Question 5

CS663 (Digital Image Processing) Assignment 3

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PART

I

Question 5

Problem 1

If a function $f(x, y)$ is real, prove that its Discrete Fourier transform $F(u, v)$ satisfies $F^*(u, v) = F(-u, -v)$. If $f(x, y)$ is real and even, prove that $F(u, v)$ is also real and even. The function $f(x, y)$ is an even function if $f(x, y) = f(-x, -y)$. [15 points]

Properties

- Given that $f(x, y)$ is real, we know that $f^*(x, y) = f(x, y)$, where * denotes complex conjugation.
- The function $f(x, y)$ is an even function if $f(x, y) = f(-x, -y)$

Section 1

Proof (a) $F^*(u, v) = F(-u, -v)$

Proof By definition, the 2D DFT of a function f(x,y) of size W_1 x W_2 is given by:

$$F(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x, y) e^{-j2\pi(\frac{ux}{W_1} + \frac{vy}{W_2})}$$

Subsection 1.1

LHS

Taking the complex conjugate of $F(u, v)$:

$$F^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f^*(x, y) e^{j2\pi\left(\frac{ux}{W_1} + \frac{vy}{W_2}\right)}$$

Since f(x, y) is real, using the above stated property 1, we can further write this as

$$F^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x, y) e^{j2\pi\left(\frac{ux}{W_1} + \frac{vy}{W_2}\right)}$$

Subsection 1.2

RHS

Now, calculating $F(-u, -v)$

$$F(-u,-v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{-j2\pi\left(\frac{-ux}{W_1}+\frac{-vy}{W_2}\right)}$$

$$F(-u,-v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{j2\pi\left(\frac{ux}{W_1}+\frac{vy}{W_2}\right)}$$

□

Therefore, LHS = RHS. Hence proved.

Section 2

Proof (b)

Now we will prove that if $f(x, y)$ is real and even, prove that $F(u, v)$ is also real and even.

$$F(u,v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{-j2\pi(\frac{ux}{W_1}+\frac{vy}{W_2})}$$

Taking the complex conjugate of $F(u, v)$ and considering f(x, y) is real:

$$F^*(u,v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{j2\pi\left(\frac{ux}{W_1}+\frac{vy}{W_2}\right)}$$

Now, let's change the indices in the equation by substituting $-x$ for x and $-y$ for y:

$$F^*(u,v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{-x=0}^{W_1-1} \sum_{-y=0}^{W_2-1} f(-x,-y) e^{j2\pi\left(\frac{u(-x)}{W_1}+\frac{v(-y)}{W_2}\right)}$$

$$F^*(u,v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=1-W_1}^{0} \sum_{y=1-W_2}^{0} f(-x,-y) e^{-j2\pi\left(\frac{ux}{W_1}+\frac{vy}{W_2}\right)}$$

Since f(x, y) is even, using property 2 we can write:

$$F^*(u,v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=1-W_1}^{0} \sum_{y=1-W_2}^{0} f(x,y) e^{-j2\pi\left(\frac{ux}{W_1}+\frac{vy}{W_2}\right)}$$

Properties Periodic property:

- For integers k_1 and k_2

$$f(x,y) = f(x + k_1W_1, y + k_2W_2)$$

$$\therefore f(x,y) = f(x - W_1, y - W_2)$$

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$$e^{-j2\pi \frac{ux}{W_1}} = e^{-j2\pi \frac{u(x-W_1)}{W_1}}$$
$$e^{-j2\pi (\frac{ux}{W_1} + \frac{vy}{W_2})} = e^{-j2\pi (\frac{u(x-W_1)}{W_1} + \frac{v(y-W_2)}{W_2})}$$

Remark Since in the above equation of $F^*(u,v)$, $x \in [1-W_1, 0]$ and $y \in [1-W_2, 0]$, therefore when we replace x by $x - W_1$ and y by $y - W_2$, we get

$$x - W_1 \in [1 - W_1, 0] \implies x \in [1, W_1]$$

and

$$y - W_2 \in [1 - W_2, 0] \implies y \in [1, W_2]$$

Now, using the periodic properties of $f(x,y)$ and $e^{-j2\pi(\frac{ux}{W_1}+\frac{vy}{W_2})}$ and replacing x by $x - W_1$ and y by $y - W_2$ in the above equation of $F^*(u,v)$, we get:

$$F^*(u,v) = \frac{1}{\sqrt{W_1W_2}} \sum_{x=1}^{W_1} \sum_{y=1}^{W_2} f(x,y) e^{-j2\pi\left(\frac{ux}{W_1}+\frac{vy}{W_2}\right)}$$

Remark Note that $f(0,0) = f(W_1, W_2)$ and considering $g(x,y) = e^{-j2\pi(\frac{ux}{W_1}+\frac{vy}{W_2})}$, $g(0,0) = g(W_1, W_2)$

Therefore, the above equation can be written as:

$$F^*(u,v) = \frac{1}{\sqrt{W_1W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{-j2\pi\left(\frac{ux}{W_1}+\frac{vy}{W_2}\right)}$$

Hence, we can see that

$$F^*(u,v) = F(u,v) \tag{2.1}$$

We also know from proof (a) that

$$F^*(u,v) = F(-u,-v) \tag{2.2}$$

Therefore, from equation (2.1) and (2.2), we say that

$$F^*(u,v) = F(u,v) = F(-u,-v) \tag{2.3}$$

Hence, from equation (2.3),

F(u,v) is real and even if f(x, y) is real and even.