EXPLORING PERFECT BINARY TREES WITH RELATION TO THE HK-PROPERTY

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Abstract. A perfect binary tree is a full binary tree in which all leaves have the same depth. A set of independent set of size k (k-independent set) in a graph, containing a fixed vertex v is called a star, and is denoted by $\mathcal{I}_G^n(v)$. We study the size of stars for different vertices in a perfect binary tree. This structure is useful in studying the Erdős-Ko-Rado theorem. Hurbert and Kumar conjectured that in trees, the largest stars are on the leaves. The conjecture was shown to be false independently by Baber, Borg, and Feghali, Johnson and Thomas. However, in some classes of trees such as caterpillars, the conjecture holds true. In this paper, we study the perfect binary trees through the lens of star centers and seek to answer if the HK-property holds for perfect binary trees. We give an formula and an inductive proof for the independence number of a perfect binary tree and show that the HK-property holds for perfect binary trees. We then show that the proof structure supports any perfect m-nary tree, where m is a natrual number. Finally, we use an algorithm by Niskanen and R. J. to generate all independent set of a perfect binary tree of depth d and compare the number of k-independent set containing a vertex v and a leaf l to see, numerically, if the HK-property holds for perfect binary trees.

1. Introduction and Preliminaries

For a given graph $G = (V, E)$, $V(G)$ and $E(G)$ denotes the vertex sets and edge sets of the graph G. For an arbitrary vertex, $v \in V(G)$, all vertices adjacent to v are called the neighbours of v and the set of neighbours of v is denoted by $N_G(v)$. The degree of a vertex $v \in V(G)$ is the cardinality of the set of neighbours of v, and is denoted by $deg_G(v)$.

For $n \in \mathbb{Z}^+$ such that $0 \leq n \leq |V(G)|$, a path of length n in G is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$ and it is called a uv-path. A cycle is an extension of a path such that the last vertex is connected to the first vertex, i.e for a path of length n, $v_n v_i \in E(G)$. As such, the length of the cycle is $n+1$.

A connected graph is a graph if for all $u, v \in V(G)$, there exists a uv-path. An independent set is a set of vertices such that no two vertices in the set are adjacent to each other. We denote a family of indpendent sets of a graph G by I_G. A family of coclique of size n in a graph G is denoted by $\mathcal{I}_G^n$. For $v \in V(G)$, the family of indpendent sets, $\mathcal{I}_G^n(v) := \{A \in \mathcal{I}_G^n : v \in A\}$ is called a star of $\mathcal{I}_G^n$ and v is called its center.

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A tree is a connected graph with no cycles, it is denoted by T. For a vertex $v \in V(T)$, if $deg_T(v) = 1$, it is called a leaf. A vertex that is not a leaf is called an interior vertex. The vertex at the top which branches 2 children is called the root vertex and is denoted by r. The depth of a vertex is defined as the cardinality of the set of vertices of the path from the root vertex to it. The parent of a vertex v is the vertex connected to v on the path to r. A child of a vertex v is a vertex of which v is the parent. We study a more particular class of trees called binary trees, denoted by T_B, where each interior vertex v has exactly 2 children and all leaves have the same depth. Further, a perfect binary tree is a binary tree in which every vertex $v \in V(T)$ has either 0 or 2 children. A perfect binary tree is denoted by T_{PB}, however in this paper we will simply drop the subscript and denote it as T for clarity. A level n of a perfect binary tree is a set of vertices such that all vertices in the set have a depth of n. A perfect m-nary tree is a tree in which every vertex $v \in V(T)$ has either 0 or m children, where $m \in \mathbb{Z}^+$.

For a perfect binary tree T of depth d, we define the depth vertex set as the set of all vertices at the same depth. A depth vertex set of depth i is denoted by $\mathcal{D}_i$, for $i \leq d$. The maximum independent set of a perfect binary tree T is denoted by $\mathcal{I}_d$. The size of the maximum independent set of a perfect binary tree T is denoted by $\alpha(T)$. Also, we define the parent vertex of a vertex v as the vertex adjacent to v having one less depth. The grandparent vertex of a vertex v is the parent of the parent vertex of v.

The star centers of a graph are interesting because they relate to the EKR theorem.

The Erdős-Ko-Rado (EKR) theorem limits the number of sets in a family of sets that can be pairwise intersecting. The theorem states that for a family of k-sets of a ground set of size n, the maximum number of sets that can be pairwise intersecting is $\binom{n-1}{k-1}$. This theorem has wide applications in combinatorics, graph theory, probability, and other areas of statistics and mathematics.

Studying the EKR theorem, [5] Hulbert and Kamat tried to narrow it down to a more reduced class of graphs called the k-EKR graphs. A graph G is said to be k-EKR if for any family of cocliques $\mathcal{F} \subset \mathcal{I}_G^k$ satisfying $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$, there is a vertex $x \in V(G)$ such that $|\mathcal{F}| \leq \mathcal{I}_G^k(v)$. They then conjectured the following:

Conjecture 1.1 (k-EKR Conjecture). *Let G be a graph, and let $\mu(G)$ be the size of its smallest maximal coclique. Then G is k-EKR for every $1 \leq k \leq \frac{\mu(G)}{2}$.*

However, this conjecture is hard to understand and prove, so they narrowed the class of graphs to be trees and gave the following conjecture:

Conjecture 1.2 (HK-Property)**.** *For any $k \geq 1$ and any tree T, there exists a leaf l of T such that $|\mathcal{I}_T^k(v)| \leq |\mathcal{I}_T^k(l)|$ for each $v \in V(T)$.*

The HK-property holds true for $k \leq 4$, but was proven false independently in [1, 2, 3]. The counterexample that that they arrived at is a type of graph that is defined as a class of trees called *lobster* [4]. A tree C is called a *caterpillar* if G removing the leaves and incident edges produces a path graph P, called the spine. A tree L is called a *lobster* if removing the leaves and incident edges produces a caterpillar C.

This is interesting for us as *lobster* graphs resemble a binary tree.

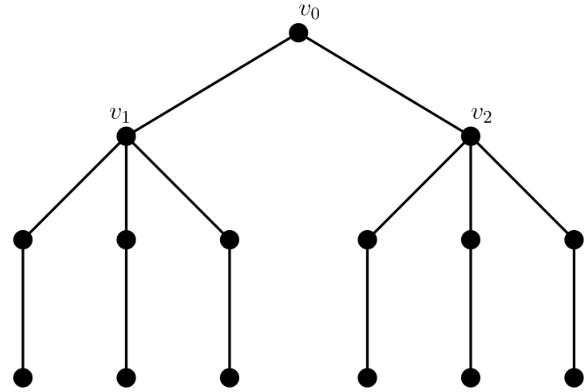


FIGURE 1. The largest k-star for $k \geq 5$ is centered at v_0 (A lobster)

They figured out that the lobsters, while not completely obeying the HK-property, almost obey the HK-property by either having the largest stars centered around the leaves or at the root of the tree.

1.1. **Some classes of graphs that are HK and a counterexample.** In the paper by Estrugo and Passtine [4], they gave a couple of classes of graphs that have the HK-property. The classes of graphs that they gave were the *caterpillars* and the *spiders*. The definition of *caterpillars* is given above. The *spiders* are trees that only have 1 vertex with degree greater than 2.

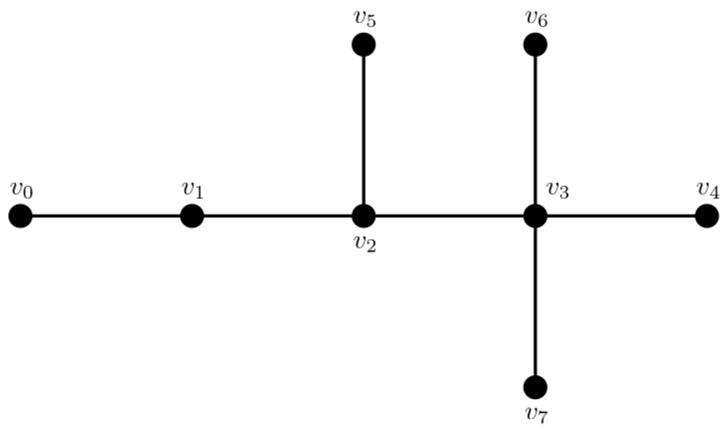


FIGURE 2. A caterpillar; Removing all the leaves, you obtain a path

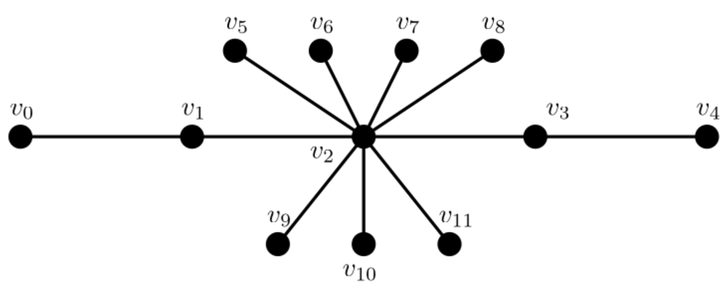


FIGURE 3. A spider; Only one vertex has degree greater than 2

2. Perfect M-Nary Trees and Their Independence Number

It is not hard to see that the perfect m-nary tree has a unique maximum coclique.

Definition 2.1 (Maximum Coclique)**.** *We denote the maximum coclique of a perfect m-nary tree of depth d by $\mathcal{I}_d$.*

Conjecture 2.2 (HK property for a perfect m-nary tree)**.** *For any $k \geq 1$ and a given perfect m-nary tree T, there exists any leaf l of T such that $|\mathcal{I}_T^k(v)| \leq |\mathcal{I}_T^k(l)|$ for each $v \in V(T)$.*

We note that the HK conjecture for perfect m-nary trees still remains open.

Theorem 2.3. *Let T be a perfect m-nary tree of depth d ($m > 1$). Then the unique maximum independent set $\mathcal{I}_d$ consists precisely of the vertices on levels*

$$d,\, d-2,\, d-4,\, \ldots,$$

and its size is given by

$$\alpha(T) = \sum_{i=1}^{\lceil d/2 \rceil} m^{\,d-2i+1}.$$

Proof.
First, notice that independent set $\mathcal{I}$ that is obtained by taking vertices on levels $d, d-2, \ldots,$ has size exactly $|\mathcal{I}| = \sum_{i=1}^{\lceil d/2 \rceil} m^{\,d-2i+1}$.

We prove the claim by induction on d.

Base Cases: For $d = 1$, the tree consists of a single vertex r, so that $\mathcal{I}_1 = \{r\}$ and $\alpha(T) = 1$. For $d = 2$, the maximum independent set is given by the set of leaves, i.e., $\mathcal{I}_2 = \{l_1, \ldots, l_m\}$, so that $\alpha(T) = m$. In both cases the formula

$$\alpha(T) = \sum_{i=1}^{\lceil d/2 \rceil} m^{\,d-2i+1}$$

holds and the independent set is unique.

Inductive Step: Assume the result holds for all perfect m-nary trees of depth up to $d-1$. Consider a tree T of depth d. Two cases arise:

Case 1: $r \in \mathcal{I}_d$ (which we will show only occurs when d is odd).

Since the root is in the independent set, none of its m children may be included. Removing r and its children disconnects T into m^2 disjoint subtrees (each of depth $d-2$). By the inductive hypothesis each such subtree has a unique maximum independent set of size

$$\sum_{i=1}^{\lceil (d-2)/2 \rceil} m^{\,(d-2)-2i+1}.$$

Thus, combining these disjoint choices and adding r, we obtain

$$= 1 + m^2 \left(\sum_{i=1}^{\lceil (d-2)/2 \rceil} m^{d-2i+1} \right).$$

In case d is even ($d = 2k$, for some $k \in \mathbb{N}$), we obtain:

$$\begin{aligned}
\alpha(T) &= 1 + m^2 \left(\sum_{i=1}^{\lceil (d-2)/2 \rceil} m^{d-2-2i+1} \right) \\
&= 1 + m^2 \left(\sum_{i=1}^{k-1} m^{2k-2i-1} \right) \\
&= 1 + \sum_{i=1}^{k-1} m^{2k-2i+1} < m + \sum_{i=1}^{k-1} m^{2k-2i+1} = \sum_{i=1}^{k} m^{2k-2i+1} = \sum_{i=1}^{\lceil d/2 \rceil} m^{d-2i+1}.
\end{aligned}$$

So if d is even, any independent set containing root has size smaller than $\mathcal{I}_d$.

If d is odd ($d = 2k+1$, for some $k \in \mathbb{N}$), then we obtain:

$$\begin{aligned}
\alpha(T) = 1 + m^2 &\left(\sum_{i=1}^{\lceil (d-2)/2 \rceil} m^{d-2-2i+1} \right) \\
&= m^{2k+1-2(k+1)+1} + \left(\sum_{i=1}^{k} m^{2k+1-2-2i+1+2} \right) \\
&= m^{2k+1-2(k+1)+1} + \sum_{i=1}^{k} m^{2k+1-2i+1} \\
&= \sum_{i=1}^{k} m^{2k+1-2i+1} \\
&= \sum_{i=1}^{\lceil d/2 \rceil} m^{d-2i+1}
\end{aligned}$$

Therefore, the only possible way an independent set containing the root has size equal to $\mathcal{I}_d$ is if d is odd.

Case 2: $r \notin \mathcal{I}_d$ (which we will show only occurs when d is even).
In this case, the root is not in the independent set, so we obtain m disjoint subtrees (each of depth $d-1$). By the inductive hypothesis each such subtree has a unique maximum independent set of size:

$$\sum_{i=1}^{\lceil (d-1)/2 \rceil} m^{(d-1)-2i+1}.$$

Thus, combining these disjoint choices, we obtain:

$$= m \left(\sum_{i=1}^{\lceil (d-1)/2 \rceil} m^{d-1-2i+1} \right).$$

In case d is odd ($d = 2k-1$, for some $k \in \mathbb{N}$), we obtain:

$$\begin{aligned}
\alpha(T) = m &\left(\sum_{i=1}^{\lceil (d-1)/2 \rceil} m^{d-1-2i+1} \right) \\
&= m \left(\sum_{i=1}^{k-1} m^{2k-1-2i} \right) \\
&= \sum_{i=1}^{k-1} m^{2k-1-2i+1} \\
&= \sum_{i=1}^{\lceil (d-1)/2 \rceil} m^{d-2i+1} < \sum_{i=1}^{\lceil d/2 \rceil} m^{d-2i+1}
\end{aligned}$$

In case d is even ($d = 2k$, for some $k \in \mathbb{N}$), we obtain:

$$\begin{aligned}
\alpha(T) &= m\left(\sum_{i=1}^{\lceil (d-1)/2\rceil} m^{d-1-2i+1}\right)\\
&= m\left(\sum_{i=1}^{k} m^{2k-1-2i}\right)\\
&= \sum_{i=1}^{k} m^{2k-1-2i+1}\\
&= \sum_{i=1}^{\lceil (d-1)/2\rceil} m^{d-2i+1}
\end{aligned}$$

Therefore, the only possible way an independent set not containing the root has size equal to $\mathcal{I}_d$ is if d is even.

In both cases the choices in the disjoint subtrees are forced (by the inductive hypothesis), ensuring the uniqueness of $\mathcal{I}_d$. Further, we now know that the root is not in $\mathcal{I}_d$ if and only if d is even.

Thus, by strong induction, the result holds for all d. □

Corollary 2.4. *By Theorem (2.3), we see that for perfect m-nary trees with odd depth d, $r \in \mathcal{I}_d$. For even depth d, $r \notin \mathcal{I}_d$.*

3. The HK-Property for Perfect M-Nary Trees

Conjecture 3.1 (Perfect m-nary trees are HK)**.** *Let T be a perfect m-nary tree with depth d. Then, T satisfies the HK-property.*

We conjecture that the HK-property holds for perfect m-nary trees. We believe that this result holds due to strong numerical evidence. However, this conjecture remains open, and below we provide some steps that we took in directions of possible proof.

For binary (and m-nary trees), Conjecture 3.1 can be proved from a slightly more restricted form.

Conjecture 3.2 (Stars around the leaves dominates stars around the root)**.** *For a given perfect binary tree $T = (V, E)$ of depth d, and any leaf ℓ of T we have $|\mathcal{I}_T^k(\ell)| \geq |\mathcal{I}_T^k(r)|$ where r is the root.*

Clearly, Conjecture 3.1 implies Conjecture 3.2 is true, and Lemma below shows that the converse is also true.

Lemma 3.3. *Assume that for every perfect m-nary tree $T = (V, E)$ with root r and any leaf ℓ we have*

$$|\mathcal{I}_T^k(\ell)| \geq |\mathcal{I}_T^k(r)|.$$

Then, for every perfect m-nary tree $T = (V, E)$, any leaf ℓ and any vertex v we have

$$|\mathcal{I}_T^k(\ell)| \geq |\mathcal{I}_T^k(v)|.$$

Before we proceed with the proof of Lemma 3.3, we need to introduce some notation and definitions.

Definition 3.4 (Enumerating Depth Vertex Sets)**.** *Recall that the depth vertex set of depth i is denoted by $\mathcal{D}_i$. We index all the vertices in $\mathcal{D}_i$ from left to right as $v_{i,k}$, where i is the depth and k is the index of the vertex in $\mathcal{D}_i$ such that $1 \leq k \leq 2^{i-1}$.*

We now show that picking any vertex v in a perfect m-nary tree is equivalent to picking an arbitrary vertex in the depth vertex set $\mathcal{D}_d$.

Lemma 3.5. *Define a function $flip_i : \mathcal{D}_i \to \mathcal{D}_i$ as follows:*

$$flip_i(v_{i,k}) = v_{i,2^{i-1}-k+1}.$$

We claim that $flip_i$ is a bijection for all i.

Proof.
Injective: Let $v_{i,k_1}, v_{i,k_2} \in \mathcal{D}_i$ such that $flip_i(v_{i,k_1}) = flip_i(v_{i,k_2})$. Then, $v_{i,2^{i-1}-k_1+1} = v_{i,2^{i-1}-k_2+1}$. This implies that $k_1 = k_2$.

Surjective: Let $v_{i,k} \in \mathcal{D}_i$. Then, $flip_i(v_{i,2^{i-1}-k+1}) = v_{i,k} \in \mathcal{D}_i$.

Hence, $flip_i$ is a bijection for all i. □

Then, by Lemma 3.5, we have that we can select any vertex v in a perfect m-nary tree T by selecting an arbitrary vertex in the depth vertex set $\mathcal{D}_i$.

We now proceed with the proof of Lemma 3.3.

Proof of lemma 3.3.

Let k be arbitrary. Let T_1 be the perfect m-nary sub-tree rooted at v. Let $x \in \{v, \ell\}$ and $\mathcal{I} \in \mathcal{I}_T^k(x)$. Let $\mathcal{I}_1 = \mathcal{I} \cap V(T_1)$ and $\mathcal{I}_0 = \mathcal{I} \cap V(T) \setminus V(T_1)$.

Now, consider fixed k_0, k_1 such that $k_0 + k_1 = k$, where $k_0 = |\mathcal{I}_0|$ and $k_1 = |\mathcal{I}_1|$. Since $|\mathcal{I}_T^k(\ell)| \geq |\mathcal{I}_T^k(r)|$ is true for T_1, k_1, ℓ, and v, there is an injection $\varphi : \mathcal{I}_{T_1}^{k_1}(v) \to \mathcal{I}_{T_1}^{k_1}(\ell)$.

Now, we consider the independent sets in T. Construct $\psi : \mathcal{I}_T^k(v) \to \mathcal{I}_T^k(\ell)$ and define it as follows:

$$\psi(\mathcal{I}) = \varphi(\mathcal{I}_1) \cup \mathcal{I}_0.$$

We now will show that the following:

Claim 1: $\psi(\mathcal{I}) \in \mathcal{I}_T^k(\ell)$.

We know that $\mathcal{I}_1 \in \mathcal{I}_{T_1}^{k_1}(\ell)$ and $\mathcal{I}_0 \in \mathcal{I}_T^{k_0}(\ell)$. Since $\mathcal{I}_1$ is a subset of $\mathcal{I}$, we have that $\mathcal{I}_1 \cup \mathcal{I}_0 = \mathcal{I} \in \mathcal{I}_T^k(\ell)$.

Claim 2: ψ is injective.

Let there be $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\psi(\mathcal{I}_1) = \psi(\mathcal{I}_2)$. Then,

$$\begin{aligned}\psi(\mathcal{I}_1) = \psi(\mathcal{I}_2) &\implies |\mathcal{I}_1 \cap V(T_1)| = |\mathcal{I}_2 \cap V(T_1)| \\ &\implies \mathcal{I}_1 \setminus V(T_1) = \mathcal{I}_2 \setminus V(T_1) \\ &\implies \mathcal{I}_1 = \mathcal{I}_2.\end{aligned}$$

Hence, ψ is injective.

Therefore, we have that $|\mathcal{I}_T^k(\ell)| \geq |\mathcal{I}_T^k(v)|$ for all k. □

Corollary 3.6. *By Lemma 3.5, we can apply Lemma 3.3 to any vertex v and any leaf ℓ in a perfect m-nary tree.*

3.1. **Generating functions approach.** In addition, we can count the number of independent sets with the help of the following generating functions.

For the rest of this subsection, m is fixed and we consider m-nary trees of depth d.

Definition 3.7 (Generating functions)**.** *For fixed d let T denote perfect m-nary tree of depth d with root r and a leaf ℓ. Let $N = \alpha(T)$ (recall (**??**)). Let*

$$P_d(t) = a_0 + a_1x + a_2x^2 + \ldots + a_Nx^N$$

be a polynomial in variable t in which coefficient $a_i = \mathcal{I}_T^i$ for all $i \in [N]$. Similarly define polynomials

$$Q_d(t) = b_0 + b_1x + b_2x^2 + \ldots + b_Nx^N,$$

$$R_d(t) = c_0 + c_1x + c_2x^2 + \ldots + c_Nx^N,$$

where $b_i = \mathcal{I}_T^i(\ell)$, and $c_i = \mathcal{I}_T^i(r)$ for all $i \in [N]$.

Polynomials P_d, Q_d, R_d are generating functions for the number of independents sets of size i, number of independents sets of size i passing through a leaf, and number of independents sets of size i passing through the root, respectively.

Lemma 3.8. *Polynomials P_n, Q_n, R_n satisfy the following recurrent formulas for $n \geq 2$ with $P_0 = 1, P_1 = 1 + x$, and $Q_0 = 0, Q_1 = x$*

$$P_n(x) = (P_{n-1}(x))^2 + x(P_{n-2}(x))^4$$
$$R_n(x) = x(P_{n-2}(x))^4$$
$$Q_n(x) = Q_{n-1}(x)P_{n-1}(x) + xQ_{n-2}(x)(P_{n-2}(x))^3$$

Proof.

We prove the recurrence relations for the generating functions $P_n(x)$, $Q_n(x)$, and $R_n(x)$ by considering the structure of the perfect m-nary tree of depth n.

Proof for $P_n(x)$

The polynomial $P_n(x)$ represents the generating function for the number of independent sets in a perfect m-nary tree of depth n. We analyze two cases based on whether the root r is included in the independent set:

- **Case 1: The root r is *not* included.** Since the children of r are independent subtrees of depth $n-1$, each contributes a factor of $P_{n-1}(x)$, leading to a total contribution of:

$$(P_{n-1}(x))^m$$

- **Case 2: The root r *is* included.** In this case, none of its immediate children can be included, but the grandchildren (depth 2) remain eligible. Since each child of r has m children, the independent sets are counted by $P_{n-2}(x)$ for each of these subtrees, giving:

$$x(P_{n-2}(x))^{m^2}$$

  The factor x accounts for the inclusion of the root.

Combining both cases, we obtain the recurrence relation:

$$P_n(x) = (P_{n-1}(x))^m + x(P_{n-2}(x))^{m^2}.$$

Proof for $R_n(x)$

The polynomial $R_n(x)$ represents the generating function for independent sets that must include the root r. Since selecting r means its children are excluded, the remaining tree consists of m independent subtrees of depth $n-2$. Therefore,

$$R_n(x) = x(P_{n-2}(x))^{m^2}.$$

Again, the factor x accounts for the inclusion of the root.

Proof for $Q_n(x)$

The polynomial $Q_n(x)$ represents the generating function for independent sets that must include a specific leaf ℓ. We consider two cases:

- **Case 1: The parent of ℓ is not in the independent set.** The remaining tree behaves as an independent subtree of depth $n-1$. Since $Q_{n-1}(x)$ counts such sets containing ℓ, the contribution is:

$$Q_{n-1}(x)P_{n-1}(x).$$

- **Case 2: The parent of ℓ *is* in the independent set.** In this case, its siblings and grandparent are excluded, leaving m independent subtrees of depth $n-2$. The contribution follows from: // TODO: How is it 3??? I get m - 1 which for the m = 2 case should be 1.

$$xQ_{n-2}(x)(P_{n-2}(x))^{m-1}.$$

  The factor x accounts for the inclusion of the parent of ℓ.

Summing both cases, we obtain:

$$Q_n(x) = Q_{n-1}(x)P_{n-1}(x) + xQ_{n-2}(x)(P_{n-2}(x))^{m-1}.$$

This completes the proof. □

The HK property (and Theorem 3.1) in the language of polynomials translates to proving that $Q_n(x) \geq_c R_n(x)$ for all n, where $Q \geq_c R \Leftrightarrow_{def} Q - R$ has only positive coefficients.

It is easier to determine several structural properties of the polynomials to compare their reduced version.

Lemma 3.9. *For odd d*

$$(P_d(x))^3 \geq_c P_{d+1}(x)(P_{d-1}(x))^2$$

For even d, the inequality is reversed, namely

$$(P_d(x))^3 \leq_c P_{d+1}(x)(P_{d-1}(x))^2$$

Proof. The proof of this Lemma is by induction. Note that $P_0(x) = 1$, $P_1(x) = 1+x$, $P_2(x) = 1 + 3x + x^2$. Then

$$P_1^3(x) - P_2(x)P_0^2(x) = (1+x)^3 - (1+3x+x^2) = 2x^2 + x^3 \geq_c 0.$$

For inductive proof, by repetitive application of Lemma 3.8

$$\begin{aligned} P_d^3 - P_{d+1}P_{d-1}^2 &= P_d^3 - (P_d^2 + xP_{d-1}^4)P_{d-1}^2 \\ &= P_d^2(P_d - P_{d-1}^2) - xP_{d-1}^6 \\ &= P_d^2(xP_{d-2}^4) - xP_{d-1}^6 \\ &= x(P_d^2P_{d-2}^4 - P_{d-1}^6) \\ &= x(P_dP_{d-2}^2 - P_{d-1}^3)(P_dP_{d-2}^2 + P_{d-1}^3). \end{aligned}$$

By inductive hypothesis, if d is odd, then $P_d P_{d-2}^2 - P_{d-1}^3$ has only non-negative coefficients. So $P_d^3 - P_{d+1}P_{d-1}^2$ has only non-negative coefficients.

Similarly, if d is event, by Inductive hypothesis, $P_d P_{d-2}^2 - P_{d-1}^3$ has only non-positive coefficients. So $P_d^3 - P_{d+1}P_{d-1}^2$ has only non-positive coefficients.

□

We were able to verify the correctness of Theorem 3.1 for all values of $n \leq 14$ (see [6]).

4. Conclusion and Future work

The HK-property for perfect binary trees is interesting because this classification of graph is very symmetric is very related to caterpillars and lobsters. Even for perfect binary trees, the HK-property is not trivial to prove. Finding an injective function that maps the stars around the leaves to the stars around the root is not trivial. However, we numerically verified that the HK-property holds for perfect binary trees of depth 5 and further presented a better deterministic algorithm using generating functions.

For future work, finding a valid injective function to map different components of a perfect binary (or even m-nary) trees to each other would be a good start.

Appendix A. cocliques Algorithm and Analysis

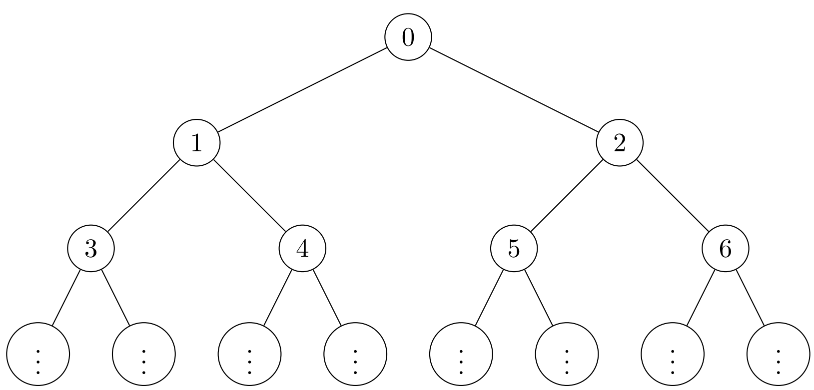
To validate our conjecture for perfect binary trees upto depth 5, we present a simple algorithm to generate all independent of a perfect binary tree of cardinality k. We then compare the number of independent set containing a vertex v and a leaf l to see if the HK-property holds for perfect binary trees [6].

To begin with, we present the following algorithm to generate a perfect binary tree of depth n:

Algorithm 1: Perfect Binary Tree Generator

```
Data: n ≥ 0, where n is the depth of the perfect binary tree
Result: A perfect binary tree graph and its leaves
Function perfect_binary_tree_generator(n):
    if n = 0 then
        return Graph()
    else
        g ← Graph();
        g.add_vertices([2^n]);
        for i in range(2^n − 1) do
            g.add_edge(i, 2 ∗ i + 1);
            g.add_edge(i, 2 ∗ i + 2);
        end
        return g
    end
```

The algorithm will generate a perfect binary tree of this form:



It is easy to see that the leaves will start with the value of $\left\lfloor \frac{2^{n+1}-1}{2} \right\rfloor$, where n is the depth of the perfect binary tree.

So to generate all the leaves of the perfect binary tree of depth n, we present the following algorithm:

Algorithm 2: Perfect Binary Tree Leaves Generator

Data: $n \geq 0$, where n is the depth of the perfect binary tree
Result: A perfect binary tree graph's leaves
Function `perfect_binary_tree_generator(`n`):`

```
    num_vertices ← 2^(n+1) − 1;
    leaves ← [];
    last_row_start ← floor(num_vertices/2);
    for vertex in range(last_row_start, num_vertices) do
        leaves.append(vertex);
    end
    return leaves
```

We then use the algorithm from [7] to generate a indepdent set of maximum cardinality for our perfect binary tree.

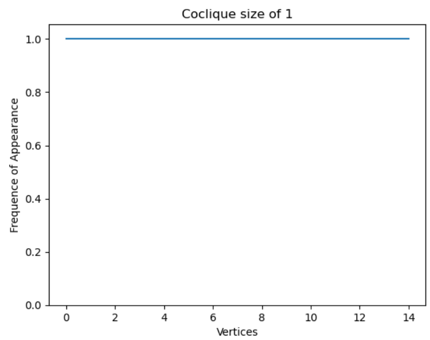
Algorithm 3: Maximum Indpendent Set Algorithm

Data: A perfect binary tree graph T
Result: A maximum independent set of T
Function `maximum_independent_set(`T`):`

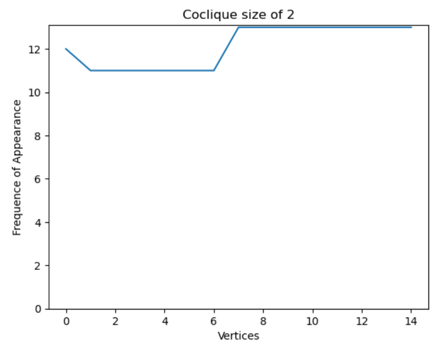
```
    cliquer ← Cliquer(T);
    return cliquer.get_maximum_independent_set()
```

The next couple of pages show the results of the algorithm for a perfect binary tree of depth 4, 5, and 6. The X-axis denotes the vertice's labels (not the actual numbers) and the Y-axis denotes the cardinality of the stars centered around the vertices.

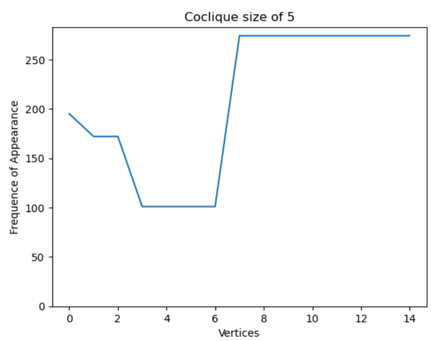
Depth 4



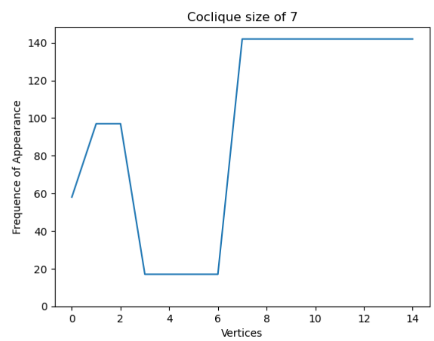
(A) independent set of size 1



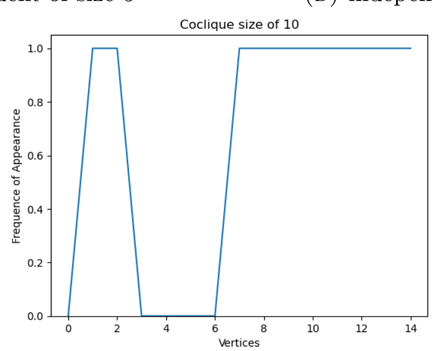
(B) independent of size 2



(C) independent of size 5



(D) independent of size 7

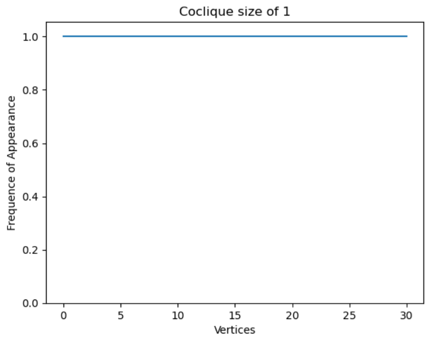


(E) independent of size 10

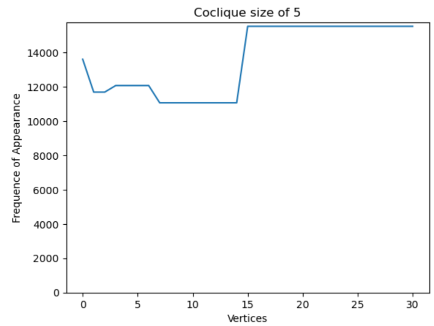
The data shown in the figures above verifies that the HK-Property holds for perfect binary trees of depth 5. The next step would be to verify this for perfect binary trees of depth 6 and 7.

However, the algorithm is very slow and inefficient and it scales exponentially. Hence, running the algorithm for perfect binary trees of depth 6 and 7 would be very computationally expensive.

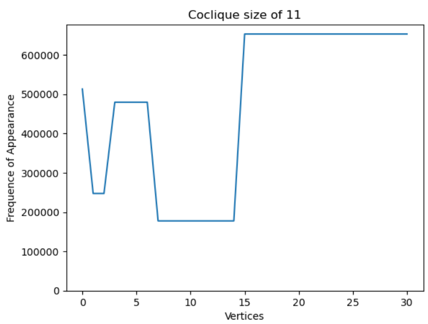
Depth 5



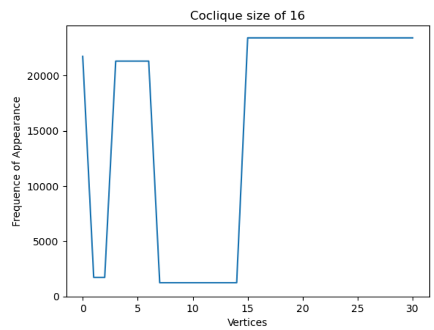
(A) independent of size 1



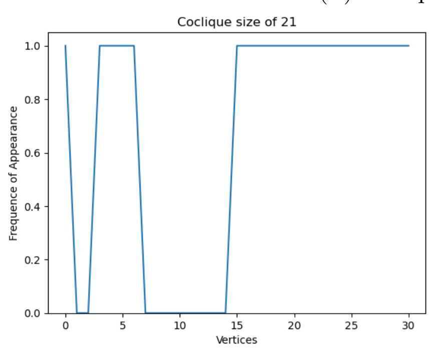
(B) coclique of size 5



(C) coclique of size 11



(D) coclique of size 16



(E) coclique of size 21

Appendix B. Generating Functions Algorithm and Analysis

To overcome the computational inefficiency of the brute force algorithm, we use generating functions to count the number of independent sets of a perfect binary tree of depth d. We then can compute the difference between the coefficients of the generated polynomials to determine if the HK-property holds for perfect binary trees numerically [6].

Recall from Lemma 3.8 that the polynomials P_d, Q_d, R_d satisfy the following recurrent formulas for $n \geq 2$ with $P_0 = 1, P_1 = 1 + x$, and $Q_0 = 0, Q_1 = x$:

$$
\begin{aligned}
P_n(x) &= (P_{n-1}(x))^2 + x(P_{n-2}(x))^4 \\
R_n(x) &= x(P_{n-2}(x))^4 \\
Q_n(x) &= Q_{n-1}(x)P_{n-1}(x) + xQ_{n-2}(x)(P_{n-2}(x))^3
\end{aligned}
$$

We present the following algorithm to generate the polynomials P_d, Q_d, R_d for a perfect binary tree of depth d:

Algorithm 3: Generate P Sequence

Input: n (integer)
Output: P sequence
$P \leftarrow [1, x + 1]$;
for $i \leftarrow 2$ ***to*** $n - 1$ **do**
  $P[i] \leftarrow P[i-1]^2 + x \cdot P[i-2]^4$;
end
return P

Algorithm 4: Generate Q Sequence

Input: n (integer)
Output: Q sequence
$Q \leftarrow [0, x]$;
for $i \leftarrow 2$ ***to*** $n - 1$ **do**
  $Q[i] \leftarrow Q[i-1] \cdot P[i-1] + x \cdot Q[i-2] \cdot P[i-2]^3$;
end
return Q

Algorithm 5: Generate R Sequence

Input: n (integer)
Output: R sequence
$R \leftarrow [0, x]$;
for $i \leftarrow 2$ ***to*** $n - 1$ **do**
  $R[i] \leftarrow x \cdot P[i-2]^4$;
end
return R

We then compare the coefficients of the polynomials Q_d and R_d to determine if the HK-property holds for perfect binary trees of depth d. The following algorithm checks if the HK-property holds for a given P, Q, R sequence:

Algorithm 6: Check HK-Property

```
Input: P, Q, R sequences
Output: Boolean
for i ← 2 to n − 1 do
    if Q[i] < R[i] then
        return False
    end
end
return True
```

For brevity, we list the first 5 values of $P_n(x)$ and $Q_n(x)$, and the first 3 values of $R_n(x)$:

n	$P_n(x)$
0	1
1	$1 + x$
2	$x^2 + 3x + 1$
3	$2x^3 + 5x^2 + 2x$
4	$x^9 + 16x^8 + 88x^7 + 242x^6 + 375x^5 + 337x^4 + 172x^3 + 46x^2 + 5x$

TABLE 1. Computed values for $P_n(x)$ for $n = 0$ to 4 with $m = 2$

n	$Q_n(x)$
0	0
1	x
2	$x^2 + x$
3	$x^3 + x^2$
4	$x^{10} + 9x^9 + 37x^8 + 84x^7 + 108x^6 + 79x^5 + 35x^4 + 9x^3 + x^2$

TABLE 2. Computed values for $Q_n(x)$ for $n = 0$ to 4 with $m = 2$

n	$R_n(x)$
2	x^2
3	$x^3 + x^2$
4	$x^{10} + 9x^9 + 37x^8 + 84x^7 + 108x^6 + 79x^5 + 35x^4 + 9x^3 + x^2$

TABLE 3. Computed values for $R_n(x)$ for $n = 2$ to 4 with $m = 2$

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