Question 3

CS663 (Digital Image Processing) Assignment 3

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PART

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Problem 1
Prove the convolution theorem for 2D Discrete fourier transforms. [10 points]

SECTION 1

Convolution theorem for 2D DFT

The convolution theorem for 2D discrete Fourier transforms (DFTs) states that the DFT of the convolution of two 2D signals is equal to the pointwise product of the DFTs of the individual signals. Mathematically, if we have two 2D signals f(x, y) and g(x, y) with their respective DFTs F(u, v) and G(u, v), then the convolution theorem can be expressed as:

$$\mathbf{F}(f * g)(u, v) = F(u, v)G(u, v)$$

PROOF To prove this theorem, let's start by calculating the convolution of two 2D signals (images) which is represented as:

$$(f * g)(x, y) = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} f(s, t)g(x - s, y - t) \tag{1.1}$$

Also, by definition, the 2D DFT of a signal f(x,y) of size W_1 x W_2 is given by:

$$F(u, v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x, y) \exp\left\{-j2\pi(\frac{ux}{W_1} + \frac{vy}{W_2})\right\} \tag{1.2}$$

Now let's calculate the DFT of the convolution:

$$\mathbf{F}(f * g)(u, v) = \mathbf{F}(\sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} f(s, t)g(x - s, y - t))$$

Formulae Linearity

$$\mathbf{F}(af+bg)(u,v) = aF(f)(u,v) + bF(g)(u,v)$$

where a and b are scalars

Using the linearity of 2D DFT, we can write the above equation as

$$\mathbf{F}(f*g)(u,v) = \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} f(s,t)\mathbf{F}(g(x-s,y-t)))$$

Formulae Translation:

$$\mathbf{F}(f(x-x_0,y-y_0))(u,v) = F(x,y)\exp\left\{-j2\pi(\frac{ux_0}{W_1}+\frac{vy_0}{W_2})\right\}$$

Following translation property of 2D DFT, we get

$$\mathbf{F}(f*g)(u,v) = \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} f(s,t)G(u,v)\exp\left\{-j2\pi(\frac{us}{W_1}+\frac{vt}{W_2})\right\}$$

After changing the limits, we can also write this as

$$\mathbf{F}(f*g)(u,v) = G(u,v)\sum_{s=0}^{W_1-1}\sum_{t=0}^{W_2-1} f(s,t)\exp\left\{-j2\pi(\frac{us}{W_1}+\frac{vt}{W_2})\right\}$$

or

$$\mathbf{F}(f*g)(u,v) = G(u,v)\sum_{x=0}^{W_1-1}\sum_{y=0}^{W_2-1} f(x,y)\exp\left\{-j2\pi(\frac{ux}{W_1}+\frac{vy}{W_2})\right\} \quad (1.3)$$

Now using equation (0.2) and (0.3) gives us

$$\mathbf{F}(f*g)(u,v) = G(u,v)F(u,v)$$

Hence proved. □