Question 7

CS663 (Digital Image Processing) Assignment 2

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PART

I

Problem 1
Prove that the Laplacian operator is rotationally invariant. For this consider a rotation of the coordinate system from (x, y) to $u = x\cos\theta - y\sin\theta, v = x\sin\theta + y\cos\theta$, and show that $f_{xx} + f_{yy} = f_{uu} + f_{vv}$ for any image f. [10 points]

Section 1

Proof

Consider the rotation of the coordinate system from (x, y) to -

$$u = x\cos\theta - y\sin\theta$$
$$v = x\sin\theta + y\cos\theta$$

Formulae Laplacian operator in 2D is defined as

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{xx} + f_{yy}$$

Intuition We aim to show that $f_{xx} + f_{yy} = f_{uu} + f_{vv}$. From the definition of the Laplacian operator above, when we change the coordinates to (u, v), the Laplacian on them becomes $f_{uu} + f_{vv}$. So, if we show that both these terms are equal when (u, v) are obtained by rotation transformation, then it means that Laplacian is rotationally invariant.

Proof Lets start with calculating the f_{xx} term of LHS -

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \tag{1.1}$$

For calculating $\frac{\partial f}{\partial x}$, using multi-variable chain rule, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \tag{1.2}$$

Note that we have rotated the system by a fixed angle θ, which means θ

is not really a variable for differentiation, therefore

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial(x\cos\theta - y\sin\theta)}{\partial x} \\
\implies \frac{\partial u}{\partial x} &= \cos\theta \\
\frac{\partial v}{\partial x} &= \frac{\partial(x\sin\theta + y\cos\theta)}{\partial x} \\
\implies \frac{\partial v}{\partial x} &= \sin\theta
\end{aligned}$$

Putting these values in 1.2,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cos\theta + \frac{\partial f}{\partial v}\sin\theta \tag{1.3}$$

Putting equation 1.3 in 1.1,

$$\begin{aligned}
f_{xx} &= \frac{\partial^2 f}{\partial x^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\cos\theta + \frac{\partial f}{\partial v}\sin\theta\right) \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial u}\cos\theta + \frac{\partial}{\partial x}\frac{\partial f}{\partial v}\sin\theta
\end{aligned} \tag{1.4}$$

Now, we have to compute $\frac{\partial}{\partial x}\frac{\partial f}{\partial u}$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial v}$. For this, we are taking the assumption that f_x, f_u, f_{xu} and f_{ux} are continuous on an open region, which means that $f_{ux} = f_{xu}$. That is, we are assuming that -

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial f}{\partial u} &= \frac{\partial}{\partial u}\frac{\partial f}{\partial x} \\
\frac{\partial}{\partial x}\frac{\partial f}{\partial v} &= \frac{\partial}{\partial v}\frac{\partial f}{\partial x}
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial f}{\partial u} &= \frac{\partial}{\partial u}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\cos\theta + \frac{\partial f}{\partial v}\sin\theta\right) \\
&= \frac{\partial^2 f}{\partial u^2}\cos\theta + \frac{\partial f}{\partial u\partial v}\sin\theta
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial v} = \frac{\partial f}{\partial v\partial u}\cos\theta + \frac{\partial^2 f}{\partial v^2}\sin\theta$$

Putting these values back in equation 1.4,

$$f_{xx} = \left(\frac{\partial^2 f}{\partial u^2}\cos\theta + \frac{\partial f}{\partial u \partial v}\sin\theta\right)\cos\theta + \left(\frac{\partial f}{\partial v \partial u}\cos\theta + \frac{\partial^2 f}{\partial v^2}\sin\theta\right)\sin\theta$$

Therefore, we have f_{xx} as -

$$f_{xx} = \frac{\partial^2 f}{\partial u^2}\cos^2\theta + \frac{\partial f}{\partial u \partial v}\sin\theta\cos\theta + \frac{\partial f}{\partial v \partial u}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial v^2}\sin^2\theta \quad (1.5)$$

Similarly, we can find the f_{yy} term to be -

$$f_{yy} = \frac{\partial^2 f}{\partial u^2}\sin^2\theta + \frac{\partial f}{\partial u \partial v}(-\cos\theta\sin\theta) + \frac{\partial f}{\partial v \partial u}(-\sin\theta\cos\theta) + \frac{\partial^2 f}{\partial v^2}\cos^2\theta \quad (1.6)$$

Finally, Adding equations 1.5 and 1.6, we get

$$\begin{aligned} f_{xx} + f_{yy} &= \frac{\partial^2 f}{\partial u^2}\cos^2\theta + \frac{\partial^2 f}{\partial v^2}\sin^2\theta + \frac{\partial^2 f}{\partial u^2}\sin^2\theta + \frac{\partial^2 f}{\partial v^2}\cos^2\theta \\ &= \frac{\partial^2 f}{\partial u^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 f}{\partial v^2}(\cos^2\theta + \sin^2\theta) \\ &= \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \\ &= f_{uu} + f_{vv} \end{aligned}$$

$$\boxed{\therefore f_{xx} + f_{yy} = f_{uu} + f_{vv}}$$

Hence Proved!

$\square$