Integer Linear Programming Formulation for the L(3,2,1)-Labeling Problem

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Given a simple graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, we create an integer variable x_i for each vertex $v_i \in V$, where $x_i \in \mathbb{Z}_{\geq 0}$. The value of variable x_i is the color of vertex v_i.

The ILP formulation is, thus, given below.

$$
\begin{array}{lll}
\text{minimize} & z & \\
\text{subject to} & x_i \leq z & \text{for all } v_i \in V, \\
& |x_i - x_j| \geq 3 & \text{for all } v_i, v_j \in V \text{ such that } d(v_i, v_j) = 1, \\
& |x_i - x_j| \geq 2 & \text{for all } v_i, v_j \in V \text{ such that } d(v_i, v_j) = 2, \\
& |x_i - x_j| \geq 1 & \text{for all } v_i, v_j \in V \text{ such that } d(v_i, v_j) = 3, \\
& x_i \in \mathbb{Z}_{\geq 0} & \text{for all } v_i \in V, \\
& z \in \mathbb{Z}_{\geq 0} &
\end{array}
$$

In order to linearize the constraints $|x_i - x_j| \geq k$, for $k \in \{3, 2, 1\}$, we first add a binary variable $b_{ij} \in \{0, 1\}$ for all $v_i, v_j \in V$ such that $d(v_i, v_j) \leq 2$ and $i \neq j$. Moreover, we remove each constraint $|x_i - x_j| \geq k$ and in its place we add two new constraints:

- **Constraint 1:** $x_i - x_j \geq k - M \cdot (1 - b_{ij})$
- **Constraint 2:** $x_j - x_i \geq k - M \cdot b_{ij}$

Here, M is a sufficiently large constant that bounds the possible differences between x_i and x_j.