Report - Question 2

Atishay Jain & Megh Gohil

August 22, 2022

1 Sums of Independent Poisson Random Variables

1.1 Empirically :

APPROACH :

Given X$\sim$Poisson(λ_x) and Y$\sim$Poisson(λ_y) as independent
Theoretically,

$$P_Z(k) = \sum_{i=0}^{k} P_X(x = i, y = (k - i)) \tag{1}$$

Since the random variable X and Y are independent the above equation boils down to

$$P_Z(k) = \sum_{i=0}^{k} P_X(i) * P_Y(k - i) \tag{2}$$

So to calculate $P_Z(k)$ empirically,
First we generated 10^6 instances of X and Y both, then we calculated the number of instances of k($0 \leq k \leq 25$) such that $k = x + y$ and appended it in the list and finally calculated the probability by diving the no of instances by 10^{12}

1.2 Theoretically :

Theoretically, $P_Z(k) = \sum_{i=0}^{k} P_X(i) * P_Y(k - i)$

$$P_X(k) = \frac{e^{\lambda} * \lambda^k}{k!} \tag{3}$$

$$P_Z(k) = \sum_{i=0}^{k} \frac{e^{-\lambda} * \lambda^i}{i!} * \frac{e^{-\lambda} * \lambda^{k-1}}{(k-i)!} \tag{4}$$

$$P_Z(k) = \sum_{i=0}^{k} \frac{e^{-\lambda_x} * \lambda_x^i}{i!} * \frac{e^{-\lambda_y} * \lambda_y^{k-1}}{(k-i)!} \tag{5}$$

$$P_Z(k) = e^{-(\lambda_x + \lambda_y)} \sum_{i=0}^{k} \frac{\lambda_x^i}{i!} * \frac{\lambda_y^{k-1}}{(k-i)!} \tag{6}$$

Multiplying numerator and denominator with k!

$$P_Z(k) = \frac{e^{-(\lambda_x+\lambda_y)}}{k!} \sum_{i=0}^{k} k! * \frac{\lambda_x^i}{i!} * \frac{\lambda_y^{k-1}}{(k-i)!} \tag{7}$$

$$P_Z(k) = \frac{e^{-(\lambda_x+\lambda_y)}}{k!} \sum_{i=0}^{k} \binom{k}{i} * \lambda_x^i * \lambda_y^{k-1} \tag{8}$$

$$P_Z(k) = \frac{e^{-(\lambda_x+\lambda_y)} * (\lambda_x+\lambda_y)^k}{k!} \tag{9}$$

Hence $Z \sim Poisson(\lambda_x + \lambda_y)$

1.3 Comparison between Emperical and Theoretical values

Below are the values that we got on running the code for the emperical values and the values using the theoretical formula for $P(Z = k)$. The values are very close to each other with the error getting close to zero as k increases.

	P(Z = k) Emperically	Theoretically
k = 0	8.998274e-04	9.118820e-04
k = 1	6.350820e-03	6.383174e-03
k = 2	2.231392e-02	2.234111e-02
k = 3	5.210853e-02	5.212925e-02
k = 4	9.117857e-02	9.122619e-02
k = 5	1.275964e-01	1.277167e-01
k = 6	1.488259e-01	1.490028e-01
k = 7	1.488468e-01	1.490028e-01
k = 8	1.303070e-01	1.303774e-01
k = 9	1.014468e-01	1.014047e-01
k = 10	7.111355e-02	7.098327e-02
k = 11	4.533687e-02	4.517117e-02
k = 12	2.649562e-02	2.634985e-02
k = 13	1.428795e-02	1.418838e-02
k = 14	7.148382e-03	7.094190e-03
k = 15	3.333340e-03	3.310622e-03
k = 16	1.454655e-03	1.448397e-03
k = 17	5.963290e-04	5.963988e-04
k = 18	2.306188e-04	2.319329e-04
k = 19	8.444582e-05	8.544896e-05
k = 20	2.943828e-05	2.990713e-05
k = 21	9.780688e-06	9.969045e-06
k = 22	3.110342e-06	3.171969e-06
k = 23	9.397610e-07	9.653818e-07
k = 24	2.705730e-07	2.815697e-07
k = 25	7.318500e-08	7.883952e-08

Figure 1: Comparison for values of $\hat{P}(Z = k)$ and $P(Z = k)$

2 Poisson Thinning Process

2.1 Empirically

APPROACH :

Theoretically,

$$P(Z|Y=j) = P_binomial(Z;j,p) \tag{10}$$

which means

$$P_Z(k) = \sum_{i=k}^{\infty} P(Y=i, Z=k) = \sum_{i=k}^{\infty} P(Z=k|Y=i)P(Y=i) \tag{11}$$

So to calculate the $P_Z(k)$ empirically, we created 10^5 instances of Y and then for each instance y we calculated the total number of success in y trials.
Then we counted the frequency of k($0 \leq k \leq 25$) in the obtained dataset and then divided it by 10^5 to get the probability corresponding to each k.

2.2 Theoretically

the equation (11) can be written as

$$P_Z(k) = \sum_{i=k}^{\infty} \frac{e^{-\lambda} * \lambda^i}{i!} * \binom{i}{k} p^k (1-p)^{i-k} \tag{12}$$

$$P_Z(k) = p^k * e^{-\lambda} \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} * \frac{i!}{k!(i-k)!} * (1-p)^{i-k} \tag{13}$$

multiplying and dividing numerator and denominator by λ^k

$$P_Z(k) = \frac{(p\lambda)^k * e^{-\lambda}}{k!} \sum_{i=k}^{\infty} \frac{1}{i!} * \frac{i!}{k!(i-k)!} * (\lambda(1-p))^{i-k} \tag{14}$$

$$P_Z(k) = \frac{(p\lambda)^k * e^{-\lambda}}{k!} e^{\lambda(1-p)} \tag{15}$$

$$P_Z(k) = \frac{(p\lambda)^k * e^{-\lambda} e^{\lambda(1-p)}}{k!} \tag{16}$$

$$P_Z(k) = \frac{(\lambda p)^k * e^{-\lambda p}}{k!} \tag{17}$$

So the Z$\sim$Poisson(λp)
Comparing P_Z with $\hat{P_Z}$

2.3 Comparison between Emperical and Theoretical values

Below are the values that we got on running the code for the emperical values and the values using the theoretical formula for $P(Z=k)$ for Poisson Thinning Process on Y which gave Z. The values are very close to each other with the error getting close to zero as k increases. Also, the Poisson thinning random generator generated values very very close to zero after k = 13 as the distribution tends to zero after it. So, the emperical value is zero, and the theoretical values are also very close to zero as well.

	P (Z = k) Emperically	Theoretically
k = 0	0.04136	4.076220e-02
k = 1	0.13090	1.304391e-01
k = 2	0.20613	2.087025e-01
k = 3	0.22396	2.226160e-01
k = 4	0.17856	1.780928e-01
k = 5	0.11217	1.139794e-01
k = 6	0.06117	6.078900e-02
k = 7	0.02881	2.778926e-02
k = 8	0.01099	1.111570e-02
k = 9	0.00420	3.952250e-03
k = 10	0.00113	1.264720e-03
k = 11	0.00053	3.679186e-04
k = 12	0.00008	9.811162e-05
k = 13	0.00001	2.415055e-05
k = 14	0.00000	5.520126e-06
k = 15	0.00000	1.177627e-06
k = 16	0.00000	2.355254e-07
k = 17	0.00000	4.433419e-08
k = 18	0.00000	7.881634e-09
k = 19	0.00000	1.327433e-09
k = 20	0.00000	2.123893e-10
k = 21	0.00000	3.236408e-11
k = 22	0.00000	4.707503e-12
k = 23	0.00000	6.549569e-13
k = 24	0.00000	8.732759e-14
k = 25	0.00000	1.117793e-14

Figure 2: Comparison for values of $\hat{P}(Z = k)$ and $P(Z = k)$