Report - Question 3

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1 Random Walkers

APPROACH :

- First generate a random number either 1 or 0,where 0 means backward step and 1 means forward step.
- Generate such 10^3 numbers and add the steps obtained from these numbers. The sum will give the final position of random walker.
- Store the final position for each of 10^3 random walkers and plot histogram **CONCLUSION :** The number of walkers with distance around 0 are higher than any other number.

1.1 Histogram : Final Locations

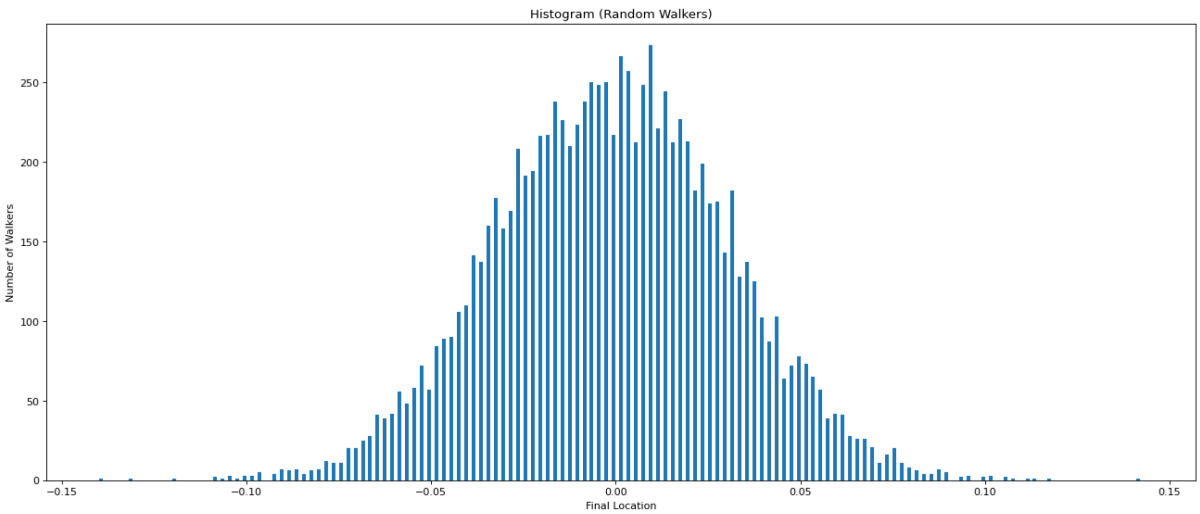

Figure 1: Histogram of Final locations of all Random Walkers

We simulated $N = 10^4$ random walkers

1.2 Space-Time Curves

APPROACH :

- While we were adding the steps, append the sum after each step to a list and use it to check the position at that given time.
- Plot the position-time graph.

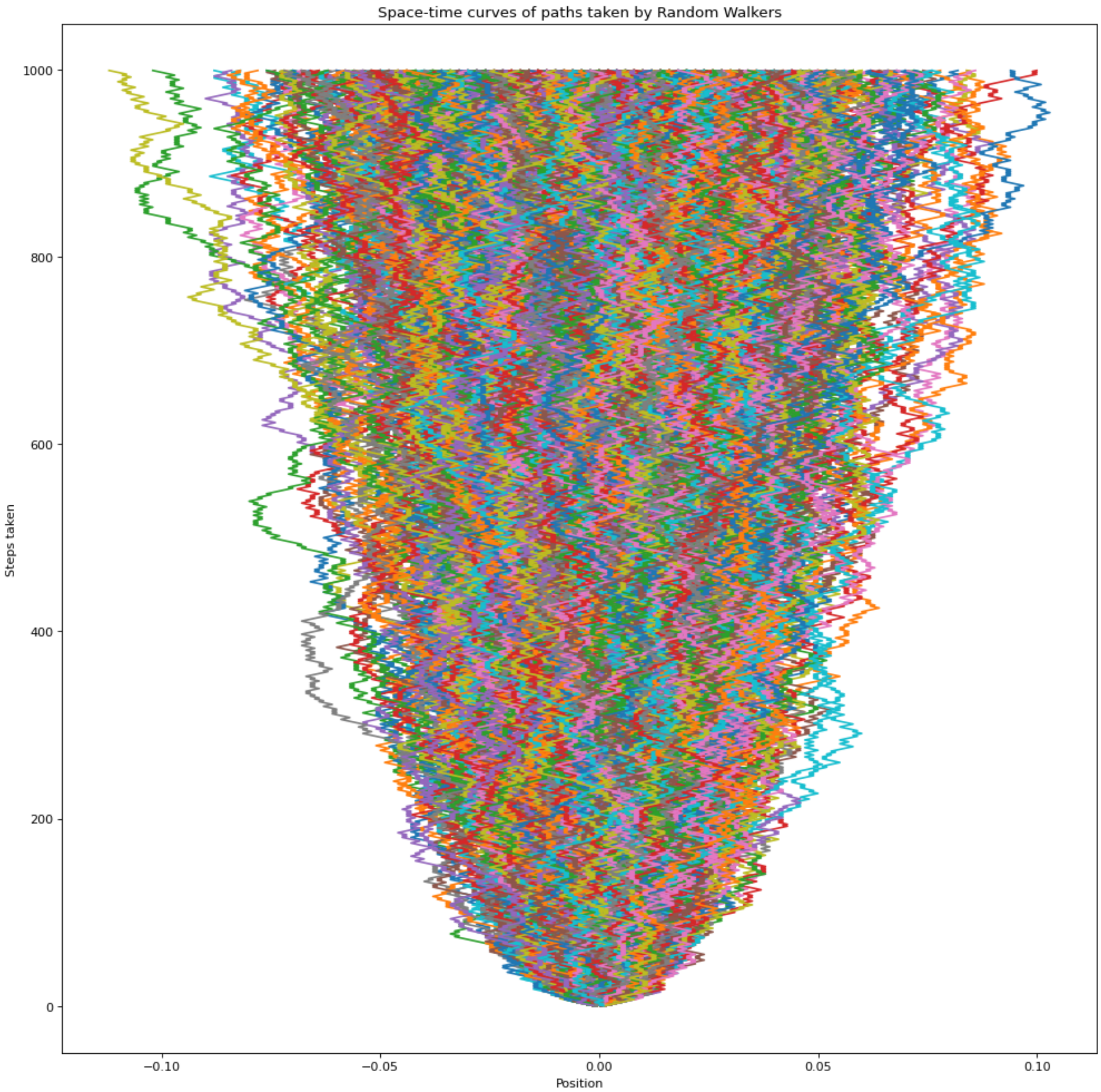

Figure 2: Space-Time Curves showing path taken by each Random Walker

2 Proofs using Law of Large Numbers

1. Use the law of large numbers to show that the random variable $\hat{M} := \frac{X_1+X_2+X_3...+X_N}{N}$. converges to the true mean $M := E[X]$ as $N \to \infty$

    Given X is a random variable,
    we construct another random variable

    $$\hat{M} = \frac{1}{N}\sum_{i=1}^{N} X_i \tag{1}$$

    where X_i is an independent draw from the distribution of X.
    since X_i are the independent draws from common distribution X, their expectation and variance will be same i.e.

    $$E[X_i] = E[X] \, \forall i \in [1, N] \tag{2}$$

    as well as

    $$Var[X_i] = Var[X] \forall i \in [1, N] \tag{3}$$

Hence we can use law of large numbers which states that,

$$P(|\hat{M} - E[X]| \geq \epsilon) \leq \frac{Var(\hat{M})}{\epsilon^2}, \frac{Var(\hat{M})}{\epsilon^2} = \frac{v}{n\epsilon^2},\ where\ v = Var(X) \tag{4}$$

$$hence\ as\ N \to \infty \implies P(|\hat{M} - E[X]| \geq \epsilon) \to 0 \tag{5}$$

2. Prove that the expected value of the random variable $\hat{V} := \frac{\sum_{i=1}^{N}(X_i - \hat{M})^2}{N}$ tends to the true variance $V := Var(X)$ as $N \to \infty$.

$$E(\hat{V}) = E(\frac{1}{N}\sum_{i=1}^{N}(X_i - \hat{M})^2) \tag{6}$$

$$E(\hat{V}) = \frac{1}{N}\sum_{i=1}^{N}E((X_i - \hat{M})^2) \tag{7}$$

We know that,

$$E(X^2) = (E(X))^2 + Var(X) \tag{8}$$

from equation 8,

$$E((X_i - \hat{M})^2) = (E(X_i - \hat{M}))^2 + Var(X_i - \hat{M}) \tag{9}$$

equation 7 becomes

$$E(\hat{V}) = \frac{1}{N}\sum_{i=1}^{N}((E(X_i - \hat{M}))^2 + Var(X_i - \hat{M})) \tag{10}$$

as $E(X_i) = E(\hat{M})$, equation (11) becomes,

$$E(\hat{V}) = \frac{1}{N}\sum_{i=1}^{N}Var(X_i - \hat{M}) \tag{11}$$

$$E(\hat{V}) = \frac{1}{N}\sum_{i=1}^{N}(Var(X_i) - Var(\hat{M})) \tag{12}$$

$$E(\hat{V}) = \frac{1}{N}\sum_{i=1}^{N}Var(X_i) - \frac{1}{N}\sum_{i=1}^{N}Var(\hat{M}) \tag{13}$$

since $Var(X_i) = Var(X)\ \forall i\ \in [1, N]$, $Var(\hat{M}) = Var(X)$
So summation of K will give constant*N and hence $\frac{1}{N}\sum_{i=1}^{N}$constant boils down to constant

$$E(\hat{V}) = Var(X) - Var(\hat{M}) \tag{14}$$

applying limits on N

$$\lim_{N\to\infty} E(\hat{V}) = Var(X) - \lim_{N\to\infty} Var(\hat{M}) \tag{15}$$

by using law of large numbers on $\hat{M}$

$$\lim_{N\to\infty} Var(\hat{M}) = 0 \tag{16}$$

$$\lim_{N\to\infty} E(\hat{V}) = Var(X) \tag{17}$$

Q.E.D

3 Report of Empirically Computed Mean and Variance

We wrote the code to report the Empirically computed mean and Variance of the final locations of random walkers. Here are the results we got -

```
Empirical-computed Mean         :  -0.0004966000000000032
Empirical-computed variance     :  0.0009970921884400082
```

Figure 3: Report of Empirically computed Mean and Variance

4 True Mean and Variance for random variable that models final location of random walkers

The final position of random walker can be thought of as Bernoulli distribution with N trials and $p = \frac{1}{2}$ and $(N - \frac{N-d}{2})$ successes where d is the final position of random walker
Why analogous to Bernoulli distribution
Suppose the random walker has a coin and he tosses it N times. If it comes head he moves forward by 1 step or vice versa.
So the final position of random walker actually depends on the number of getting heads and hence we can think of it as Bernoulli distribution.
Lets say the random walker takes N steps and each step has length l
So the values the final position of random walker takes depends on the number of forward steps.
So lets say the random walker takes q forward steps, hence the final position has value $(q-(N-q))*l = (2q-N)*l$

4.1 Mean

$$E = \sum_{x=0}^{N} P(x) * d \, , d = final\, position \, , x = total\, success \tag{18}$$

$$P(x) = \binom{N}{x} p^x (1-q)^x \tag{19}$$

$$E = \sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x} * d \tag{20}$$

Now since $p = 1 - p = \frac{1}{2}$

$$E = \sum_{x=0}^{N} \binom{N}{x} p^x p^{N-x} * d \tag{21}$$

$$E = \sum_{x=0}^{N} \binom{N}{x} p^N * d \tag{22}$$

since d can be written as $(2x - N) * l$ where x is number of forward steps

$$E = \sum_{x=0}^{N} \binom{N}{x} p^N * (2x - N) * l \tag{23}$$

$$E = p^N l \sum_{x=0}^{N} \binom{N}{x} * (2x - N) \tag{24}$$

$$E = p^N l \left(\sum_{x=0}^{N} \binom{N}{x} * (2x) - \sum_{x=0}^{N} \binom{N}{x} N \right) \tag{25}$$

We know that,

$$\sum_{x=0}^{N} \binom{N}{x} = 2^N$$

$$\sum_{i=0}^{N} \binom{N}{x} * x = N2^{N-1}$$

So equation (8) becomes,

$$E = p^N l(2 * N * 2^{N-1} - N * 2^N) = 0 \tag{26}$$

hence the true mean of the the final position of random walkers is 0.

4.2 Variance

$$Var = \sum_{x=0}^{N} \binom{N}{x} p^x (1-q)^{N-x} * (d - mean)^2 \tag{27}$$

Since mean is 0 and $p = 1 - p = \frac{1}{2}$

$$Var = \sum_{x=0}^{N} \binom{N}{x} p^x (p)^{N-x} * (d)^2 \tag{28}$$

$$Var = \sum_{x=0}^{N} \binom{N}{x} p^N * ((2x - N) * l)^2 \tag{29}$$

$$Var = p^N l^2 \sum_{x=0}^{N} \binom{N}{x} * (4x^2 + N^2 - 4xN) \tag{30}$$

$$Var = p^N l^2 \sum_{x=0}^{N} \binom{N}{x} * (4x^2 + N^2 - 4xN) \tag{31}$$

$$Var = p^N l^2 \left(\sum_{x=0}^{N} \binom{N}{x} * (4x^2) + \sum_{x=0}^{N} \binom{N}{x} (N^2) - \sum_{x=0}^{N} \binom{N}{x} (4xN) \right) \tag{32}$$

$$\sum_{x=0}^{N} \binom{N}{x} * x^2 = N * 2^{N-1} + N * (N-1) * 2^{N-2}$$

$$Var = p^N l^2 ((N * 2^{N+1} + N * 2^N - N * 2^N) + N^2 * 2^N - N^2 * 2^{N+1}) \tag{33}$$

$$Var = 2^N p^N l^2 N \tag{34}$$

$p = \frac{1}{2}$ therefore $p^N * 2^N = 1$

$$Var = l^2 N \tag{35}$$

5 Report of error between True and Empirical Values

We wrote the code to find error between true and empirical values of mean and variance in the code folder. Here are the results of error we got -

```
The errors between true mean and empirical-computed mean         :  0.0004966000000000032
The errors between true variance and empirical-computed variance :  2.9078115599918616e-06
```

Figure 4: Report of Error