Question 1

CS663 (Digital Image Processing) Assignment 2

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Question 1

Problem 1

Consider a 1D convolution mask given as $(w_0, w_1, ..., w_6)$. Express the convolution of the mask with a 1D image f as the multiplication of a suitable matrix with the image vector f. What are the properties of this matrix? What could be a potential application of such a matrix-based construction? [10 points]

SECTION 1

Convolution of 1D image

Given, 1D convolution mask $w = (w_0, w_1, w_2, w_3, w_4, w_5, w_6)$ and let the image be

$$f = (f_0, f_1, f_2, ..., f_{n-1})$$

with the output image g after applying w be

$$g = (g_0, g_1, g_2, ..., g_{m-1})$$

We aim to find a suitable matrix A such that

$$g = \mathbf{A}f$$

Remark Note that the size of given mask w is 7. According to slides, for a mask of size k (in 1D), we should pad with $k-1$ zeros on either side of the image. So, we are taking padding of 6 zeros on either side of image f.

Remark Zero padding of 6 on either side will increase the dimension of output image g by 6, that is, $m = n + 6$ and $g = (g_0, g_1, ..., g_{n+5})$. This is just an assumption, we could have padded with 3 zeros on either side to maintain the dimensions of g same as f, but that won't be a problem in convolution.

Method For finding the 1D convolution, the mask is first rotated by 180 degrees (since we are solving in 1D, the mask is just reversed) and then moved over the image by computing sum of products.

So, the mask which we are going to pass over the image after reversing w becomes $w = (w_6, w_5, w_4, w_3, w_2, w_1, w_0)$

Lets see the passing of this mask over the image -

0 0 0 0 0 0	f_0	f_1	f_2	.	.	.	f_{n-2}	f_{n-1}	0 0 0 0 0 0
$w_6w_5w_4w_3w_2w_1$	w_0								
$w_6w_5w_4w_3w_2$	w_1	w_0							
$w_6w_5w_4w_3$	w_2	w_1	w_0						
$w_6w_5w_4$	w_3	w_2	w_1	w_0					
w_6w_5	w_4	w_3	w_2	w_1	w_0				
w_6	w_5	w_4	w_3	w_2	w_1	w_0			
	w_6	w_5	w_4	w_3	w_2	w_1			
		.	.	.	.	.	.		
			.	.	.	.	.		
							w_6	w_5	$w_4w_3w_2w_1w_0$
								w_6	$w_5w_4w_3w_2w_1w_0$

Each row of the above table denotes each iteration of convolution pass over the image. That is, each row represents the value of g_i, which can be found by multiplying the w_k present in that row with the f_j column in which it is and summing them. Thus, the g_i's which we get are -

$$g_0 = w_0 f_0$$

$$g_1 = w_1 f_0 + w_0 f_1$$

$$g_2 = w_2 f_0 + w_1 f_1 + w_0 f_2$$

and so on, upto

$$g_{m-1} = w_6 f_{n-1}$$

Let f be represented as a column vector of dimensions $n \times 1$ -

$$f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_{n-1} \end{bmatrix}_{n\times 1}$$

Similarly we can represent output image g also as a column vector of dimension $m \times 1$.

$$g = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ \vdots \\ g_{m-1} \end{bmatrix}_{m\times 1}$$

Notice that the value of g_2 can be written as the following matrix multiplication -

$$g_2 = \begin{bmatrix} w_2 & w_1 & w_0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{1\times n} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{bmatrix}_{n\times 1}$$

From observing a similar kind of calculation in each value of g_i, we can see that the final result of convolution can be represented as a matrix multiplication as follows -

$$\mathbf{g}_{m\times 1} = \begin{bmatrix} w_0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ w_1 & w_0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_5 & w_4 & w_3 & w_2 & w_1 & w_0 & 0 & \cdots & \cdots & 0 \\ w_6 & w_5 & w_4 & w_3 & w_2 & w_1 & w_0 & 0 & \cdots & 0 \\ 0 & w_6 & w_5 & w_4 & w_3 & w_2 & w_1 & w_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & w_6 & w_5 & w_4 & w_3 & w_2 & w_1 & w_0 \\ 0 & \cdots & \cdots & 0 & w_6 & w_5 & w_4 & w_3 & w_2 & w_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & w_6 & w_5 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & w_6 \end{bmatrix}_{m\times n} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ \vdots \\ f_{n-1} \end{bmatrix}_{n\times 1}$$

Thus, we have constructed a matrix A for 1D convolution as $g = A \cdot f$, where A is the above $m \times n$ matrix.

Section 2

Properties and Application of such Matrix

The properties of matrix A are -

Properties

- As most of the entries of this matrix are 0, the matrix is **Sparse**
- Each descending diagonal from left to right is constant, and such a matrix is also called **Toeplitz** matrix or **diagonal-constant** matrix in Linear Algebra
- Mathematically, in a Toeplitz matrix A if i, j element is denoted by $A_{i,j}$, then we have

$$A_{i,j} = A_{i+1,j+1}$$

Subsection 2.1

Potential Application

- A potential application of such matrix-based construction could be in fields that use convolution operations many times like image filtering, Convolutional Neural Networks, signal processing, etc
- The matrix-based convolution can be applied in various signal-processing tasks, such as filtering and feature extraction in audio, image, or time-series data.
- CNNs use convolution operations extensively for feature extraction. Expressing convolution as matrix multiplication can help accelerate convolution operations on GPUs and parallel processors, leading to faster training and inference times.
- For a particular convolution mask, this matrix can be computed once and can be used several times on different images/inputs whenever needed, thus improving efficiency.