Edge-magic labellings

Atilio Luiz

MO804 - Tópicos em Teoria de Grafos
UNICAMP, Brasil

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Magic Square

- A *magic square* is a $n \times n$ square grid filled with distinct positive integers in the range $1, \ldots, n^2$ such that each cell of the grid contains a different integer and the sum of the integers in each row, column and diagonal is equal.

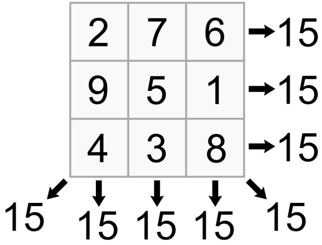


Figure: (Lo Shu square) The smallest (and unique up to rotation and reflection) magic square of dimension 3×3.

Magic Square in Art

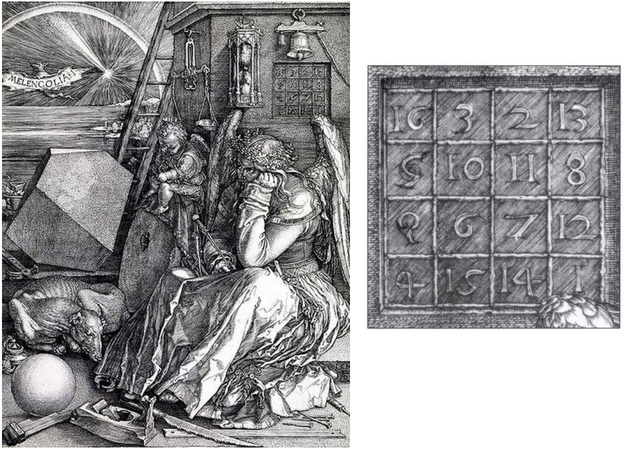


Figure: (Melencolia I, Albert Durer, 1514) The sum 34 can be found in the rows, columns, diagonals, each of the quadrants, the center four squares, and the corner squares (of the 4×4 as well as the four contained 3×3 grids).

Srinivasa Ramanujan's magic square

The Indian mathematician Srinivasa Ramanujan created a square where the first row shows his date of birth, 22nd Dec. 1887.

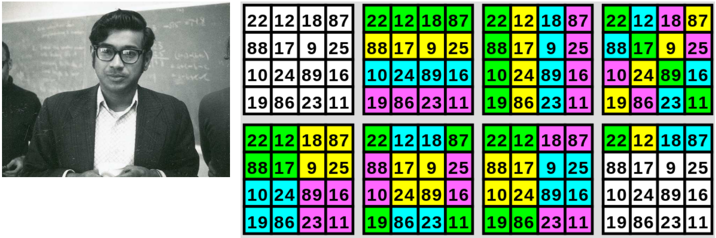
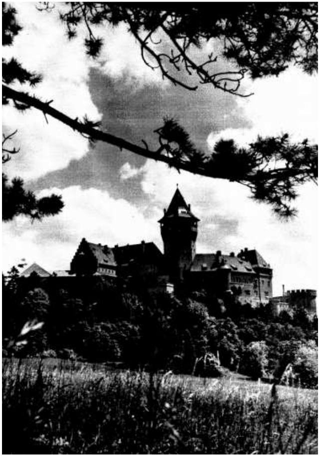
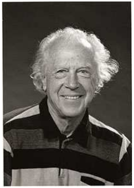


Figure: This magic square has 24 groups of four fields with the sum of 139 and in the first row - shown at bottom-right - Ramanujan's date of birth.

Smolenice 1963



A.Kotzig



G. Ringel

J. Sedlácek

Figure: Smolenice Symposium in Graph Theory, Czechoslovakia, 1963.

19. Except the trivial case of a graph with two vertices and five edges no regular plane Hamilton[3]) graph of degree 5 is known. Do there exist such graphs?

A. KOTZIG

20. Does there exist an integer $n > 1$ such that the complete $2n$-gon is not Hamiltonian?[3])

A. KOTZIG

21. To construct all regular Hamilton[3]) graphs of degree 3.

A. KOTZIG

22. Let K be a given Hamilton[3]) circuit of a complete $2n$-gon G_{2n}. Denote by Λ_n (and $\Lambda_n^\bullet$, Λ_n°, respectively) the number of distinct linear factors of G_{2n} such that the composition of K with any of these factors yields a Hamilton (or bipartite Hamilton or plane Hamilton, respectively) graph of degree 3. Determine the functions Λ_n, $\Lambda_n^\bullet$, Λ_n°.

A. KOTZIG

23. Find the number $\Omega(2n + 1)$ of all non-isomorphic orientations of edges of the complete $(2n + 1)$-gon G_{2n+1} with the property that in each of these orientations there are precisely n edges ending in any vertex of G_{2n+1}.

A. KOTZIG

24. Let G be a non-directed graph without loops and multiple edges, with the set V of vertices. Characterize those subsets X (respectively Y) of V which arise from an orientation of G as the set of those vertices at which there is no incoming (respectively outgoing) edge.

A. KOTZIG

25. It is conjectured that the complete $(2n + 1)$-gon can be decomposed into $2n + 1$ subgraphs which are all isomorphic to a given tree with n edges.

G. RINGEL

26. *Graphs assigned to groups.* Let H be a finite group with elements $a, b, c, \ldots$ and the identity element e. If $a, b \in H$, an element $c \in H$ is uniquely determined so that $abc = e$; then also $bca = cab = e$.

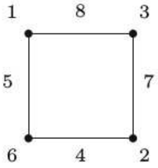
A triple x, y, z of elements from H with $xyz = e$ will be termed an e-triple; two e-triples are equal if one arises from the other by a cyclic permutation. An e-triple (x, y, z) is called regular if $x \neq y \neq z \neq x$, otherwise it is singular. Analogously, a pair $\{x, y\}$ of elements in H is regular if it determines a regular e-triple, otherwise it is singular (especially $\{x, x\}$ is singular for $x \in H$).

[3]) Hamiltonian in sense of KOTZIG. See p. 63.

Problem 27

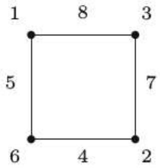
- [Jiri Sedlácek, 1963] A simple connected graph G is called magic if there is a real-valued labelling of the edges of G such that:
  (i) distinct edges have distinct values;
  (ii) the sum of values assigned to all edges incident to a given vertex $v \in V(G)$ is the same for all vertices of G.
- Find necessary and sufficient conditions for a graph to be magic.

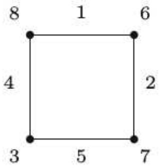
- [Kotzig and Rosa, 1970] Let G be an (n, m)-graph. An edge-magic labelling of G is a bijection $f\colon V(G) \cup E(G) \to [1, m+n]$ such that $f(u) + f(uv) + f(v) = k$, for all $uv \in E(G)$.
- If G has an edge-magic labelling, then G is called edge-magic.
- The constant k is called the valence of the labelling f.



Complementary labelling

- The complementary labelling of an edge-magic labelling f, denoted $\overline{f}$, is defined by $\overline{f}(x) = n + m + 1 - f(x)$, for all $x \in V(G) \cup E(G)$.
- $\overline{f}$ is also an edge-magic labelling and its valence is $\overline{k} = 3(n + m + 1) - k$.





Warm up

Theorem

Let G be a (n, m)-graph. If G has m even and $n + m \equiv 2 \pmod 4$, and every vertex of G has odd degree, then G has no edge-magic labelling.

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Corollary

The complete graph K_n is not edge-magic when $n \equiv 4 \pmod 8$. The wheel W_n with $n + 1$ vertices is not edge-magic when $n \equiv 3 \pmod 4$.

Stars

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Theorem

There are $3 \cdot 2^n$ edge-magic labellings of $K_{1,n}$ up to equivalence.

Trees

Conjecture [Kotzig and Rosa, 1970]

Every tree is edge-magic.

Theorem [Kotzig and Rosa, 1970]

Every caterpillar is edge-magic.

Theorem

The one-factor F_{2n} is edge-magic if and only if n is odd.

Cycles

Theorem

All cycles C_n with $n \geq 3$ are edge-magic.

References I

[1] J. A. Gallian.
A dynamic survey of graph labeling.
The Electronic Journal of Combinatorics, 2016.