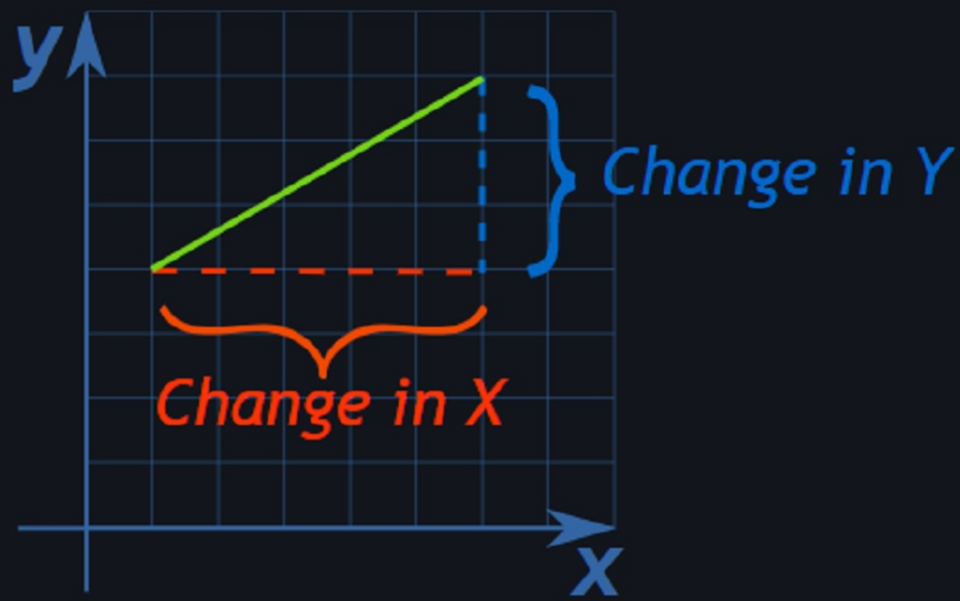
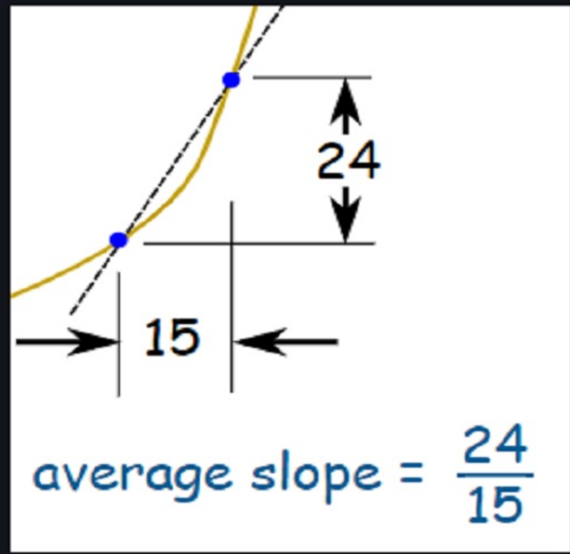
Introduction to Derivatives

It is all about slope!

$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}}$$

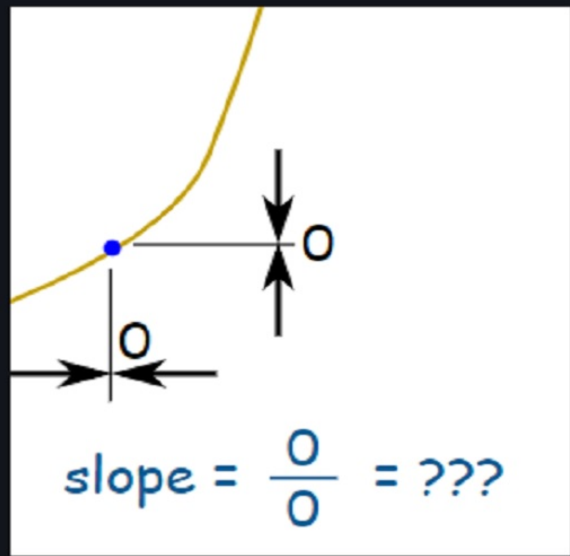


We can find an **average** slope between two points.



But how do we find the slope **at a point**?

There is nothing to measure!



But with derivatives we use a small difference ...

... then have it **shrink towards zero**.

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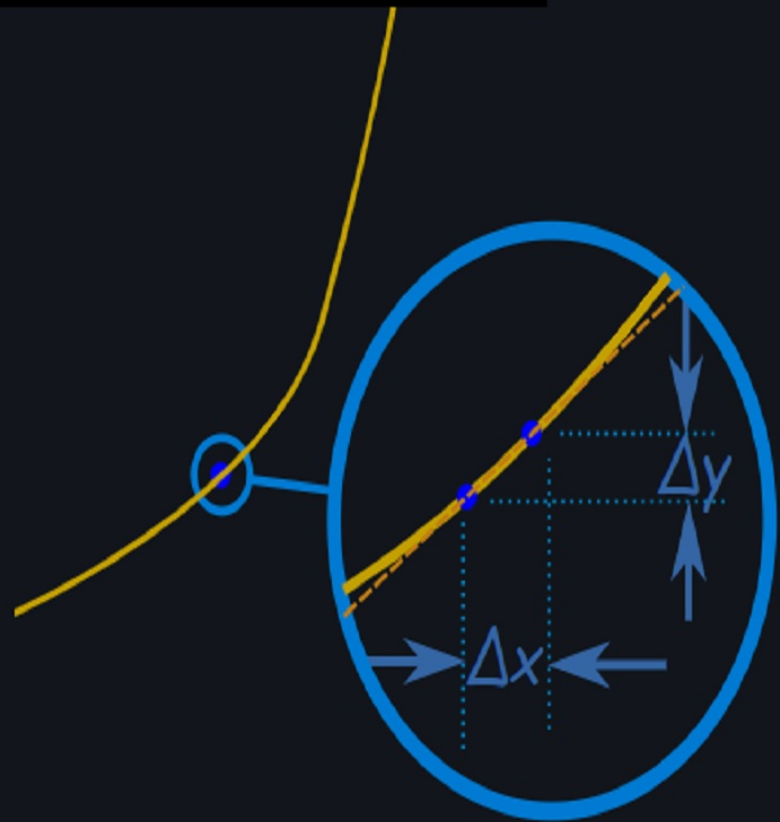
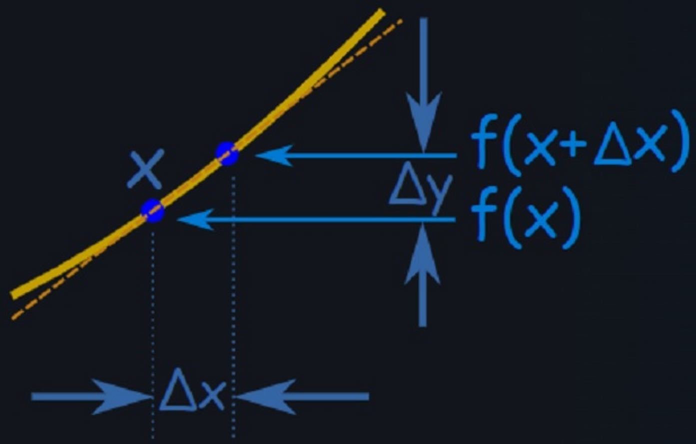


Let us Find a Derivative!

To find the derivative of a function $y = f(x)$ we use the slope formula:

$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}} = \frac{\Delta y}{\Delta x}$$

And (from the diagram) we see that:



x changes from x to $x+\Delta x$

y changes from $f(x)$ to $f(x+\Delta x)$

Now follow these steps:

- Fill in this slope formula: $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$
- Simplify it as best we can
- Then make **Δx** shrink towards zero.

Like this:

Example: the function $f(x) = x^2$

We know $f(x) = x^2$, and we can calculate $f(x+\Delta x)$:

Start with: $f(x+\Delta x) = (x+\Delta x)^2$

Expand $(x + \Delta x)^2$: $f(x+\Delta x) = x^2 + 2x\,\Delta x + (\Delta x)^2$

The slope formula is: $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Put in $f(x+\Delta x)$ and $f(x)$: $\dfrac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$

Simplify (x^2 and $-x^2$ cancel): $\dfrac{2x\,\Delta x + (\Delta x)^2}{\Delta x}$

Simplify more (divide through by Δx): $= 2x + \Delta x$

Then, **as Δx heads towards 0** we get: $= 2x$

Result: the derivative of x^2 is 2x

In other words, the slope at x is **2x**

We write **dx** instead of **"Δx heads towards 0"**.

And "the derivative of" is commonly written $\frac{d}{dx}$ like this:

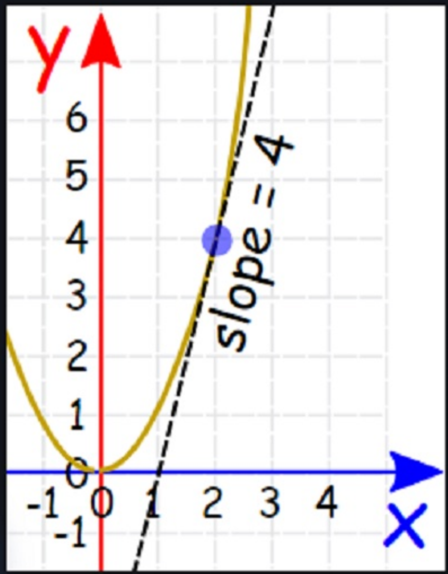
$$\frac{d}{dx}x^2 = 2x$$

"The derivative of $\boldsymbol{x^2}$ *equals* $\boldsymbol{2x}$*"*

or simply *"d dx of* $\boldsymbol{x^2}$ *equals* $\boldsymbol{2x}$*"*

So what does $\frac{d}{dx}x^2 = 2x$ mean?

It means that, for the function x^2, the slope or "rate of change" at any point is **2x**.



So when **x=2** the slope is **2x = 4**, as shown here:

Or when **x=5** the slope is **2x = 10**, and so on.

Note: $f'(x)$ can also be used for "the derivative of":

$$f'(x) = 2x$$

"The derivative of f(x) equals 2x"

or simply *"f-dash of x equals 2x"*

Let's try another example.

Example: What is $\frac{d}{dx}x^3$?

We know $\mathbf{f(x) = x^3}$, and can calculate $\mathbf{f(x+\Delta x)}$:

Start with: $\mathbf{f(x+\Delta x) = (x+\Delta x)^3}$

Expand $(x + \Delta x)^3$: $\mathbf{f(x+\Delta x) = x^3 + 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3}$

The slope formula: $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Put in $\mathbf{f(x+\Delta x)}$ and $\mathbf{f(x)}$: $\dfrac{x^3 + 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

Simplify (x^3 and $-x^3$ cancel): $\dfrac{3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3}{\Delta x}$

Simplify more (divide through by Δx): $3x^2 + 3x\,\Delta x + (\Delta x)^2$

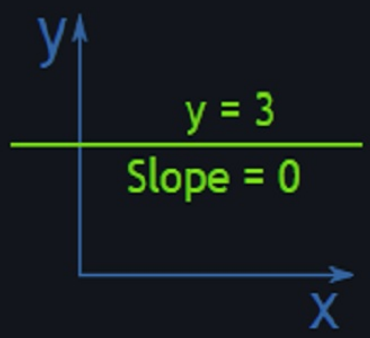
Then, **as Δx heads towards 0** we get: $3x^2$

Result: the derivative of $\mathbf{x^3}$ is $\mathbf{3x^2}$

Derivative Rules

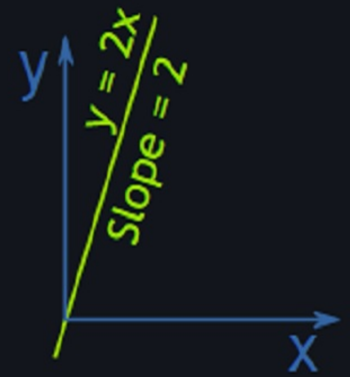
The Derivative tells us the slope of a function at any point.

There are **rules** we can follow to find many derivatives.



For example:

- The slope of a **constant** value (like 3) is always 0
- The slope of a **line** like 2x is 2, or 3x is 3 etc
- and so on.



Here are useful rules to help you work out the derivatives of many functions (with examples below). Note: the little mark **'** means **derivative of**, and f and g are functions.

Common Functions	Function	Derivative
Constant	c	0
Line	x	1
	ax	a
Square	x^2	2x
Square Root	√x	$(½)x^{-½}$
Exponential	e^x	e^x
	a^x	$\ln(a)\ a^x$
Logarithms	ln(x)	1/x
	$\log_a(x)$	1 / (x ln(a))
Trigonometry (x is in radians)	sin(x)	cos(x)
	cos(x)	−sin(x)
	tan(x)	$\sec^2(x)$
Inverse Trigonometry	$\sin^{-1}(x)$	$1/\sqrt{(1-x^2)}$
	$\cos^{-1}(x)$	$-1/\sqrt{(1-x^2)}$
	$\tan^{-1}(x)$	$1/(1+x^2)$

Rules	Function	Derivative
Multiplication by constant	cf	cf'
Power Rule	x^n	nx^{n-1}
Sum Rule	f + g	f' + g'
Difference Rule	f - g	f' − g'
Product Rule	fg	f g' + f' g
Quotient Rule	f/g	$\frac{f'g - g'f}{g^2}$
Reciprocal Rule	1/f	$-f'/f^2$
Chain Rule (as "Composition of Functions")	f º g	(f' º g) × g'
Chain Rule (using ')	f(g(x))	f'(g(x))g'(x)
Chain Rule (using $\frac{d}{dx}$)	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	

"The derivative of" is also written $\frac{d}{dx}$

So $\frac{d}{dx}\sin(x)$ and $\sin(x)'$ both mean "The derivative of sin(x)"

Examples

Example: what is the derivative of sin(x) ?

From the table above it is listed as being **cos(x)**

It can be written as:

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Or:

$$\sin(x)' = \cos(x)$$

Power Rule

Example: What is $\frac{d}{dx}x^3$?

The question is asking "what is the derivative of x^3 ?"

We can use the Power Rule, where n=3:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}x^3 = 3x^{3-1} = \mathbf{3x^2}$$

(In other words the derivative of x^3 is $3x^2$)

So it is simply this:

$$x^{3} \xrightarrow{-1} 3x^{2}$$

"multiply by power
then reduce power by 1"

It can also be used in cases like this:

Example: What is $\frac{d}{dx}(1/x)$?

1/x is also $\mathbf{x^{-1}}$

We can use the Power Rule, where n = −1:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}x^{-1} = -1x^{-1-1}$$

$$= -x^{-2}$$

$$= \frac{-1}{x^2}$$

So we just did this:

$$x^{-1} \xrightarrow{-1} -1x^{-2}$$

which simplifies to $\mathbf{-1/x^2}$

Multiplication by constant

Example: What is $\frac{d}{dx}5x^3$?

$$\text{the derivative of } cf = cf'$$

$$\text{the derivative of } 5f = 5f'$$

We know (from the Power Rule):

$$\frac{d}{dx}x^3 = 3x^{3-1} = 3x^2$$

So:

$$\frac{d}{dx}5x^3 = 5\frac{d}{dx}x^3 = 5 \times 3x^2 = \mathbf{15x^2}$$

Sum Rule

Example: What is the derivative of x^2+x^3 ?

The Sum Rule says:

$$\text{the derivative of } f + g = f' + g'$$

So we can work out each derivative separately and then add them.

Using the Power Rule:

- $\frac{d}{dx}x^2 = 2x$
- $\frac{d}{dx}x^3 = 3x^2$

And so:

$$\text{the derivative of } x^2 + x^3 = \mathbf{2x + 3x^2}$$

Difference Rule

What we differentiate with respect to doesn't have to be **x**, it could be anything. In this case **v**:

Example: What is $\frac{d}{dv}(v^3-v^4)$?

The Difference Rule says

$$\text{the derivative of } f - g = f' - g'$$

So we can work out each derivative separately and then subtract them.

Using the Power Rule:

- $\frac{d}{dv}v^3 = 3v^2$
- $\frac{d}{dv}v^4 = 4v^3$

And so:

$$\text{the derivative of } v^3 - v^4 = \mathbf{3v^2 - 4v^3}$$

Sum, Difference, Constant Multiplication And Power Rules

Example: What is $\frac{d}{dz}(5z^2 + z^3 - 7z^4)$?

Using the Power Rule:

- $\frac{d}{dz}z^2 = 2z$
- $\frac{d}{dz}z^3 = 3z^2$
- $\frac{d}{dz}z^4 = 4z^3$

And so:

$$\frac{d}{dz}(5z^2 + z^3 - 7z^4) = 5 \times 2z + 3z^2 - 7 \times 4z^3$$

$$= \mathbf{10z + 3z^2 - 28z^3}$$

Product Rule

Example: What is the derivative of cos(x)sin(x) ?

The Product Rule says:

$$\text{the derivative of } fg = f\,g' + f'\,g$$

In our case:

- $f = \cos$
- $g = \sin$

We know (from the table above):

- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$

So:

$$\text{the derivative of } \cos(x)\sin(x) = \cos(x)\cos(x) - \sin(x)\sin(x)$$

$$= \cos^2(x) - \sin^2(x)$$

Quotient Rule

To help you remember:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

The derivative of "High over Low" is:

"Low dHigh minus High dLow, over the line and square the Low"

Example: What is the derivative of cos(x)/x ?

In our case:

- $f = \cos$
- $g = x$

We know (from the table above):

- $f' = -\sin(x)$
- $g' = 1$

So:

$$\text{the derivative of } \frac{\cos(x)}{x} = \frac{\text{Low dHigh minus High dLow}}{\text{square the Low}}$$

$$= \frac{x(-\sin(x)) - \cos(x)(1)}{x^2}$$

$$= -\frac{x\sin(x) + \cos(x)}{x^2}$$

Reciprocal Rule

Example: What is $\frac{d}{dx}(1/x)$?

The Reciprocal Rule says:

$$\text{the derivative of } \frac{1}{f} = \frac{-f'}{f^2}$$

With f(x)= x, we know that f′(x) = 1

So:

$$\text{the derivative of } \frac{1}{x} = \frac{-1}{x^2}$$

Which is the same result we got above using the Power Rule.

Chain Rule

Example: What is $\frac{d}{dx}\sin(x^2)$?

$\sin(x^2)$ is made up of **sin()** and **x^2**:

- $f(g) = \sin(g)$
- $g(x) = x^2$

The Chain Rule says:

$$\text{the derivative of } f(g(x)) = f'(g(x))g'(x)$$

The individual derivatives are:

- $f'(g) = \cos(g)$
- $g'(x) = 2x$

So:

$$\frac{d}{dx}\sin(x^2) = \cos(g(x))\ (2x)$$

$$= 2x\cos(x^2)$$

Another way of writing the Chain Rule is: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Let's do the previous example again using that formula:

Example: What is $\frac{d}{dx}\sin(x^2)$?

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Let $u = x^2$, so $y = \sin(u)$:

$$\frac{d}{dx}\sin(x^2) = \frac{d}{du}\sin(u)\frac{d}{dx}x^2$$

Differentiate each:

$$\frac{d}{dx}\sin(x^2) = \cos(u)\ (2x)$$

Substitute back $u = x^2$ and simplify:

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2)$$

Same result as before (thank goodness!)

Another couple of examples of the Chain Rule:

Example: What is $\frac{d}{dx}(1/\cos(x))$?

$1/\cos(x)$ is made up of **$1/g$** and **$\cos()$**:

- $f(g) = 1/g$
- $g(x) = \cos(x)$

The Chain Rule says:

$$\text{the derivative of } f(g(x)) = f'(g(x))g'(x)$$

The individual derivatives are:

- $f'(g) = -1/(g^2)$
- $g'(x) = -\sin(x)$

So:

$$(1/\cos(x))' = \frac{-1}{g(x)^2}(-\sin(x))$$

$$= \frac{\sin(x)}{\cos^2(x)}$$

Note: $\frac{\sin(x)}{\cos^2(x)}$ is also $\frac{\tan(x)}{\cos(x)}$ or many other forms.

Example: What is $\frac{d}{dx}(5x-2)^3$?

The Chain Rule says:

$$\text{the derivative of } f(g(x)) = f'(g(x))g'(x)$$

$\mathbf{(5x-2)^3}$ is made up of $\mathbf{g^3}$ and $\mathbf{5x-2}$:

- $f(g) = g^3$
- $g(x) = 5x-2$

The individual derivatives are:

- $f'(g) = 3g^2$ (by the Power Rule)
- $g'(x) = 5$

So:

$$\frac{d}{dx}(5x-2)^3 = (3g(x)^2)(5) = 15(5x-2)^2$$

Derivatives of Other Functions

We can use the same method to work out derivatives of other functions (like sine, cosine, logarithms, etc).

Example: what is the derivative of sin(x) ?

On Derivative Rules it is listed as being cos(x)

Done.

But using the rules can be tricky!

Example: what is the derivative of cos(x)sin(x) ?

We get a **wrong** answer if we try to multiply the derivative of cos(x) by the derivative of sin(x) ... !

Instead we use the "Product Rule" as explained on the Derivative Rules page.

And it actually works out to be $\cos^2(x) - \sin^2(x)$

So that is your next step: learn how to use the rules.

Notation

"Shrink towards zero" is actually written as a limit like this:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

"The derivative of f equals
the limit as Δx goes to zero of f(x+Δx) - f(x) over Δx"

Or sometimes the derivative is written like this (explained on Derivatives as dy/dx):

$$\frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx}$$

The process of finding a derivative is called "differentiation".

You **do** differentiation ... to **get** a derivative.

$$f(x, y) = x^3 + y^2$$

$$\partial f / \partial x = 3x^2 + 0 = 3x^2$$

$$\partial f / \partial y = 0 + 2y = 2y$$

Slope	Derivative
Used for linear equation	Used for non linear equation
It is a constant	It is a function