CS215 Assignment 2
Question 1
Sampling within an Euclidian plane

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1 Generating Uniform Points in Ellipse

1.1 Algorithm for Sampling Points

In polar coordinates, the equation of ellipse is given by

$$r(\theta) = \frac{ab}{\sqrt{(b\cos\theta)^2 + (a\sin\theta)^2}}$$

For any random θ, the probability of sampling it will be proportional to the length of the line at angle θ (i.e. r). So, we can say that

$$P(\theta) \propto r$$

Or we can say

$$P(\theta)d\theta = \frac{dA}{A}$$

where dA is area of small segment at angle θ and width $d\theta$ and A is the total area of Ellipse with major and minor axes as a and b respectively. Using $dA = \frac{1}{2}r^2 d\theta$ and $A = \pi ab$, we can calculate the **Marginal PDF** of θ :

$$P(\theta)d\theta = \frac{1}{2}\frac{r^2 d\theta}{\pi ab} \implies P(\theta) = \frac{ab}{2\pi((b\cos\theta)^2 + (a\sin\theta)^2)}$$

Now, we have PDF of θ. For generating random numbers, we used *Inverse CDF technique* which is that generating a uniform random number between $[0, 1]$ and calculating inverse CDF at that point gives uniform random numbers from the desired Distribution.
The CDF of the above PDF is :

$$\begin{aligned} F(\theta) &= \int_0^\theta \frac{ab}{2\pi((b\cos\theta)^2 + (a\sin\theta)^2)} d\theta \\ &= \frac{1}{2\pi}\tan^{-1}\frac{a\tan\theta}{b} \end{aligned}$$

Inverse CDF function is inverse of $F(\theta)$:

$$\theta = \tan^{-1}\frac{b}{a}\tan 2\pi x$$

where x is a Uniform random number generated between 0 to 1
Since range of $\tan^{-1}$ function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we need to modify the angle θ that we get so as to cover all the angles in the ellipse **uniformly**. Suppose random number generated from $[0,1]$ is $rand$, then we did this as -

- if $rand \in [0, 0.5]$, add π to θ so that $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ now
- else, leave as it is, so that $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus covering all angles uniformly

Therefore we get a uniform θ and then we can calculate a uniform point inside ellipse by choosing a random number between 0 to r again using inverse CDF technique for R, where r is the distance from origin to boundary point of ellipse in the direction of θ and R is the distance from origin to uniform point sampled in ellipse.
For a given θ, the probability of sampling a particular R will be proportional to r.

$$P(r = R) \propto r \implies F(r \leq R) \propto r^2$$

So, inverse CDF will be proportional to square root of distance. On scaling it properly and generating a random number $rand$ from 0 to 1, random R from inverse CDF will be scaled $r(\theta)$ times and will be generated as:

$$R = r(\theta)\sqrt{rand}$$

$$R = \frac{ab}{\sqrt{(b\cos\theta)^2 + (a\sin\theta)^2}}\sqrt{rand}$$

Thus, we sample uniformly points in the Elliptical distribution using uniform random number generation in an 2D Euclidian Plane.

1.2 Plot

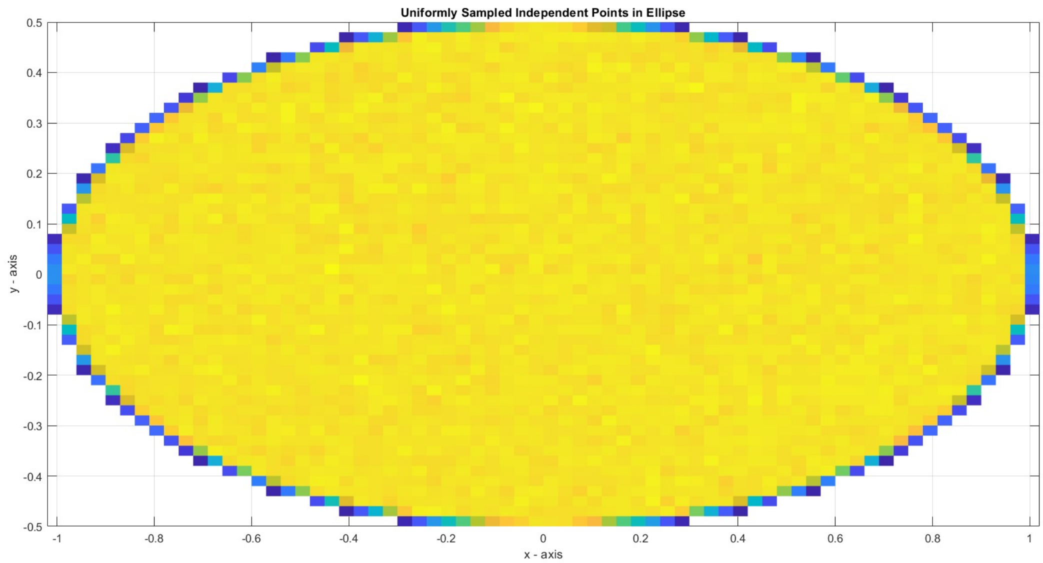

Figure 1: Histogram of 2-D uniform points sampled in Elliptical Distribution

The above plot shows sampling of $N = 10^7$ independent uniform points inside an Ellipse using the algorithm explained in section 1.1

2 Generating uniform points in Triangle

2.1 Algorithm for Sampling Points

The same idea as used for ellipse is also implemented here.
For any point $X \in [0, \pi]$ on x-axis ,

$$P(X) \propto y_{max}$$

where y_{max} is the height of the triangle at $x = X$. For convenience, we scaled the PDF by area of triangle as it is proportional to y_{max} and y_{max} is proportional to area. Thus, the scaled CDF will have values from 0 to area, hence we generated uniform random numbers from 0 to area for sampling x coordinates. The scaled PDF is given by -

$$P(X) = \begin{cases} \frac{3ex}{\pi} & , 0 \leq x \leq \frac{\pi}{3} \\ \frac{-3e(x-\pi)}{2\pi} & , \frac{\pi}{3} < x \leq \pi \end{cases}$$

The corresponding CDF(scaled) is -

$$F(X) = \int_0^x P(X)dx$$

$$F(X) = \begin{cases} \frac{3ex^2}{2\pi} & , 0 \leq x \leq \frac{\pi}{3} \\ ax^2 + bx + c & , \frac{\pi}{3} < x \leq \pi \end{cases}$$

where coefficients of Quadratic equations are $a = -\frac{3e}{4\pi}, b = \frac{3e}{2}, c = \frac{-e\pi}{4}$ Thus, inverse scaled CDF function for generating x coordinates is -

$$X = \begin{cases} \sqrt{\frac{2x\pi}{3e}} & , 0 \leq x \leq \frac{e\pi}{6} \\ \frac{-b+\sqrt{b^2-4a(c-x)}}{2a} & , \frac{e\pi}{6} < x \leq \frac{e\pi}{2} \end{cases}$$

We generated uniform random numbers $x \in [0, \frac{e\pi}{2}]$ (as total area of triangle is $\frac{e\pi}{2}$), and by putting them in above function X, we obtained x-coordinates of points.
For y-coordinates, once we have the x-coordinates, we generated uniform random numbers for each x such that the random number generated lies in 0 to y_{max}, where y_{max} is the maximum height of the triangle at $x = X$, i.e. the value of $y(X)$.
Thus, we sampled uniform points in a Triangular distribution.

2.2 Plot

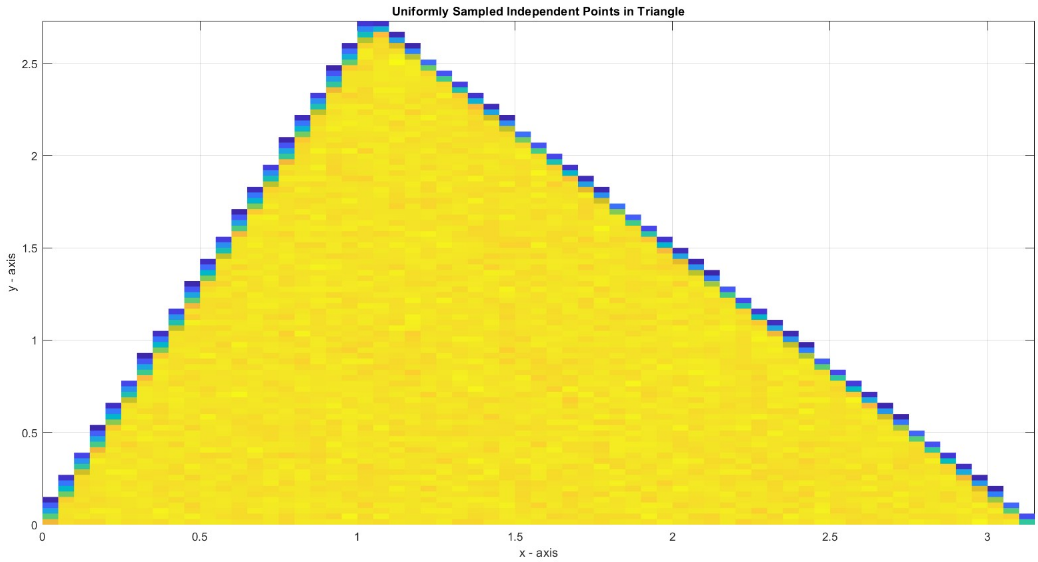

Figure 2: Histogram of 2-D uniform points sampled in Triangular Distribution

The above plot shows sampling of $N = 10^7$ independent uniform points inside an Triangle using the algorithm explained in section 2.1