**Integer Programming Formulation for the Graceful Coloring Problem**

Given a simple graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, we create an integer variable $x_i$ for each vertex $v_i \in V$, where $x_i \in \mathbb{N}_{\geq 1}$. The value of variable $x_i$ is the color of vertex $v_i$.

The ILP formulation is, thus, given below.

$$
\begin{array}{lll}
\text{minimize} & z & \\
\text{subject to} & x_i \leq z & \text{for all } v_i \in V, \\
& |x_i - x_j| \geq 1 & \text{for all } v_i, v_j \in V \text{ such that } d(v_i, v_j) \leq 2 \text{ and } i \neq j, \\
& |(x_i + x_k) - 2x_j| \geq 1 & \text{for all } v_iv_j, v_jv_k \in E, i \neq k, i \neq j, k \neq j, \\
& x_i \in \mathbb{N}_{\geq 1} & \text{for all } v_i \in V, \\
& z \in \mathbb{N}_{\geq 1} &
\end{array}
$$

In order to linearize the constraints $|x_i - x_j| \geq 1$, we first add a binary variable $b_{ij} \in \{0, 1\}$ for all $v_i, v_j \in V$ such that $d(v_i, v_j) \leq 2$ and $i \neq j$. Moreover, we remove each constraint $|x_i - x_j| \geq 1$ and in its place we add two new constraints:

- **Constraint 1:** $x_i - x_j \geq 1 - M \cdot (1 - b_{ij})$
- **Constraint 2:** $x_j - x_i \geq 1 - M \cdot b_{ij}$

Here, $M$ is a sufficiently large constant that bounds the possible differences between $x_i$ and $x_j$. Since $x_i$ and $x_j$ are vertex labels, we know that $1 \leq x_i, x_j \leq 2\Delta^2 - \Delta + 1$, which implies that $|x_i - x_j| \leq |(2\Delta^2 - \Delta + 1) - 1|$. Therefore, a good choice for $M$ is $M = 2\Delta^2 - \Delta + 1$.

Note that if $b_{ij} = 0$, then $x_j - x_i \geq 1$, which implies that $x_j > x_i$. In this case, $x_i - x_j \geq 1 - M$. With $M = 2\Delta^2 - \Delta + 1$, the first constraint is always fulfilled.

On the other hand, if $b_{ij} = 1$, then $x_i - x_j \geq 1$, which implies that $x_i > x_j$. In this case, $x_j - x_i \geq 1 - M$. With $M = 2\Delta^2 - \Delta + 1$, the second constraint is always fulfilled. In both cases, we obtain that $|x_i - x_j| \geq 1$, as required.

In order to linearize the constraints $|(x_i - x_k) - 2x_j| \geq 1$, we first add a binary variable $d_{ijk} \in \{0, 1\}$ for all $v_i v_j, v_j v_k \in E$. Moreover, we remove each constraint $|(x_i - x_k) - 2x_j| \geq 1$ and in its place we add two new constraints:

- **Constraint 1:** $((x_i + x_k) - 2x_j) \geq 1 - L \cdot (1 - d_{ijk})$
- **Constraint 2:** $(2x_j - (x_i + x_k)) \geq 1 - L \cdot d_{ijk}$

Here, $L$ is a sufficiently large constant that bounds the possible differences between $(x_i + x_k)$ and $2x_j$. Since $x_i, x_j, x_k$ are vertex labels, we know that $1 \leq x_i, x_j, x_k \leq 2\Delta^2 - \Delta + 1$. This fact and the fact that $x_i \neq x_k$ imply that $3 \leq x_i + x_k \leq 4\Delta^2 - 2\Delta + 1$. Moreover, we obtain that $2 \leq 2x_j \leq 4\Delta^2 - 2\Delta + 2$. These facts imply that $|(x_i + x_k) - 2x_j| \leq (4\Delta^2 - 2\Delta + 1) - 2 = 4\Delta^2 - 2\Delta - 1$. Therefore, a good choice for $L$ is $L = 4\Delta^2 - 2\Delta$.

Note that if $d_{ijk} = 0$, then the second constraint is satisfied, that is, $(2x_j - (x_i + x_k)) \geq 1$, which implies that $2x_j > x_i + x_k$. With $L = 4\Delta^2 - 2\Delta$, the first constraint is always fulfilled.

On the other hand, if $d_{ijk} = 1$, then the first constraint is satisfied, that is, $((x_i + x_k) - 2x_j) \geq 1$, which implies that $x_i + x_k > 2x_j$. With $L = 4\Delta^2 - 2\Delta$, the second constraint is always fulfilled. In both cases, we obtain that $|x_i - x_j| \geq 1$, as required.