ReferenceslemmaLemmaLemmas

1 Theorems and Lemmas

Definition 1.1 (Depth). *The depth starts from 1-index. A single vertex is of depth 1.*

Definition 1.2 (Maximum Coclique). *We denote the maximum coclique of a perfect binary tree of depth d by $\mathcal{I}_d$.*

Conjecture 1.3 (HK property for a perfect binary tree). *For any given perfect binary tree T, the maximum number of cocliques lie in the leaves. The number of cocliques is denoted by k. $\alpha(T)$ denotes the independence number of a tree T.*

Lemma 1.4. *For a given perfect binary tree T with depth d and maximum number of cocliques possible, i.e. $k = \alpha(T)$ we have:*

$$\alpha(T) = \begin{cases} 2^{d-1} + 2^{d-3} + \ldots + 1 \text{ for odd } d \\ 2^{d-1} + 2^{d-3} + \ldots + 2 \text{ for even } d \end{cases}$$

And for odd d, Root $\in \mathcal{I}_d$, otherwise Root $\notin \mathcal{I}_d$.
Or, in summation notation:

$$\alpha(T) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} 2^{d-2i-1}$$

Proof.
We shall proceed by inducing on d.
We will have two cases, one for d being odd and another for d being even.
Case 1: d is odd

For our base case, consider the trivial case of $d = 1$. Here, $\alpha(T) = 1$. Hence, the base case holds.

Now, say that the statement holds for all d. We now have to show that it holds for $d + 1$. That is, we need to show that it holds for a given perfect binary tree T of depth $d + 1$. Note that if d is odd, then $d + 1$ is even. Hence, we expect that:

$$\alpha(T) = 2^d + 2^{d-2} + \ldots + 2$$

Consider these 2 cases:

Case 1: $Root \in \mathcal{I}_{d+1}$

Let the left and right child of the root be v_0 and v_1 respectively.

Let C_1, C_2, C_3, C_4 be the four perfect binary tree components generated by $T \setminus \{Root, v_0, v_1\}$.

If the root $\in \mathcal{I}_{d+1}$, then $v_0 \notin \mathcal{I}_d$ and $v_1 \notin \mathcal{I}_d$. This means that the remaining vertices of $\mathcal{I}_d$ is in one of the 4 components. Since the current depth from the *Root* is $d+1$, then $T \setminus \{Root, v_0, v_1\}$ will have depth of $d+1-2 = d-1$. Since d is odd, then $d+1$ is even which implies that $d-1$ is also even.

By symmetry, it is enough to consider one of the 4 components' maximum coclique for our calculations. Since the 4 components are disjoint, then we can add their independence numbers together along with the *Root* and obtain the following:

$$\alpha(T_{d+1}) = 4(\alpha(T_{d-1})) + 1$$

Then, from our induction hypothesis, we get that:

$$\begin{aligned} \alpha(T_{d+1}) &= 4\underbrace{(2^{d-2} + 2^{d-4} + \ldots + 2)}_{\frac{d-1}{2}\text{ terms}} + 1 \\ &= 4\sum_{i=1}^{\frac{d-1}{2}} 2^{d-2i} + 1 \end{aligned} \tag{1}$$

Case 2: Root $\notin \mathcal{I}_{d+1}$

If *Root* $\notin \mathcal{I}_d$, then the remaining elements of $\mathcal{I}_d$ are from $T \setminus \{Root\}$.

Let $T' = T \setminus \{Root\}$. Then T' is a forest of 2 disjoint and distinct components. Let C_1 and C_2 be the 2 components of T'. Since T was a perfect binary tree of depth $d+1$, then C_1 and C_2 are also perfect binary trees of depth:

$$\begin{aligned} &= (d+1) - 1 \\ &= d \end{aligned}$$

Since d is odd, and C_1 and C_2 are disjoint and distrinct perfect binary trees,

$$\alpha(T') = \alpha(C_1) + \alpha(C_1)$$

By symmetry we get that,

$$\alpha(T') = 2\alpha(C_1)$$

Since $Root \notin \mathcal{I}_d$,

$$\alpha(T) = \alpha(T') = 2\alpha(C_1)$$

Then, by our induction hypothesis,

$$\begin{aligned}\alpha(T) &= \underbrace{2(2^{d-1} + 2^{d-3} + \cdots + 1)}_{\lceil \frac{d}{2} \rceil \text{terms}} \\ &= 2\left(\sum_{i=1}^{\lceil \frac{d}{2} \rceil} 2^{d-(2i-1)}\right)\end{aligned}$$

Note that $\lceil \frac{d}{2} \rceil = \frac{d+1}{2}$, since d is odd, then,

$$\begin{aligned}\alpha(T) &= 2\left(\sum_{i=1}^{\lceil \frac{d}{2} \rceil} 2^{d-(2i-1)}\right) \\ &= 2\left(\sum_{i=1}^{\frac{d+1}{2}} 2^{d-(2i-1)}\right) \qquad (2)\end{aligned}$$

Since $\alpha(T)$ is the maximum coclique,

$$\alpha(T) = \max(1, 2)$$

Remember that our objective is to show that $\alpha(T) = (2)$ as this aligns with our inductive hypothesis.

From (2), we can simplify it as the following:

$$
\begin{aligned}
\alpha(T) &= 2\left(\sum_{i=1}^{\frac{d+1}{2}} 2^{d-(2i-1)}\right) \\
&= 2\left(2^{d-2\left(\frac{d+1}{2}\right)+1} + \sum_{i=1}^{\frac{d-1}{2}} 2^{d-(2i-1)}\right) \\
&= 2\left(2^{d-d-1+1} + \sum_{i=1}^{\frac{d-1}{2}} 2^{d-(2i-1)}\right) \\
&= 2\left(2^{0} + \sum_{i=1}^{\frac{d-1}{2}} 2^{d-(2i-1)}\right) \\
&= 2\left(1 + \sum_{i=1}^{\frac{d-1}{2}} 2^{d-(2i-1)}\right) \\
&= 2\left(1 + \sum_{i=1}^{\frac{d-1}{2}} 2^{d-2i+1)}\right) \\
&= 2\left(1 + 2\sum_{i=1}^{\frac{d-1}{2}} 2^{d-2i)}\right) \\
&= 2 + 4\sum_{i=1}^{\frac{d-1}{2}} 2^{d-2i} \\
&= 4\sum_{i=1}^{\frac{d-1}{2}} 2^{d-2i} + 2 \\
&> 4\sum_{i=1}^{\frac{d-1}{2}} 2^{d-2i} + 1 \\
&= (1)
\end{aligned}
$$

Which implies that,

$$\alpha(T) = \max(1, 2) = (2) \text{ as required.}$$

Case 2: d is even

We will proceed similarly to the odd case.

For our base case, consider the case of $d = 2$. Let v_0 and v_1 be the leaves of the perfect binary tree of depth 2. We claim that $\mathcal{I}_2 = \{v_0, v_1\}$. This implies that $Root \notin \mathcal{I}_2$.

We will prove the claim by using contradiction.

For contradiction, say that $Root \in \mathcal{I}_2$, then $v_0 \notin \mathcal{I}_2$ and $v_0 \notin \mathcal{I}_2$ since both v_0 and v_1 are neighbours of $Root$. This then implies that $\alpha(T) = 1$. However, the set of leaves $\{v_0, v_1\}$ has cardinality $2 > 1 = \alpha(T)$ which is a contradiction.

Hence, for a perfect binary tree T of depth 2, $\alpha(T) = 2$. Thus, our base case holds.

Now, say that the statement holds for all d. We now have to show that it holds for $d + 1$. That is, we need to show that it holds for a given perfect binary tree T of depth $d+1$. Note that if d is even, then $d + 1$ is odd. Hence, we expect that:

$$\alpha(T) = 2^d + 2^{d-2} + \ldots + 1$$

which follows from our previous case.

□

Lemma 1.5. *Following from the previous lemma 1.4, we claim that if $|\mathcal{I}_d| = \alpha(T)$, then $\mathcal{I}_d$ is unique.*

Proof.
Let $\mathcal{I}_d$ be the maximum coclique, we will first construct a set of vertices that build $\mathcal{I}_d$ and then show that no other vertex can be added to the set nor removed which will satisfy uniqueness.

We shall proceed by cases.

Case 1: d is odd

From 1.4, we know that if d is odd, then $Root \in \mathcal{I}_d$.

Let v_0 and v_1 be the children of $Root$,

If $Root \in \mathcal{I}_d$, then we know that $v_0 \notin \mathcal{I}_d$ and $v_1 \notin \mathcal{I}_d$.

Let T_1, T_2, T_3, T_4 be the disjoint components obtained from $T \backslash \{v_0, v_1, Root\}$. Note that $T_1, T_2, T_3,$ and T_4 are all perfect binary trees of depth $d - 2$.

Since d was odd, then $d - 2$ is also odd.

□