Question 3

CS663 (Digital Image Processing) Assignment 5

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Contents

Question 3

PART

I

Problem 1

Consider a matrix $\boldsymbol{A}$ of size $m \times n, m \leq n$. Define $\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T$. (Note: all matrices, vectors and scalars involved in this question are real-valued).

1. Prove that for any vector $\boldsymbol{y}$ with appropriate number of elements, we have $\boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} \geq 0$. Similarly show that $\boldsymbol{z}^t\boldsymbol{Q}\boldsymbol{z} \geq 0$ for a vector $\boldsymbol{z}$ with appropriate number of elements. Why are the eigenvalues of $\boldsymbol{P}$ and $\boldsymbol{Q}$ non-negative?

2. If $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue λ, show that $\boldsymbol{A}\boldsymbol{u}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue λ. If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue μ, show that $\boldsymbol{A}^T\boldsymbol{v}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue μ. What will be the number of elements in $\boldsymbol{u}$ and $\boldsymbol{v}$?

3. If $\boldsymbol{v}_i$ is an eigenvector of $\boldsymbol{Q}$ and we define $\boldsymbol{u}_i \triangleq \dfrac{\boldsymbol{A}^T\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$. Then prove that there will exist some real, non-negative γ_i such that $\boldsymbol{A}\boldsymbol{u}_i = \gamma_i\boldsymbol{v}_i$.

4. It can be shown that $\boldsymbol{u}_i^T\boldsymbol{u}_j = 0$ for $i \neq j$ and likewise $\boldsymbol{v}_i^T\boldsymbol{v}_j = 0$ for $i \neq j$ for correspondingly distinct eigenvalues. (You did this in HW4 where you showed that the eigenvectors of symmetric matrices are orthonormal.) Now, define $\boldsymbol{U} = [\boldsymbol{v}_1|\boldsymbol{v}_2|\boldsymbol{v}_3|...|\boldsymbol{v}_m]$ and $\boldsymbol{V} = [\boldsymbol{u}_1|\boldsymbol{u}_2|\boldsymbol{u}_3|...|\boldsymbol{u}_m]$. Now show that $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T$ where $\boldsymbol{\Gamma}$ is a diagonal matrix containing the non-negative values $\gamma_1, \gamma_2, ..., \gamma_m$. With this, you have just established the existence of the singular value decomposition of any matrix $\boldsymbol{A}$. This is a key result in linear algebra and it is widely used in image processing, computer vision, computer graphics, statistics, machine learning, numerical analysis, natural language processing and data mining.
[7.5 + 7.5 + 7.5 + 7.5 = 30 points]

SECTION 1
Part 1

We are given that $\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T$.

PROOF We want to prove that $\boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} \geq 0$ for any vector $\boldsymbol{y}$ and $\boldsymbol{z}^t\boldsymbol{Q}\boldsymbol{z} \geq 0$ for a vector $\boldsymbol{z}$.
Substituting $\boldsymbol{P}$ with $\boldsymbol{A}^T\boldsymbol{A}$, we get

$$\boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} = \boldsymbol{y}^t\boldsymbol{A}^t\boldsymbol{A}\boldsymbol{y} = (\boldsymbol{A}\boldsymbol{y})^t\boldsymbol{A}\boldsymbol{y} = \|\boldsymbol{A}\boldsymbol{y}\|^2$$

Similarly, Substituting $\boldsymbol{Q}$ with $\boldsymbol{A}\boldsymbol{A}^T$, we get

$$\boldsymbol{z}^t\boldsymbol{Q}\boldsymbol{z} = \boldsymbol{z}^t\boldsymbol{A}\boldsymbol{A}^t\boldsymbol{z} = \boldsymbol{z}^t\boldsymbol{A}(\boldsymbol{z}^t\boldsymbol{A})^t = \|\boldsymbol{z}^t\boldsymbol{A}\|^2$$

Here, $\|.\|^2$ is the squared Euclidean norm of the matrices, which is always non-negative. Therefore, $\boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} \geq 0$ for any vector $\boldsymbol{y}$ and $\boldsymbol{z}^t\boldsymbol{Q}\boldsymbol{z} \geq 0$ for any vector $\boldsymbol{z}$ with appropriate number of elements. $\square$

The eigenvalues of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are non-negative because:

- For $\boldsymbol{P}$, we have $\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{A}$. The eigenvalues of $\boldsymbol{P}$ are the non-negative square singular values (or singular values squared) of $\boldsymbol{A}$. This is because, for any real matrix (like $\boldsymbol{A}$), the eigenvalues of a positive semidefinite (PSD) matrix (like $\boldsymbol{P}$) are square singular values of $\boldsymbol{A}$ (non-negative). Therefore, all eigenvalues of $\boldsymbol{P}$ are non-negative.
- For $\boldsymbol{Q}$, we have $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T$. The eigenvalues of $\boldsymbol{Q}$ are the non-negative square singular values (or singular values squared) of $\boldsymbol{A}^T$. Therefore similarly, all eigenvalues of $\boldsymbol{Q}$ are non-negative.

SECTION 2
Part 2

Part 2a :-

Given: $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue λ i.e, $\boldsymbol{P}\boldsymbol{u} = \lambda\boldsymbol{u}$ where $\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{A}$

Show: $\boldsymbol{Q}\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{A}\boldsymbol{u}$

Substituting $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T$ in $\boldsymbol{Q}\boldsymbol{A}\boldsymbol{u}$, we get

$$\boldsymbol{Q}\boldsymbol{A}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{u} \tag{2.1}$$

$$\because \boldsymbol{P}\boldsymbol{u} = \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u} \tag{2.2}$$

$\therefore$ from equation (2.1) and (2.2), we get

$$\boldsymbol{Q}\boldsymbol{A}\boldsymbol{u} = \boldsymbol{A}\lambda\boldsymbol{u}$$

$$\boldsymbol{Q}\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{A}\boldsymbol{u}$$

$\implies$ $\boldsymbol{A}\boldsymbol{u}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue λ.

Part 2b :-

Given: $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue μ i.e, $\boldsymbol{Q}\boldsymbol{v} = \mu\boldsymbol{v}$ where $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T$

Show: $\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{v} = \mu\boldsymbol{A}^T\boldsymbol{v}$

Substituting $\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{A}$ in $\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{v}$, we get

$$\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{v} = \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{v} \tag{2.3}$$

$$\because \boldsymbol{Q}\boldsymbol{v} = \boldsymbol{A}\boldsymbol{A}^T\boldsymbol{v} = \mu\boldsymbol{v} \tag{2.4}$$

$\therefore$ from equation (2.3) and (2.4), we get

$$\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{v} = \boldsymbol{A}^T\mu\boldsymbol{v}$$

$$\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{v} = \mu\boldsymbol{A}^T\boldsymbol{v}$$

$\implies$ $\boldsymbol{A}^T\boldsymbol{v}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue μ.

Part 2c :-

- Since $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{P}$, it has the same number of elements as in a column of $\boldsymbol{P}$, which is n. Therefore, $\boldsymbol{v}$ is an n-dimensional vector (n elements).
- Similarly, since $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{Q}$, it has the same number of elements as in a column of $\boldsymbol{Q}$, which is m. Therefore, $\boldsymbol{v}$ is an m-dimensional vector (m elements).

SECTION 3

Part 3

Given: $\boldsymbol{Q}\boldsymbol{v_i} = \lambda \boldsymbol{v}_i$ (λ is the corresponding eigenvalue of $\boldsymbol{Q}$) where $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T$ and $\boldsymbol{u}_i \triangleq \dfrac{\boldsymbol{A}^T\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$

To prove: $\boldsymbol{A}\boldsymbol{u_i} = \gamma_i \boldsymbol{v}_i$ (for some real, non-negative γ_i)

PROOF We know that

$$\boldsymbol{Q}\boldsymbol{v_i} = \boldsymbol{A}\boldsymbol{A}^T\boldsymbol{v_i} = \lambda \boldsymbol{v}_i \tag{3.1}$$

Also,

$$\boldsymbol{A}\boldsymbol{u}_i = \frac{\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2} \tag{3.2}$$

From equation (3.1) and (3.2):

$$\boldsymbol{A}\boldsymbol{u}_i = \frac{\lambda\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$$

$$\therefore \boldsymbol{A}\boldsymbol{u}_i = \gamma_i \boldsymbol{v}_i$$

where $\gamma_i = \dfrac{\lambda}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$

γ_i is real and non-negative because

- λ is an eigenvalue of $\boldsymbol{Q}$ and in Part 1 we showed that eigenvalues of $\boldsymbol{Q}$ are non-negative and real.
- $\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2$ (L_2 norm) is always non-negative and real.

□

SECTION 4

Part 4

Given: $\boldsymbol{U} = [\boldsymbol{v}_1|\boldsymbol{v}_2|\boldsymbol{v}_3|...|\boldsymbol{v}_m]_{m\times m}$ and $\boldsymbol{V} = [\boldsymbol{u}_1|\boldsymbol{u}_2|\boldsymbol{u}_3|...|\boldsymbol{u}_m]_{n\times m}$
Show: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T$

$$\boldsymbol{\Gamma} = \begin{bmatrix} \gamma_1 & 0 & \cdots\cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & & & \\ 0 & \cdots & \cdots & \gamma_m \end{bmatrix}_{m\times m}$$

Now, we can write

$$\boldsymbol{AV} = [\boldsymbol{Au}_1|\boldsymbol{Au}_2|\boldsymbol{Au}_3|...|\boldsymbol{Au}_m]$$

Also, in part 3, we showed that $\boldsymbol{Au}_i = \gamma_i \boldsymbol{v}_i$. Therefore,

$$[\boldsymbol{Au}_1|\boldsymbol{Au}_2|\boldsymbol{Au}_3|...|\boldsymbol{Au}_m] = [\boldsymbol{\gamma}_1\boldsymbol{v}_1|\boldsymbol{\gamma}_2\boldsymbol{v}_2|\boldsymbol{\gamma}_3\boldsymbol{v}_3|...|\boldsymbol{\gamma}_m\boldsymbol{v}_m]$$

$$\implies \boldsymbol{AV} = [\boldsymbol{\gamma}_1\boldsymbol{v}_1|\boldsymbol{\gamma}_2\boldsymbol{v}_2|\boldsymbol{\gamma}_3\boldsymbol{v}_3|...|\boldsymbol{\gamma}_m\boldsymbol{v}_m] \tag{4.1}$$

Now expanding $\boldsymbol{U\Gamma}$,

$$\boldsymbol{U\Gamma} = [\boldsymbol{\gamma}_1\boldsymbol{v}_1|\boldsymbol{\gamma}_2\boldsymbol{v}_2|\boldsymbol{\gamma}_3\boldsymbol{v}_3|...|\boldsymbol{\gamma}_m\boldsymbol{v}_m] \tag{4.2}$$

From equation (4.1) and (4.2), we get

$$\boldsymbol{AV} = \boldsymbol{U\Gamma}$$

Postmultiplying both sides by $\boldsymbol{V}^T$

$$\boldsymbol{AVV}^T = \boldsymbol{U\Gamma V}^T$$

$\because \boldsymbol{V}$ is a orthonormal matrix, we can write $\boldsymbol{VV}^T = \boldsymbol{I}$

$$\boldsymbol{A} = \boldsymbol{U\Gamma V}^T$$

Hence proved.