Wick's theorem, we can write this as

$$\langle \mathbf{p}_3, \mathbf{p}_4 |\, N\{\phi_x\phi_x\phi_x\phi_x + \text{contractions}\}\,|\mathbf{p}_1, \mathbf{p}_2\rangle\,. \qquad (3.6.52)$$

Since the external states are not $|0\rangle$, terms that are not fully contracted do not necessarily vanish: e.g. we can use an annihilation operator from ϕ_x to annihilate a particle in the initial state:

$$\begin{aligned}\phi^-(x)|\mathbf{p}\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\,a_\mathbf{k}e^{-ik\cdot x}\,\sqrt{2E_\mathbf{p}}\,a^\dagger_\mathbf{p}\,|0\rangle \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\,e^{-ik\cdot x}\,\sqrt{2E_\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})\,|0\rangle \\ &= e^{-ip\cdot x}\,|0\rangle\,. \qquad (3.6.53)\end{aligned}$$

An uncontracted ϕ-operator inside the normal-ordered product has two terms: ϕ^- on the far right and ϕ^+ on the far left. We get one contribution to the S-matrix element from each way of commuting the a of ϕ^- past an initial-state $a^\dagger$, and one contribution for each way of commuting the $a^\dagger$ of ϕ^+ past a final-state a. To keep track of this, we define contractions of the field operators with the external states:

$$\phi(x)|\mathbf{p}\rangle = e^{-ip\cdot x}|0\rangle\,, \qquad \langle\mathbf{p}|\phi(x) = \langle 0|e^{+ip\cdot x}\,. \qquad (3.6.54)$$

To evaluate the S-matrix element, we then simply write down all possible full contractions of the ϕ-operators *and* the external-state particles.

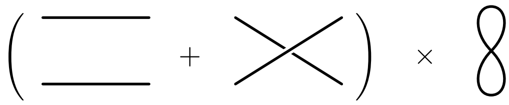
Let us apply this to (3.6.52). The normal-ordered product contains three type of terms:

$$\phi_x\phi_x\phi_x\phi_x\,, \quad \phi_x\phi_x\phi_x\phi_x\,, \quad \phi_x\phi_x\phi_x\phi_x\,. \tag{3.6.55}$$

The first term, with all ϕ-operators contracted with each other, corresponds to

$$\begin{aligned}\mathcal{S}_1^a &\equiv -i\,\frac{\lambda}{4!}\int \mathrm{d}^4x\,\langle \mathbf{p}_3,\mathbf{p}_4|\,\phi_x\phi_x\phi_x\phi_x\,|\mathbf{p}_1,\mathbf{p}_2\rangle \\ &= -i\,\frac{\lambda}{4!}\,\langle \mathbf{p}_3,\mathbf{p}_4|\,\mathbf{p}_1,\mathbf{p}_2\rangle \times \int \mathrm{d}^4x\,\Delta_F^2(x-x)\,. \end{aligned}\tag{3.6.56}$$

The corresponding spacetime picture is

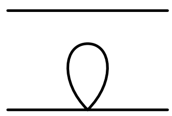


This is the trivial part of the S-matrix times a *vacuum bubble diagram*.

Next, we consider the second term in (3.6.55), in which two of the four ϕ-operators are contracted. The remaining two ϕ-operators are contracted with two of the external particles (one initial and one final). Finally, the last two external particles are contracted with each other. This leads to

$$\begin{aligned}\mathcal{S}_1^b &\equiv -i\,\frac{\lambda}{4!}\int \mathrm{d}^4x\,\langle \mathbf{p}_3,\mathbf{p}_4|\,\phi_x\phi_x\phi_x\phi_x\,|\mathbf{p}_1,\mathbf{p}_2\rangle \;+\; 3 \text{ other terms} \\ &= -i\,\frac{\lambda}{4!}\,\langle \mathbf{p}_4|\,\mathbf{p}_2\rangle \times \int \mathrm{d}^4x\,\Delta_F(x-x)\,e^{i(p_3-p_1)\cdot x} \;+\; 3 \text{ other terms}\,. \end{aligned}\tag{3.6.57}$$

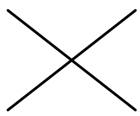
The spacetime picture corresponding to this expression is



$+\quad 3$ other diagrams.

Note that the integral over x in (3.6.57) produces a momentum-conserving delta function, so these diagrams again describe trivial processes in which the initial and final states are identical.

Next, we consider the first term in (3.6.55), where none of the ϕ-operators are contracted with each other. All ϕ-operators therefore must be contracted with external states. The corresponding spacetime diagram is



To evaluate this contribution, we contract two of the ϕ-operators with $|\mathbf{p}_1,\mathbf{p}_2\rangle$ and the other two with $\langle\mathbf{p}_3,\mathbf{p}_4|$. There are 4! ways of doing this. We therefore obtain

$$\begin{aligned}\mathcal{S}_1^c &\equiv 4! \times \left(-i\frac{\lambda}{4!}\right)\int \mathrm{d}^4x\;e^{i(p_3+p_4-p_1-p_2)\cdot x} \\ &= -i\lambda\,(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\,. \end{aligned}\tag{3.6.58}$$

Notice that the factor of 4! (corresponding to the number of permutations that give the same result) precisely canceled the 1/4! in the coupling. This is not a coincidence, but it is the reason the 1/4! was introduced in the Lagrangian in the first place. We see that the result (3.6.58) is of the form (3.4.31), with

$$i\mathcal{A}_1 = -i\lambda \,, \tag{3.6.59}$$

i.e. the tree-level scattering amplitude is simply given by the coupling λ.

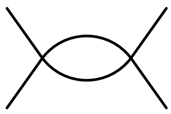
- At order λ^2, we need to evaluate

$$\langle \mathbf{p}_3, \mathbf{p}_4 | \, T\{(\phi\phi\phi\phi)_x(\phi\phi\phi\phi)_y\} \, |\mathbf{p}_1, \mathbf{p}_2\rangle \,. \tag{3.6.60}$$

Many different contractions are now possible. For example, we can have

$$\frac{1}{2}\left(-i\,\frac{\lambda}{4!}\right)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \; \langle \mathbf{p}_3, \mathbf{p}_4 | \, (\phi\phi\phi\phi)_x(\phi\phi\phi\phi)_y \, |\mathbf{p}_1, \mathbf{p}_2\rangle \,, \tag{3.6.61}$$

and permutations thereof. The spacetime picture corresponding to this contribution is



$$\tag{3.6.62}$$

Note that, in (3.6.61), we contracted one $\phi(y)$ with the external state $|\mathbf{p}_1\rangle$, and another $\phi(y)$ with $|\mathbf{p}_2\rangle$. There are $4 \times 3 = 12$ ways of doing this. Similarly, we contracted one $\phi(x)$ with $|\mathbf{p}_3\rangle$, and another $\phi(x)$ with $|\mathbf{p}_4\rangle$. There are again $4 \times 3 = 12$ ways of doing this. The remaining two $\phi(x)$ are contracted with the remaining $\phi(y)$. There are 2 ways of doing this. An equal contribution is obtained by exchanging $x \leftrightarrow y$. This gives an additional factor of 2. Adding all of this up, we get

$$\mathcal{S}_2 = \frac{(-i\lambda)^2}{2} \int \mathrm{d}^4x \int \mathrm{d}^4y \, \Big[\Delta_F(x-y)\Delta_F(x-y)\Big] \, e^{-i(p_1+p_2)\cdot y} \, e^{+i(p_3+p_4)\cdot x} \,. \tag{3.6.63}$$

We see that this time the factor of $(1/4!)^2$ in (3.6.61) didn't cancel completely when we added all equivalent permutations. The remaining factor of 1/2 is called a *symmetry factor*. This arises because the diagram in (3.6.62) has a discrete symmetry corresponding to the exchange of the two internal lines. Symmetry factors can usually be determined a priori by counting the number of discrete symmetries of the spacetime graph corresponding to a certain contribution of the Dyson expansion. Having said that, in this course, we won't obsess about symmetry factors. At most they will change our answers by small factors (like 2).

To evaluate (3.6.63), we substitute the momentum space representation of the Feynman propagator:

$$\begin{aligned}\mathcal{S}_2 = \frac{(-i\lambda)^2}{2} &\int \mathrm{d}^4x \int \mathrm{d}^4y \, e^{-i(p_1+p_2)\cdot y} \, e^{+i(p_3+p_4)\cdot x} \\ &\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \int \frac{\mathrm{d}^4\tilde{k}}{(2\pi)^4} \frac{i}{\tilde{k}^2 - m^2 + i\epsilon} e^{-i\tilde{k}\cdot(x-y)} \,. \end{aligned} \tag{3.6.64}$$

Integrating over x leads to $(2\pi)^4\delta^{(4)}(p_1+p_2+k+\tilde{k})$. Similarly, the integral over y gives $(2\pi)^4\delta^{(4)}(p_3+p_4+k+\tilde{k})$. This allows us to perform the integral over $\tilde{k}$. We obtain

$$\begin{aligned}\mathcal{S}_2 &= (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\\ &\quad\times\frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(p_1+p_2-k)^2-m^2+i\epsilon}\,. \end{aligned}\tag{3.6.65}$$

Stripping off the overall momentum-conserving delta function, we get the following result for the scattering amplitude

$$i\mathcal{A}_2=\frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(p_1+p_2-k)^2-m^2+i\epsilon}\,. \tag{3.6.66}$$

Notice that the amplitude involves an integral over the loop momentum k. At large internal momentum, the integral scales as $\int^\Lambda \mathrm{d}^4k/k^4$ and therefore diverges as $\log\Lambda$. Dealing with this divergence requires *renormalization*, which you will learn about in future courses.

3.7 Feynman Diagrams

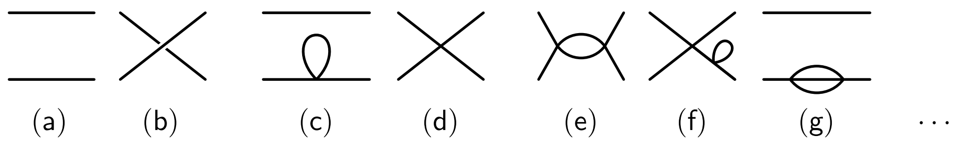
Using Wick's theorem to compute higher-order contributions to scattering amplitudes quickly becomes very complicated. Fortunately, there is a better way. This involves drawing *Feynman diagrams* to represent each term in the expansion of the S-matrix elements and assigning an amplitude to each diagram according to simple rules.

The recipe for drawing Feynman diagrams is:

- Draw an external line for each particle in the initial and final states.
- Join the external lines using the vertices of the interaction Hamiltonian.

Every diagram corresponds to a term in the expansion of $\langle f|S|i\rangle$.

The following are a few examples of Feynman diagrams for $2\to 2$ scattering in ϕ^4 theory up to order λ^2:



(3.7.67)

We will only consider so-called *connected* diagrams. Those are diagrams in which every part of the diagram is connected to at least one external leg, such as (d) and (e), but not (c) and (g), in (3.7.67). We will also not consider diagrams with loops on external lines, such as (f) in (3.7.67). Diagrams in which all loops on the external legs have been cut off are called *amputated*. Recall that we have taken the asymptotic in and out states to be states of the free theory. A proper treatment of the one-particle states in the interacting theory will remove all disconnected and un-amputated Feynman diagrams from the S-matrix. We will have to remove these contributions by hand.

3.8 Feynman Rules

Feynman introduced a wonderfully simple algorithm for computing all contributions to the S-matrix at a given order in perturbation theory by drawing all allowed (connected and amputated) Feynman diagrams and assigning amplitudes to them according to a certain set of Feynman rules.

Position-space rules. The following rules assign an S-matrix element to each diagram:

- Each vertex contributes a factor of $-i\lambda$.
- Each internal line (running from x to y) contributes a factor of $\Delta_F(x-y)$.
- Each external line contributes $e^{-ip\cdot x}$ (if incoming) or $e^{+ip\cdot x}$ (if outgoing).
- Integrate the positions of the vertices over all spacetime.
- Divide by the symmetry factor of each diagram.

Momentum-space rules. It is usually more convenient to work directly in momentum space. In particular, the following rules then allow us to write down expressions for scattering amplitudes directly:

- Each vertex contributes a factor of $-i\lambda$.
- Each internal line (with momentum k) contributes a factor of $\Delta_F(k)$.
- Each external line contributes a factor of 1.
- Impose momentum conservation at each vertex.
- Integrate over all unconstrained internal momenta.
- Divide by the symmetry factor of each diagram.

Doing the problems at the end of this chapter will give you some practice in applying the Feynman rules to a variety of scattering problems.

3.9 Scalar Yukawa Theory

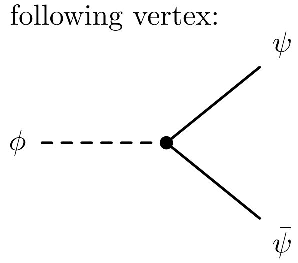
Tree level scattering in ϕ^4 theory is rather trivial as it includes only a single contact diagram. As a slightly more nontrivial example let us consider scattering in the scalar Yukawa theory (3.1.5):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - M^2\psi^*\psi - g\psi^*\psi\phi\,. \tag{3.9.68}$$

We now have two types of particles: "photons" ϕ and "electrons" ψ (as well as "positrons" which we will denote by $\bar{\psi}$). We will represent them by two types of lines:

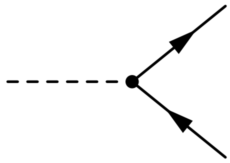
photon = , electron = . (3.9.69)

The interaction $\psi^*\psi\phi$ leads to the following vertex:



(3.9.70)

Instead of labelling the fermion legs, we can distinguish between electrons and positrons by following the flow of negative charge. We will do this be attaching arrows to the fermion legs. The vertex in (3.9.70) then becomes

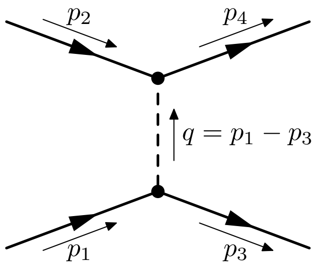


(3.9.71)

We can now combine the ingredients in (3.9.69) and (3.9.71) to draw Feynman diagrams for all kinds of scattering processes and then evaluate the corresponding amplitudes. We will study one example explicitly, then guess the Feynman rules of the theory and apply them to some further examples.

Electron scattering

Consider $\psi\psi \to \psi\psi$ scattering. At order g^2, we can have the following diagram:



(3.9.72)

This represents the scattering of two electrons mediated by the exchange of a photon with momentum $q = p_1 - p_3$. We would like to identify and evaluate the corresponding term in the Dyson series.

Let us write the initial and final states as

$$
\begin{aligned}
|i\rangle &= \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}\, b^\dagger_{\mathbf{p}_1} b^\dagger_{\mathbf{p}_2}|0\rangle \equiv |\mathbf{p}_1, \mathbf{p}_2\rangle\,, \\
|f\rangle &= \sqrt{4E_{\mathbf{p}_3}E_{\mathbf{p}_4}}\, b^\dagger_{\mathbf{p}_3} b^\dagger_{\mathbf{p}_4}|0\rangle \equiv |\mathbf{p}_3, \mathbf{p}_4\rangle\,.
\end{aligned}
\tag{3.9.73}
$$

The relevant term in the Dyson expansion of the S-matrix is

$$
\langle f|S|i\rangle \;=\; \frac{(-ig)^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y\; \langle \mathbf{p}_3, \mathbf{p}_4|\, T\{(\psi^\dagger\psi\phi)_x(\psi^\dagger\psi\phi)_y\}|\mathbf{p}_1, \mathbf{p}_2\rangle\,. \tag{3.9.74}
$$

The contraction corresponding the diagram in (3.9.72) is

$$
\langle \mathbf{p}_3, \mathbf{p}_4|\,(\psi^\dagger\psi\phi)_x(\psi^\dagger\psi\phi)_y|\mathbf{p}_1, \mathbf{p}_2\rangle \;+\; \{x \leftrightarrow y\}\,. \tag{3.9.75}
$$

Using

$$
\psi(x)|\mathbf{p}\rangle = e^{-ip\cdot x}|0\rangle\;, \qquad \langle\mathbf{p}|\psi^\dagger(x) = \langle 0|e^{+ip\cdot x}\;, \qquad \phi(x)\phi(y) = \Delta_F(x-y)\,, \tag{3.9.76}
$$

we get

$$
\begin{aligned}
\langle f|S|i\rangle \;\subset\; &(-ig)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y\; e^{-ip_1\cdot y - ip_2\cdot x}\, e^{+ip_3\cdot y + ip_4\cdot x}\, \Delta_F(x-y) \\
=\; &(-ig)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y\; e^{-i(p_1-p_3)\cdot y}\, e^{-i(p_2-p_4)\cdot x} \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{ie^{ik\cdot(x-y)}}{k^2-m^2}\,.
\end{aligned}
\tag{3.9.77}
$$

Performing the integrals over x and y, we get

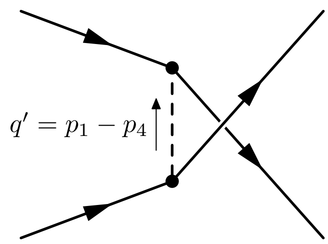
$$\begin{aligned}\langle f|S|i\rangle \subset (-ig)^2 &\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i(2\pi)^8}{k^2-m^2}\,\delta^{(4)}(p_1-p_3+k)\,\delta^{(4)}(p_2-p_4-k)\\ &= i(-ig)^2\,\frac{1}{(p_1-p_3)^2-m^2}\,(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\,.\end{aligned} \tag{3.9.78}$$

The corresponding scattering amplitude is

$$\mathcal{A}_t = (-ig)^2\,\frac{1}{(p_1-p_3)^2-m^2}\,, \tag{3.9.79}$$

where the subscript 't' will be explained in a moment.

Another diagram with the same external states is



(3.9.80)

I leave it to you to show that this diagram corresponds to the following contraction

$$\langle \mathbf{p}_3, \mathbf{p}_4|\,(\psi^\dagger\psi\phi)_x(\psi^\dagger\psi\phi)_y|\mathbf{p}_1,\mathbf{p}_2\rangle \,+\, \{x\leftrightarrow y\}\,, \tag{3.9.81}$$

and that the associated scattering amplitude is

$$\mathcal{A}_u = (-ig)^2\,\frac{1}{(p_1-p_4)^2-m^2}\,. \tag{3.9.82}$$

The total amplitude, at order g^2, then is

$$\mathcal{A} = (-ig)^2\left[\frac{1}{(p_1-p_3)^2-m^2}+\frac{1}{(p_1-p_4)^2-m^2}\right]. \tag{3.9.83}$$

Mandelstam variables.—The combination of momenta in (3.9.83) appears frequently in scattering calculation, so sometimes the following notation is used

$$s \equiv (p_1+p_2)^2\,, \tag{3.9.84}$$
$$t \equiv (p_1-p_3)^2\,, \tag{3.9.85}$$
$$u \equiv (p_1-p_4)^2\,, \tag{3.9.86}$$

where s, t, u are called the *Mandelstam variables*. The result in (3.9.83) can then be written as

$$\mathcal{A} = (-ig)^2\left[\frac{1}{t-m^2}+\frac{1}{u-m^2}\right], \tag{3.9.87}$$

and the two contributions are called *t-channel* and *u-channel*, respectively.

The rules that would have given us the amplitude (3.9.83) are straightforward generalizations of our previous Feynman rules:

- Each vertex contributes a factor of $-ig$.
- Each internal photon line contributes a factor of

$$\Delta_F(k) = \frac{i}{k^2 - m^2} ,$$

where k is the momentum flowing through the line.
- Each external electron line contributes a factor of 1.

We could now study more complicated diagrams to determine the complete set of Feynman rules of the theory. Instead, I will simply state the Feynman rules and then apply them to more examples.

Feynman rules

The complete set of Feynman rules for scalar Yukawa theory are:

- Each vertex contributes a factor of $-ig$.
- Each internal line contributes a factor of

$$\text{- - - - - - - -} \quad = \quad \frac{i}{k^2 - m^2} ,$$

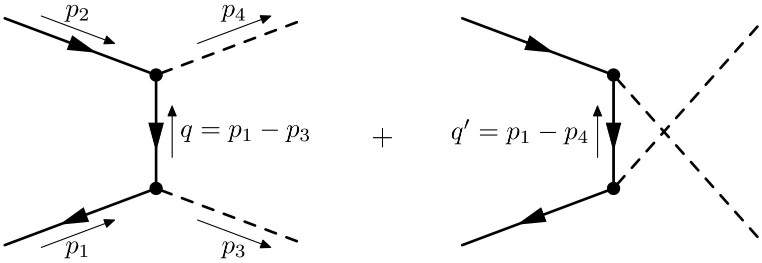
$$\text{―――――――} \quad = \quad \frac{i}{k^2 - M^2} ,$$

where k is the momentum flowing through the line.
- Each external line contributes a factor of 1.
- Impose momentum conservation at each vertex.
- Integrate over all unconstrained internal momenta.
- Divide by the symmetry factor of each diagram.

Further examples

To get some practice, let us now apply these rules to a few more scattering processes in scalar Yukawa theory:

- **Electron-to-photon scattering.** We first look at the annihilation of an electron-positron pair into two photons, $\psi\bar{\psi} \to \phi\phi$. The simplest Feynman diagrams for this process are

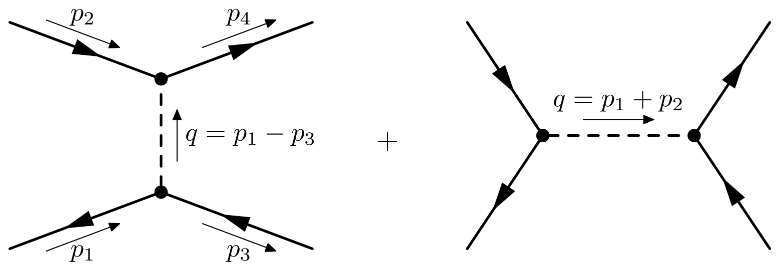


(3.9.88)

The virtual exchange particle in these diagrams is now the electron ψ. Otherwise, the computation is the same as in the previous example. The associated scattering amplitude is

$$\mathcal{A} = (-ig)^2 \left[\frac{1}{t - M^2} + \frac{1}{u - M^2} \right] . \tag{3.9.89}$$

- **Electron-positron scattering.** The scattering of an electron and a positron, $\psi\bar{\psi} \to \psi\bar{\psi}$, is represented by the following Feynman diagrams:



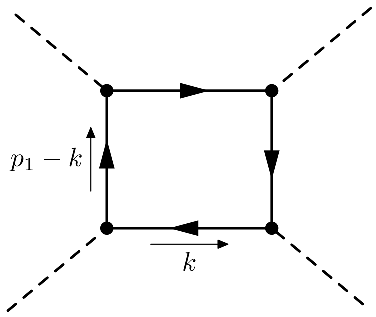
(3.9.90)

and the corresponding scattering amplitude is

$$\mathcal{A} = (-ig)^2 \left[\frac{1}{t - m^2} + \frac{1}{s - m^2 + i\epsilon} \right] . \tag{3.9.91}$$

The second contribution in (3.9.90) is called the *s-channel* contribution.

- **Photon scattering.** For $\phi\phi \to \phi\phi$, the simplest Feynman diagram involves a loop



(3.9.92)

The scattering amplitude is written as an integral over the loop momentum k:

$$\begin{aligned} \mathcal{A} = (-ig)^4 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, & \frac{1}{(k^2 - M^2 + i\epsilon)((k - p_1)^2 - M^2 + i\epsilon)} \\ & \frac{1}{((k - p_1 - p_2)^2 - M^2 + i\epsilon)((k - p_3)^2 - M^2 + i\epsilon)} . \end{aligned} \tag{3.9.93}$$

For large k, this integral scales as $\int \mathrm{d}^4 k / k^8$, which converges for $k \to \infty$.

3.10 Relation to Observables

Experiments measure *decay rates*, *cross sections* and *correlation functions*. In this section, we will provide the link between these observables and the scattering amplitudes that we computed above. In the interest of time, I will cite results without proof.

Phase space. Let the initial state have n particles. We will consider both $n = 1$ (decay) and $n = 2$ (scattering). This state evolves into a final state with N particles. In quantum mechanics, determining the probability of the process involves summing over all final states. This includes integrating over the allowed momenta of the particles in the final state. To write this momentum integral it is convenient to define the *Lorentz-invariant phase space element*

$$\mathrm{d}\Pi \equiv (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{n=1}^{N} \frac{\mathrm{d}^3 p_n}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}_n}} \,, \tag{3.10.94}$$

where p_f and p_i are the total four-momentum of the final and initial states, respectively.

Example.—Consider the decay of a particle of mass M into two particles of masses m_1 and m_2. Since the phase space element is Lorentz invariant, we can compute it any frame. The simplest choice is the rest frame of the initial particle. In that case,

$$\mathrm{d}\Pi = (2\pi)^4 \delta(M - E_1 - E_2) \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2) \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \tag{3.10.95}$$

$$= \frac{1}{(2\pi)^2} \frac{1}{4E_1 E_2} \delta(M - E_1 - E_2)\, \mathrm{d}^3 p_1 \,, \tag{3.10.96}$$

where we have used the delta function $\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2)$ to perform the integration over $\mathrm{d}^3 p_2$. Writing $\mathrm{d}^3 p_1 = p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega$, we get

$$\mathrm{d}\Pi = \frac{1}{(2\pi)^2} \mathrm{d}\Omega \int_0^\infty \frac{1}{4E_1 E_2}\, p_1^2 \mathrm{d}p_1\, \delta\Big(M - \sqrt{p_1^2 + m_1^2} - \sqrt{p_1^2 + m_2^2}\Big) \,. \tag{3.10.97}$$

Using $\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|$, where x_0 is the zero of $f(x)$, we find

$$\mathrm{d}\Pi = \frac{1}{32\pi^2 M^2} \Big[M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)\Big]^{1/2} \mathrm{d}\Omega \,. \tag{3.10.98}$$

In the limit $m_1 = m_2 \equiv m$, this reduces to

$$\mathrm{d}\Pi = \frac{1}{32\pi^2} \sqrt{1 - \frac{4m^2}{M^2}}\, \mathrm{d}\Omega \,. \tag{3.10.99}$$

This result assumes that the two particles in the final state are distinguishable. Otherwise, the phase space is reduced by a factor of 1/2!.

Decay rate. Consider the decay of a particle of mass M. The decay rate in the rest frame of the particle is

$$\Gamma = \frac{1}{2M} \sum_f \int |\mathcal{A}|^2 \, \mathrm{d}\Pi \,. \tag{3.10.100}$$

The inverse of the decay rate gives the half-life of the particle, $\tau = 1/\Gamma$. In a general frame, the factor of $1/2M$ in (3.10.100) gets replaced by $1/2E_{\mathbf{p}}$. For a particle with velocity v, we have $E_{\mathbf{p}} = \gamma M$, where $\gamma(v)$ is the Lorentz factor. The decay rate then increases by a factor of γ as expected by relativistic time dilation.

Cross section. The LHC collides beams of particles. The beams have a *luminosity* L, defined as the flux of particles per unit area per unit time. Sometimes the particles in the beams will scatter off each other, sometimes they will pass through each other. The rate of scattering is

$$R = \sigma L \,, \tag{3.10.101}$$

where σ is the (scattering) cross section. QFT allows us to compute σ in terms of the scattering amplitude $\mathcal{A}$. The cross section for $2 \to N$ scattering can be written as

$$\sigma = \frac{1}{4I} \sum_f \int |\mathcal{A}|^2 \, \mathrm{d}\Pi \,, \tag{3.10.102}$$

where $I \equiv \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}$ is a flux factor defined in terms of the four-momenta and masses of the particles in the initial state. It is convenient to work in the *center-of-mass frame*, where the incoming particles have four-momenta $p_1 = (E_1, \mathbf{p})$ and $p_2 = (E_2, -\mathbf{p})$, with $E_i^2 = \mathbf{p}^2 + m_i^2$. The center-of-mass energy is $E_1 + E_2 = \sqrt{s}$ and the flux factor is $I = |\mathbf{p}|\sqrt{s}$. In this course, we will always have $N = 2$ particles in the final state. We denote their four-momenta by p_3 and p_4. In the center-of-frame, we have $p_3 = (E_3, \mathbf{p}')$ and $p_3 = (E_4, -\mathbf{p}')$. Energy conservation implies

$$|\mathbf{p}'| = \frac{1}{2\sqrt{s}} \Big[s^2 + (m_3^2 - m_4^2)^2 - 2s(m_3^2 + m_4^2) \Big]^{1/2} . \tag{3.10.103}$$

The Lorentz-invariant phase space element is

$$\mathrm{d}\Pi = \frac{1}{(2\pi)^2} \frac{|\mathbf{p}'|}{4\sqrt{s}} \, \mathrm{d}\Omega \,, \tag{3.10.104}$$

where $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi$. The differential cross section $\mathrm{d}\sigma$ per unit solid angle $\mathrm{d}\Omega$ therefore becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}'|}{|\mathbf{p}|} |\mathcal{A}|^2 . \tag{3.10.105}$$

If all four particles are identical (or at least are equal-mass), then this becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{A}|^2}{64\pi^2 s} . \tag{3.10.106}$$

This simple form of the differential cross section applies to many important examples.

Correlation functions. For many applications in cosmology and condensed matter physics we are interested in (time-ordered) correlation functions and not in scattering amplitudes. A general relation between correlation functions and S-matrix elements was obtained by Lehmann, Symanzik and Zimmermann and is known as the *LSZ reduction formula*:

$$\begin{aligned} G^{(n+2)}(p_a, p_b, q_1, \dots, q_n) &\equiv \prod_{i=a,b} \int \mathrm{d}^4 x_i \, e^{ip_i x_i} \prod_{j=1}^{n} \int \mathrm{d}^4 y_j \, e^{iq_j y_j} \, \langle \Omega | T(\phi_a \phi_b \phi_1 \dots \phi_n) | \Omega \rangle \\ &= \prod_{i=a,b} \frac{i}{p_i^2 - m^2} \prod_{j=1}^{n} \frac{i}{q_j^2 - m^2} \, \langle q_1, \dots, q_n | \, S \, | p_a, p_b \rangle . \end{aligned} \tag{3.10.107}$$

For a proper derivation and further discussion of this result I refer you to the book by Peskin and Schroeder (or your favourite equivalent).

3.11 From Amplitudes to Forces

In the non-relativistic limit, the results of our scattering calculations should reproduce the corresponding results of non-relativistic quantum mechanics, where the interaction between particles is described by a potential $V(\mathbf{r})$. In the *Born approximation*, the scattering amplitude for a particle with incoming momentum $\mathbf{p}$ and outgoing momentum $\mathbf{p}'$ is:

$$\langle \mathbf{p}'|V(\mathbf{r})|\mathbf{p}\rangle = -i\int \mathrm{d}^3r\, V(\mathbf{r})\, e^{-i\mathbf{q}\cdot\mathbf{r}} = -iV(\mathbf{q})\,, \tag{3.11.108}$$

where $\mathbf{q} \equiv \mathbf{p}-\mathbf{p}'$ is the momentum transfer. We would like to compare this to the non-relativistic limit of our QFT calculations.

As a concrete example, consider the case of “electron”-“electron” scattering in scalar Yukawa theory, $\psi\psi \to \psi\psi$, whose amplitude was given in (3.9.87):

$$\mathcal{A} = (-ig)^2\left[\frac{1}{t-m^2} + \frac{1}{u-m^2}\right]. \tag{3.11.109}$$

In the center-of-mass frame, we write $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$ and $\mathbf{p}' = \mathbf{p}_3 = -\mathbf{p}_4$, with $|\mathbf{p}| = |\mathbf{p}'| \ll M$ in the non-relativistic limit. The amplitude (3.11.109) then becomes

$$\mathcal{A} \approx g^2\left[\frac{1}{(\mathbf{p}-\mathbf{p}')^2+m^2} + \frac{1}{(\mathbf{p}+\mathbf{p}')^2+m^2}\right]. \tag{3.11.110}$$

Taking into account that there is a relative factor of $(2M)^2$ between $\mathcal{A}$ and $\langle \mathbf{p}|V(\mathbf{r})|\mathbf{p}'\rangle$ (due to the relativistic normalization of $|\mathbf{p}_1,\mathbf{p}_2\rangle$), we have $V(\mathbf{q}) = -\mathcal{A}/(2M)^2$. Notice that the two contributions in (3.11.110) are related by $\mathbf{p}' \to -\mathbf{p}'$. Focusing on the first term, we get

$$V(\mathbf{q}) = -\frac{\lambda^2}{|\mathbf{q}|^2+m^2}\,, \tag{3.11.111}$$

where we have defined the dimensionless coupling $\lambda \equiv g/2M$. Performing the inverse Fourier transform, we get

$$V(\mathbf{r}) = -\frac{\lambda^2}{4\pi|\mathbf{r}|}e^{-m|\mathbf{r}|}\,. \tag{3.11.112}$$

This is the *Yukawa potential* describing the force mediated by a particle of mass m. We see that, for $m \neq 0$, the force has a finite range $r \sim m^{-1}$.

Derivation.—The inverse Fourier transform of (3.11.111) can be written as

$$\begin{aligned} V(\mathbf{r}) = -\int \frac{\mathrm{d}^3q}{(2\pi)^3}\, e^{i\mathbf{q}\cdot\mathbf{r}}\, \frac{\lambda^2}{|\mathbf{q}|^2+m^2} &= -\frac{\lambda^2}{(2\pi)^2}\int_0^\infty \mathrm{d}q\, \frac{q^2}{q^2+m^2}\frac{2\sin(qr)}{qr} \\ &= -\frac{\lambda^2}{(2\pi)^2 r}\int_{-\infty}^{+\infty}\mathrm{d}q\, \frac{q\sin(qr)}{q^2+m^2} \\ &= -\frac{\lambda^2}{2\pi r}\mathrm{Re}\left[\int_{-\infty}^{+\infty}\frac{\mathrm{d}q}{2\pi i}\, \frac{qe^{iqr}}{q^2+m^2}\right]. \end{aligned} \tag{3.11.113}$$

We compute this last integral by closing the contour in the upper half plane $q \to i\infty$, picking up the pole at $q = +im$. This gives

$$V(\mathbf{r}) = -\frac{\lambda^2}{4\pi r}e^{-mr}\,, \tag{3.11.114}$$

which is the result (3.11.112).

The negative sign in (3.11.112) means that the corresponding force is attractive. The same conclusion would have been reached for "electron"-"anti-electron" scattering (see §3.12). This is a special feature of scattering mediated by scalars: the force is universally attractive. In real electrodynamics, the force is mediated by a spin-1 particle (the photon) and the force is attractive or repulsive depending on the charges of the particles. Interestingly, forces mediated by spin-2 particles—e.g. gravity—are again universally attractive.

3.12 Problems

1. Consider $2 \to 2$ scattering. The particles in the initial state have masses m_1, m_2 and four-momenta p_1, p_2, while those in the final state have masses m_3, m_4 and four-momenta p_3, p_4. The Mandelstam variables are $s \equiv (p_1 + p_2)^2$, $t \equiv (p_1 - p_3)^2$ and $u \equiv (p_1 - p_4)^2$.

i) What values of s are physical?

ii) Show that

$$s + t + u = \sum_{i=1}^{4} m_i^2 .$$

iii) Show that the following relations hold in the center-of-mass frame:

$$|\mathbf{p}_1| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2} ,$$
$$|\mathbf{p}_3| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_3^2 + m_4^2)s + (m_3^2 - m_4^2)^2} ,$$
$$t = m_1^2 + m_3^2 - 2E_1E_3 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta ,$$

where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$. Use this to show that the differential cross section can be written as

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s|\mathbf{p}_1|^2}|\mathcal{A}|^2 .$$

Explain how this result can be used to compute the total cross section.

2. Consider ϕ^3 theory given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3 .$$

i) Draw all connected and amputated Feynman diagrams for the process $\phi\phi \to \phi\phi$, up to order λ^4. Describe the Feynman rules of the theory and use them to compute the scattering amplitude at order λ^2. Express your result in terms of the Mandelstam variables s, t and u.

- Show that the following relations hold in the center-of-mass frame

$$t = -\frac{1}{2}(s - 4m^2)(1 - \cos\theta) ,$$
$$u = -\frac{1}{2}(s - 4m^2)(1 + \cos\theta) ,$$

where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$.

- Show that the non-relativistic limit $|\mathbf{p}_1| \ll m$ corresponds to $s - 4m^2 \ll m^2$, and that the scattering amplitude in that limit is

$$\mathcal{A} \approx \frac{5}{3}\frac{\lambda^2}{m^2}\left[1 - \frac{8}{15}\frac{s - 4m^2}{m^2} + \frac{5}{18}\left(1 + \frac{27}{25}\cos^2\theta\right)\left(\frac{s - 4m^2}{m^2}\right)^2 + \cdots\right] .$$

- Show that the result in the ultra-relativistic limit, $s \gg m^2$, is

$$\mathcal{A} \approx \frac{\lambda^2}{s\sin^2\theta}\left[3 + \cos^2\theta - \left(\frac{(3 + \cos^2\theta)^2}{\sin^2\theta} - 16\right)\frac{m^2}{s} + \cdots\right] .$$

Comment on the behaviour of this result for $\theta = 0$ and $\theta = \pi$.

- Derive the total cross section in the non-relativistic limit.

*ii**) Draw the Feynman diagrams for the tree-level and one-loop contributions to the decay $\phi \to \phi\phi$ and determine the corresponding amplitudes.

Now start over and write down the one-loop result in position space, in terms of integrals over the intermediate points and Wick contractions, represented by factors of Δ_F. Show that this reduces to the previous answer after you apply the LSZ formula.

3. Consider the following theory

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2 + (\partial\Phi)^2 - M^2\Phi^2\right] + g\phi^2\Phi\,,$$

where ϕ and Φ are real scalar fields.

i) Compute the s-channel amplitude for the process $\phi\phi \to \phi\phi$ at tree level. How do you interpret the result for $M > 2m$?

ii) Draw the Feynman diagram corresponding to the one-loop correction to the propagator of Φ. This can be interpreted as a correction to the mass of Φ. Show that the result develops an imaginary part for $M > 2m$. The *optical theorem* states that this imaginary part is related to decay rate of the process $\Phi \to \phi\phi$:

$$\Gamma = \frac{1}{M}\,\mathrm{Im}[\mathcal{A}]\,.$$

Hint: You may use without proof that $\mathrm{Im}[(x+i\epsilon)^{-1}] = -\pi\delta(x)$.

iii) Verify the correctness of the above relation by computing Γ explicitly, using

$$\Gamma = \frac{1}{2M}\int \frac{|\mathcal{A}|^2}{32\pi^2}\sqrt{1-4m^2/M^2}\,\mathrm{d}\Omega\,.$$

4. Using the Born approximation, derive the leading-order potential for $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering in scalar Yukawa theory. Compare this to the potential for $\psi\psi \to \psi\psi$ scattering. What does your finding tell you about the nature of scalar interactions?

Compute the leading-order potential for $\phi\phi \to \phi\phi$ scattering in ϕ^4 theory. What is the physical meaning of your result?

4 Fermions

In Chapter 2, we have seen that the quantization of scalar fields leads to spin-0 particles. Much of the complexity of the world, however, arises from particles with spin: for example, the structure of the periodic table is a consequence of electrons being spin-$\frac{1}{2}$ particles and forces are mediated by spin-1 particles.

In this chapter, we will introduce spinor fields whose quantization will lead to spin-$\frac{1}{2}$ particles and anti-particles. The Pauli exclusion principle will arise as a consequence of the anti-commutation relations satisfied by these spinors. In the next chapter, we will show that spin-1 particles, like the photon, are quantum excitations of vector fields.

4.1 Lorentz Representations

Particles can be identified as irreducible unitary representations of the Poincaré group. That's a lot of jargon! To unpack these words, we will have to take a small detour into group theory (see Appendix A for further background).

Groups and representations A *group* is a set of elements $\{g_i\}$ with a composition rule $g_i \times g_j = g_k$. A *representation* of the group is a particular embedding of the group elements g_i into operators that act on a vector space. For finite-dimensional representation, this corresponds to a mapping of the g_i onto matrices. Often people refer to the vectors on which the matrices act as being the representations, but strictly speaking the matrix embedding is the representation. A representation is called *irreducible* if there is no subset of group elements that transform among themselves.

Vector representation Recall that the Lorentz group is the set of rotations and boosts that preserve the Minkowski metric, $\Lambda^T \eta \Lambda = \eta$. Under these transformations, spacetime four-vectors transform as

$$V_\mu \to \Lambda^\nu{}_\mu V_\nu \,. \tag{4.1.1}$$

The matrices Λ in these transformations define the so-called *vector representation* of the Lorentz group. Explicitly, the matrices corresponding to rotations around the x_1, x_2 and x_3-axes are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} , \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} , \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 & 0 \\ 0 & -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{4.1.2}$$

while boosts in the x_1, x_2 and x_3 directions are represented by

$$\begin{pmatrix} \cosh\beta_1 & \sinh\beta_1 & 0 & 0 \\ \sinh\beta_1 & \cosh\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cosh\beta_2 & 0 & \sinh\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\beta_2 & 0 & \cosh\beta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cosh\beta_3 & 0 & 0 & \sinh\beta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta_3 & 0 & 0 & \cosh\beta_3 \end{pmatrix}. \tag{4.1.3}$$

For infinitesimal rotation angles θ_i and boost rapidities β_i, the transformation (4.1.1) can be written as

$$\begin{aligned} \delta V_0 &= \beta_i V_i \,, \\ \delta V_i &= \beta_i V_0 - \epsilon_{ijk}\theta_j V_k \,. \end{aligned} \tag{4.1.4}$$

Alternatively, we can write this as

$$\delta V_\mu = i\left[\sum_{i=1}^{3}\theta_i (J_i)^\nu{}_\mu + \beta_i (K_i)^\nu{}_\mu\right] V_\nu \,, \tag{4.1.5}$$

where

$$J_1 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_3 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1.6}$$

$$K_1 = i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1.7}$$

The matrices J_i and K_i are the *generators* of the Lorentz group in the vector representation. They generate the group in the sense that any element of the group can be written as

$$\Lambda = \exp(i\theta_i J_i + i\beta_i K_i) \,. \tag{4.1.8}$$

This result is general: the group elements can always be written as an exponential of matrices. The matrices will simply be different for the different representations of the group.

It is easy to check that the generators J_i and K_i satisfy the *Lorentz algebra* $SO(1,3)$:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \,, \tag{4.1.9}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k \,, \tag{4.1.10}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \,. \tag{4.1.11}$$

Again, this result is general and holds for all representations of the Lorentz group.

It is sometimes useful to combine the rotation and boost generators into the following matrix

$$V^{\rho\sigma} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}. \tag{4.1.12}$$

Note that each element of $V^{\rho\sigma}$ is itself a 4×4 matrix, e.g. $V^{23} = J_1$. A Lorentz transformation can then be written as

$$\Lambda_V = \exp\left(\frac{1}{2}\theta_{\rho\sigma}V^{\rho\sigma}\right), \tag{4.1.13}$$

for six numbers $\theta_{\rho\sigma}$. Equation (4.1.13) is equivalent to (4.1.8) if we make the identifications $\theta_{ij} = -\epsilon_{ijk}\theta_k$ and $\theta_{0i} = -\beta_i$.

Exercise.—Show that the elements of the matrices $V^{\rho\sigma}$ can be written as

$$(V^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}, \tag{4.1.14}$$

and that the Lorentz algebra implies

$$[V^{\mu\nu}, V^{\rho\sigma}] = i(\eta^{\nu\rho}V^{\mu\rho} - \eta^{\mu\rho}V^{\nu\sigma} - \eta^{\nu\sigma}V^{\mu\rho} + \eta^{\mu\sigma}V^{\nu\rho}). \tag{4.1.15}$$

Note that although we arrived at this result in the Weyl representation, it holds generally.

General representations Consider the following linear combinations of the Lorentz generators

$$J_i^{\pm} \equiv \frac{1}{2}(J_i \pm iK_i). \tag{4.1.16}$$

In terms of these generators, the Lorentz algebra becomes

$$[J_i^+, J_j^+] = i\epsilon_{ijk}J_k^+, \tag{4.1.17}$$

$$[J_i^-, J_j^-] = i\epsilon_{ijk}J_k^-, \tag{4.1.18}$$

$$[J_i^+, J_j^-] = 0. \tag{4.1.19}$$

We see that the Lorentz algebra has two commuting $SU(2)$ subalgebras. The representations of $SO(1,3)$ can therefore be determined as representations of $SU(2)\oplus SU(2)$. The representations of $SU(2) = SO(3)$ are discussed in introductory quantum mechanics courses (although maybe not in this language). There you would have learned that each irreducible representation of $SU(2)$ is labelled by a half-integer quantum number j. These representations act on a vector space with $2j+1$ elements. The representations of the Lorentz group can therefore be labelled by two half-integer quantum numbers j^+ and j^-. The (j^+, j^-) representation has $(2j^+ + 1)(2j^- + 1)$ degrees of freedom. The rotation group $SO(3)$ is a subgroup of the Lorentz group. Each representation of the Lorentz group should therefore also be labelled by the total spin j. Writing the generator of $SO(3)$ as $J_i = J_i^+ + J_i^-$, we see that the possible values of j are constrained by the rules for adding spins in quantum mechanics. This gives $j = |j^+ - j^-|, |j^+ - j^-| + 1, \ldots, j^+ + j^-$. For example, the vector representation[1] V_μ corresponds to an irreducible $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2) \oplus SU(2)$, but is a reducible representation of the subgroup $SO(3)$, with $j = 0, 1$. We will encounter this fact again in §5.2.

Spinor representation Spin-$\frac{1}{2}$ particles will be associated with the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group. To construct these representations, we need to find 2×2 matrices

[1] I have done it! I just referred to the vector, not the matrix embedding as the representation.

that satisfy $SU(2)$ algebra. These are the Pauli matrices

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.1.20}$$

I invite you to show that these matrices satisfy $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. We therefore make the following identifications:

$$(\tfrac{1}{2}, 0): \qquad J_i^+ = \tfrac{1}{2}\sigma_i\,, \quad J_i^- = 0\,, \tag{4.1.21}$$

$$(0, \tfrac{1}{2}): \qquad J_i^+ = 0\,, \qquad J_i^- = \tfrac{1}{2}\sigma_i\,. \tag{4.1.22}$$

For the original Lorentz generators, this implies

$$(\tfrac{1}{2}, 0): \qquad J_i = \tfrac{1}{2}\sigma_i\,, \quad K_i = \tfrac{i}{2}\sigma_i\,, \tag{4.1.23}$$

$$(0, \tfrac{1}{2}): \qquad J_i = \tfrac{1}{2}\sigma_i\,, \quad K_i = -\tfrac{i}{2}\sigma_i\,. \tag{4.1.24}$$

The elements of the vector space on which the spin-$\frac{1}{2}$ representations act are called *spinors*. The $(\frac{1}{2}, 0)$ spinors are called[2] *left-handed Weyl spinors* and are denoted by ψ_L. Similarly, $(0, \frac{1}{2})$ spinors are called *right-handed Weyl spinors* and are denoted by ψ_R. These two types of spinors transform as

$$(\tfrac{1}{2}, 0): \qquad \psi_L \to \Lambda_L \psi_L = \exp\left\{\tfrac{1}{2}(i\theta_j\sigma_j - \beta_j\sigma_j)\right\}\psi_L\,,$$

$$(0, \tfrac{1}{2}): \qquad \psi_R \to \Lambda_R \psi_R = \exp\left\{\tfrac{1}{2}(i\theta_j\sigma_j + \beta_j\sigma_j)\right\}\psi_R\,,$$

where the rotation angles θ_i and the boost angles β_i are real numbers. For infinitesimal transformations, we have

$$\delta\psi_L = \tfrac{1}{2}(i\theta_j - \beta_j)\sigma_j\,\psi_L\,, \tag{4.1.25}$$

$$\delta\psi_R = \tfrac{1}{2}(i\theta_j + \beta_j)\sigma_j\,\psi_R\,. \tag{4.1.26}$$

In the next section, we will construct the unique Lorentz-invariant Lagrangian for the fields $\psi_L(x)$ and $\psi_R(x)$ that propagates the right number of degrees of freedom, namely two.

4.2 Weyl Spinors

A natural guess for the Lagrangian of a Weyl spinor would be

$$\mathcal{L} \stackrel{?}{=} \partial_\mu\psi_R^\dagger\partial^\mu\psi_R - m^2\psi_R^\dagger\psi_R\,. \tag{4.2.27}$$

However, it is easy to see, using (4.1.26) and (4.1.25), that these terms are not Lorentz invariant. For example,

$$\begin{aligned}\delta(\psi_R^\dagger\psi_R) &= \frac{1}{2}\psi_R^\dagger[(i\theta_i + \beta_i)\sigma_i\psi_R] + \frac{1}{2}[\psi_R^\dagger(-i\theta_i + \beta_i)\sigma_i]\psi_R \\ &= \beta_i\psi_R^\dagger\sigma_i\psi_R \neq 0\,. \end{aligned} \tag{4.2.28}$$

[2]The reason for calling these spinors "left-handed" and "right-handed" will become clear later.

To write a Lorentz-invariant mass term, we must combined the two spinors ψ_R and ψ_L. This allows us to write

$$\mathcal{L} \subset -m\left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\right), \tag{4.2.29}$$

which is Lorentz invariant since

$$\delta(\psi_L^\dagger \psi_R) = \frac{1}{2}\psi_L^\dagger[(i\theta_i + \beta_i)\sigma_i \psi_R] + \frac{1}{2}[\psi_L^\dagger(-i\theta_i - \beta_i)\sigma_i]\psi_R \,=\, 0\,. \tag{4.2.30}$$

For the kinetic terms, we could try

$$\mathcal{L} \overset{?}{\subset} \partial_\mu \psi_L^\dagger \partial^\mu \psi_R + \partial_\mu \psi_R^\dagger \partial^\mu \psi_L\,. \tag{4.2.31}$$

However, this would simply lead to the Lagrangian of four coupled scalars, which is not what we are looking for.

Consider instead $V_R^\mu \equiv \psi_R^\dagger \sigma^\mu \psi_R$, with $\sigma^\mu \equiv (1, \sigma^i)$. I claim that this transforms as a four-vector. We already determined, in (4.2.28), that $\delta V_{R,0} = \beta_i V_{R,i}$. It therefore just remains to show that V_R^i transforms appropriately. Using (4.1.26), we find

$$\begin{aligned} \delta V_{R,i} &= \frac{1}{2}\psi_R^\dagger \sigma_i[(i\theta_j + \beta_j)\sigma_j \psi_R] + \frac{1}{2}[\psi_R^\dagger(-i\theta_j + \beta_j)\sigma_j]\sigma_i \psi_R \\ &= \frac{\beta_j}{2}\psi_R^\dagger(\sigma_i\sigma_j + \sigma_j\sigma_i)\psi_R + \frac{i\theta_j}{2}\psi_R^\dagger(\sigma_i\sigma_j - \sigma_j\sigma_i)\psi_R \\ &= \beta_i \psi_R^\dagger \psi_R - \theta_j \epsilon_{ijk} \psi_R^\dagger \sigma_k \psi_R\,, \\ &= \beta_i V_{R,0} - \theta_j \epsilon_{ijk} V_{R,k}\,. \end{aligned} \tag{4.2.32}$$

Comparison with (4.1.4) shows that V_R^μ indeed transforms as a four-vector. A Lorentz-invariant kinetic term therefore is[3]

$$\mathcal{L} \subset i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R\,, \tag{4.2.33}$$

where we have included a factor of i to make the term Hermitian, i.e.

$$(i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R) = -i(\partial_\mu \psi_R)^\dagger \sigma^\mu \psi_R = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R\,. \tag{4.2.34}$$

A similar exercise shows that $\psi_L^\dagger \bar\sigma^\mu \psi_L$, with $\bar\sigma^\mu \equiv (1, -\sigma^i)$, transforms as a four-vector. The kinetic term for the left-handed Weyl spinor therefore is

$$\mathcal{L} \subset i\psi_L^\dagger \bar\sigma^\mu \partial_\mu \psi_L\,. \tag{4.2.35}$$

Combining the above terms, we find that the complete Lagrangian is

$$\boxed{\mathcal{L} = i\psi_L^\dagger \bar\sigma^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)}\,. \tag{4.2.36}$$

In the limit $m \to 0$, the left-handed and right-handed Weyl spinors evolve independently. The corresponding equations of motion are

$$\sigma^\mu \partial_\mu \psi_R = 0\,, \tag{4.2.37}$$

$$\bar\sigma^\mu \partial_\mu \psi_L = 0\,. \tag{4.2.38}$$

These are the *Weyl equations* for right-handed and left-handed spinors. In momentum space, these equations can be written as

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\,\psi_R = \psi_R\,, \quad \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\,\psi_L = -\psi_L\,. \tag{4.2.39}$$

This shows that right-handed and left-handed Weyl spinors are eigenstates of the *helicity* operators $h \equiv \boldsymbol{\sigma}\cdot\mathbf{p}/|\mathbf{p}|$, with eigenvalues $+1$ and -1, respectively.

[3]We didn't use $\partial_\mu(\psi_R^\dagger \sigma^\mu \psi_R)$ because it is a total derivative.

4.3 Dirac Equation

The Lagrangian (4.2.36) can be written in a simpler form if we combine the two Weyl spinors ψ_L and ψ_R into the four-component *Dirac spinor*

$$\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} . \tag{4.3.40}$$

If we also define the following gamma matrices

$$\gamma^\mu \equiv \begin{pmatrix} & \sigma_\mu \\ \bar{\sigma}_\mu & \end{pmatrix} \qquad \Leftrightarrow \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \tag{4.3.41}$$

and the Dirac adjoint $\bar{\psi} \equiv \psi^\dagger \gamma^0 = (\psi_R^\dagger , \psi_L^\dagger)$, then equation (4.2.36) becomes[4]

$$\boxed{\mathcal{L}_{\rm Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi} . \tag{4.3.42}$$

The corresponding equation of motion is the famous *Dirac equation*

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0} . \tag{4.3.43}$$

Notice that the Dirac equation is first order in derivatives, but still Lorentz invariant. It's the magic of the gamma matrices.

Slash notation.—Contractions with the gamma matrices are so common that it will be useful to introduce the shorthand $\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$. The Dirac equation then reads $(i\partial\!\!\!/ - m)\psi = 0$.

Degrees of freedom.—How many degrees of freedom are described by the Dirac spinor ψ? Naively, we count 8 degrees of freedom associated with the 4 complex components of ψ. But this is not the right way to count. In classical mechanics, the number of degrees of freedom is half the dimension of the phase space. A real scalar field ϕ has a single degree of freedom corresponding to one type of particle after quantization. A complex field has two degrees of freedom, a particle and an antiparticle in the quantum theory. The counting changes for the Dirac spinor since its equation of motion is first order rather than second order. The momentum conjugate to ψ is

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger . \tag{4.3.44}$$

Crucially, it is not proportional to $\dot{\psi}$. The phase space for a spinor is therefore parameterized by ψ and $\psi^\dagger$, while for a scalar it is parameterized by ϕ and $\pi = \dot{\phi}$. The phase space of the Dirac spinor therefore has 8 real dimensions and the number of real degrees of freedom is therefore 4. After quantization, this counting manifests itself as two degrees of freedom (spin up and down) for the particle, and two for the corresponding anti-particle.

[4] It should really be $\bar{\psi}(i\gamma^\mu \partial_\mu - m \times \mathbf{1})\psi$, where $\mathbf{1}$ is the 4×4 identity matrix. We will throughout suppress writing the identity matrix.

Exercise.—Show that the gamma matrices defined in (4.3.41) satisfy the following *Clifford algebra*

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,, \tag{4.3.45}$$

where I haven't written the 4×4 identity matrix on the right-hand side. Note that although we arrived at (4.3.45) in the *Weyl representation*, it holds generally.

Note that the Dirac equation mixes the different components of ψ through the matrices γ^μ. However, each individual component still solves the Klein-Gordon equation. To see this, we act with $(i\gamma^\nu\partial_\nu + m)$ on the Dirac equation

$$\begin{aligned} 0 = (i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi &= -(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi \\ &= -\left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2\right)\psi \\ &= -(\partial_\mu\partial^\mu + m^2)\psi\,, \end{aligned} \tag{4.3.46}$$

where we have used (4.3.45) in the final equality. Since the Klein-Gordon equation has no γ^μ matrices, it applies to each component of ψ^α, with $\alpha = 1, 2, 3, 4$.

4.3.1 Meaning of Spin-$\frac{1}{2}$

In §4.1, we showed that the transformation of Weyl spinors is

$$\psi_L(x) \;\to\; \psi'_L(x) = \Lambda_L\, \psi(\Lambda_V^{-1}x)\,, \tag{4.3.47}$$

$$\psi_R(x) \;\to\; \psi'_R(x) = \Lambda_R\, \psi(\Lambda_V^{-1}x)\,, \tag{4.3.48}$$

where $\Lambda_{R,L} \equiv \exp\left\{\frac{1}{2}(i\theta_j\sigma_j \pm \beta_j\sigma_j)\right\}$ in the Weyl representation and $\Lambda_V = \exp(i\theta_j J_j + i\beta_j K_j)$ in the vector representation. The action on a Dirac spinor then is

$$\psi(x) \;\to\; \psi'(x) = \Lambda_S\, \psi(\Lambda_V^{-1}x)\,, \quad \Lambda_S \equiv \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}. \tag{4.3.49}$$

We showed, in §4.1, that the vector representation of the Lorentz transformation can be written as

$$\Lambda_V = \exp\left(\frac{1}{2}\theta_{\rho\sigma}V^{\rho\sigma}\right), \tag{4.3.50}$$

where $(V^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$. Similarly, the matrices Λ_D can be written as

$$\Lambda_S = \exp\left(\frac{1}{2}\theta_{\rho\sigma}S^{\rho\sigma}\right), \tag{4.3.51}$$

where we have defined

$$S^{\rho\sigma} \equiv \frac{1}{4}[\gamma^\rho, \gamma^\sigma]\,, \tag{4.3.52}$$

and made the following identifications: $\theta_{ij} = -\epsilon_{ijk}\theta_k$ and $\theta_{0i} \equiv -\beta_i$. Although we have derived (4.3.51) in the Weyl representation, this expression holds in general. Note that in order for (4.3.50) to be a consistent transformation, the transformation parameters $\theta_{\rho\sigma}$ must be identical in Λ_V and Λ_S.

Exercise.—Show that $\bar{\psi}\psi$ and $\bar{\psi}\partial\!\!\!/\psi$ are invariant under the transformation (4.3.49). This confirms that the Dirac Lagrangian is indeed Lorentz invariant.

It is interesting to consider the special case of spatial rotations. The generators of rotations are

$$S^{ij} = \frac{1}{2}\gamma^i\gamma^j = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} . \tag{4.3.53}$$

We then find

$$\Lambda_S = \exp\left(\frac{1}{2}\theta_{ij}S^{ij}\right) = \begin{pmatrix} \exp(+\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}) & 0 \\ 0 & \exp(+\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}) \end{pmatrix} . \tag{4.3.54}$$

Consider now a rotation by $\Delta\theta = 2\pi$ around the x^3-axis, i.e. choose $\boldsymbol{\theta} = (0, 0, 2\pi)$. This gives

$$\Lambda_D = \begin{pmatrix} \exp(+i\pi\sigma^3) & 0 \\ 0 & \exp(+i\pi\sigma^3) \end{pmatrix} = -1 \,, \tag{4.3.55}$$

and, hence,

$$\psi(x) \,\to\, -\psi(x) \,. \tag{4.3.56}$$

We need to rotate by $\Delta\theta = 4\pi$ in order for ψ to return to its original state. This is what will make the excitations of the field ψ particles of spin 1/2.

4.3.2 Symmetries

The Dirac equation contains a number of important symmetries. In this section, I will briefly review these symmetries and compute the corresponding Noether currents.

Spacetime symmetries

We first consider spacetime symmetries:

- **Translations.** Consider the spacetime translation $x^\mu \to (x')^\mu = x^\mu + \epsilon^\mu$. In §1.2, we showed that a scalar field ϕ transforms as $\delta\phi = -\epsilon^\mu\partial_\mu\phi$ under this transformation. A spinor ψ transforms in the same way

  $$\delta\psi = -\epsilon^\mu\partial_\mu\psi \,. \tag{4.3.57}$$

  Following the Noether procedure, cf. eq. (1.2.44), leads to the following energy-momentum tensor: $T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}$. In fact, imposing the equation of motion (4.3.43) allows us to set $\mathcal{L} = 0$ and we get

  $$\boxed{T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi} \,. \tag{4.3.58}$$

  The total energy is

  $$E = \int \mathrm{d}^3x\, T^{00} = \int \mathrm{d}^3x\, i\bar{\psi}\gamma^0\dot{\psi} = \int \mathrm{d}^3x\, \psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi \,, \tag{4.3.59}$$

  where in the last equality we have again used (4.3.43).

- **Lorentz transformations.** Consider the Lorentz transformation

$$x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu x^\nu = (\delta^\mu_\nu + \omega^\mu{}_\nu)x^\nu \,. \tag{4.3.60}$$

In §1.2, we showed that a scalar field ϕ transforms as $\delta\phi = -\omega^\mu{}_\nu \, x^\nu \partial_\mu \phi$ under this transformation. The transformation of a Dirac spinor ψ will receive an extra contribution from the factor $\Lambda_S = \exp(\frac{1}{2}\theta_{\rho\sigma}S^{\rho\sigma})$ in (4.3.49):

$$\delta\psi^\alpha = -\omega^\mu{}_\nu x^\nu \partial_\mu \psi^\alpha + \frac{1}{2}\theta_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta \, \psi^\beta \,, \tag{4.3.61}$$

where we have introduced the spinor indices for clarity. Using that $\omega^{\mu\nu} \equiv \frac{1}{2}\theta_{\rho\sigma}(V^{\rho\sigma})^{\mu\nu} = \theta^{\mu\nu}$, we can write this as

$$\delta\psi^\alpha = -\omega_{\rho\sigma}\left[x^\sigma \partial^\rho \psi^\alpha - \frac{1}{2}(S^{\rho\sigma})^\alpha{}_\beta \, \psi^\beta\right] . \tag{4.3.62}$$

The associated Noether current is

$$\boxed{(\mathcal{J}^\mu)^{\rho\sigma} = (x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}) - i\bar\psi\gamma^\mu S^{\rho\sigma}\psi} \,. \tag{4.3.63}$$

The first term is the same as for a scalar field, cf. eq. (1.2.61), but the second term is new. This new term is what makes the excitations of ψ particles of spin 1/2.

Internal symmetries

The Dirac equation also has two important internal symmetries:

- **Phase rotations.** It is easy to see that the Dirac Lagrangian is invariant under a rotation of the phase of the spinor,

$$\psi(x) \to e^{i\alpha}\psi(x) \,. \tag{4.3.64}$$

The associated Noether current is

$$\boxed{J^\mu_V = \bar\psi\gamma^\mu\psi} \,, \tag{4.3.65}$$

where the subscript V stands for "vector". To check that this current is indeed conserved, we use the equations of motion:

$$\begin{aligned} \partial_\mu J^\mu_V &= (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu\partial_\mu\psi \\ &= (im\bar\psi)\psi + \bar\psi(-im\psi) = 0 \,. \end{aligned} \tag{4.3.66}$$

The conserved charge is

$$Q = \int \mathrm{d}^3x \, \bar\psi\gamma^0\psi = \int \mathrm{d}^3x \, \psi^\dagger\psi \,, \tag{4.3.67}$$

where we have used $(\gamma^0)^2 = 1$ in the second equality.

- **Chiral rotations.** In the limit $m = 0$, the Dirac Lagrangian inherits an additional internal symmetry

$$\psi(x) \to e^{i\alpha\gamma^5}\psi(x) \,, \tag{4.3.68}$$

where[5]

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{4.3.69}$$

This *chiral transformation* rotates left-handed and right-handed fermions in opposite directions. The associated *axial vector current* is

$$\boxed{J^\mu_A = \bar\psi\gamma^\mu\gamma^5\psi} \, , \tag{4.3.70}$$

where the subscript A stands for "axial". The equations of motion, for general m, imply

$$\begin{aligned} \partial_\mu J^\mu_A &= (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi \\ &= 2im\,\bar\psi\gamma^5\psi \, , \end{aligned} \tag{4.3.71}$$

which indeed vanishes for $m = 0$.

Discrete symmetries*

The Dirac equation also has a few discrete symmetries. Although we won't be discussing these symmetries very much in the rest of the course, let me mention them here for completeness:

- **Charge conjugation.** Consider the following transformation of the Dirac spinor

$$C : \quad \psi \to -i\gamma_2\,\psi^* \, . \tag{4.3.72}$$

This transformation is called *charge conjugation.* As we will see below, it *takes particles to antiparticles and flips the spin.* In the Weyl representation, we have $\gamma_2^* = -\gamma_2$ and $\gamma_2^T = \gamma_2$, so that

$$C : \quad \psi^* \to -i\gamma_2\psi \, , \tag{4.3.73}$$

which implies that $C^2 = 1$. The transformation of the Dirac mass term $\bar\psi\psi$ is

$$\begin{aligned} C : \quad \bar\psi\psi &\to (-i\gamma_2\psi)^T\gamma_0(-i\gamma_2\psi^*) \\ &= -\psi^T\gamma_2^T\gamma_0\gamma_2\psi^* = -\psi^T\gamma_0\psi^* = -(\gamma_0)_{\alpha\beta}\psi_\alpha\psi^*_\beta \, . \end{aligned} \tag{4.3.74}$$

The last expression is equal to $\bar\psi\psi$ if (and only if) $\psi_\alpha\psi^*_\beta = -\psi^*_\beta\psi_\alpha$. We will show below that spinors are indeed anticommuting fields. We have thus found that

$$C : \quad \bar\psi\psi \to \bar\psi\psi \, . \tag{4.3.75}$$

A similar argument shows that

$$C : \quad \bar\psi\partial\!\!\!/\psi \to \bar\psi\partial\!\!\!/\psi \, . \tag{4.3.76}$$

This proves that the Dirac Lagrangian (4.3.43) is invariant under charge conjugation.

- **Parity.** Consider the spatial parity transformation, $\mathbf{x} \to \mathbf{x}' = -\mathbf{x}$. This is a symmetry of the Dirac Lagrangian if the Dirac spinor transforms as

$$P : \quad \psi(t,\mathbf{x}) \to \gamma_0\,\psi(t,-\mathbf{x}) \, . \tag{4.3.77}$$

[5]The 5 on γ^5 is a relic of an old notation in which γ^0 was called γ^4.

This transformation exchanges ψ_R and ψ_L. It is easy to the parity transformation (4.3.77) leaves the Dirac mass term $\bar{\psi}\psi$ invariant:

$$P: \quad \bar{\psi}\psi(t,-\mathbf{x}) \to \psi^\dagger\gamma_0\gamma_0\gamma_0\psi(t,-\mathbf{x}) = \psi^\dagger\gamma_0\psi(t,-\mathbf{x}) = \bar{\psi}\psi(t,-\mathbf{x})\,, \tag{4.3.78}$$

where we have used that $(\gamma_0)^2 = 1$. Similarly, one can show that

$$P: \quad \bar{\psi}\gamma_0\psi(t,-\mathbf{x}) \to +\bar{\psi}\gamma_0\psi(t,-\mathbf{x})\,, \tag{4.3.79}$$

$$P: \quad \bar{\psi}\gamma_i\psi(t,-\mathbf{x}) \to -\bar{\psi}\gamma_i\psi(t,-\mathbf{x})\,. \tag{4.3.80}$$

This means that $\bar{\psi}\gamma_\mu\psi$ transforms as a four-vector and hence the Dirac Lagrangian is parity invariant.

- **Time reversal.** The most subtle of the discrete symmetries is time reversal. The problem is that we need to make $i\bar{\psi}\partial\!\!\!/\psi$ invariant. However, since $T: \partial_t \to -\partial_t$, we must have $T: \bar{\psi}\gamma^0\psi = \psi^\dagger\psi \to -\psi^\dagger\psi$, i.e. a positive definite quantity must change sign. This is impossible for a linear transformation. Time reversal is therefore defined as an anti-linear transformation, so that

$$T: \quad i \to -i\,. \tag{4.3.81}$$

This fixes the problem because of the i in the kinetic term. In the Weyl representation, T acts on the gamma matrices as follows

$$T: \quad \gamma_{0,1,3} \to \gamma_{0,1,3}\,, \quad \gamma_2 \to -\gamma_2\,, \tag{4.3.82}$$

since only γ_2 is imaginary. Under time reversal, solutions of the Dirac equation should then be mapped to solutions of the complex conjugated equation. In the Weyl representation, this is achieved by asking the Dirac spinors to transform as follows under time reversal

$$T: \quad \psi(t,\mathbf{x}) \to \gamma_1\gamma_3\,\psi(-t,\mathbf{x})\,. \tag{4.3.83}$$

As we will see, this *flips the spin of particles, but does not turn particles into antiparticles.* Note that T reverses momentum because $\mathbf{p} = i\boldsymbol{\nabla}$ and T flips the sign of i. Time reversal therefore makes particles go forward in time, but with their momenta and spins flipped.

- **CPT.** The combination of charge conjugation, parity and time reversal is called CPT. This transformation flips the sign of all spacetime coordinates and performs a complex conjugation. One can show the combined action of C, P and T on Dirac spinors is

$$CPT: \quad \psi(x) \to -i\gamma_0\gamma_1\gamma_2\gamma_3\,\psi^*(-x) = -\gamma_5\,\psi^*(-x)\,, \tag{4.3.84}$$

and that the Dirac Lagrangian is invariant under this transformation. This invariance is a special case of a deeper fact: the *CPT-theorem* states that *all* relativistic QFTs must be invariant under the CPT-transformation, although they don't have to be invariant under C, P and T separately.

4.3.3 Plane Wave Solutions

Before quantizing the Dirac equation, let us find its classical solutions. Since the spinor ψ also has to satisfy the Klein-Gordon equation, we must be able to write the solution of the Dirac equation as a linear combination of plane waves. We therefore make the following ansatz

$$\psi(x) = u(\mathbf{p})e^{-ipx}\,, \tag{4.3.85}$$

where $u(\mathbf{p})$ is a four-component basis spinor that is independent of x, but depends on the three-momentum $\mathbf{p}$. The function (4.3.85) is called a "positive frequency" solution, because $\psi \propto \exp(-iEt)$. Inserting this ansatz into the Dirac equation, we obtain

$$(\not{p} - m)u(\mathbf{p}) = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} u(p) = 0\,, \tag{4.3.86}$$

where $\not{p} \equiv \gamma^\mu p_\mu$. Writing $u \equiv (u_1, u_2)^T$, where u_1 and u_2 are two-component spinors, we get

$$(p \cdot \sigma)u_2 = m u_1\,, \tag{4.3.87}$$

$$(p \cdot \bar{\sigma})u_1 = m u_2\,, \tag{4.3.88}$$

where $p \cdot \sigma \equiv p_\mu \sigma^\mu$ and $p \cdot \bar{\sigma} \equiv p_\mu \bar{\sigma}^\mu$. These equations aren't independent from each other, since

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_0^2 - p_i p_j \sigma^i \sigma^j = p_0^2 - p_i p_j \delta^{ij} = p_\mu p^\mu = m^2\,. \tag{4.3.89}$$

Consider the ansatz $u_1 = (p \cdot \sigma)\xi'$, for some spinor ξ'. Equation (4.3.88) then immediately tells us that $u_2 = m\xi'$. Any spinor of the form

$$u(\mathbf{p}) = A \begin{pmatrix} (p \cdot \sigma)\xi' \\ m\xi' \end{pmatrix}, \tag{4.3.90}$$

with constant A, is therefore a solution of (4.3.86). To make this more symmetric, we choose $A = 1/m$ and $\xi' = \sqrt{p \cdot \bar{\sigma}}\,\xi$, with constant spinor ξ, whose normalization is $\xi^\dagger \xi = 1$. We then get

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\,\xi \\ \sqrt{p \cdot \bar{\sigma}}\,\xi \end{pmatrix}, \tag{4.3.91}$$

where the square root of a matrix can be defined by changing to the diagonal basis, taking the (positive) square root of the eigenvalues, and then changing back to the original basis.

Further solutions to the Dirac equation follow from the ansatz

$$\psi(x) = v(\mathbf{p})e^{+ipx}\,, \tag{4.3.92}$$

where $v(\mathbf{p})$ is a second set of four-component basis spinors. These solutions oscillate as $\psi \propto \exp(+iEt)$ and are therefore called "negative frequency" solutions. Note that both (4.3.92) and (4.3.85) are classical solutions with *positive* total energy (4.3.59). The Dirac equation requires that the spinor $v(\mathbf{p})$ satisfies

$$(\not{p} + m)v(\mathbf{p}) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} v(\mathbf{p}) = 0\,, \tag{4.3.93}$$

Following the same reasoning as above, it is easy to show that this equation is solved by

$$v(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\,\eta \\ -\sqrt{p \cdot \bar{\sigma}}\,\eta \end{pmatrix}, \tag{4.3.94}$$

where η is an arbitrary two-compenent spinor with normalization $\eta^\dagger \eta = 1$.

Spin-up and spin-down

In the rest frame, $p^\mu = (m, 0, 0, 0)$, the solutions (4.3.91) and (4.3.94) reduce to

$$u(\mathbf{p}) = \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix}, \quad v(\mathbf{p}) = \sqrt{m}\begin{pmatrix}\eta \\ -\eta\end{pmatrix}. \tag{4.3.95}$$

The meaning of the spinors ξ and η becomes apparent when we act with the generator of rotations (4.3.54) on the solutions. We see that ξ and η transform under rotations as ordinary two-component spinors and therefore determine the spin orientation of the solution. For example, we can choose the eigenstates of σ^3 as a basis for the two-component spinors

$$\xi_\uparrow = \begin{pmatrix}1 \\ 0\end{pmatrix}, \quad \xi_\downarrow = \begin{pmatrix}0 \\ 1\end{pmatrix}, \quad \eta_\uparrow = \begin{pmatrix}1 \\ 0\end{pmatrix}, \quad \eta_\downarrow = \begin{pmatrix}0 \\ 1\end{pmatrix}. \tag{4.3.96}$$

Four linearly independent solutions to the Dirac equation are then given by

$$u_\uparrow = \sqrt{m}\begin{pmatrix}1 \\ 0 \\ 1 \\ 0\end{pmatrix}, \quad u_\downarrow = \sqrt{m}\begin{pmatrix}0 \\ 1 \\ 0 \\ 1\end{pmatrix}, \quad v_\uparrow = \sqrt{m}\begin{pmatrix}1 \\ 0 \\ -1 \\ 0\end{pmatrix}, \quad v_\downarrow = \sqrt{m}\begin{pmatrix}0 \\ 1 \\ 0 \\ -1\end{pmatrix}. \tag{4.3.97}$$

After quantization, these solutions will become spin-up and spin-down particles and antiparticles, respectively.

Performing a boost along the z-direction, with $p^\mu = (E, 0, 0, p_z)$, the solutions in (4.3.97) become

$$u_\uparrow = \begin{pmatrix}\sqrt{E - p_z} \\ 0 \\ \sqrt{E + p_z} \\ 0\end{pmatrix}, \quad u_\downarrow = \begin{pmatrix}0 \\ \sqrt{E - p_z} \\ 0 \\ \sqrt{E + p_z}\end{pmatrix}, \tag{4.3.98}$$

$$v_\uparrow = \begin{pmatrix}\sqrt{E - p_z} \\ 0 \\ -\sqrt{E + p_z} \\ 0\end{pmatrix}, \quad v_\downarrow = \begin{pmatrix}0 \\ \sqrt{E - p_z} \\ 0 \\ -\sqrt{E + p_z}\end{pmatrix}. \tag{4.3.99}$$

In the ultra-relativistic limit, or for massless particles, we get $p_z = \pm E$ and the solutions will only have one non-zero component. For example, for $p_z = E$, we get

$$u_\uparrow = \sqrt{2E}\begin{pmatrix}0 \\ 0 \\ 1 \\ 0\end{pmatrix}, \quad u_\downarrow = \sqrt{2E}\begin{pmatrix}0 \\ 0 \\ 0 \\ 1\end{pmatrix}, \quad v_\uparrow = \sqrt{2E}\begin{pmatrix}0 \\ 0 \\ -1 \\ 0\end{pmatrix}, \quad v_\downarrow = \sqrt{2E}\begin{pmatrix}0 \\ 0 \\ 0 \\ -1\end{pmatrix}. \tag{4.3.100}$$

These solution correspond to spin-up and spin-down right-handed Weyl spinors. For $p_z = -E$, we would have gotten the corresponding left-handed spinors.

Helicity

The solutions in (4.3.100) are eigenstates of the *helicity operator*

$$h \equiv \frac{i}{2}\epsilon_{ijk}\hat{p}^i S^{ij} = \frac{1}{2}\hat{p}_i\begin{pmatrix}\sigma^i & 0 \\ 0 & \sigma^i\end{pmatrix}, \tag{4.3.101}$$

where S^{ij} are the rotation generators defined in (4.3.52). The field corresponding to $u_\uparrow$ has helicity 1/2 and is said to be right-handed, while the one corresponding to $u_\downarrow$ has helicity $-1/2$ and is called left-handed. Notice that the helicity of a massive particle depends on the frame of reference, since one can always boost to a frame in which its momentum is in the opposite direction, but its spin is unchanged. For a massless particle which travels at the speed of light one cannot perform such a boost. This also explains the origin of the notation $\psi_{L,R}$ for Weyl spinors. The solutions of the Weyl equations (4.2.38) and (4.2.37) are states of definite helicity, corresponding to left- and right-handed particles, respectively.

Inner and outer products

Let us derive a few technical results that will be useful later. For convenience, we define the basis spinors ξ_s and η_s, with $s = 1, 2$, such that

$$\xi_s^\dagger \xi_{s'} = \delta_{ss'} \,, \quad \eta_s^\dagger \eta_{s'} = \delta_{ss'} \,. \tag{4.3.102}$$

Consider then the following *inner product* of the four-component basis spinors

$$\begin{aligned}
\bar{u}_s(\mathbf{p}) u_{s'}(\mathbf{p}) = u_s^\dagger(\mathbf{p}) \gamma^0 u_{s'}(\mathbf{p}) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_{s'} \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_{s'} \end{pmatrix} \\
&= \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}^\dagger \begin{pmatrix} \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & \\ & \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} \end{pmatrix} \begin{pmatrix} \xi_{s'} \\ \xi_{s'} \end{pmatrix} .
\end{aligned} \tag{4.3.103}$$

Since this expression is Lorentz invariant, we can evaluate it in the rest frame. This leads to

$$\bar{u}_s(\mathbf{p}) u_{s'}(\mathbf{p}) = 2m\, \delta_{ss'} \,, \tag{4.3.104}$$

where we have used $\xi_s^\dagger \xi_{s'} = \delta_{ss'}$. A similar analysis for $v_s(p)$ gives

$$\bar{v}_s(\mathbf{p}) v_{s'}(\mathbf{p}) = -2m\, \delta_{ss'} \,. \tag{4.3.105}$$

It will also be useful to compute the following non-Lorentz-invariant inner product

$$u_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}^\dagger \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_{s'} \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_{s'} \end{pmatrix} = 2E_{\mathbf{p}}\, \xi_s^\dagger \xi_{s'} = 2E_{\mathbf{p}}\, \delta_{ss'} \,, \tag{4.3.106}$$

and similarly, $v_s^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) = 2E_{\mathbf{p}}\, \delta_{ss'}$.

Exercise.—Show that

$$\bar{u}_s(\mathbf{p}) v_{s'}(\mathbf{p}) = \bar{v}_s(\mathbf{p}) u_{s'}(\mathbf{p}) = 0 \,, \tag{4.3.107}$$

$$u_s^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) = v_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}) \neq 0 \,, \tag{4.3.108}$$

$$u_s^\dagger(\mathbf{p}) v_{s'}(-\mathbf{p}) = v_s^\dagger(\mathbf{p}) u_{s'}(-\mathbf{p}) = 0 \,. \tag{4.3.109}$$

When evaluating Feynman diagrams, we will often need to sum over the spin states of a fermion. This will involve the following *outer product*

$$\begin{aligned}\sum_{s=1,2} u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = \sum_{s=1,2} u_s(\mathbf{p})u_s^\dagger(\mathbf{p})\gamma^0 &= \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s\end{pmatrix}^\dagger\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\ &= \begin{pmatrix}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}\end{pmatrix} \\ &= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m\end{pmatrix}, \end{aligned} \tag{4.3.110}$$

Notice that the two spinors appearing on the left-hand side of (4.3.110) are not contracted. On the right-hand side, we have used that

$$\sum_{s=1,2} \xi_s\xi_s^\dagger = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{4.3.111}$$

The result can be written as

$$\sum_{s=1,2} u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = \not{p} + m\,. \tag{4.3.112}$$

A similar analysis for $v_s(p)$ gives

$$\sum_{s=1,2} v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = \not{p} - m\,. \tag{4.3.113}$$

Recall that the shorthand $\not{p}$ stands for the matrix $\gamma^\mu p_\mu$.

4.4 Quantization

We are finally ready to quantize the Dirac Lagrangian (4.3.42). The procedure will be the same as in §2.3, but with a few extra technical complications.

The momentum conjugate to the field ψ is

$$\pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = i\bar\psi\gamma^0 = i\psi^\dagger\,, \tag{4.4.114}$$

where we used that $(\gamma^0)^2 = 1$. If we would treat the quantization of ψ as if it were a scalar field, we would now impose the commutation relation[6]

$$\left[\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\,. \tag{4.4.115}$$

However, this would lead to a Hamiltonian that is unbounded from below (see §4.7). The problem does not arise is instead we impose the following anti-commutation relation[7]

$$\left\{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\right\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\,. \tag{4.4.116}$$

[6]For the moment, we are back in the Schrödinger picture. Alternatively, this can be viewed as the equal time commutation relation in the Heisenberg picture.

[7]Imposing an anti-commutation relation rather than a commutation relation is also required in order for the S-matrix to be Lorentz-invariant and the evolution to be causal.

The difference between (4.4.116) and (4.4.115) is the difference between fermions and bosons. We write the solution for the free Dirac field as a sum over the plane wave solutions presented in §4.3.3:

$$\psi(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(b^s_{\mathbf{p}} u_s(\mathbf{p}) e^{+i\mathbf{p}\cdot\mathbf{x}} + c^{s\dagger}_{\mathbf{p}} v_s(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \tag{4.4.117}$$

and, hence,

$$\psi^\dagger(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(b^{s\dagger}_{\mathbf{p}} u^\dagger_s(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} + c^s_{\mathbf{p}} v^\dagger_s(\mathbf{p}) e^{+i\mathbf{p}\cdot\mathbf{x}} \right). \tag{4.4.118}$$

The anti-commutation relation (4.4.116) then implies

$$\begin{aligned} \left\{ b^s_{\mathbf{p}}, b^{r\dagger}_{\mathbf{q}} \right\} &= (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) \delta^{sr} \,, \\ \left\{ c^s_{\mathbf{p}}, c^{r\dagger}_{\mathbf{q}} \right\} &= (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) \delta^{sr} \,, \end{aligned} \tag{4.4.119}$$

with all other anti-commutators vanishing. Had we instead used the commutation relation (4.4.115), we would have found

$$\begin{aligned} \left[b^s_{\mathbf{p}}, b^{r\dagger}_{\mathbf{q}} \right] &= (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) \, \delta^{sr} \,, \\ \left[c^s_{\mathbf{p}}, c^{r\dagger}_{\mathbf{q}} \right] &= -(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) \, \delta^{sr} \,. \end{aligned} \tag{4.4.120}$$

Notice the minus sign in the second commutation relation.

Derivation.—Let us show that the commutators (4.4.119) indeed lead to (4.4.116):

$$\begin{aligned} \left\{ \psi(\mathbf{x}), \psi^\dagger(\mathbf{y}) \right\} &= \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 q}{(2\pi)^6} \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{s,r} \Big(\left\{ b^s_{\mathbf{p}}, b^{r\dagger}_{\mathbf{q}} \right\} u^s(\mathbf{p}) u^{r\dagger}(\mathbf{q}) \, e^{i(\mathbf{x}\cdot\mathbf{p} - \mathbf{y}\cdot\mathbf{q})} \\ &\qquad\qquad + \left\{ c^{s\dagger}_{\mathbf{p}}, c^r_{\mathbf{q}} \right\} v^s(\mathbf{p}) v^{r\dagger}(\mathbf{q}) \, e^{-i(\mathbf{x}\cdot\mathbf{p} - \mathbf{y}\cdot\mathbf{q})} \Big) \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left(u^s(\mathbf{p}) \bar{u}^s(\mathbf{p}) \gamma^0 \, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + v^s(\mathbf{p}) \bar{v}^s(\mathbf{p}) \gamma^0 \, e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right). \end{aligned} \tag{4.4.121}$$

Using the outer products (4.3.112) and (4.3.113), we get

$$\left\{ \psi(\mathbf{x}), \psi^\dagger(\mathbf{y}) \right\} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left((\not{p} + m) \gamma^0 \, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + (\not{p} - m) \gamma^0 \, e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right) \tag{4.4.122}$$

Sending $\mathbf{p} \to -\mathbf{p}$ in the second term, this can be written as

$$\begin{aligned} \left\{ \psi(\mathbf{x}), \psi^\dagger(\mathbf{y}) \right\} &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Big((p_0 \gamma^0 + p_i \gamma^i + m) \gamma^0 + (p_0 \gamma^0 - p_i \gamma^i - m) \gamma^0 \Big) \, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \delta^{(3)}(\mathbf{x}-\mathbf{y}) \,, \end{aligned} \tag{4.4.123}$$

which confirms that (4.4.119) indeed lead to (4.4.116).

The Hamiltonian of the Dirac theory is

$$H = \int \mathrm{d}^3 x \left[\pi_\psi \dot{\psi} - \mathcal{L} \right] = \int \mathrm{d}^3 x \; \bar{\psi} (-i\gamma^i \partial_i + m) \psi \,, \tag{4.4.124}$$

which agrees with the conserved energy derived via Noether's theorem in (4.3.59). Substituting the mode expansion (4.4.117), we find

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left[b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} + c^{s\dagger}_{\mathbf{p}} c^s_{\mathbf{p}} - (2\pi)^3 \delta^{(3)}(0) \right] . \tag{4.4.125}$$

As before, the delta function contribution can be removed by normal ordering. Before we drop the delta function, however, we note that it comes with a negative sign while the contribution from bosonic fields comes with positive sign. Supersymmetry, a symmetry between bosonic and fermionic degrees of freedom, enforces a precise cancellation of these contributions. Unfortunately supersymmetry is not an exact symmetry of Nature, so this doesn't work as a solution to the cosmological constant problem.

Derivation.—Consider

$$(-i\gamma^i \partial_i + m)\psi = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \Big[b^s_{\mathbf{p}} (-\gamma^i p_i + m) u^s(\mathbf{p})\, e^{+i\mathbf{p}\cdot\mathbf{x}} + c^{s\dagger}_{\mathbf{p}} (+\gamma^i p_i + m) v^s(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}} \Big] . \tag{4.4.126}$$

There's a small subtlety with the minus signs in deriving this equation that arises from the use of the Minkowski metric in contracting indices, so that $\mathbf{p}\cdot\mathbf{x} = \sum_i x^i p^i = -x^i p_i$. Using $(\gamma^\mu p_\mu - m)u^s(\mathbf{p}) = 0$ and $(\gamma^\mu p_\mu + m)v^s(\mathbf{p}) = 0$, we can write this as

$$(-i\gamma^i \partial_i + m)\psi = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{E_{\mathbf{p}}}{2}} \gamma^0 \sum_s \left[b^s_{\mathbf{p}} u^s(\mathbf{p})\, e^{+i\mathbf{p}\cdot\mathbf{x}} - c^{s\dagger}_{\mathbf{p}} v^s(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}} \right] . \tag{4.4.127}$$

Substituting this into the Hamiltonian operator, we get

$$\begin{aligned} H &= \int \mathrm{d}^3 x\, \psi^\dagger \gamma^0 (-i\gamma^i \partial_i + m)\psi \\ &= \int \frac{\mathrm{d}^3 x\, \mathrm{d}^3 p\, \mathrm{d}^3 q}{(2\pi)^6} \sqrt{\frac{E_p}{2E_q}} \sum_{r,s} \left[b^{r\dagger}_{\mathbf{q}} u^{r\dagger}(\mathbf{q})\, e^{-i\mathbf{q}\cdot\mathbf{x}} + c^r_{\mathbf{q}} v^{r\dagger}(\mathbf{q})\, e^{+i\mathbf{q}\cdot\mathbf{x}} \right] \\ &\qquad \left[b^s_{\mathbf{p}} u^s(\mathbf{p})\, e^{+i\mathbf{p}\cdot\mathbf{x}} - c^{s\dagger}_{\mathbf{p}} v^s(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}} \right] \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2} \sum_{r,s} \Big[b^{r\dagger}_{\mathbf{p}} b^s_{\mathbf{p}}\, u^{r\dagger}(\mathbf{p}) u^s(\mathbf{p}) - c^r_{\mathbf{p}} c^{s\dagger}_{\mathbf{p}}\, v^{r\dagger}(\mathbf{p}) v^s(\mathbf{p}) \\ &\qquad - b^{r\dagger}_{\mathbf{p}} c^{s\dagger}_{-\mathbf{p}}\, u^{r\dagger}(\mathbf{p}) v^s(-\mathbf{p}) + c^r_{\mathbf{p}} b^s_{-\mathbf{p}}\, v^{r\dagger}(\mathbf{p}) u^s(-\mathbf{p}) \Big] . \end{aligned} \tag{4.4.128}$$

Using the inner products (4.3.106) and (4.3.109),

$$u^{r\dagger}(\mathbf{p}) u^s(\mathbf{p}) = v^{r\dagger}(\mathbf{p}) v^s(\mathbf{p}) = 2E_{\mathbf{p}} \delta^{rs} , \tag{4.4.129}$$

$$u^{r\dagger}(\mathbf{p}) v^s(-\mathbf{p}) = v^{r\dagger}(\mathbf{p}) u^s(-\mathbf{p}) = 0 , \tag{4.4.130}$$

we get

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left[b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} - c^s_{\mathbf{p}} c^{s\dagger}_{\mathbf{p}} \right] . \tag{4.4.131}$$

After normal-ordering, this is

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left[b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} + c^{s\dagger}_{\mathbf{p}} c^s_{\mathbf{p}} \right] , \tag{4.4.132}$$

where we have used (4.4.119). If we had used (4.4.120) instead of (4.4.119), we would have found

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left[b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} - c^{s\dagger}_{\mathbf{p}} c^s_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(0) \right] . \tag{4.4.133}$$

This is unbounded from below, so it is not a stable quantum theory.

Particles and antiparticles

Although b and c satisfy anti-commutation relations, we still have

$$\left[H, b_{\mathbf{p}}^{s\dagger}\right] = +E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger}\,, \quad \left[H, c_{\mathbf{p}}^{s\dagger}\right] = +E_{\mathbf{p}} c_{\mathbf{p}}^{s\dagger}\,, \tag{4.4.134}$$

We can therefore still construct a tower of energy eigenstates by acting with $b_{\mathbf{p}}^{s\dagger}$ and $c_{\mathbf{p}}^{s\dagger}$ on the vacuum. As in the case of a complex scalar field, $b_{\mathbf{p}}^{s\dagger}$ creates particles and $c_{\mathbf{p}}^{s\dagger}$ creates the corresponding antiparticles.

In §4.3.2, we introduced the vector current $J_V^\mu = \bar{\psi}\gamma^\mu\psi$, as well as the corresponding conserved charge

$$Q = \int \mathrm{d}^3x\, \psi^\dagger(\mathbf{x})\psi(\mathbf{x})\,. \tag{4.4.135}$$

Substituting the mode expansion (4.4.117), and normal-ordering, this becomes

$$Q = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_s \left[b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s\right]. \tag{4.4.136}$$

We see that $b_{\mathbf{p}}^{s\dagger}$ creates fermions with charge $+1$, while $c_{\mathbf{p}}^{s\dagger}$ creates antifermions with charge -1. In QED, these charges will be identified with the electric charges of fermions and antifermions (see Chapter 6).

Spin and statistics

In §4.3.2, we derived the Noether charge associated to spatial rotations, cf. eq. (4.3.63). We saw that it contained an orbital contribution and a spin term. In the rest frame, only the spin term contributes and the conserved charge is

$$Q^{ij} = \int \mathrm{d}^3x\, (\mathcal{J}^0)^{ij} = -i \int \mathrm{d}^3x\, \psi^\dagger S^{ij}\psi \;\equiv\; -\epsilon^{ijk} S^k\,, \tag{4.4.137}$$

where we have introduced the vector

$$\boldsymbol{S} = \frac{1}{2}\int \mathrm{d}^3x\, \psi^\dagger \boldsymbol{\Sigma}\psi\,, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{4.4.138}$$

To show that a Dirac particle has spin 1/2, we apply S_3 to the state $b_{\mathbf{0}}^{s\dagger}|0\rangle$. Since S_3 annihilates the vacuum state, we can write this as

$$S_3 b_{\mathbf{0}}^{s\dagger}|0\rangle = [S_3, b_{\mathbf{0}}^{s\dagger}]|0\rangle\,, \tag{4.4.139}$$

where the commutator is non-zero only for the terms in S_3 that contain annihilation operators at $\mathbf{p} = \mathbf{0}$. Using the mode expansion (4.4.117), we have

$$\begin{aligned} S_3 = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p\, \mathrm{d}^3p'}{(2\pi)^6} \frac{1}{\sqrt{4E_{\mathbf{p}}E'_{\mathbf{p}}}}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \\ \sum_{s,s'} \left(b_{\mathbf{p}}^{s\dagger} u_s^\dagger(\mathbf{p}) + c_{-\mathbf{p}}^{s} v_s^\dagger(-\mathbf{p})\right) \frac{\Sigma^3}{2} \left(b_{\mathbf{p}'}^{s'} u_{s'}(\mathbf{p}') + c_{-\mathbf{p}'}^{s'\dagger} v_{s'}(-\mathbf{p}')\right). \end{aligned} \tag{4.4.140}$$

Substituting this into (4.4.139), we get

$$S_3 b_{\mathbf{0}}^{s\dagger}|0\rangle = \frac{1}{2m}\sum_s \left(u_s^\dagger(\mathbf{0})\frac{\Sigma^3}{2}u_r(\mathbf{0})\right) b_{\mathbf{0}}^{r\dagger}|0\rangle = \sum_r \left(\xi_s^\dagger \frac{\sigma^3}{2}\xi_r\right) b_{\mathbf{0}}^{r\dagger}|0\rangle\,, \tag{4.4.141}$$

where we have used the explicit form of $u_s(\mathbf{0})$, cf. eq. (4.3.91), to obtain the final expression. Let us choose the basis spinors ξ_r to be eigenstates of σ^3. We then find that for $\xi_1^T = (1,0)$, the state is an eigenstate of S_3 with eigenvalue $+1/2$, while for $\xi_2^T = (0,1)$, it is an eigenstate of S_3 with eigenvalue $-1/2$. This confirms that fermions are spin-$\frac{1}{2}$ particles. The calculation for an antifermion state is similar, except we get an overall sign change in the final answer. This means that $c_{\mathbf{0}}^{s\dagger}|0\rangle$ is an eigenstate of S_3 with eigenvalue $-1/2$ for $\xi_1^T = (1,0)$ and $+1/2$ for $\xi_2^T = (0,1)$. In summary, the angular momentum of fermions in their rest frame is given by

$$S_3 b_{\mathbf{0}}^{s\dagger}|0\rangle = \pm\frac{1}{2} b_{\mathbf{0}}^{s\dagger}|0\rangle\,, \quad S_3 c_{\mathbf{0}}^{s\dagger}|0\rangle = \mp\frac{1}{2} c_{\mathbf{0}}^{s\dagger}|0\rangle\,, \tag{4.4.142}$$

where the upper sign is for $\xi_1^T = (1,0)$ and the lower sign for $\xi_2^T = (0,1)$. This shows that fermionic (anti)particles have spin 1/2.

An important consequence of the anti-commutations relations for the operators b and c is the fact that fermionic particles obey Fermi-Dirac statistics. For example, a two-particle state satifies

$$|\mathbf{p}_1, s_1; \mathbf{p}_2, s_2\rangle \equiv b_{\mathbf{p}_1}^{s_1\dagger} b_{\mathbf{p}_2}^{s_2\dagger}|0\rangle = -b_{\mathbf{p}_2}^{s_2\dagger} b_{\mathbf{p}_1}^{s_1\dagger}|0\rangle = -|\mathbf{p}_2, s_2; \mathbf{p}_1, s_1\rangle\,. \tag{4.4.143}$$

This implies the *Pauli exclusion principle*, i.e. no two fermions can occupy the same state

$$|\mathbf{p}, s; \mathbf{p}, s\rangle = 0\,. \tag{4.4.144}$$

Note that in quantum mechanics, Fermi-Dirac statistics is an observational input, while in quantum field theory it is a theoretical output.

4.5 Fermion Propagator

In Chapter 3, we have seen that the Feynman propagator plays an important role in perturbative quantum field theory. To derive the Feynman propagator for fermions we move to the Heisenberg picture. The mode expansion of the field ψ becomes[8]

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(b_{\mathbf{p}}^s u_s(p) e^{-ip\cdot x} + c_{\mathbf{p}}^{s\dagger} v_s(p) e^{+ip\cdot x}\right). \tag{4.5.145}$$

The Feynman propagator for fermionic fields is then defined as[9]

$$S_F(x-y) \equiv \langle 0|T\{\psi(x)\bar\psi(y)\}|0\rangle\,, \tag{4.5.146}$$

where the time ordering is

$$T\{\psi(x)\bar\psi(y)\} = \begin{cases} +\psi(x)\bar\psi(y) & x^0 > y^0 \\ -\bar\psi(y)\psi(x) & x^0 < y^0 \end{cases} \tag{4.5.147}$$

[8] In an abuse of notation, we will henceforth denote the basis spinors by $u(p)$ and not $u(\mathbf{p})$.

[9] Notice the distinction between $\bar\psi\psi = \bar\psi_\alpha(x)\psi_\alpha(x) = \mathrm{Tr}(\bar\psi_\alpha(x)\psi_\beta(x))$ in the Lagrangian and $\bar\psi(x)\psi(y) = \bar\psi_\alpha(x)\psi_\beta(y)$ in the time-ordered product. The former is a number, while the later is a 4×4 matrix. The Feynman propagator for fermions is therefore a 4×4 matrix.

Notice the crucial minus sign in the time-ordered product.[10] Substituting the mode expansion (4.5.145), we find

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not{p}+m)\,e^{-ip\cdot(x-y)}\,, \tag{4.5.148}$$

$$\langle 0|\bar{\psi}(y)\psi(x)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not{p}-m)\,e^{+ip\cdot(x-y)}\,, \tag{4.5.149}$$

and hence obtain the following expression for the Feynman propagator, in momentum space,

$$\boxed{S_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}}\,. \tag{4.5.150}$$

The derivation of this result is given in the following insert.

Derivation.—Substituting the mode expansion (4.5.145) into $\langle 0|\psi(x)\bar{\psi}(y)|0\rangle$ gives

$$\begin{aligned}\langle 0|\psi(x)\bar{\psi}(y)|0\rangle &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{s,r}\langle 0|\left(b^s_{\mathbf{p}}u_s(p)e^{-ip\cdot x}+c^{s\dagger}_{\mathbf{p}}v_s(p)e^{ip\cdot x}\right)\\ &\qquad\qquad\times\left(b^{r\dagger}_{\mathbf{q}}\bar{u}_r(q)e^{iq\cdot y}+c^r_{\mathbf{q}}v_r(q)e^{-iq\cdot y}\right)|0\rangle\\ &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{s,r}u_s(p)\bar{u}_r(q)\,\langle 0|\,b^s_{\mathbf{p}}b^{r\dagger}_{\mathbf{q}}\,|0\rangle e^{-ipx+iqy}\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s u_s(p)\bar{u}_s(p)e^{-ip(x-y)}\,.\end{aligned} \tag{4.5.151}$$

Using the outer product (4.3.112), we get

$$\begin{aligned}\langle 0|\psi(x)\bar{\psi}(y)|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not{p}+m)\,e^{-ip\cdot(x-y)}\\ &= (i\not{\partial}_x+m)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\,e^{-ip\cdot(x-y)}\,.\end{aligned} \tag{4.5.152}$$

A similar computation for $\langle 0|\bar{\psi}(y)\psi(x)|0\rangle$, gives

$$\begin{aligned}\langle 0|\bar{\psi}(y)\psi(x)|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not{p}-m)\,e^{+ip\cdot(x-y)}\\ &= -(i\not{\partial}_x+m)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\,e^{+ip\cdot(x-y)}\,.\end{aligned} \tag{4.5.153}$$

Substituting these results into the time-ordered product,

$$T\{\psi(x)\bar{\psi}(y)\} = \theta(x^0-y^0)\,\psi(x)\bar{\psi}(y) - \theta(y^0-x^0)\,\bar{\psi}(y)\psi(x)\,, \tag{4.5.154}$$

we find

$$\begin{aligned}S_F(x-y) &= (i\not{\partial}_x+m)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)\,e^{-ip\cdot(x-y)}+\theta(y^0-x^0)\,e^{+ip\cdot(x-y)}\right],\\ &= (i\not{\partial}_x+m)\,\Delta_F(x-y)\,,\end{aligned} \tag{4.5.155}$$

[10]When $(x-y)^2<0$, there is no invariant way to determine whether $x^0>y^0$ or $x^0<y^0$. In this case the minus sign is necessary to make the two definitions agree since $\{\psi(x),\bar{\psi}(y)\}=0$ outside the light-cone.

where $\Delta_F(x-y)$ is the scalar Feynman propagator, cf. eq. (2.4.85). Using (2.4.92), we therefore get

$$\begin{aligned} S_F(x-y) &= (i\partial\!\!\!/_x + m)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)} \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i(p\!\!\!/+m)}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)}\,, \end{aligned} \tag{4.5.156}$$

which is the four-dimensional Fourier transform of (4.5.150).

The minus sign in (4.5.147) also occurs for any string of operators inside any time-ordered product. While bosonic operators commute inside T, fermionic operators anti-commute. The same applied to normal-ordered products, with fermionic operators picking up minus signs when they are moved past each other. With the understanding that all fermionic operators anti-commute inside time- and normal-ordered products, Wick's theorem then proceeds as in the bosonic case (see §3.5). The contraction of two fermionic fields is the Feynman propagator.

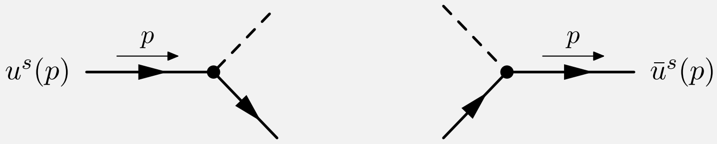
4.6 Yukawa Theory

In §3.9, we studied scalar Yukawa theory as a simple model for electron-photons interactions. We can now make the model more realistic by representing the electrons by a spinor field ψ. The *Yukawa Lagrangian* is

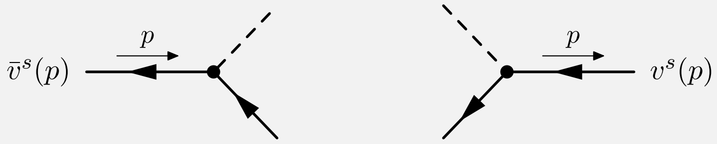
$$\mathcal{L}_{\text{Yukawa}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar\psi(i\partial\!\!\!/ - m)\psi - g\,\phi\bar\psi\psi\,. \tag{4.6.157}$$

We will state the Feynman rules for this theory and then apply them to a few examples.

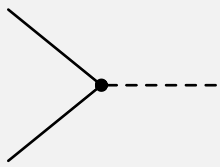
- To each incoming fermion with momentum p and spin s, we associate a spinor $u^s(p)$. To outgoing fermions we associate $\bar u^s(p)$.



- To each incoming anti-fermion with momentum p and spin s, we associate a spinor $\bar v^s(p)$. To outgoing anti-fermions we associate $v^s(p)$.



- Each vertex contributes

 $= \quad -ig$

- Each internal line gives a factor of the relevant propagator:

$$
\begin{aligned}
&= \frac{i}{p^2 - \mu^2 + i\epsilon} \, . \\
&= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \, .
\end{aligned}
$$

- Impose momentum conservation at each vertex.
- Integrate over undetermined loop momenta.
- Determine the overall sign of the diagram.

To determine the sign of the diagram, we need to keep track of minus signs that arise from the re-ordering of fermionic fields. The following rules will do this for you:

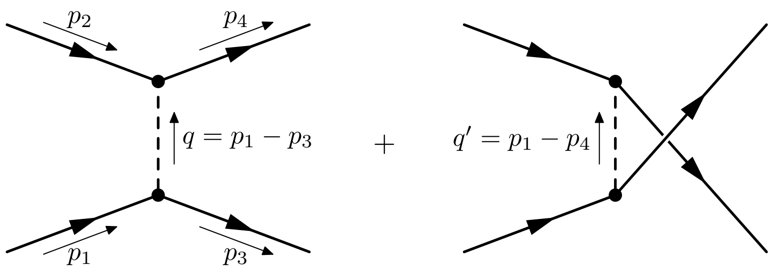
1. Add a minus sign for every closed fermion loop.
2. Add a minus sign for every open fermionic line which connects an outgoing anti-fermion to an incoming anti-fermion.
3. Add a minus sign for every crossing of the fermionic lines.

We will encounter all of these rules in the examples below.

Examples

Let us go through the same scattering processes as in §3.9, but this time including the spin of the electrons.

- **Electron scattering.** We start with $\psi\psi \to \psi\psi$ scattering. At order g^2, we have the same t- and u-channel contributions as in (3.9.72) and (3.9.80):



(4.6.158)

The above Feynman rules then give

$$\mathcal{A}_t = +(-ig)^2\, \bar{u}^{s_4}(p_4)\cdot u^{s_2}(p_2)\, \frac{1}{(p_1-p_3)^2-\mu^2}\, \bar{u}^{s_3}(p_3)\cdot u^{s_1}(p_1)\,, \tag{4.6.159}$$

$$\mathcal{A}_u = -(-ig)^2\, \bar{u}^{s_3}(p_3)\cdot u^{s_2}(p_2)\, \frac{1}{(p_1-p_4)^2-\mu^2}\, \bar{u}^{s_4}(p_4)\cdot u^{s_1}(p_1)\,. \tag{4.6.160}$$

The momentum in the "photon" propagators is determined by conservation at each vertex. Note that the two contributions have opposite signs. This relative sign arises because the two diagrams in (4.6.158) involve the exchange of the two fermions in the final state, which creates a sign change because the fermion fields are anti-commuting (see **rule 3** above). To see this in more detail, we go back to the contractions corresponding to the two diagram. The contraction corresponding to the t-channel diagram is of the form

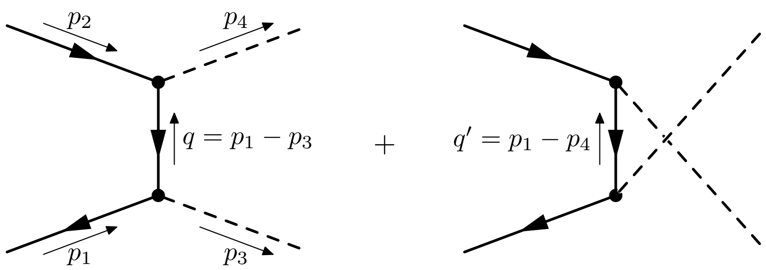
$$\langle 0| a^{s_4}_{\mathbf{p}_4} a^{s_3}_{\mathbf{p}_3} \bar{\psi}_x \psi_x \bar{\psi}_y \psi_y \, a^{s_1\dagger}_{\mathbf{p}_1} a^{s_2\dagger}_{\mathbf{p}_2} |0\rangle \,. \tag{4.6.161}$$

This contraction can be untangled by moving $\bar{\psi}_y$ two spaces to the left, picking up a factor of $(-1)^2 = 1$ in the process. The contraction corresponding to the u-channel diagram, on the other hand, is of the form

$$\langle 0| a^{s_4}_{\mathbf{p}_4} a^{s_3}_{\mathbf{p}_3} \bar{\psi}_x \psi_x \bar{\psi}_y \psi_y \, a^{s_1\dagger}_{\mathbf{p}_1} a^{s_2\dagger}_{\mathbf{p}_2} |0\rangle \,. \tag{4.6.162}$$

Untangelling this contraction this time only requires moving $\bar{\psi}_y$ one space to the left, giving a factor of -1.

- **Electron-positron annihilation.** Next, we consider $\psi\bar{\psi} \to \phi\phi$. At order g^2, the relevant Feynman diagrams were presented in (3.9.88)



(4.6.163)

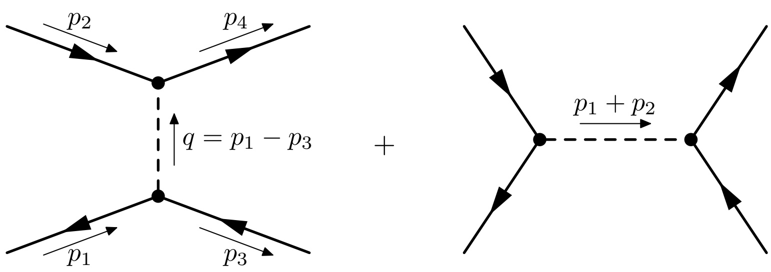
The corresponding scattering amplitudes are

$$\mathcal{A}_t = (-ig)^2 \, \bar{v}^{s_1}(p_1) \, \frac{(\not{p}_1 - \not{p}_3) + m}{(p_1 - p_3)^2 - m^2} \, u^{s_2}(p_2) \,, \tag{4.6.164}$$

$$\mathcal{A}_u = (-ig)^2 \, \bar{v}^{s_1}(p_1) \, \frac{(\not{p}_1 - \not{p}_4) + m}{(p_1 - p_4)^2 - m^2} \, u^{s_2}(p_2) \,. \tag{4.6.165}$$

Now the exchange statistics applies to the final "photon" states. These are bosons and, hence, there is no relative minus sign between the two diagrams.

- **Electron-positron scattering.** For $\psi\bar{\psi} \to \psi\bar{\psi}$, the lowest order Feynman diagrams are



(4.6.166)

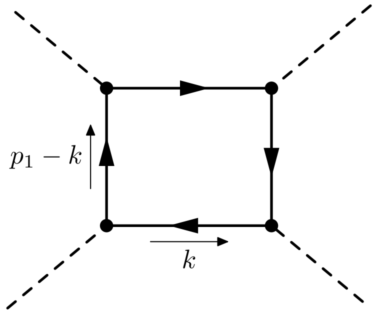
and the corresponding scattering amplitudes are

$$\mathcal{A}_t = -(-ig)^2 \, \bar{u}^{s_4}(p_4) \cdot u^{s_2}(p_2) \, \frac{1}{(p_1 - p_3)^2 - \mu^2} \, \bar{v}^{s_1}(p_1) \cdot v^{s_3}(p_3) \,, \tag{4.6.167}$$

$$\mathcal{A}_s = +(-ig)^2 \, \bar{v}^{s_1}(p_1) \cdot u^{s_2}(p_2) \, \frac{1}{(p_1 + p_2)^2 - \mu^2} \, \bar{u}^{s_4}(p_4) \cdot v^{s_3}(p_3) \,. \tag{4.6.168}$$

The two diagrams differ by a relative sign since they are related by a single exchange of fermionic particles (those with momenta p_1 and p_4). The overall minus sign of the first diagram is due to the fermionic line which connects an outgoing anti-fermion (with momentum p_3) to an incoming anti-fermion (with momentum p_1) (see **rule 2** above).

- **Photon scattering.** As in the scalar Yukawa theory, $\phi\phi \to \phi\phi$ scattering occurs through a loop diagram



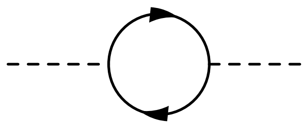
(4.6.169)

The corresponding amplitude now is

$$i\mathcal{A} = -(-i\lambda)^4 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \frac{\not{k} - \not{p}_1 + m}{(k-p_1)^2 - m^2 + i\epsilon} \times \frac{\not{k} - \not{p}_1 - \not{p}_2 + m}{(k-p_1-p_2)^2 - m^2 + i\epsilon} \frac{\not{k} - \not{p}_3 + m}{(k-p_3)^2 - m^2 + i\epsilon}\right] . \tag{4.6.170}$$

For large k, this integral goes as $\int \mathrm{d}^4k/k^4$, which diverges logarithmically for $k \to \infty$. Notice the overall minus sign of the amplitude (4.6.170) relative to the amplitude in (3.9.93). This sign arises because this time there is a fermion running in the loop (see **rule 1** above).

To illustrate the origin of the minus sign, let us consider the simpler diagram



(4.6.171)

This involves the following contractions

$$\begin{aligned} \bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y) &= -\,\psi_\beta(y)\bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(y) \\ &= -\mathrm{Tr}[S_F(y-x)S_F(x-y)] \,. \end{aligned} \tag{4.6.172}$$

After passing the fermionic fields through each other, an overall minus sign has appeared.

The differences in the signs of fermion and boson loops plays an important role in *supersymmetry*. A supersymmetric theory contains an exact symmetry between fermions and bosons which can lead to precise cancellations of many loop contributions.

Yukawa potential

In §3.11, we saw that the exchange of a real scalar particles leads to a universally attractive Yukawa force between spin-0 particles and antiparticles. We will now show that the same conclusion applies for the force between spin-1/2 particles.

As before, we will take the non-relativistic limit of the relevant scattering amplitudes and compare the result, via the Born approximation, to the scattering potential in quantum mechanics.

In §4.3.3, we have seen that the non-relativistic limit of the spinors $u(p)$ and $v(p)$ is

$$u(p) \to \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \quad v(p) \to \sqrt{m} \begin{pmatrix} \eta \\ -\eta \end{pmatrix}, \tag{4.6.173}$$

which implies that

$$\bar{u}^s(p) \cdot u^{s'}(p') = +2m\, \delta^{ss'}, \tag{4.6.174}$$

$$\bar{v}^s(p) \cdot v^{s'}(p') = -2m\, \delta^{ss'}. \tag{4.6.175}$$

This allows us to write the amplitude for $\psi\psi \to \psi\psi$ scattering [see (4.6.159) and (4.6.160)] as

$$\mathcal{A}_{\psi\psi} \approx +g^2\,(2m)^2 \frac{\delta^{s_4 s_2}\delta^{s_3 s_1}}{|\mathbf{q}|^2 + \mu^2} - g^2\,(2m)^2 \frac{\delta^{s_3 s_2}\delta^{s_4 s_1}}{|\mathbf{q}'|^2 + \mu^2}, \tag{4.6.176}$$

where $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_3$ and $\mathbf{q}' = \mathbf{p}_1 - \mathbf{p}_4$. The Kronecker delta's enforce that the spin of each particle is separately conserved in the nonrelativistic limit. In the Born approximation, we have the following relation to the scattering potential

$$-\frac{\mathcal{A}_{\psi\psi}}{(2m)^2} = V_{\psi\psi}(\mathbf{q})\,\delta^{s_4 s_2}\delta^{s_3 s_1} - V_{\psi\psi}(\mathbf{q}')\,\delta^{s_3 s_2}\delta^{s_4 s_1}, \tag{4.6.177}$$

where the two terms account for the permutations of the identical fermions. We therefore have

$$V_{\psi\psi}(\mathbf{q}) = -\frac{g^2}{|\mathbf{q}|^2 + \mu^2}. \tag{4.6.178}$$

Similarly, the amplitude for $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering [see (4.6.167) and (4.6.168)] becomes

$$\mathcal{A}_{\psi\bar{\psi}} \approx +g^2\,(2m)^2 \frac{\delta^{s_4 s_2}\delta^{s_3 s_1}}{|\mathbf{q}|^2 + \mu^2}, \tag{4.6.179}$$

where we have dropped the s-channel contribution since it is sub-dominant in the non-relativistic limit. Notice that the relative sign in the inner products of the u and v spinors has cancelled the relative sign of the t-channel amplitudes (4.6.159) and (4.6.167). The Born approximation therefore gives

$$V_{\psi\bar{\psi}}(\mathbf{q}) = -\frac{\mathcal{A}_{\psi\bar{\psi}}}{(2m)^2} = -\frac{g^2}{|\mathbf{q}|^2 + \mu^2} = V_{\psi\psi}(\mathbf{q}). \tag{4.6.180}$$

The sign and momentum dependence is the same as in the bosonic case, cf. eq. (3.9.87), so that we get the same Yukawa potential as before

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-\mu r}}{r}. \tag{4.6.181}$$

This means that the Yukawa force is again universally attractive. To get a repulsive force we have to wait for the exchange of spin-1 particles in quantum electrodynamics (cf. §6.4).

4.7 Problems

1. Using just the algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (i.e. without resorting to a particular representation), prove the following results:

i. $\mathrm{Tr}(\gamma^\mu) = 0$

ii. $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$

iii. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$

iv. $(\gamma^5)^2 = 1$

v. $\mathrm{Tr}(\gamma^5) = 0$

vi. $\not{p}\,\not{q} = 2\,p\cdot q - \not{q}\,\not{p} = p\cdot q + 2S^{\mu\nu}p_\mu q_\nu$

vii. $\mathrm{Tr}(\not{p}\,\not{q}) = 4\,p\cdot q$

viii. $\mathrm{Tr}(\not{p}_1 \ldots \not{p}_n) = 0$, if n is odd

ix. $\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4\,[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4)]$

x. $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2) = 0$

xi. $\gamma_\mu\not{p}\gamma^\mu = -2\not{p}$

xii. $\gamma_\mu\not{p}_1\not{p}_2\gamma^\mu = 4p_1\cdot p_2$

xiii. $\gamma_\mu\not{p}_1\not{p}_2\not{p}_3\gamma^\mu = -2\not{p}_3\not{p}_2\not{p}_1$

xiv. $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4i\epsilon_{\mu\nu\,\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$

where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^4$, $\not{p} \equiv p_\mu\gamma^\mu$ and $S^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu, \gamma^\nu]$.

[*Hint*: useful tricks include the cyclicity of the trace and inserting $(\gamma^5)^2 = 1$ into a trace.]

2. In the lectures, we introduced the Lorentz-invariant Dirac mass term $(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$. However, this is not the unique possibility. Show that $i(\psi_L^\dagger\sigma_2\psi_L^* - \psi_L^T\sigma_2\psi_L)$ is also Lorentz-invariant. This is the so-called *Majorana mass.*

3. The purpose of this question is to give you a glimpse into the *spin-statistics theorem.* This theorem roughly says that if you try to quantize a field with the wrong statistics, bad things will happen. In this problem, we will see what goes wrong if you try to quantize a spin-$\frac{1}{2}$ field as a boson. We start with the usual decomposition

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s \left(b_{\mathbf{p},s}u_s(\mathbf{p})e^{-ip\cdot x} + c_{\mathbf{p},s}^\dagger v_s(\mathbf{p})e^{ip\cdot x}\right).$$

This time we choose bosonic commutation relations for the annihilation and creation operators,

$$\left[b_\mathbf{p}^r, b_\mathbf{q}^{s\dagger}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,\delta^{rs}\,,$$
$$\left[c_\mathbf{p}^r, c_\mathbf{q}^{s\dagger}\right] = -(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,\delta^{rs}\,.$$

Note the strange minus sign in the second commutation relation. Show that these are equivalent to

$$\left[\psi_\alpha(t,\mathbf{x}), \psi_\beta^\dagger(t,\mathbf{y})\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y})\,\delta_{\alpha\beta}\,.$$

Show that, after normal ordering, the Hamitonian is given by

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left[b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s \right] .$$

This Hamiltonian is not bounded from below: you can lower the energy indefinitely by creating more and more c-particles. This is the reason a theory of bosonic spin-$\frac{1}{2}$ particles is sick.

4. The Lagrangian for a *pseudoscalar Yukawa interaction* is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\phi\bar{\psi}\gamma^5\psi .$$

Write down the Feynman rules for this theory. Use this to determine the amplitude at order g^2 for $\psi\psi \to \psi\psi$ scattering and $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering.

5 Gauge Fields

In this chapter, we will quantize the Maxwell Lagrangian. This will lead to massless spin-1 particles. As we will see in the next chapter, these particles mediate long range forces between charged particles.

5.1 Wigner's Classification

In the previous chapter, we showed how particles can be identified with unitary irreducible representations of the Poincaré group. These representations are labeled by the mass m and the spin j of the particles. Since the mass is a Lorentz invariant, it is obviously a good quantum number to describe particles. To understand the role of spin, we go to the rest frame of the particle, where $p^\mu = (m, 0, 0, 0)$, for $m > 0$. Transformations associated with the rotation group $SO(3)$ (also called the *little group*) leave the four-momentum invariant. The quantum number associated with the group $SO(3)$ is the spin j. States within each representation are labeled by $j_z = -j, -j+1, \dots, +j$. This means that massive particles of spin j have $2j+1$ degrees of freedom.

For massless particles, on the other hand, the rest frame does not exist. Choosing the direction of propagation to be the z-axis, we can write $p^\mu = (E, 0, 0, E)$. The little group that leaves the four-momentum invariant is $SO(2)$, the group of rotations in the (x, y) plane. The generator of the little group is J_z. The eigenvalues of J_z represent the angular momentum in the direction of propagation of the particles (here, chosen to be the z direction), also called the *helicity*. It can be shown that these eigenvalues are quantized, $h = 0, \pm\frac{1}{2}, \pm 1, \dots$. The two massless states with helicity ± 1 describe the photon and those with helicity ± 2 correspond to the graviton. Neutrinos and antineutrinos have helicity $h = -\frac{1}{2}$ and $h = +\frac{1}{2}$, respectively.[1]

Particles with integer spin j are naturally identified with tensor fields $T_{\mu_1 \dots \mu_j}$. However, in general, the tensor $T_{\mu_1 \dots \mu_j}$ has 4^j components, while a massive spin-j particle has $2j+1$ degrees of freedom (and a massless particle with spin has only two degrees of freedom). The tensor $T_{\mu_1 \dots \mu_j}$ is not yet an irreducible representation of spin-j particles. We need to impose extra constraints to isolate the spin-j component of the field. We will illustrate this for the case of spin-1 particles.

[1]Strictly speaking, a massless particles with helicity $+h$ and a massless particle with helicity $-h$ are different particles. However, we may write the helicity operator as $h = \hat{\mathbf{p}} \cdot \mathbf{J}$, where $\hat{\mathbf{p}}$ is the unit vector in the direction of propagation. Helicity is therefore a pseudo-scalar, i.e. it changes sign under parity. If the interaction conserved parity, then particle of helicity $+h$ and $-h$ must appear symmetrically in the theory. It is then natural to treat the two helicities as two degrees of freedom of the same particle, e.g. the two polarizations of the photon and the graviton.

5.2 Rediscovering Maxwell

Our goal is to construct a Lagrangian for A_μ that propagates the right number of degrees of freedom, i.e. three for a massive field and two for a massless field.

Massless spin-1 particles

Lorentz invariance and locality allow two possible kinetic terms for a vector field

$$\mathcal{L}_{\rm kin} = a_1 \mathcal{L}_1 + a_2 \mathcal{L}_2 \,, \tag{5.2.1}$$

where

$$\mathcal{L}_1 = \partial_\mu A^\nu \partial^\mu A_\nu \,, \tag{5.2.2}$$
$$\mathcal{L}_2 = \partial_\mu A^\mu \partial_\nu A^\nu = \partial_\nu A^\mu \partial_\mu A^\nu + \text{boundary terms} \,. \tag{5.2.3}$$

Let us write $A_\mu \equiv \hat{A}_\mu + \partial_\mu \phi$, where $\partial^\mu \hat{A}_\mu = 0$. The fields $\hat{A}_\mu$ and ϕ are referred to as the transverse mode and the longitudinal mode, respectively. The Lagrangian (5.2.1) implies that the longitudinal mode satisfies

$$\mathcal{L}^\phi_{\rm kin} = (a_1 + a_2)(\Box\phi)^2 \,, \tag{5.2.4}$$

which is equivalent to

$$\mathcal{L}^\phi_{\rm kin} = (a_1 + a_2)\left(\tilde{\phi}\Box\phi - \frac{1}{4}\tilde{\phi}^2\right)^2 \,, \tag{5.2.5}$$

after integrating out the Lagrange multiplier field $\tilde{\phi} = 2\Box\phi$. Defining $\phi \equiv \phi_1 + \phi_2$ and $\tilde{\phi} \equiv \phi_1 - \phi_2$, this becomes

$$\mathcal{L}^\phi_{\rm kin} = (a_1 + a_2)\left(+\phi_1\Box\phi_1 - \phi_2\Box\phi_2 - \frac{1}{4}(\phi_1 - \phi_2)\right)^2 \,. \tag{5.2.6}$$

We see that one of the two fields is always a ghost, unless $a_1 + a_2 = 0$. In that case, the longitudinal mode is non-dynamical and the transverse mode satisfies

$$\mathcal{L}^{\hat{A}}_{\rm kin} = a_1 (\partial_\mu \hat{A}_\nu)^2 \,, \tag{5.2.7}$$

which has the right normalization for $a_1 = \frac{1}{2}$. The only allowed kinetic term therefore is

$$\mathcal{L}_{\rm kin} = \frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A^\mu \partial_\nu A^\nu\right) = -\frac{1}{4}F_{\mu\nu}^2 \,, \tag{5.2.8}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

What we have just discovered is rather remarkable: A massless spin-1 field *must* satisfy the *Maxwell Lagrangian*

$$\boxed{\mathcal{L}_{\rm Maxwell} = -\frac{1}{4}F_{\mu\nu}^2} \,, \tag{5.2.9}$$

if we want the theory to be ghost-free.

Notice that the field strength $F_{\mu\nu}$, and hence the Lagrangian (5.2.9), is invariant under the transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha \,, \tag{5.2.10}$$

for any function $\alpha(x)$. This is called a *gauge symmetry* of the theory. Field configurations $A_\mu(x)$ that differ by the derivative of a scalar function are physically equivalent. As we will now show, the gauge symmetry removes two degrees of freedom from the vector field A_μ, leaving precisely the two degrees of freedom of a massless spin-1 field.

The equation of motion associated with (5.2.9) is

$$\Box A_\mu - \partial_\mu(\partial_\nu A_\nu) = 0\,. \tag{5.2.11}$$

It is useful to separate this into the 0 and i components:

$$-\partial_j^2 A_0 + \partial_0\partial_j A_j = 0\,, \tag{5.2.12}$$

$$\Box A_i - \partial_i(\partial_0 A_0 - \partial_j A_j) = 0\,. \tag{5.2.13}$$

Under the gauge symmetry (5.2.10), we have $\partial_j A_j \to \partial_j A_j + \partial_j^2\alpha$. This allows us to set $\partial_j A_j = 0$. This choice is called *Coulomb gauge* and removes one degree of freedom. The equation of motion (5.2.12) then becomes

$$\partial_j^2 A_0 = 0\,. \tag{5.2.14}$$

The field component A_0 transforms as $A_0 \to A_0 + \partial_0\alpha$. Note that this doesn't change the equation of motion (5.2.14), as long as $\partial_j^2\alpha = 0$, so that we remain in Coulomb gauge. We can use this residual gauge freedom to set $A_0 \equiv 0$. The equation of motion for A_i then reduces to

$$\Box A_i = 0\,, \quad \partial_i A_i = 0\,. \tag{5.2.15}$$

This describes the two degrees of freedom associated with a massless spin-1 particle.

Massive spin-1 particles

A masssive spin-1 field must satisfy the *Proca Lagrangian*

$$\boxed{\mathcal{L}_{\rm Proca} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2}\,, \tag{5.2.16}$$

where the presence of a mass term does not change the fact that the kinetic has been uniquely fixed by the requirement of the absence of a ghost. The corresponding equation of motion is

$$\partial^\mu F_{\mu\nu} = m^2 A_\nu\,. \tag{5.2.17}$$

Acting on this with ∂^ν, the left-hand side vanishes, $\partial^\nu\partial^\mu F_{\mu\nu}$ because the field strength $F_{\mu\nu}$ is anti-symmetric. The right-hand side then implies the following constraint

$$\partial^\nu A_\nu = 0\,. \tag{5.2.18}$$

This removes one degree of freedom from A_μ, so that the theory has the expected three propagating degrees of freedom. This can also be seen by substituting $A_\mu \equiv \hat{A}_\mu + m^{-1}\partial_\mu\phi$ into (5.2.16), which gives

$$\mathcal{L}_{\rm Proca} = \frac{1}{2}(\partial_\mu\hat{A}_\nu)^2 + \frac{1}{2}m^2\hat{A}_\mu^2 + \frac{1}{2}(\partial_\mu\phi)^2\,. \tag{5.2.19}$$

We see that the longitudinal mode has been revived by the mass term. A massive vector field thus propagates three degrees of freedom: two transverse modes $\hat{A}_\mu$ and a longitudinal mode ϕ.

5.3 Gauge Symmetry

It is worth saying a few more words about the gauge symmetry of the Maxwell Lagrangian. First of all, "gauge symmetry" is a bad term, since it isn't really a symmetry at all, but rather a *redundancy of description*. Ordinary symmetries map a physical state into another physical state with the same properties. States that are related by gauge symmetries, on the other hand, are to be identified: they are the same physical state.

It is often helpful to describe the physics by selecting a representative field configuration from those related by the gauge symmetry. However, this procedure, called *gauge fixing*, is not unique. Different gauge choices can be useful for different purposes. We will present the quantization of the theory in two different gauges:

- **Lorentz gauge**: We can always pick a representative field configuration satisfying

$$\partial_\mu A^\mu = 0\,. \tag{5.3.20}$$

To see this, suppose that the gauge field $A'_\mu(x)$ does not satisfy (5.3.20), but instead we have $\partial^\mu A'_\mu = f(x)$, for some function $f(x)$. Applying the gauge transformation (5.2.10) leads to (5.3.20) if we can chose $\alpha(x)$ such that

$$\partial_\mu \partial^\mu \alpha = -f\,. \tag{5.3.21}$$

This equation is guaranteed to have a solution, so we can always impose the Lorentz gauge condition (5.3.20). In fact, the constraint (5.3.21) still does not pick a unique gauge, because we are always free to make further gauge transformations with

$$\partial_\mu \partial^\mu \alpha = 0\,, \tag{5.3.22}$$

which also has non-trivial solutions. This residual gauge symmetry can sometimes lead to confusions. As we will see, it leads to subtleties in the quantization of the theory. On the other hand, an advantage of working in Lorentz gauge is that it is manifestly Lorentz invariant.

- **Coulomb gauge**: The residual gauge transformations (5.3.22) allow us to set $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$. By the equations of motion, this implies

$$A_0 = 0\,. \tag{5.3.23}$$

A disadvantage of working on Coulomb gauge is that it breaks Lorentz invariance. Although the final answer for any physical observables will be Lorentz invariant, this will not be obvious at intermediate steps in the calculation. On the other hand, the physical degrees of freedom are most transparent in Coulomb gauge: the three components of the vector $\mathbf{A}$ satisfy $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, which leaves two physical degrees of freedom which after quantization become the two polarization states of the photon.

5.4 Quantization*

We will present the quantization of the Maxwell Lagrangian twice: first in Coulomb gauge and then in Lorentz gauge. We will get the same answers, but in each case we will face different subtleties.

To prepare for the quantization procedure, we derive the momentum Π^μ conjugate to A_μ:

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0\,, \tag{5.4.24}$$

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i\,. \tag{5.4.25}$$

The fact that Π^0 vanishes is consistent with the fact that A_0 is a non-dynamical field, but it highlights the problem of maintaining covariance in the quantization procedure. The remaining components of the momentum are simply the electric field E^i. We derive the Hamiltonian in the standard way

$$\begin{aligned} H &= \int \mathrm{d}^3x \left(\Pi^i \dot{A}_i - \mathcal{L}\right) \\ &= \int \mathrm{d}^3x \left[\frac{1}{2}(E_i^2 + B_i^2) - A_0(\boldsymbol{\nabla}\cdot\mathbf{E})\right], \end{aligned} \tag{5.4.26}$$

where $E_i \equiv \partial_0 A_i - \partial_i A_0$ and $B_i \equiv \epsilon_{ijk}\partial_j A_k$. This agrees with the conserved energy derived via Noether's theorem in (1.2.48). We see that A_0 plays the role of a Lagrange multiplier imposing Gauss' law, $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$.

Coulomb gauge

In §5.2, we showed that the Maxwell equation in Coulomb gauge becomes

$$\Box \mathbf{A} = 0\,, \tag{5.4.27}$$

subject to the constraint $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. Solutions to this equation can be written as

$$\mathbf{A}(x) = \boldsymbol{\epsilon}(\mathbf{p})\, e^{-ipx}\,, \tag{5.4.28}$$

which is analogous to the positive frequency spinor solutions discussed in the previous chapter, cf. eq. (4.3.85). In order for the gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ to hold, the polarization vectors $\boldsymbol{\epsilon}(\mathbf{p})$ must satisfy $\mathbf{p}\cdot\boldsymbol{\epsilon} = 0$, i.e. they are transverse polarizations. For $\mathbf{p} = (0,0,p)$, the two independent polarization vectors can be chosen to be $\boldsymbol{\epsilon}_1 = \frac{1}{\sqrt{2}}(1,+i,0)$ and $\boldsymbol{\epsilon}_2 = \frac{1}{\sqrt{2}}(1,-i,0)$.

The mode expansion of the field operator and its conjugate momentum therefore are

$$\mathbf{A}(\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_\lambda \left[\boldsymbol{\epsilon}_\lambda(\mathbf{p})\, a_\mathbf{p}^\lambda\, e^{+i\mathbf{p}\cdot\mathbf{x}} + \boldsymbol{\epsilon}_\lambda^*(\mathbf{p})\, a_\mathbf{p}^{\lambda\dagger}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\right], \tag{5.4.29}$$

$$\mathbf{E}(\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}(-i)\sqrt{\frac{E_\mathbf{p}}{2}}\sum_\lambda \left[\boldsymbol{\epsilon}_\lambda(\mathbf{p})\, a_\mathbf{p}^\lambda\, e^{+i\mathbf{p}\cdot\mathbf{x}} - \boldsymbol{\epsilon}_\lambda^*(\mathbf{p})\, a_\mathbf{p}^{\lambda\dagger}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\right]. \tag{5.4.30}$$

where the sum runs over the two polarizations $\lambda = 1, 2$. We impose the usual commutation relations on the operators $a_\mathbf{p}^\lambda$ and $a_\mathbf{p}^{\lambda\dagger}$:

$$\left[a_\mathbf{p}^\lambda, a_\mathbf{q}^{\lambda'\dagger}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,\delta^{\lambda\lambda'}\,, \tag{5.4.31}$$

$$\left[a_\mathbf{p}^\lambda, a_\mathbf{q}^{\lambda'}\right] = \left[a_\mathbf{p}^{\lambda\dagger}, a_\mathbf{q}^{\lambda'\dagger}\right] = 0\,. \tag{5.4.32}$$

Note that this implies a somewhat unusual commutation relation for A^i and $\Pi^i = E^i$, namely

$$[A^i(t,\mathbf{x}), E^j(t,\mathbf{y})] = i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left(\delta^{ij} - \frac{p^i p^j}{|\mathbf{p}|^2}\right) \;\equiv\; i\,\delta_\perp^{ij}(\mathbf{x}-\mathbf{y})\,, \tag{5.4.33}$$

where $\delta_\perp^{ij}(\mathbf{x}-\mathbf{y})$ is called the "transverse" Dirac delta function. This would be an ordinary delta function if we didn't have the $p^i p^j/\mathbf{p}^2$ term in the integrand. To see that this extra term is necessary, let us take the divergence of both sides of (5.4.33) with respect to $\mathbf{x}$. On the left-hand side, we find $[\boldsymbol{\nabla}\cdot\mathbf{A}(t,\mathbf{x}),\mathbf{E}(t,\mathbf{y})]$ which vanishes because $\boldsymbol{\nabla}\cdot\mathbf{A}=0$ in Coulomb gauge. On the right-hand side, the integrand becomes $p^i(\delta^{ij}-p^ip^j/\mathbf{p}^2) = p^j - p^j = 0$. We see that without the $p^ip^j/\mathbf{p}^2$ term in the integrand the commutator would not be consistent with the constraint $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. Taking the divergence with respect to $\mathbf{y}$ would give $[\mathbf{A}(t,\mathbf{x}),\boldsymbol{\nabla}\cdot\mathbf{E}(t,\mathbf{y})]$ on the left-hand side, which also has to vanish because of Gauss' law $\boldsymbol{\nabla}\cdot\mathbf{E}=0$.

Exercise.—Prove the following identity

$$\sum_\lambda \epsilon_\lambda^i(\mathbf{p})\epsilon_\lambda^j(\mathbf{p}) = \delta^{ij} - \frac{p^ip^j}{|\mathbf{p}|^2}\,. \tag{5.4.34}$$

Use it to show that (5.4.31) and (5.4.32) imply (5.4.33).

To find the Hamiltonian operator, we substitute the mode expansions (5.4.29) and (5.4.30) into (5.4.26). After normal ordering, this gives

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_\lambda E_{\mathbf{p}}\, a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}\,. \tag{5.4.35}$$

Similarly, the momentum operator becomes

$$\mathbf{P} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_\lambda \mathbf{p}\, a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}\,, \tag{5.4.36}$$

where $|\mathbf{p}| = E_{\mathbf{p}}$. This shows that $a_{\mathbf{p},\lambda}^\dagger|0\rangle$ creates particles with momentum $\mathbf{p}$ and energy $E_{\mathbf{p}} = |\mathbf{p}|$. With a bit more work, we could also show that these particles carry spin 1 and helicity $\lambda=\pm1$. We have discovered photons!

Finally, let us consider the Feynman propagator for the fields $A^i(x)$:

$$\Delta_\perp^{ij}(x-y) \equiv \langle 0|T\{A^i(x)A^j(y)\}|0\rangle\,. \tag{5.4.37}$$

which sometimes is referred to as the "transverse" propagator. Substituting the mode expansion (5.4.29) into (5.4.37), we find

$$\Delta_\perp^{ij}(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}\left(\delta^{ij}-\frac{p^ip^j}{|\mathbf{p}|^2}\right)e^{-ip\cdot(x-y)}\,. \tag{5.4.38}$$

The lack of manifest Lorentz symmetry[2] of this form of the propagator is the price that we have to pay for working in Coulomb gauge. However, in §5.5, we will see that this propagator can be massaged into a much nicer, and more manifestly Lorentz-invariant form, once we couple the theory to matter.

[2]To prove that the theory secretly is still Lorentz invariant, we could express all generators of the Poincaré group in terms of the creation and annihilation operators (we already did this explicitly for the four-momentum). Using the commutation relations for the creation and annihilation operators, we could then show that these generators indeed satisfy the Lorentz algebra. This proves covariance of the quantization in Coulomb gauge.

Lorentz gauge

To maintain Lorentz invariance throughout the quantization procedure, we must be able to treat all components of $A^\mu(x)$ on an equal footing and impose the following commutation relations

$$[A^\mu(t,\mathbf{x}),\Pi^\nu(t,\mathbf{y})] = i\eta^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{5.4.39}$$

$$[A^\mu(t,\mathbf{x}),A^\nu(t,\mathbf{y})] = [\Pi^\mu(t,\mathbf{x}),\Pi^\nu(t,\mathbf{y})] = 0\,. \tag{5.4.40}$$

However, this is in conflict with the fact that we have $\Pi^0 = 0$ for the Maxwell Lagrangian, cf. eq. (5.4.24). The solution will be to modify the Lagrangian in such a way that $\Pi^0 \neq 0$. This procedure will introduce spurious degrees of freedom that in the end have to be removed by hand from the spectrum of states.

Consider the following Lagrangian

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}^2 - \frac{\xi}{2}(\partial_\mu A^\mu)^2\,, \tag{5.4.41}$$

where ξ is a parameter that, in principle, can be chosen freely. Classically, the Lagrangian $\mathcal{L}'$ reduces to the Maxwell Lagrangian if we impose the Lorentz gauge condition $\partial_\mu A^\mu = 0$. The extra contribution in (5.4.41) is called a *gauge-fixing* term. It plays the role of a Lagrange multiplier that imposed the Lorentz constraint on the field. We will see, however, the quantum-mechanically, the Lorentz condition cannot hold as a operator statement, i.e. we are forced to $\partial_\mu \hat{A}^\mu(x) \neq 0$. Instead, we will impose a weaker condition on the physical states, namely, $\langle\Psi|\partial_\mu \hat{A}^\mu(x)|\Psi\rangle = 0$.

The conjugate momentum now is

$$\Pi^\mu = \frac{\partial\mathcal{L}'}{\partial\dot{A}^\mu} = -F^{0\mu} - \xi\eta^{0\mu}(\partial_\nu A^\nu)\,. \tag{5.4.42}$$

In particular, the field A_0 has the conjugate partner $\Pi^0 = -\xi(\partial_\nu A^\nu)$. The equation of motion corresponding to $\mathcal{L}'$ is

$$\Box A^\mu - (1-\xi)\partial^\mu(\partial_\nu A^\nu) = 0\,. \tag{5.4.43}$$

Classically, this is equivalent to the Maxwell equation in Lorentz gauge, i.e. $\Box A^\mu = 0$, with $\partial_\mu A^\mu = 0$.

In the following, we will specialize to the case $\xi = 1$ (sometimes confusingly called *Feynman gauge*). The Lagrangian can then be brought into a particularly simple form after integration by parts in the action integral:

$$\begin{aligned}\mathcal{L}' &= -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu)\\ &= -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu\left[A_\nu(\partial^\nu A^\mu) - (\partial_\nu A^\nu)A^\mu\right]\,.\end{aligned} \tag{5.4.44}$$

The last term is a total divergence and therefore does not contribute to the equation of motion. The dynamics can therefore be described by

$$\mathcal{L}'' = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu\,, \tag{5.4.45}$$

and the conjugate momentum is

$$\Pi^\mu = \frac{\partial\mathcal{L}''}{\partial\dot{A}^\mu} = -\dot{A}^\mu\,. \tag{5.4.46}$$

It is now manifest that the time and space components of the field A^μ appear on an equal footing. The Hamiltonian density is

$$\mathcal{H}'' = \Pi^\mu \dot{A}_\mu - \mathcal{L}'' = -\frac{1}{2}\Pi^\mu\Pi_\mu + \frac{1}{2}\partial_i A_\mu \partial^i A^\mu \,. \tag{5.4.47}$$

Written out in components, this becomes

$$\mathcal{H}'' = -\frac{1}{2}\left[(\dot{A}^0)^2 + (\nabla A^0)^2\right] + \frac{1}{2}\left[(\dot{A}^i)^2 + (\nabla A^i)^2\right] . \tag{5.4.48}$$

Notice the wrong sign for the scalar part A^0! The Hamiltonian isn't positive definite. This shouldn't come as a surprise. In §5.2, we worked hard to show that the Maxwell Lagrangian is the unique Lagrangian for a massless vector field with positive definite Hamiltonian. By adding the extra term in (5.4.41) we undid all that good work. As we will see, imposing the gauge condition will fix the problem and ensure that the Hamiltonian of the physical degrees of freedom is positive definite. The ghost degree of freedom in (5.4.48) will be removed from the spectrum of allowed states.

Turning the classical fields into operators, we impose the commutation relation (5.4.39), or

$$[A^\mu(t,\mathbf{x}), \dot{A}^\nu(t,\mathbf{y})] = -i\eta^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \,. \tag{5.4.49}$$

For the spatial components A^i these just look like three copies of the commutation relations for a scalar field, $[\phi^\mu(t,\mathbf{x}), \dot{\phi}(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$. The commutation relation for A^0, on the other hand, has the wrong sign! This is unavoidable if the commutation relation is written in covariant form: the right-hand side must contain the metric tensor $\eta^{\mu\nu}$ which contains both signs. States associates with the operator A^0 will have negative norm. It is a good thing that we already decided that we will get ride of these states eventually, i.e. once the gauge condition is imposed.

Fearlessly, we march on. Expanding the operator $A^\mu(x)$ in plane wave solutions, we get

$$A^\mu(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=0}^{3}\left[\epsilon^\mu_\lambda(\mathbf{p})\, a^\lambda_\mathbf{p}\, e^{-ip\cdot x} + \epsilon^{\mu *}_\lambda(\mathbf{p})\, a^{\lambda\dagger}_\mathbf{p}\, e^{ip\cdot x}\right] . \tag{5.4.50}$$

This time we have four polarization vectors $\epsilon^\mu_\lambda(\mathbf{p})$ instead of the two polarization vectors we encountered in Coulomb gauge. This is because we are working with all four components of the redundant field A^μ and haven't yet imposed a gauge condition. We choose ϵ^μ_0 to be timelike and $\epsilon^\mu_{1,2,3}$ to be spacelike. We select ϵ^μ_1 and ϵ^μ_2 as the two transverse polarizations

$$p_\mu\epsilon^\mu_1 = p_\mu\epsilon^\mu_2 = 0 \,. \tag{5.4.51}$$

The polarization vector ϵ^μ_3 is then longitudinal. Finally, we normalize the polarization vectors, so that

$$\eta_{\mu\nu}\epsilon^\mu_\lambda\epsilon^\nu_{\lambda'} = \eta_{\lambda\lambda'} \,. \tag{5.4.52}$$

Substituting (5.4.50) into (5.4.49), we get

$$\left[a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_\mathbf{q}\right] = -\eta^{\lambda\lambda'}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \,. \tag{5.4.53}$$

As we already anticipated, the signs are weird. For $\lambda = 1,2,3$, everything looks fine, but for $\lambda = 0$, we get the wrong sign

$$\begin{aligned}\left[a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_\mathbf{q}\right] &= \delta^{\lambda\lambda'}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \,, \quad \lambda,\lambda' = 1,2,3, \\ \left[a^0_\mathbf{p}, a^{0\dagger}_\mathbf{q}\right] &= -(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \,.\end{aligned} \tag{5.4.54}$$

Naively, this spells disaster for the stability of the theory. Consider the vacuum and one-particle excitations defined in the usual way

$$\begin{aligned} a^\lambda_{\mathbf{p}}|0\rangle &= 0\,, \\ |\mathbf{p},\lambda\rangle &\equiv a^{\lambda\dagger}_{\mathbf{p}}|0\rangle\,. \end{aligned} \tag{5.4.55}$$

For states with timelike polarization, we then get

$$\langle \mathbf{p},0|\mathbf{p},0\rangle = \langle 0|a^0_{\mathbf{p}}a^{0\dagger}_{\mathbf{p}}|0\rangle = -(2\pi)^3\delta^{(3)}(0)\,. \tag{5.4.56}$$

What is the negative sign doing there? A Hilbert space with negative norm means negative probability which doesn't make any sense.

Gupta-Bleuler resolution

Imposing the Lorentz condition $\partial_\mu A^\mu = 0$ will come to our rescue. How this is done, however, is a bit subtle. Asking for the operator A^μ to satisfy $\partial_\mu A^\mu = 0$ would not be consistent with the commutation relations. Its the old issue that $\Pi^0 = -(\partial_\mu A^\mu)$ must be non-zero. A weaker condition would be to impose

$$\partial_\mu A^\mu|\Psi\rangle = 0\,, \tag{5.4.57}$$

and hope that the states Ψ that satisfy this would exclude the negative norm states. Unfortunately, this also doesn't work. To see this, let us split the operator $A_\mu(x)$ into positive and negative frequency more, i.e. $A_\mu(x) = A^+_\mu(x) + A^-_\mu(x)$, where $A^+_\mu(x)|0\rangle = 0$. The problem is then apparent: since $\partial^\mu A^-_\mu|0\rangle \neq 0$, not even the vacuum would be a physically allowed state if we impose the condition (5.4.57). To keep the vacuum alive, we try the so-called *Gupta-Bleuler* condition

$$\partial^\mu A^+_\mu|\Psi\rangle = 0\,, \tag{5.4.58}$$

Note that this implies $\langle\Psi|\partial^\mu A^+_\mu = 0$, so that the operator $\partial_\mu A^\mu$ has vanishing matrix elements between physical states

$$\langle\Psi'|\partial_\mu A^\mu|\Psi\rangle = 0\,. \tag{5.4.59}$$

Unfortunately, even the Gupta-Bleuler condition doesn't remove the negative norm states from the physical Hilbert space. But it is close, so let's not give up prematurely.

Let us split the states of the Hilbert space into states $|\psi_\perp\rangle$ containing only transverse photons (created by $a^{1\dagger}_{\mathbf{p}}$ and $a^{2\dagger}_{\mathbf{p}}$) and states $|\phi\rangle$ including both timelike photons (created by $a^{0\dagger}_{\mathbf{p}}$) and the longitudinal photons (created by $a^{3\dagger}_{\mathbf{p}}$). The Gupta-Bleuler condition (5.4.58) requires that

$$(a^3_{\mathbf{p}} - a^0_{\mathbf{p}})|\phi\rangle = 0\,. \tag{5.4.60}$$

In words, physical states must contain combinations of timelike and longitudinal photons: a state with a timelike photon of momentum $\mathbf{p}$, must also contain a longitudinal photon with the same momentum. A general state $|\phi\rangle$ will be a linear combination of states ϕ_n with n pairs of timelike and longitudinal photons:

$$|\phi\rangle = \sum_{n=0}^{\infty} c_n|\phi_n\rangle\,, \tag{5.4.61}$$

where $|\phi_0\rangle = |0\rangle$. For $n > 0$, the states $|\phi_n\rangle$ have zero norm

$$\langle\phi_n|\phi\rangle = \delta_{0n}\delta_{0m}\,. \tag{5.4.62}$$

This is progress, since the inner product of all states is now positive semi-definite. However, nobody has ever seen timelike or longitudinal photons, so what are we to make of the state zero-norm states $|\phi_n\rangle$? The answer is that they are gauge equivalent to the vacuum state. In other words, two states that differ only by $|\phi\rangle$ are physically equivalent, i.e. $|\psi_\perp\rangle + |\phi\rangle \sim |\psi_\perp\rangle$. This means that we will get the same expectation values for all physical observables if we simply ignore the states $|\phi\rangle$. For example, consider the Hamiltonian

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}|\mathbf{p}| \left(\sum_{\lambda=1}^{3} a_{\mathbf{p}}^{\lambda\dagger} a_{\mathbf{p}}^{\lambda} - a_{\mathbf{p}}^{0\dagger} a_{\mathbf{p}}^{0} \right) . \tag{5.4.63}$$

The Gupta-Bleuler condition (5.4.60) implies $\langle\Psi|a_{\mathbf{p}}^{3\dagger} a_{\mathbf{p}}^{3}|\Psi\rangle = \langle\Psi|a_{\mathbf{p}}^{0\dagger} a_{\mathbf{p}}^{0}|\Psi\rangle$, so that the contribution from the timelike and longitudinal photons cancel in the expectation value of the Hamiltonian. This also makes the Hamiltonian positive definite. Life is good again.

5.5 Photon Propagator

The Feynman propagator of the vector field A^μ is defined as

$$\Delta_F^{\mu\nu}(x-y) \equiv \langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle \, . \tag{5.5.64}$$

In this section, we compute this propagator in both Lorentz and Coloumb gauge.

Lorentz gauge

Direct substitution of the mode expansions (5.4.50) into (5.5.64) gives

$$\Delta_F^{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{-i\eta^{\mu\nu}}{p^2+i\epsilon} e^{-ip\cdot(x-y)} \, , \tag{5.5.65}$$

or, in momentum space,

$$\boxed{\Delta_F^{\mu\nu}(p) = \frac{-i\eta^{\mu\nu}}{p^2+i\epsilon}} \, . \tag{5.5.66}$$

Unlike the propagator in Coulomb gauge, this form of the propagator is manifestly Lorentz invariant.

Coulomb gauge*

In §6.1, we will see that gauge invariance demands that the coupling between light and matter is through couplings of the vector potential A_μ to conserved currents J^μ. Including this interaction into the Maxwell Lagrangian, we obtain

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - A_\mu J^\mu + \cdots \, , \tag{5.5.67}$$

where J^μ is some function of the matter fields (to be discussed in the next chapter) and the ellipses denote additional terms that characterize the dynamics of the matter fields (e.g. the Dirac Lagrangian for fermions). Let us use this Lagrangian to show that the ugly looking transverse propagator in Coulomb gauge, cf. eq. (5.4.38), can be written in a nicer form.

We first note that, in the Coulomb gauge, the Lagrangian (5.5.67) becomes

$$\mathcal{L} = \frac{1}{2}\dot{A}_i^2 - \frac{1}{2}(\partial_j A_i)^2 - \frac{1}{2}A_0\nabla^2 A^0 - A_0 J^0 + A_i J^i \,. \tag{5.5.68}$$

Unlike before, we are not allowed to set $A_0 = 0$ in the interacting theory. Instead, the equation of motion implies

$$\nabla^2 A_0 = -J^0 \quad \Rightarrow \quad A_0(t,\mathbf{x}) = \int \mathrm{d}^3 y\, \frac{J^0(t,\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \,. \tag{5.5.69}$$

Since A_0 is non-dynamical, we can plug this back into the action to get

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu A_i)^2 + A_i J^i - \frac{1}{2}A_0 J^0 \\ &= \frac{1}{2}(\partial_\mu A_i)^2 + A_i J^i + \frac{1}{2}\int \mathrm{d}^3 y\, \frac{J^0(t,\mathbf{x})J^0(t,\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \,.\end{aligned} \tag{5.5.70}$$

The last there is called the *Coulomb term*. It looks nonlocal, but this is an artefact of working in Coulomb gauge and will not show up in physical observables.

We could capture the Coulomb term by defining the following propagator for the field $A^0(x)$:[3]

$$\Delta_F^{00}(x-y) = \frac{\delta(x^0-y^0)}{4\pi|\mathbf{x}-\mathbf{y}|} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{|\mathbf{p}|^2} \,. \tag{5.5.71}$$

This can be combined with the propagator for the field $A^i(x)$, cf. eq. (5.4.38), by writing

$$\Delta_F^{\mu\nu}(p) = \begin{cases} \dfrac{i}{|\mathbf{p}|^2} & \mu=\nu=0 \\[2ex] \dfrac{i}{p^2+i\epsilon}\left(\delta^{ij} - \dfrac{p^i p^j}{|\mathbf{p}|^2}\right) & \mu=i,\nu=j \\[2ex] 0 & \text{otherwise} \end{cases} \tag{5.5.72}$$

or, more compactly,

$$\Delta_F^{\mu\nu}(p) = \frac{i}{p^2+i\epsilon}\left(P_\perp^{\mu\nu}(p) + \frac{p^2}{|\mathbf{p}|^2}n^\mu n^\nu\right), \tag{5.5.73}$$

where $P_\perp^{0\nu} \equiv 0$, $P_\perp^{ij} \equiv \delta^{ij} - p^i p^j/|\mathbf{p}|^2$, and $n^\mu \equiv (1,0,0,0)$. To massage this further, let us define $\tilde{p}^\mu$ as the vector consisting of only the spatial components of p^μ, i.e. $\tilde{p}^\mu \equiv (0, p^i)$. We can then write (5.5.73) as

$$\begin{aligned}\Delta_F^{\mu\nu}(p) &= \frac{i}{p^2+i\epsilon}\left(-\eta^{\mu\nu} + n^\mu n^\nu - \frac{\tilde{p}^\mu\tilde{p}^\nu}{|\mathbf{p}|^2} + \frac{(p^0)^2-|\mathbf{p}|^2}{|\mathbf{p}|^2}n^\mu n^\nu\right) \\ &= \frac{i}{p^2+i\epsilon}\left(-\eta^{\mu\nu} - \frac{\tilde{p}^\mu\tilde{p}^\nu}{|\mathbf{p}|^2} + \frac{(p^0 n^\mu)(p^0 n^\nu)}{|\mathbf{p}|^2}\right) \\ &= \frac{i}{p^2+i\epsilon}\left(-\eta^{\mu\nu} - \frac{\tilde{p}^\mu\tilde{p}^\nu}{|\mathbf{p}|^2} + \frac{(p^\mu-\tilde{p}^\mu)(p^\nu-\tilde{p}^\nu)}{|\mathbf{p}|^2}\right) \\ &= \frac{i}{p^2+i\epsilon}\left(-\eta^{\mu\nu} + \frac{p^\mu p^\nu - p^\mu\tilde{p}^\nu - \tilde{p}^\mu p^\nu}{|\mathbf{p}|^2}\right).\end{aligned} \tag{5.5.74}$$

[3]The field A^0 mediates the Coulomb interaction and sometime is referred to as the *Coulomb photon*.

The crucial feature of the second term in (5.5.74) is that it is proportional to p^μ, so it doesn't contribute when contracted with a conserved current:

$$J^*_\mu(p)\Delta_F^{\mu\nu}(p)J_\nu(p) = \frac{-i}{p^2}J^*_\mu(p)J^\mu(p)\,. \tag{5.5.75}$$

As we will see, physical observables will always involve these types of contractions, so without loss of generality we can work with the simpler propagator

$$\boxed{\Delta_F^{\mu\nu}(p) = \frac{-i\eta^{\mu\nu}}{p^2 + i\epsilon}}\,. \tag{5.5.76}$$

Of course, this is the same propagator we found more directly in Lorentz gauge.

5.6 Problems

1. A natural guess for the Lagrangian of a massive spin-1 field would be

$$\mathcal{L} \stackrel{?}{=} -\frac{1}{2}\partial_\nu A_\mu \partial^\nu A^\mu + \frac{1}{2}m^2 A_\mu^2\,,$$

where $A_\mu^2 \equiv A_\mu A^\mu$. Show that this would lead to a Hamiltonian that is unbounded from below. In the quantum theory this would lead to a catastrophic instability.

Then, try the Lagrangian

$$\mathcal{L} \stackrel{?}{=} -\frac{1}{2}\partial_\nu A_\mu \partial^\nu A^\mu + \frac{b}{2}A_\mu \partial_\mu \partial_\nu A_\nu + \frac{1}{2}m^2 A_\mu^2\,,$$

where b is an arbitrary constant. Determine the value of b for which this theory propagates exactly three degrees of freedom. Show that the corresponding Hamiltonian is bounded from below.

2. Consider the *Proca Lagrangian* for a massive vector field

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2\,.$$

i) Verify that this theory is not gauge invariant and show that the equations of motion derived from this Lagrangian are

$$(\Box + m^2)A^\mu = 0\,, \quad \partial_\mu A^\mu = 0\,.$$

ii) Perform the canonical quantization of the Proca theory and verify that it leads to massive particles of spin 1.

6 Quantum Electrodynamics

After the legwork performed in the previous chapters, we are now in a position to apply QFT to the interactions of light and matter. The resulting theory is called *quantum electrodynamics*, or QED, and is one of the greatest intellectual achievements in the history of human civilization.

We will start by showing that the gauge symmetry of massless vector fields uniquely fixes their couplings to matter. We then derive the Feynman rules for QED and apply them to a number of famous examples.

6.1 Light and Matter

In classical electrodynamics, the interactions between light and matter are captured by adding a current as a source to the inhomogeneous Maxwell equation

$$\partial_\mu F^{\mu\nu} = J^\nu \, . \tag{6.1.1}$$

Since $F^{\mu\nu}$ is anti-symmetric, we have $\partial_\nu \partial_\mu F^{\mu\nu} = 0$ and hence $\partial_\nu J^\nu = 0$, i.e. the current is conserved. The conservation of the current can also be understood as a consequence of gauge invariance. The action that gives rise to the equation of motion (6.1.1) is

$$S = -\int \mathrm{d}^4x \left(\frac{1}{4} F_{\mu\nu}^2 + J^\mu A_\mu \right) . \tag{6.1.2}$$

Under a gauge transformation, $A_\mu \to A_\mu + \partial_\mu \alpha$, the term $J^\mu A_\mu$ changes by $J^\mu \partial_\mu \alpha$. Integrating by parts and dropping the boundary term, the change in the action is

$$\delta S = \int \mathrm{d}^4x \, \alpha \, \partial_\mu J^\mu \, . \tag{6.1.3}$$

The action is therefore gauge invariant if and only if the current is conserved, $\partial_\mu J^\mu = 0$.

As we discussed in §5.2, the gauge symmetry of the free field theory was required in order not to excite more than the two degrees of freedom that are expected for a massless vector field A_μ. In order not to affect our counting of degrees of freedom, the interactions of A_μ must also respect gauge invariance. This will severely restrict the allowed interactions between light and matter.

Coupling to fermions

Consider the Dirac action

$$S = \int \mathrm{d}^4x \; \bar{\psi}(i\partial\!\!\!/ - m)\psi \, . \tag{6.1.4}$$

In §4.3.2, we saw that this is invariant under a global phase rotation, $\psi \to e^{i\beta}\psi$, for constant β. Moreover, we learned that this leads to a globally conserved charge. However, we also expect charge conservation to hold locally. Let us therefore try to make the phase rotation of the Dirac spinor a *local* symmetry,

$$\psi \to e^{i\beta(x)}\psi \,. \tag{6.1.5}$$

The mass term in (6.1.4) clearly is invariant under this transformation as well. But, what about the kinetic term? Using

$$\bar\psi \;\to\; \bar\psi\, e^{-i\beta(x)}\,, \tag{6.1.6}$$

$$\partial_\mu\psi \;\to\; \partial_\mu(e^{i\beta(x)}\psi(x)) = e^{i\beta(x)}\partial_\mu\psi(i\partial_\mu\beta)e^{i\beta(x)}\psi(x)\,, \tag{6.1.7}$$

we find that the action transforms as

$$\delta S = -\int \mathrm{d}^4x\; (\partial_\mu\beta)\,\bar\psi\psi\,. \tag{6.1.8}$$

We see that this is only invariant if $\partial_\mu\beta = 0$, i.e. the symmetry is global. Not all is lost however. Notice that the change in the action is proportional to $\partial_\mu\beta$ which looks like the gauge transformation of a massless vector field. Could a coupling between A_μ and ψ cancel the offensive term in (6.1.8)?

Let us write the Dirac action as

$$S = \int \mathrm{d}^4x\; \bar\psi(i\not{D} - m)\psi\,, \tag{6.1.9}$$

where we have introduced the *covariant derivative*

$$D_\mu\psi \equiv (\partial_\mu + iqA_\mu)\,\psi\,. \tag{6.1.10}$$

The coupling q defines the strength of the interaction between A_μ and ψ. Consider now the following combined transformation

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu\alpha(x)\,,\\ \psi &\to e^{i\beta(x)}\psi\,. \end{aligned} \tag{6.1.11}$$

The covariant derivative transforms as

$$D_\mu\psi \to e^{i\beta(x)}D_\mu\psi + (i\partial_\mu\beta)e^{i\beta(x)}\psi(x) + (iq\,\partial_\mu\alpha)e^{i\beta(x)}\psi\,. \tag{6.1.12}$$

If we choose $\beta = -q\,\alpha$, then the last two terms chancel and the transformation of the covariant derivative take a particularly nice form

$$D_\mu\psi \to e^{-iq\,\alpha(x)}D_\mu\psi\,. \tag{6.1.13}$$

Combining this with $\bar\psi \to e^{+iq\,\alpha(x)}\bar\psi$ proves the gauge invariance of the action (6.1.14).

Writing out the Lagrangian in (6.1.14),

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi - qA_\mu\bar\psi\gamma^\mu\psi\,. \tag{6.1.14}$$

we see a unique coupling between A_μ and ψ. We recognize this as the coupling between A_μ and the vector current $J_V^\mu \equiv \bar\psi\gamma^\mu\psi$. The conservation of the vector current, $\partial_\mu J_V^\mu = 0$, implies gauge invariance of the action by the same argument as in classical electrodynamics.

Comparison with (6.1.2) suggests that the total conserved charge Q is

$$Q = q \int \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi = q \int \mathrm{d}^3x\, \psi^\dagger\psi\,. \tag{6.1.15}$$

Substituting the mode expansion of the Dirac spinor, we find

$$Q = q \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_s \left[b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s \right]. \tag{6.1.16}$$

This shows that the coupling q has the interpretation of the electric change of the ψ particles. We see that particles and antiparticles are required to have opposite charges. We will mostly study the interactions with electron, for which we write $q = e$. The elementary charge e is also often expressed in terms of a dimensionless ratio α, known as the fine structure constant and defined as

$$\alpha \equiv \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}\,. \tag{6.1.17}$$

Setting $\hbar = c = 1$, this implies $e = \sqrt{4\pi\alpha} \approx 0.3$. The fact that this coupling is smaller than one is what makes perturbation theory possible.

Coupling to complex scalars

Above we have defined the gauge-invariant coupling between photons and fermions by promoting the partial derivative in the Dirac Lagrangian to a covariant derivative. This procedure is called *minimal coupling*. This trick works for any theory, so let us see how this works for the coupling to scalar fields.

A real scalar field doesn't have a suitable conserved current, so we don't expect that it couples to a gauge field. We therefore consider the coupling to a complex scalar field φ. (To avoid confusion with spinors, we will, from now on, use φ instead of ψ for complex scalars.) We write the Klein-Gordon Lagrangian as

$$\mathcal{L} = D_\mu\varphi^* D^\mu\varphi - m^2\varphi^*\varphi\,, \tag{6.1.18}$$

where $D_\mu\varphi = (\partial_\mu + iqA_\mu)\varphi$ is the covariant derivative of the scalar field. As before, the covariant derivative transforms as $D_\mu\varphi \to e^{-iq\,\alpha} D_\mu\varphi$ under the gauge transformation $A_\mu \to A_\mu + \partial_\mu\alpha$ and $\varphi \to e^{-iq\,\alpha(x)}\varphi$. Writing out (6.1.19), we find

$$\mathcal{L} \;=\; \underbrace{|\partial_\mu\varphi|^2 - m^2|\varphi|^2}_{\mathcal{L}_0} \;+\; \underbrace{ieA_\mu(\varphi\partial^\mu\varphi^* - \varphi^*\partial^\mu\varphi) + e^2|\varphi|^2 A_\mu A^\mu}_{\mathcal{L}_{\rm int}}\,. \tag{6.1.19}$$

The linear coupling to A_μ is indeed of the form $A_\mu J^\mu$, with the current J^μ given by the Noether current found in (1.2.71). However, the minimal coupling procedure has also led to an additional higher-order coupling, $|\varphi|^2 A_\mu A^\mu$. In the Standard Model, this coupling is what gives rise to the masses of vector fields via the Higgs mechanism.

6.2 Feynman Rules

Quantum electrodynamics deals with the interactions of photons and electrons. In the previous section, we showed that theses interactions are uniquely fixes by gauge symmetry. The QED

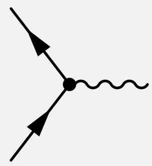
Lagrangian was determined to be

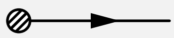
$$\begin{aligned}\mathcal{L}_{\mathrm{QED}} &= -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(i\not{D} - m)\psi \\ &= \underbrace{-\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi}_{\mathcal{L}_0} - \underbrace{qA_\mu\bar{\psi}\gamma^\mu\psi}_{\mathcal{L}_{\mathrm{int}}}\,.\end{aligned} \tag{6.2.20}$$

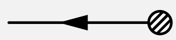
This Lagrangian implies the following Feynman rules:

Fermion propagator : $= \dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

Photon propagator : $= \dfrac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$

QED vertex :  $= -ie\gamma^\mu$

External fermions :
 $= u^s(p)$ (incoming)
 $= \bar{u}^s(p)$ (outgoing)

External antifermions :
 $= \bar{v}^s(p)$ (incoming)
 $= v^s(p)$ (outgoing)

External photons :
 $= \epsilon_\mu^\lambda(p)$ (incoming)
 $= \epsilon_\mu^{\lambda *}(p)$ (outgoing)

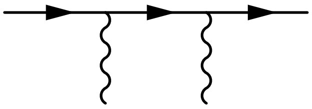
- Momentum is conserved at each vertex.
- Undetermined loop momenta are integrated over.
- Fermion loops receive a factor of (-1).
- Each diagram can potentially have a symmetry factor.

Before applying these rules, we make a few general comments:

• The index μ on the γ^μ of the vertex will get contracted with the μ of the photon, which will either be in the $\eta_{\mu\nu}$ of the propagator (if the photon is internal) or the ϵ_μ of the polarization vector (if the photon is external).

• The matrix $\gamma^\mu = \gamma^\mu_{\alpha\beta}$ will always get sandwiched between spinors, as in $\bar{u}\gamma^\mu u = \bar{u}_\alpha\gamma^\mu_{\alpha\beta}u_\beta$ for e^-e^- scattering. The barred spinor always goes on the left.

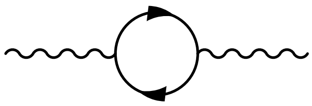
• If there is an internal fermion line between the end, the fermion propagator goes between the end spinors:



$$= \quad (-ie)^2\, \bar{u}(p_3)\gamma^\mu \frac{i(\not{p}_2 + m)}{p_2^2 - m^2 + i\epsilon}\gamma^\nu u(p_1)\, \epsilon^2_\mu(q_2)\epsilon^1_\nu(q_1)\,, \qquad (6.2.21)$$

where $q_1 \equiv p_2 - p1$ and $q_2 = p_3 - p_2$.

• Connecting the ends of the diagram in (6.2.21), we get a loop:



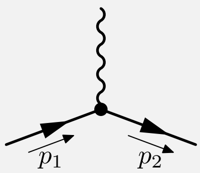
(6.2.22)

This diagram is called *vacuum polarization* and you will study it more in your next QFT course. Since any possible intermediate states are allowed in the loop, we must integrate over the momenta of the virtual spinors *and* sum over their possible spins. Replacing $\bar{u}_\alpha u_\beta$ in (6.2.21) by the propagator, in fact, automatically sums over all possible spins, since $(\not{p}_2 + m)_{\beta\alpha} = \sum_s u^s_\beta \bar{u}^s_\alpha$.

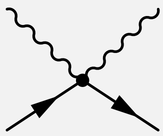
We also considered the coupling of photons to complex scalar fields. The resulting theory is called *scalar QED* and has the following Lagrangian

$$\begin{aligned} \mathcal{L}_{\rm sQED} &= -\frac{1}{4}F^2_{\mu\nu} + |D_\mu\varphi|^2 - m^2|\varphi|^2 \\ &= \underbrace{-\frac{1}{4}F^2_{\mu\nu} + |\partial_\mu\varphi|^2 - m^2|\varphi|^2}_{\mathcal{L}_0} + \underbrace{ieA_\mu(\varphi\partial^\mu\varphi^* - \varphi^*\partial^\mu\varphi) + e^2|\varphi|^2 A_\mu A^\mu}_{\mathcal{L}_{\rm int}}\,. \end{aligned} \qquad (6.2.23)$$

The two new interactions in $\mathcal{L}_{\rm int}$ have the following Feynman rules attached to them:



$$= \quad -ie(p_1 + p_2)_\mu$$



$$= \quad +2e^2\eta_{\mu\nu}$$

The momentum dependence of the cubic vertex arises from the derivative in $\varphi\partial^\mu\varphi^* A_\mu$.

6.3 Scattering in QED

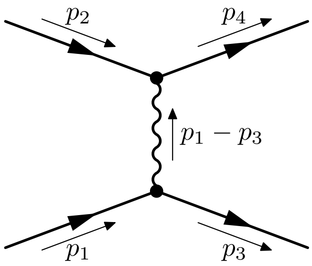
Let us revisit the scattering processes from §3.9 and §4.6, this time in full QED, i.e. with the spins of the electrons and the polarizations of the photons properly taken into account.

Amplitudes

Given the Feynman rules of the previous section, it is now straightforward to write down the amplitudes for a number of QED scattering processes.

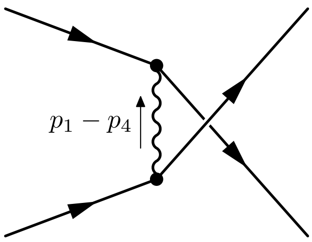
- **Electron-electron scattering.** The t- and u-channel amplitudes for the process $e^-e^- \to e^-e^-$ are given by

$$i\mathcal{A}_t \quad =$$



$$\begin{aligned}
&= \quad \bar{u}^{s_4}(p_4)(-ie\gamma^\mu)u^{s_2}(p_2) \times \frac{-i\eta_{\mu\nu}}{(p_1-p_3)^2} \times \bar{u}^{s_3}(p_3)(-ie\gamma^\nu)u^{s_1}(p_1) \\
&= \quad +\, ie^2\,\frac{[\bar{u}^{s_4}(p_4)\gamma^\mu u^{s_2}(p_2)][\bar{u}^{s_3}(p_3)\gamma_\mu u^{s_1}(p_1)]}{(p_1-p_3)^2}\,, \qquad (6.3.24)
\end{aligned}$$

$$i\mathcal{A}_u \quad =$$

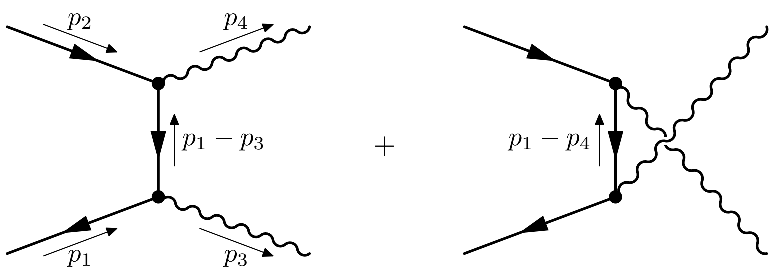


$$\begin{aligned}
&= \quad -\, \bar{u}^{s_3}(p_3)(-ie\gamma^\mu)u^{s_2}(p_2) \times \frac{-i\eta_{\mu\nu}}{(p_1-p_4)^2} \times \bar{u}^{s_4}(p_4)(-ie\gamma^\nu)u^{s_1}(p_1) \\
&= \quad -\, ie^2\,\frac{[\bar{u}^{s_3}(p_3)\gamma^\mu u^{s_2}(p_2)][\bar{u}^{s_4}(p_4)\gamma_\mu u^{s_1}(p_1)]}{(p_1-p_4)^2}\,. \qquad (6.3.25)
\end{aligned}$$

The total amplitude is the sum of the two contributions, $\mathcal{A} = \mathcal{A}_t + \mathcal{A}_u$.

- **Electron-positron annihilation.** The amplitude for $e^-e^+ \to \gamma\gamma$ is

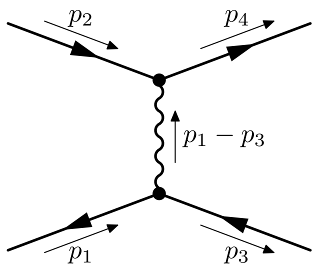
$$i\mathcal{A} \quad =$$



$$= \quad -\, ie^2\, \bar{v}^{s_1}(p_1)\,\frac{\gamma_\mu(\not{p}_1 - \not{p}_3 + m)\gamma_\nu}{(p_1-p_3)^2 - m^2}\, u^{s_2}(p_2)\,\epsilon^\nu_{\lambda_3}(p_3)\epsilon^\mu_{\lambda_4}(p_4) \;+\; (3 \leftrightarrow 4)\,. \qquad (6.3.26)$$

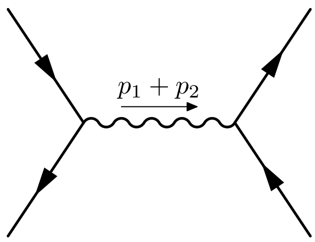
- **Electron-positron scattering.** The t- and s-channel amplitudes for $e^-e^+ \to e^-e^+$ scattering (also called Bhabha scattering) are

$$i\mathcal{A}_t =$$



$$\begin{aligned} &= \bar{u}^{s_4}(p_4)(-ie\gamma^\mu)u^{s_2}(p_2) \times \frac{-i\eta_{\mu\nu}}{(p_1-p_3)^2} \times \bar{v}^{s_1}(p_1)(-ie\gamma^\nu)v^{s_3}(p_3) \\ &= +\, ie^2\, \frac{[\bar{u}^{s_4}(p_4)\gamma^\mu u^{s_2}(p_2)][\bar{v}^{s_1}(p_1)\gamma_\mu v^{s_3}(p_3)]}{(p_1-p_3)^2}\,, \end{aligned} \tag{6.3.27}$$

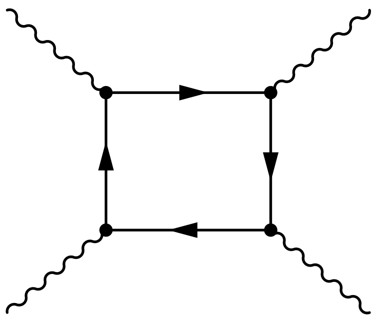
$$i\mathcal{A}_s =$$



$$\begin{aligned} &= -\,\bar{u}^{s_4}(p_4)(-ie\gamma^\mu)v^{s_3}(p_3) \times \frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2} \times \bar{v}^{s_1}(p_1)(-ie\gamma^\nu)u^{s_2}(p_2) \\ &= -\, ie^2\, \frac{[\bar{u}^{s_4}(p_4)\gamma^\mu v^{s_3}(p_3)][\bar{v}^{s_1}(p_1)\gamma_\mu u^{s_2}(p_2)]}{(p_1+p_2)^2}\,. \end{aligned} \tag{6.3.28}$$

The total amplitude is the sum of the two contributions, $\mathcal{A} = \mathcal{A}_t + \mathcal{A}_s$.

- **Photon scattering.** Maxwell's equations are linear, so we expect light waves to pass through each other unimpeded. This is no longer the case in QED. At one loop, we can have the following diagram for photon scattering



Previously, we found the amplitude for this process to be logarithmically divergent. In QED, however, gauge invariance makes the amplitude finite.

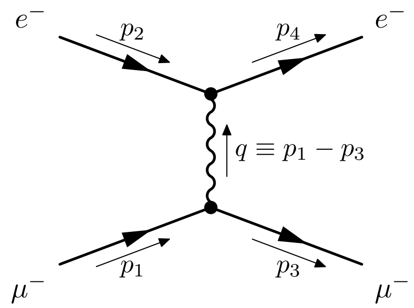
Cross sections and spin sums

To relate the results of the previous section to observables, we need to square the corresponding amplitudes. Usually we have an unpolarised beam and target and do not measure the polarisation of the outgoing particles. Thus we calculate the squared amplitudes for each possible spin combination then average over initial spin states and sum over final spin states. Note that we square and then sum since the different spin configurations are, in principle, distinguishable. In contrast, if several Feynman diagrams contribute to the same process you have to sum the amplitudes first.

We will illustrate this for the case of electron-muon scattering, $e^-\mu^- \to e^-\mu^-$. Further practice can be gained by working through the problems at the end of this chapter.

Only the t-channel contributes to electron-muon scattering:

$i\mathcal{A} \;=$



$$= \; i\,\frac{e^2}{q^2}\,\bar{u}_e(p_4)\gamma^\mu u_e(p_2)\,\bar{u}_m(p_3)\gamma_\mu u_m(p_1)\,, \tag{6.3.29}$$

where the subscripts e and m denote whether the spinors satisfy the Dirac equation for electrons or for muons. For clarity, we have dropped the spin labels on the spinors. We will restore them when needed. The square of the amplitude is

$$|\mathcal{A}|^2 = \frac{e^4}{q^4} L^{\mu\nu}_{(e)} L_{(m),\mu\nu}\,, \tag{6.3.30}$$

where we have defined

$$L^{\mu\nu}_{(e)} \equiv \bar{u}_e(p_4)\gamma^\mu u_e(p_2)\,\bar{u}_e(p_2)\gamma^\nu u_e(p_4)\,, \tag{6.3.31}$$

and similarly for $L^{\mu\nu}_{(m)}$. To perform the sum over spins, we use

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m\,. \tag{6.3.32}$$

We find

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}} |\mathcal{A}|^2 &= \frac{e^4}{4q^4}\Big[\gamma^\mu_{\alpha\beta}(\not{p}_2 + m_e)_{\beta\gamma}\gamma^\nu_{\gamma\delta}(\not{p}_4 + m_e)_{\delta\alpha}\Big]\Big[\gamma_{\mu,\alpha\beta}(\not{p}_1 + m_\mu)_{\beta\gamma}\gamma_{\nu,\gamma\delta}(\not{p}_3 + m_\mu)_{\delta\alpha}\Big] \\ &= \frac{e^4}{4q^2}\,\mathrm{Tr}\Big(\gamma^\mu(\not{p}_2 + m_e)\gamma^\nu(\not{p}_4 + m_e)\Big)\,\mathrm{Tr}\Big(\gamma_\mu(\not{p}_1 + m_\mu)\gamma_\nu(\not{p}_3 + m_\mu)\Big)\,. \end{aligned} \tag{6.3.33}$$

In the first line, we have written the spinor indices explicitly in order to show how the traces appear in the second line. We see that the calculation requires the evaluation of traces of products of gamma matrices. To compute this we use the anticommutation relations of the gamma matrices, together with the invariance of the trace under a cyclic change of its arguments. This leads to the following "trace theorems"

$$\mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b \tag{6.3.34}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\big(a \cdot b\, c \cdot d - a \cdot c\, b \cdot d + a \cdot d\, b \cdot c\big) \tag{6.3.35}$$

$$\mathrm{Tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_n}) = 0 \quad \text{for} \quad n = \text{odd} \tag{6.3.36}$$

Using these results, and expressing the answer in terms of the Mandelstam variables of equation, we find

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{A}|^2 = \frac{2e^4}{t^2}\Big(s^2 + u^2 - 4(m_e^2 + m_\mu^2)(s+u) + 6(m_e^2 + m_\mu^2)^2\Big)\,. \tag{6.3.37}$$

In the high-energy limit $s, |u| \gg m_e, m_\mu$, this leads to the following differential cross section in the center-of-mass frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 s}\frac{s^2 + u^2}{t^2}\,. \tag{6.3.38}$$

Other calculations of cross sections or decay rates follow the same steps. You draw the diagrams, write down the amplitude, square it and evaluate the traces.

Examples.—The squared amplitude for electron-electron scattering, $e^-e^- \to e^-e^-$, in the high-energy limit, is

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{A}|^2 = 2e^4\left(\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + \frac{2s^2}{tu}\right). \tag{6.3.39}$$

Similarly, the squared amplitude for electron-positron scattering, $e^-e^+ \to e^-e^+$, in the high-energy limit, is

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{A}|^2 = 2e^4\left(\frac{s^2 + u^2}{t^2} + \frac{u^2 + t^2}{s^2} + \frac{2u^2}{ts}\right). \tag{6.3.40}$$

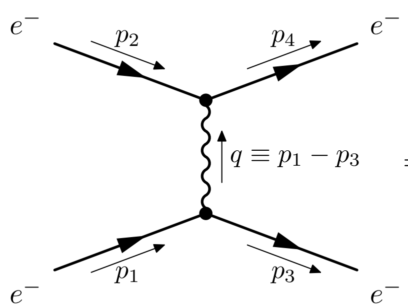
Note that these results are related by crossing symmetry, i.e. by performing the interchange $s \leftrightarrow u$.

6.4 Coulomb Potential

We will end this course by returning to something familiar: Coulomb's law. We will now show that it follows from the non-relativistic limit of scattering in QED.

We will repeat the computation that led to the Yukawa force in §3.11 and §4.6. First, we consider electron-electron scattering. To determine the scattering potential it suffices to look at the t-channel contribution

$\mathcal{A}_{e^-e^-} =$



$$= +\,e^2\,\bar{u}(p_4)\gamma^\mu u(p_2)\,\frac{-\eta_{\mu\nu}}{q^2}\,\bar{u}(p_3)\gamma^\nu u(p_1)\,. \tag{6.4.41}$$

In §4.3.3, we have seen that the non-relativistic limit of the spinor is

$$u(p) \to \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix} . \tag{6.4.42}$$

This implies that

$$\bar{u}^{s'}(p')\gamma^0 u^s(p) \to 2m\,\delta^{ss'} , \tag{6.4.43}$$

$$\bar{u}^{s'}(p')\gamma^i u^s(p) \to 0 . \tag{6.4.44}$$

We see that only the zeroth component of the gauge field contributes to the amplitude in the non-relativistic limit. The amplitude (6.4.41) then becomes

$$\mathcal{A}_{e^-e^-} \;\to\; -\,e^2(2m)^2\,\frac{\delta^{s_1s_3}\delta^{s_2s_4}}{|\mathbf{q}|^2} , \tag{6.4.45}$$

which has the same momentum dependence (for $\mu = 0$), but the opposite sign as the amplitude in (4.6.176). This extra minus sign relative to the Yukawa result can be traced to extra minus sign in the zero component of the propagator, $-\eta_{00} = -1$. The potential therefore is

$$V_{e^-e^-}(r) = +\frac{e^2}{4\pi r} , \tag{6.4.46}$$

which corresponds to a *repulsive* force.

Similarly, the leading amplitude for electron-positron scattering is

$$\mathcal{A}_{e^-e^+} \;=\; (-1)\times e^2\,\bar{u}(p_4)\gamma^\mu u(p_2)\,\frac{-\eta_{\mu\nu}}{q^2}\,\bar{v}(p_1)\gamma^\nu v(p_3) , \tag{6.4.47}$$

where the overall minus sign comes from treating the fermions correctly. In the non-relativistic limit, we have

$$\bar{v}^{s'}(p')\gamma^0 v^s(p) \to +2m\,\delta^{ss'} , \tag{6.4.48}$$

$$\bar{v}^{s'}(p')\gamma^i v^s(p) \to 0 . \tag{6.4.49}$$

We see that again only the zeroth component of the gauge field contributes to the amplitude in the non-relativistic limit. Moreover, comparing (6.4.48) to (4.6.175), we note that the insertion of the γ^0 matrix gives an extra minus sign relative to the Yukawa case. This minus sign cancels the extra minus sign from the zeroth component of the gauge field propagator, and the amplitude becomes

$$\mathcal{A}_{e^-e^+} \;=\; +\,e^2(2m)^2\,\frac{\delta^{s_1s_3}\delta^{s_2s_4}}{|\mathbf{q}|^2} \;=\; -\,\mathcal{A}_{e^-e^-} . \tag{6.4.50}$$

The potential between opposite charges therefore is

$$V_{e^-e^+}(r) = -\frac{e^2}{4\pi r} , \tag{6.4.51}$$

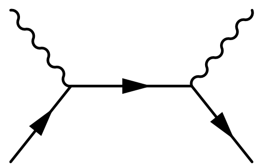
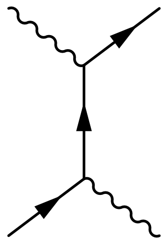
which corresponds to an *attractive* force.

We have therefore reproduced the classic result that like charges repel while unlike charges attract. To appreciate why this is a significant achievement, recall that in non-relativistic quantum mechanics the Coulomb potential is simply chosen from a range of many possibilities. In contrast, in quantum field theory the force between charged fermions is much more constrained. In particular, the gauge symmetry of QED enforced a unique interaction vertex, leaving us with no freedom to select the potential. There is no other way to couple a charged fermion to the electromagnetic field, and the only freedom is to change the value of the coupling by changing the charge of the particle. In this sense, quantum field theory provides a much deeper explanation of Coulomb's law. This may be a good place to end the course.

6.5 Problems

1. The scattering of photons (in particular X-rays) off electrons $e^-\gamma \to e^-\gamma$ is known as *Compton scattering.* Historically, the change in wavelength of the photon in the scattering process was one of the conclusive pieces of evidence that light could behave as a particle.

The two lowest order diagrams for Compton scattering are:

$\mathcal{A}_s =$  $\qquad \mathcal{A}_u =$ 

The amplitudes for these diagrams may be evaluated to give the *Klein-Nishina formula.* However, the full derivation is rather lengthy, so instead you will be asked to compute the amplitudes in the highly relativistic limit where you can ignore the mass of the electron.

- By averaging over the initial spin states and photon polarizations and summing over those of the final states, show that

$$\frac{1}{4}\sum |\mathcal{A}_s|^2 = -2e^4\,\frac{u}{s}\,, \qquad \frac{1}{4}\sum |\mathcal{A}_u|^2 = -2e^4\,\frac{s}{u}\,,$$

  where s and u are the Mandelstam variables.

There is no interference term, so the total squared amplitude is the sum of the s- and u-channel contributions.

2. If the energy is sufficiently high, electron-positron annihilation can lead to the production of muons and anti-muons, $e^-e^+ \to \mu^-\mu^+$.

- Draw the relevant Feynman diagrams at lower order in the coupling e.
- Show that the corresponding amplitude is

$$\mathcal{A} = (-ie)^2\,\frac{[\bar{v}_e^{s_1}(p_1)\gamma_\mu u_e^{s_2}(p_2)][\bar{u}_m^{s_4}(p_4)\gamma^\mu v_m^{s_3}(p_3)]}{(p_1+p_2)^2}\,,$$

  where the subscripts e and m denote whether the spinors satisfy the Dirac equation for electrons or for muons.
- Show that the squared amplitude for unpolarized scattering is

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{A}|^2 = \frac{2e^4}{s^2}\Big[t^2 + u^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2\Big]\,,$$

  where s, t, u are the Mandelstam variables.
- Taking ultra-high-energy limit, $m_e = 0$ and $m_\mu = 0$, show that the differential cross section in the center-of-mass frame is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}(1 + \cos^2\theta)\,.$$

3. The scattering of electrons off atomic nuclei, e.g. $e^-p^+ \to e^-p^+$, is known as *Rutherford scattering.*

- Using *crossing symmetry*, show that the result of the previous problem implies that the squared amplitude for Rutherford scattering is

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{A}|^2 = \frac{2e^4}{t^2}\Big[u^2 + s^2 + 4t(m_e^2 + m_p^2) - 2(m_e^2 + m_p^2)^2\Big]\,.$$

- Taking the non-relativistic limit, show that the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{e^4 m_e^2}{64\pi^2}\frac{1}{p^4\sin^4(\theta/2)}\,.$$

This result is known as *Rutherford's formula.*

Outlook

This course has only scratched the surface of quantum field theory. There are many important topics that we did not have time for.

First of all, we only studied QFT at tree level. Many new phenomena arise when *loops* are considered. These loops are typically divergent and the theory is made finite through an intricate procedure called *renormalization*. This is based on the realization that the parameters appearing in the Lagrangian, such as the bare mass and charge of the electron, aren't physically observable, but get dressed by a cloud of other virtual particles, which interact with the original electron. Taking these interactions into account shows that the electron in fact behaves as if it had a different mass and charge. The theory is perfectly finite when expressed in terms only of physically observable quantities. An important consequence of renormalization is that the values of physical parameters can depend on the energy scale (or length scale) at which they are being measured. As this scale is varied, the masses and couplings of a theory can "run" (or "flow") according to the renormalization group equations. The modern way of thinking about renormalization is within the framework of *effective field theory*.

Second, our treatment of QFT was based on canonical quantization, and we did not have time for any discussion of *path integral quantization*. This is a shame since path integral quantization is arguably more elegant. For certain applications, such as the quantization of massless gauge fields, it is also simpler. I hope that you will have the chance to learn about the path integral approach to QFT some time later in your life.

Finally, we spent most of our time and effort on the conceptual foundations of QFT and therefore were a bit slim on *applications*. Let me therefore reiterate: QFT is the language in which the fundamental laws of Nature are written, and its range of applications span all branches of theoretical physics. I hope that this course has given you the foundation to study some of these applications in your own time.

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A Mathematical Preliminaries

Quantum field theory doesn't require a lot of sophisticated mathematics. In this appendix, I will collect some of the mathematical background that we will need in this course.

A.1 Fourier Transforms

Nonlinear equations are hard to solve. Most progress in physics therefore relies on systems whose equations are approximately linear (and nonlinearities can be treated as perturbative corrections). The evolution equation may involve variations in both time and space. It is then solved most efficiently by first transforming the equation to Fourier space.

Given a function $f(t,\mathbf{x})$ in d spatial dimensions, we define its Fourier transform as

$$f(t,\mathbf{k}) \equiv \int \mathrm{d}^d x \, e^{-i\mathbf{k}\cdot\mathbf{x}} f(t,\mathbf{x}) \,, \tag{A.1.1}$$

where the minus sign in the exponential is our convention. The inverse Fourier transform is

$$f(t,\mathbf{x}) \equiv \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, e^{i\mathbf{k}\cdot\mathbf{x}} f(t,\mathbf{k}) \,, \tag{A.1.2}$$

where the factor of $(2\pi)^d$ is our convention. We refer to the expression (A.1.2) as the Fourier expansion of the field $f(t,\mathbf{x})$.

Exercise.—Show that

$$f(\mathbf{x}) = \frac{1}{|\mathbf{x}|} e^{-m|\mathbf{x}|} \quad \leftrightarrow \quad f(\mathbf{k}) = \frac{1}{|\mathbf{k}|^2 + m^2} \,, \tag{A.1.3}$$

$$f(\mathbf{x}) = \delta^{(d)}(\mathbf{x}) \quad \leftrightarrow \quad f(\mathbf{k}) = 1 \,. \tag{A.1.4}$$

To illustrate the utility of the Fourier representation, consider the Klein-Gordon equation for a massless scalar field (cf. Chapter 1)

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) f(t,\mathbf{x}) = 0 \,. \tag{A.1.5}$$

Acting with spatial derivatives on the Fourier expansion (A.1.2) gives

$$\boldsymbol{\nabla} f(t,\mathbf{x}) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, e^{i\mathbf{k}\cdot\mathbf{x}} \big[i\mathbf{k} f(t,\mathbf{k}) \big] \,, \tag{A.1.6}$$

$$\nabla^2 f(t,\mathbf{x}) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, e^{i\mathbf{k}\cdot\mathbf{x}} \big[-|\mathbf{k}|^2 f(t,\mathbf{k}) \big] \,. \tag{A.1.7}$$

We see that each derivative has pulled down a factor of $i\mathbf{k}$ from the exponential $e^{i\mathbf{k}\cdot\mathbf{x}}$. Substituting the Fourier expansion (A.1.2) in the Klein-Gordon equation (A.1.8), we therefore find

$$\left(\frac{\partial^2}{\partial t^2} + |\mathbf{k}|^2\right) f(t,\mathbf{k}) = 0\,. \tag{A.1.8}$$

This has turned the PDE for $f(t,\mathbf{x})$ into an ODE for $f(t,\mathbf{k})$. Moreover, because the original equation is linear in $f(t,\mathbf{x})$, the different Fourier modes $f(t,\mathbf{k})$ don't mix and the equation can be solved for each value of $\mathbf{k}$ separately.

A.2 Group Theory

Symmetries play an important role in physics in general and in quantum field theory in particular. The study of symmetries is group theory.

Rotations

A spatial rotation can be written as

$$\mathbf{x}' = R\,\mathbf{x} \quad \text{or} \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{A.2.9}$$

where R is an orthogonal 3×3 matrix, i.e. $R^T R = 1$, so that the rotation preserves the magnitude of the vector, i.e. $\mathbf{x}^T\mathbf{x}$ is invariant. The matrices R form the group $\mathbf{O(3)}$: if R_1 and R_2 are orthogonal then so is R_1R_2. Imposing $\det R = 1$ removes reflections and the group becomes $\mathbf{SO(3)}$. As an explicit example, consider the rotation[1] of a vector $\mathbf{v}$ around the z-axis:

$$\begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} = R_z(\theta) \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \tag{A.2.10}$$

where

$$R_z(\theta) \equiv \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.2.11}$$

Similarly, for rotations around the x and y-axes, we have

$$R_x(\phi) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}, \tag{A.2.12}$$

$$R_y(\psi) \equiv \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}. \tag{A.2.13}$$

[1] Considered as an active rotation (i.e. rotating the vector, but keeping the coordinate axes fixed), the rotation is left-handed, while as a passive rotation (i.e. rotating the axes and leaving the vector fixed) it is right-handed.

The group is called *non-Abelian* because the matrices R_i do not commute, e.g. $R_x(\phi)R_z(\theta) \neq R_z(\theta)R_x(\phi)$. Infinitesimal rotations can be written as

$$R_x(\delta\phi) = 1 + iJ_x\delta\phi\,, \quad R_y(\delta\psi) = 1 + iJ_y\delta\psi\,, \quad R_z(\delta\theta) = 1 + iJ_z\delta\theta\,, \tag{A.2.14}$$

where J_i are the *generators* of the group. Explicitly, we have

$$\begin{aligned} J_x &\equiv \frac{1}{i}\frac{dR_x(\phi)}{d\phi}\bigg|_{\phi=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\ J_y &\equiv \frac{1}{i}\frac{dR_y(\psi)}{d\psi}\bigg|_{\psi=0} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \\ J_z &\equiv \frac{1}{i}\frac{dR_z(\theta)}{d\theta}\bigg|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{A.2.15}$$

It is easy to verify that

$$[J_x, J_y] = iJ_z\,, \quad [J_z, J_x] = iJ_y\,, \quad [J_y, J_z] = iJ_x\,, \tag{A.2.16}$$

or, more compactly,

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,. \tag{A.2.17}$$

This is the *Lie algebra* of the group.

A finite rotation can be built up from many infinitesimal rotations. For example, a rotation by $\theta = N\delta\theta$ around the z-axis is

$$R_z(\theta) = [R_z(\delta\theta)]^N = (1 + iJ_z\delta\theta)^N = (1 + iJ_z\theta/N)^N \xrightarrow{N\to\infty} \exp(iJ_z\theta)\,. \tag{A.2.18}$$

It is easy to confirm that this indeed corresponds to the form given in (A.2.11):

$$\begin{aligned} \exp(iJ_z\theta) &= 1 + iJ_z\theta - J_z^2\frac{\theta^2}{2!} - iJ_z^3\frac{\theta^3}{3!} + \cdots \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\theta^2}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{A.2.19}$$

Another important group is $\mathbf{SU(2)}$, the group of unitary 2×2 matrices with unit determinant:

$$U^\dagger U = 1\,, \quad \det U = 1\,. \tag{A.2.20}$$

The elements of the group can be written as

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2 + |b|^2 = 1\,, \tag{A.2.21}$$

for complex a and b. We can think of the U's as acting on a two-dimensional complex space with basis spinors

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} . \tag{A.2.22}$$

The generators of $SU(2)$ are a set of three linearly independent, traceless 2×2 Hermitian matrices. These generators are usually taken to be the famous *Pauli matrices*:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.2.23}$$

The exponentiation of the Pauli matrices, $U_i = e^{i\theta_i \sigma_i/2}$, gives the following group elements

$$\begin{aligned} U_x &= \begin{pmatrix} \cos(\phi/2) & i\sin(\phi/2) \\ i\sin(\phi/2) & \cos(\phi/2) \end{pmatrix} , \quad U_y = \begin{pmatrix} \cos(\psi/2) & \sin(\psi/2) \\ -\sin(\psi/2) & \cos(\psi/2) \end{pmatrix} , \\ U_z &= \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} . \end{aligned} \tag{A.2.24}$$

It is easy to verify that the Pauli matrices satisfy the following algebra

$$[\sigma_x, \sigma_y] = 2i\sigma_z , \quad [\sigma_z, \sigma_x] = 2i\sigma_y , \quad [\sigma_y, \sigma_z] = 2i\sigma_x , \tag{A.2.25}$$

or, more compactly,

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k . \tag{A.2.26}$$

This is the same as the algebra of $SO(3)$ if we identify $J_i = \sigma_i/2$, cf. eq. (A.2.17). In fact, this is the first sign of a deeper connection between $SU(2)$ and $SO(3)$:

An SU(2) transformation on $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is isomorphic to an SO(3) transformation on $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

To see this, let us define $x \equiv \frac{1}{2}(\xi_1^2 - \xi_2^2)$, $y \equiv \frac{1}{2i}(\xi_1^2 + \xi_2^2)$ and $z \equiv \xi_1\xi_2$. The transformations of the spinor ξ then correspond to transformations of the vector $\mathbf{x}$. It is straightforward to verify that the relation between the group elements is

$$U_x(\phi) \leftrightarrow R_x(\phi) , \quad U_y(\psi) \leftrightarrow R_y(\psi) , \quad U_z(\theta) \leftrightarrow R_z(\theta) . \tag{A.2.27}$$

The mapping, however, isn't one-to-one. Instead, we see from (A.2.11) and (A.2.24) that a rotation by $\Delta\theta_i = 2\pi$ gives $R_i \to R_i$ and $U_i \to -U_i$. This implies that both U and $-U$ map to R. The group $SU(2)$ is said to be a *double cover* of $SO(3)$.

Lorentz transformations

In §B.1, we review the concept of Lorentz transformations. Here, we highlight a few group theoretic aspects of these transformations. Recall that the transformations of the spacetime coordinates can be written as

$$V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu} . \tag{A.2.28}$$

The 4×4 matrices Λ form the Lorentz group $\mathbf{SO(1,3)}$. The six Lorentz transformations separate into three spatial rotations and three boosts. For example, a boost along the x-axis mixes the x and t coordinates in the following way

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \tag{A.2.29}$$

where $\eta = \tanh^{-1}(v/c)$ is called the rapidity of the boost. Embedded this into a 4×4 matrix leads to

$$B_x(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.2.30}$$

The generator of the boost then is

$$K_x \equiv \frac{1}{i}\frac{dB_x(\eta)}{d\eta}\bigg|_{\eta=0} = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.2.31}$$

Similarly, the generators of boosts along the y and z-axes are

$$K_y = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A.2.32}$$

The generators of the three spatial rotations are trivially constructed by embedding the generators of $SO(3)$, cf. eq. (A.2.15), into 4×4 matrices:

$$J_x = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_y = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_z = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.2.33}$$

It is now straightforward to verify that the six generators of the Lorentz group satisfy the following algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,, \tag{A.2.34}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k\,, \tag{A.2.35}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k\,. \tag{A.2.36}$$

We see that pure boosts do not form a group: the generators K_i do not form a close algebra under commutations.

It is convenient to define the following linear combinations of the Lorentz generators

$$J_i^{\pm} \equiv \frac{1}{2}(J_i \pm iK_i)\,, \tag{A.2.37}$$

so that the Lorentz algebra becomes

$$[J_i^+, J_j^+] = i\epsilon_{ijk} J_k^+ \,, \tag{A.2.38}$$

$$[J_i^-, J_j^-] = i\epsilon_{ijk} J_k^- \,, \tag{A.2.39}$$

$$[J_i^+, J_j^-] = 0 \,. \tag{A.2.40}$$

We see that the Lorentz algebra can be written as two copies of our old friend **SU(2)**. Representations of the Lorentz group can therefore be constructed from representations of $SU(2)$. The latter are familiar from elementary quantum mechanics (see §B.2).

A.3 Complex Analysis

Complex analysis is a beautiful and rich subject. One powerful result is Cauchy's theorem which will allow us to perform many of the integrals that we encounter in this course. To explain the theorem, we need to introduce a few preliminary concepts.

Analytic functions Let $z \equiv x + iy$ be a complex number. A function $f(z)$ is called *analytic* in a region close to a point z, if its derivative at every point in that region is well-defined. We define the derivative of f with respect to z as

$$\frac{df}{dz} \equiv \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \,. \tag{A.3.41}$$

Importantly, for the function to be analytic, the derivative shouldn't depend on the way the interval in the complex plane z is selected. Extending a function on the real axis, $f(x)$, to the complex plane, $f(z)$, is called *analytic continuation.*

Poles Near a point z_0, a function $f(z)$ can be written as a *Laurent series*

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots \,. \tag{A.3.42}$$

If $b_n = 0$, for all n, then the function is analytic at $z = z_0$. If $b_n \neq 0$, for any n, then it is singular at z_0. The singularity is referred to as a *pole* of order n, if $b_n \neq 0$ and $b_{m>n} = 0$. A pole of order 1 is called a *simple pole.* The coefficient b_1 is called the *residue* of $f(z)$ at $z = z_0$.

Example.—The Feynman propagator of a massive scalar field is

$$\Delta_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \,. \tag{A.3.43}$$

It is helpful to write this as a function of the complex variable p^0. We then have

$$\Delta_F(p) = \frac{i}{2E_{\mathbf{p}}} \left[\frac{1}{p^0 - E_p + i\epsilon} - \frac{1}{p^0 + E_p - i\epsilon} \right] . \tag{A.3.44}$$

We see that the Feynman propagator has simple poles at $p^0 = E_{\mathbf{p}} - i\epsilon$ and $p^0 = -E_{\mathbf{p}} + i\epsilon$.

Residues The residue plays an important role in evaluating complex integrals. You can find a residue $R(z_0)$ at the pole z_0 by writing a Laurent series, where there are also more direct methods. For example, when we have a simple pole, we can write

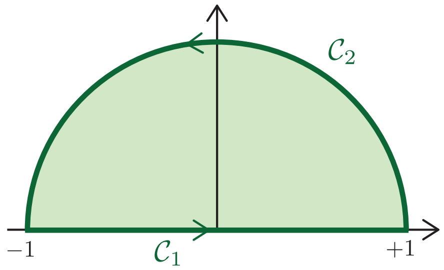
$$R(z_0) \equiv \lim_{z \to z_0} (z - z_0) f(z) \,. \tag{A.3.45}$$

Contours A contour $\mathcal{C}$ is a closed path in the complex plane with a finite number of corners which doesn't cross itself. Integrals around such contours have a number of useful properties.

Example.—Consider the integral

$$\mathcal{I} \equiv \oint_{\mathcal{C}} \mathrm{d}z\, z^2\,, \tag{A.3.46}$$

evaluated along the following contour:



We have split the contour into two parts: a straight line along the real axis, $\mathcal{C}_1$, and a semi-circle in the upper half-plane, $\mathcal{C}_2$. Writing $z \equiv re^{i\theta}$, the first part of the integral then is

$$\mathcal{I}_1 = \int_{-1}^{1} \mathrm{d}r\, r^2 = \left[\frac{r^3}{3}\right]_{-1}^{1} = \frac{2}{3}\,. \tag{A.3.47}$$

Using $z = e^{i\theta}$ and $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta$, the second part of the integral becomes

$$\mathcal{I}_2 = \int_0^{\pi} \mathrm{d}\theta\, ie^{3i\theta} = \left[\frac{e^{3i\theta}}{3}\right]_0^{\pi} = -\frac{2}{3}\,. \tag{A.3.48}$$

We therefore get

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 = 0\,. \tag{A.3.49}$$

This is a special case of Cauchy's theorem.

Cauchy's theorem If $f(z)$ is analytic on and inside $\mathcal{C}$, then

$$\oint_{\mathcal{C}} \mathrm{d}z\, f(z) = 0\,,$$

i.e. the integral of an analytic function along a closed contour vanishes.

Residue theorem If $f(z)$ has singularities at points z_i inside $\mathcal{C}$, then

$$\oint_{\mathcal{C}} \mathrm{d}z\, f(z) = 2\pi i \sum_i R(z_i)\,,$$

where $R(z_i)$ are the residues of the poles at z_i and the integral around $\mathcal{C}$ is performed in the anti-clockwise direction. (Performing the integral in the clockwise direction would change the sign of the answer.)

Often we want to do difficult integrals over real variables. These may be turned into easier integrals if we form a contour in the complex plane which includes the original domain of integration and use the rules given above. The art is in choosing the best contour to do the integral.

B Physical Preliminaries

Quantum field theory (QFT) is the union of special relativity (SR) and quantum mechanics (QM). We better know something about SR and QM before we begin with our study of QFT. In this appendix, we will give a lightning review of elementary concepts in SR and QM. This is only meant to serve as a reminder and not as a substitute for proper courses on these subjects.

B.1 Special Relativity

The special theory of relativity is based on the observation that the speed of light c is the same in every inertial frame (i.e. frames moving with constant velocity). Let S' be a frame moving with velocity $\mathbf{v} = (v, 0, 0)$ relative to a frame S. The coordinates of an event in the two frames are related by the following *Lorentz transformation*

$$\begin{aligned} ct' &= \gamma(ct - \beta x)\,, \\ x' &= \gamma(x - \beta\, ct)\,, \\ y' &= y\,, \\ z' &= z\,, \end{aligned} \tag{B.1.1}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$ and $\beta \equiv v/c$. Because the speed of light sets the scale for all speeds, we will choose units such that $c \equiv 1$.

Spacetime

We see that Lorentz transformations mix time and space. It is therefore convenient to combine the time and space coordinates into a spacetime four-vector $x^\mu = (t, x^i)$. A compact way of writing the Lorentz transformation of the spacetime coordinate then is

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu\,, \tag{B.1.2}$$

where we have used the Einstein summation convention which instructs us to sum over repeated indices. For the transformation in (B.1.1) the components of the tensor $\Lambda^\mu{}_\nu$ are

$$\Lambda^\mu{}_\nu \equiv \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.1.3}$$

Similar results apply to boosts along the y and z axes.

Certain quantities are invariant under Lorentz transformations. Consider, for example, the spacetime interval,

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\,, \tag{B.1.4}$$

where we have introduced the Minkowski metric

$$\eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{B.1.5}$$

To see that this is invariant under Lorentz transformations, you will have to convince yourself that $\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$. As a general rule, any object with fully contracted spacetime indices is a Lorentz-invariant quantity.

Four-momentum

Lorentz transformations also mix the energy E and spatial momentum p^i of a particle. The easiest way to describe this is to introduce the four-vector $p^\mu = (E, p^i)$. I will refer to p^i as the momentum and p^μ as the four-momentum. Under a Lorentz transformation, the four-momentum p^μ transforms as

$$p'^\mu = \Lambda^\mu{}_\nu \, p^\nu \,, \tag{B.1.6}$$

just like (B.1.2). Writing this out in components would give the relation between the original and the transformed energies and momenta. The square of the four-momentum defines an invariant constant

$$p^2 \equiv \eta_{\mu\nu} p^\mu p^\nu = m^2 \,, \tag{B.1.7}$$

which is the rest mass of the particle. Writing this out in components, we get the famous relation

$$E^2 = \mathbf{p}^2 + m^2 \,, \tag{B.1.8}$$

in units where $c \equiv 1$.

Electrodynamics

Physical laws that are consistent with relativity must transform appropriately under Lorentz transformations. In practice, this means that we must be able to write them in terms of spacetime tensors (four-vectors in the simplest cases). Some old laws, like the laws of Newtonian dynamics and Coulomb's law of electrostatics, had to be upgraded to become consistent with the principle of relativity. Maxwell's equations of electrodynamics, on the other hand, happened to have relativity already built into them.

In Heaviside-Lorentz units, $\mu_0 = \epsilon_0 = c \equiv 1$, the Maxwell equations are

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \rho \,, \qquad & \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \,, \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \,, \qquad & \boldsymbol{\nabla} \times \mathbf{B} &= +\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \,, \end{aligned} \tag{B.1.9}$$

where all quantities have their usual meanings. Since "div curl = 0", the equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ is solved automatically if we define $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential. Since "curl grad = 0", the equation $\boldsymbol{\nabla} \times (\mathbf{E} + \dot{\mathbf{A}}) = 0$ is solved automatically if we define $\mathbf{E} = -\dot{\mathbf{A}} - \boldsymbol{\nabla}\phi$, where ϕ is the scalar potential. The remaining two inhomogeneous Maxwell equations can be unified by defining the four-vector current and the four-vector potential, $J_\mu = (\rho, \mathbf{J})$ and $A_\mu = (\phi, \mathbf{A})$. This leads to

$$\partial^\nu F_{\mu\nu} = J_\mu \,, \tag{B.1.10}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. The anti-symmetry of the field strength tensor requires that the current is conserved, $\partial^\mu J_\mu = 0$. In going from (B.1.9) to (B.1.10), we have been able to massage the Maxwell equations into a manifestly Lorentz covariant form. This shows that relativity is hidden in the Maxwell equations.

B.2 Quantum Mechanics

In quantum mechanics, the state of a system, $(\mathbf{q}, \mathbf{p})$, is represented by a vector $|\psi\rangle$ on a Hilbert space $\mathcal{H}$. Functions $f(\mathbf{q}, \mathbf{p})$ on phase space become linear operators $\hat{F}$ on $\mathcal{H}$, and Poisson brackets $\{f, g\}$ become commutators $-i\hbar^{-1}[\hat{F}, \hat{G}]$.

The evolution of a state is determined by the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle \;=\; \hat{H}\,|\psi(t)\rangle \,, \tag{B.2.11}$$

where $\hat{H}$ is the Hamiltonian. The probability of finding a system described by $|\psi\rangle$ in a new state $|\phi\rangle$ is given by the square of the inner product $|\langle\phi|\psi\rangle|^2$. For this probability interpretation to hold, we require that the norm $\langle\psi|\psi\rangle$

- is positive,
- can be normalized to 1,
- is conserved.

For conservation to hold, the Hamiltonian must be Hermitian, $\hat{H}^\dagger = \hat{H}$. The time evolution of a state can be expressed as

$$|\psi(t_2)\rangle \;=\; U(t_2, t_1)\,|\psi(t_1)\rangle \,, \tag{B.2.12}$$

where $U(t_2, t_1)$ is a unitary operator. The matrix element $\langle\psi|\hat{F}|\psi\rangle$ describes the expectation value of the operator $\hat{F}$ in the state $|\psi\rangle$. It obeys the following evolution equation

$$\frac{d}{dt}\langle\psi|\hat{F}|\psi\rangle = \langle\psi| \left(\frac{\partial \hat{F}}{\partial t} - \frac{1}{i\hbar}[\hat{H}, \hat{F}] \right) |\psi\rangle \,, \tag{B.2.13}$$

which is similar to the classical time evolution if we return to Poisson brackets.

Quantum mechanics and relativity

Attempts to develop a relativistic version of quantum mechanics fail for subtle reasons. I will briefly sketch why.

First, let us recall the Schrödinger equation for a free particle in the non-relativistic limit, where $E = \mathbf{p}^2/(2m)$. Expanding the state $|\psi\rangle$ in position eigenstates $|\mathbf{q}\rangle$, we get

$$\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\left(\frac{\partial}{\partial \mathbf{q}}\right)^2 \right] \psi(t, \mathbf{q}) = 0 \,, \tag{B.2.14}$$

where $\psi(t,\mathbf{q}) \equiv \langle \mathbf{q}|\psi(t)\rangle$ is the wavefunction. We would like to upgrade this equation to the relativistic limit, where we have $E^2 = \mathbf{p}^2 + m^2$. A natural guess for the corresponding Schrödinger equation is the Klein-Gordon equation

$$\left[\left(-\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial \mathbf{q}}\right)^2 - m^2\right]\psi(t,\mathbf{q}) = 0\,. \tag{B.2.15}$$

This equation has several conceptual problems:

- **Probability** Because the equation is second order in time derivatives, the norm $\langle\psi|\psi\rangle$ is not conserved. We therefore can't interpret $\langle\psi|\psi\rangle$ as a probability. The real conserved quantity is

$$Q \equiv \frac{i}{2m}\left(\langle\psi|\dot\psi\rangle - \langle\dot\psi|\psi\rangle\right), \tag{B.2.16}$$

  but it isn't positive definite.

- **Causality** The probability amplitude for a particle moving from $(t_1, \mathbf{q}_1)$ to $(t_2, \mathbf{q}_2)$ is

$$\langle \mathbf{q}_2|U(t_2,t_1)|\mathbf{q}_1\rangle\,. \tag{B.2.17}$$

  It may be shown that this quantity is non-zero even for spacelike separated points, in apparent violation of causality.

- **Instability** Being second order in time derivatives, equation (B.2.15) allows for solutions of both positive and negative energy. The presence of negative-energy solutions suggest that the theory has no stable vacuum state. An unlimited amount of energy can be released by positive-energy particles transitioning to negative-energy states.

The Dirac equation was an attempt to overcome the above problems. The following ansatz is first order in time and space derivatives

$$\frac{\partial}{\partial t}\psi = \alpha^i \frac{\partial}{\partial q^i}\psi + m\beta\,\psi\,, \tag{B.2.18}$$

where ψ is now a multi-component spinor. For a suitable choice of the matrices α^i and β the Dirac equation implies the Klein-Gordon equation, so that its solutions will satisfy $E^2 = \mathbf{p}^2 + m^2$. The norm $\langle\psi|\psi\rangle$ is now conserved and positive definite. However, the equation still allows for negative-energy solutions, so taken at face value it isn't a solution of the problems of the Klein-Gordon equation. The solution will require interpreting ψ as a quantum field rather than a probability amplitude. The negative-energy solutions will be reinterpreted as positive energy antiparticles.

Quantum harmonic oscillator

In Chapter 2, we show that the Fourier modes of a free field obey the equation of motion of a *simple harmonic oscillator* (SHO). The quantization of the field therefore follows from the quantum mechanics of the SHO. This appendix will remind you how this works.

Consider a particle of mass m attached to a spring with spring constant κ. Let q be the deviation of the particle from its equilibrium position. The kinetic and potential energies of the particle are $\frac{1}{2}m\dot{q}^2$ and $\frac{1}{2}\kappa q^2$, respectively. The equation of motion of the particle is

$$\ddot{q} + \omega^2 q = 0\,, \tag{B.2.19}$$

where $\omega \equiv \sqrt{\kappa/m}$ is the angular frequency of the oscillator. We would like to find the energy spectrum of the system. We could do this by solving the time-independent Schrödinger equation, $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, for the energy eigenstates. Alternatively, we can use the method of "ladder operators" to determine the spectrum without solving the Schrödinger equation.

Let us define the following non-Hermitian operators

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right), \tag{B.2.20}$$

$$\hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right). \tag{B.2.21}$$

These are called creation and annihilation operators (also known as raising/lowering operators, or sometimes ladder operators). This implies the following representations of the operators $\hat{q}$ and $\hat{p}$:

$$\hat{q} = \sqrt{\frac{\hbar}{2}\frac{1}{m\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \tag{B.2.22}$$

$$\hat{p} = -i\sqrt{\frac{\hbar}{2}m\omega}\left(\hat{a} - \hat{a}^\dagger\right). \tag{B.2.23}$$

The canonical commutation relation then implies

$$[\hat{q}, \hat{p}] = i\hbar \qquad \Leftrightarrow \qquad [\hat{a}, \hat{a}^\dagger] = 1\,. \tag{B.2.24}$$

The Hamiltonian of the system becomes

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 \;=\; \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{B.2.25}$$

The commutators between the Hamiltonian and the creation and annihilation operators are

$$[\hat{H}, \hat{a}^\dagger] = +\hbar\omega\,\hat{a}^\dagger\,, \tag{B.2.26}$$

$$[\hat{H}, \hat{a}] = -\hbar\omega\,\hat{a}\,. \tag{B.2.27}$$

This cements the roles of $\hat{a}^\dagger$ and $\hat{a}$ as raising and lowering operators. Let $|\psi_n\rangle$ be an eigenstate with energy E_n, so that $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$. Acting with $\hat{a}$ and $\hat{a}^\dagger$ produces new eigenstates with lower and higher energy:

$$\hat{H}\,\hat{a}^\dagger|\psi_n\rangle = (E_n + \hbar\omega)\,\hat{a}^\dagger|\psi_n\rangle\,, \tag{B.2.28}$$

$$\hat{H}\,\hat{a}|\psi_n\rangle = (E_n - \hbar\omega)\,\hat{a}|\psi_n\rangle\,. \tag{B.2.29}$$

If the energy is bounded from below, there must be a *ground state* which satisfies $\hat{a}|0\rangle = 0$. This state has ground state energy (also known as zero-point energy),

$$E_0 = \frac{1}{2}\hbar\omega\,. \tag{B.2.30}$$

Excited states then arise from repeated application of the raising operator

$$|\psi_n\rangle = (\hat{a}^\dagger)^n|0\rangle\,, \tag{B.2.31}$$

where we have ignored the normalization of the states, so that $\langle\psi_n|\psi_n\rangle \neq 1$. The energy of these states is

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega\,. \tag{B.2.32}$$

Of course, there is much more to quantum mechanics, but this will be enough to get us going in our study of quantum field theory.

D Guide to the Literature

The literature on quantum field theory is vast. The following is a sample of the books, lectures notes and videos that I have found useful in preparing this course.

Books

- Peskin and Schroeder, *An Introduction to Quantum Field Theory*
  This course will cover the first five chapters of Peskin and Schroeder, although my presentation will follow a slightly different order.
- Schwartz, *Quantum Field Theory and the Standard Model*
  My treatment in Chapter 4 was inspired by Chapters 8 and 10 in Schwartz's book.
- Maggiore, *A Modern Introduction to QFT*
  My treatment of the quantization of massless gauge fields partly follows Chapter 4 of this book.
- Lancaster and Blundell, *QFT for the Gifted Amateur*
  I am a big fan of this book. It contains many nice intuitive explanations. My introduction to the Feynman propagator is from Lancaster and Blundell's book.
- Ryder, *Quantum Field Theory*
  I remember liking this book when I studied QFT for the first time. I don't remember why.
- Zee, *Quantum Field Theory in a Nutshell*
  An entertaining introduction to the subject.
- Greiner and Reinhardt, *Field Quantization*
  The book is full of many explicit calculations and therefore ideally suited for self-study.
- Srednicki, *Quantum Field Theory*
  The book is written in a very modular way and therefore ideal for dipping in and out.
- Weinberg, *The Quantum Theory of Fields*
  A classic book by a master of the subject. Unfortunately, if you don't know QFT already, Weinberg's book is probably incomprehensible. If you think you do know QFT, however, then you can find many additional insights here.

Lecture notes

- Tong, *Quantum Field Theory*
  David Tong's notes form the backbone of these lectures.
- Beisert, *Quantum Field Theory*
  Niklas Beisert's notes are a nice complement to the lectures by Tong.

- Amin, *Introduction to Quantum Field Theory*
  Mustafa Amin's course is similar in spirit to these lectures.

- Kaplan, *Quantum Field Theory*
  My treatment of lattice QFT was based on Jared Kaplan's notes.

Popular books

- Close, *The Infinity Puzzle*
  This is nice description of the history of quantum field theory.

- Weinberg, *Dreams of a Final Theory*
  At some point in your life you should read this. Why not now?

Videos

- Tong, *PSI Lectures*
  A set of lectures given at the PSI course of the Perimeter Institute.

- Coleman, *Harvard Lectures*
  A classic set of lectures by another master of the subject.

- Arkani-Hamed, *Messenger Lectures*
  Series of public lectures by the past, present and future of quantum field theory.

- Tong, *Quantum Fields: The Real Building Block of the Universe*
  Public lecture at the Royal Institution.