ReferenceslemmaLemmaLemmas

1 Theorems and Lemmas

Definition 1.1 (Depth)**.** *The depth starts from 1-index. A single vertex is of depth 1.*

Definition 1.2 (Max Independent Set)**.** *We denote the maximum independent set of a perfect binary tree of depth d by $\mathcal{I}_d$.*

Conjecture 1.3 (HK property for a perfect binary tree)**.** *For any given perfect binary tree T, the maximum number of cocliques lie in the leaves. The number of cocliques is denoted by k. $\alpha(T)$ denotes the independence number of a tree T.*

Lemma 1.4. *For a given perfect binary tree T with depth d and maximum number of cocliques possible, i.e. $k = \alpha(T)$ we have:*

$$\alpha(T) = \begin{cases} 2^{d-1} + 2^{d-3} + \ldots + 1 \text{ for odd } d \\ 2^{d-1} + 2^{d-3} + \ldots + 2 \text{ for even } d \end{cases}$$

Or, in summation notation:

$$\alpha(T) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} 2^{d-2i}$$

Proof.
We shall proceed by inducing on d.
We will have two cases, one for d being odd and another for d being even.

Case 1: d is odd

For our base case, consider the trivial case of $d = 1$. Here, $\alpha(T) = 1$. Hence, the base case holds.

Now, say that the statement holds for all d. We now have to show that it holds for $d + 1$.

We now consider 2 cases:

Case 1: *Root* $\in \mathcal{I}_{d+1}$

Let the left and right child of the root be v_0 and v_1 respectively.
Let C_1, C_2, C_3, C_4 be the four perfect binary tree components generated by $T \setminus \{Root, v_0, v_1\}$.
If the root $\in \mathcal{I}_{d+1}$, then $v_0 \notin \mathcal{I}_d$ and $v_1 \notin \mathcal{I}_d$. This means that the remaining vertices of $\mathcal{I}_d$ is in one of the 4 components. Since the current depth from the *Root* is $d + 1$, then

$T \setminus \{Root, v_0, v_1\}$ will have depth of $d+1-2 = d-1$. Since d is odd, then $d+1$ is even which implies that $d-1$ is also even.

By symmetry, it is enough to consider one of the 4 components' maximum independent set for our calculations. Since the 4 components are disjoint, then we can add their independence numbers together along with the *Root* and obtain the following:

$$\alpha(T_{d+1}) = 4(\alpha(T_{d-1})) + 1$$

Then, from our induction hypothesis, we get that:

$$\begin{aligned} \alpha(T_{d+1}) &= 4(2^{d-2} + 2^{d-4} + \ldots + 2) + 1 && (1) \\ &= \underbrace{2^d + 2^{d-2} + \ldots + 2^3 + 1}_{\frac{d-1}{2}\text{terms}} && (2) \end{aligned}$$

Case 2: Root $\notin \mathcal{I}_d$

If *Root* $\notin \mathcal{I}_d$, then the remaining elements of $\mathcal{I}_d$ are from $T \setminus \{Root\}$.

Let $T' = T \setminus \{Root\}$. Then T' is a forest of 2 disjoint and distinct components. Let C_1 and C_2 be the 2 components of T'. Since T was a perfect binary tree of depth $d+1$, then C_1 and C_2 are also perfect binary trees of depth:

$$\begin{aligned} &= (d+1) - 1 \\ &= d \end{aligned}$$

Since d is odd, and C_1 and C_2 are disjoint and distrinct perfect binary trees,

$$\alpha(T') = \alpha(C_1) + \alpha(C_1)$$

By symmetry we get that,

$$\alpha(T') = 2\alpha(C_1)$$

Since *Root* $\notin \mathcal{I}_d$,

$$\alpha(T) = \alpha(T') = 2\alpha(C_1)$$

Then, by our induction hypothesis,

$$\begin{aligned} \alpha(T) &= 2(2^{d-1} + 2^{d-3} + \cdots + 1) && (3) \\ \alpha(T) &= \underbrace{2^d + 2^{d-2} + \cdots + 2}_{\frac{d}{2}\text{terms}} && (4) \end{aligned}$$

Since $\alpha(T)$ is the maximum independent set,

$$\alpha(T) = \max(2, 4)$$

From (2) and (4),
Note that 2 can be rewritten in summation notation as the following:

$$= \sum_{i=1}^{\frac{d-1}{2}} 2^{d-2i} \tag{5}$$

And that 4 can be rewritten in summation notation as the following:

$$= \sum_{i=1}^{\lceil \frac{d}{2} \rceil} 2^{d-(2i-1)} \tag{6}$$

Since $\lceil \frac{d}{2} \rceil = \frac{d+1}{2}$, then the upper bound on the summation can be rewritten as:

$$= \sum_{i=1}^{\frac{d+1}{2}} 2^{d-(2i-1)} \tag{7}$$

Case 2: d is even

□

Lemma 1.5. *Following from the previous lemma 1.4, we claim that the star whose cardinality is $\alpha(T)$ is unique, i.e there is a unique maximum family of cocliques.*