Exploring perfect binary trees with relation to the HK-property

MXML Presentation

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Outline

Introduction

- Providing an overview of the Erdős-Ko-Rado (EKR) theorem and its relevance to intersecting families of sets.
- Introducing perfect binary trees and their relation to the HK-property.
- Objectives of this presentation:
  - Exploring properties of perfect binary trees.
  - Discussing the HK-property.
  - Investigating potential connections between perfect binary trees and the HK-property.

EKR Theorem

- [1] The Erdős-Ko-Rado (EKR) theorem, named after mathematicians Paul Erdős, Chao Ko, and Richard Rado, is a fundamental result in extremal set theory.
- The theorem deals with intersecting families of sets, which are collections of sets that share a common non-empty intersection.
- Specifically, the EKR theorem provides conditions under which the size of the largest intersecting family of sets can be determined.
- [2] This result has applications in combinatorics, graph theory, probability and other areas of statistics and mathematics.

[1] **Erds1961INTERSECTIONTF**.
[2] **MR0892525**.

EKR Theorem

Definition (Intersecting family)

A family of subsets $\mathcal{F}$ of some set is **intersecting** if any two members of $\mathcal{F}$ have a non-empty intersection.

- The **Erdős-Ko-Rado** theorem limits the number of sets in an intersecting family.

Theorem (EKR Theorem)

[a] *If $\mathcal{F}$ is an intersecting family of k-subsets of an n-set (cardinality of the set is n), then*

- $|\mathcal{F}| \leq \binom{n-1}{k-1}$
- *If equality holds, $\mathcal{F}$ consists of the k-subsets that contain i, for some i in the n-set.*

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[a] **Godsil˙Meagher˙2015**.

HK-property

Some definitions before we get into the property:

Definition (Cocliques)

- A **coclique** in a graph is a set of vertices such that no two vertices in the set are adjacent.
- The maximum size of a coclique in a graph is called the **indepdence number** of the graph. For a graph G, it is denoted by $\alpha(G)$.

Definition (Stars and Stars Center)

- Let $G = (V, E)$ be a graph, and $v \in V(G)$. The family $\mathcal{I}_G^k(v) = A \in \mathcal{I}_G^k : v \in A$ is called a **star** of $\mathcal{I}_G^k$ and v is called it's **star center**.

Definition (k-EKR graph)

A graph is said to be k-EKR if for any family of indepdent sets $\mathcal{I}_G^k$ of size k, the intersection of any two sets in $\mathcal{I}_G^k$ is non-empty and that $|\mathcal{F}| \leq \mathcal{I}_G^k(v)$, for a vertex $v \in V(G)$.

HK-property

Studying the EKR theorem,[3] Holroyd and Talbot made the following two conjectures:

Conjecture (k-EKR Conjecture)

Let G be a graph, and let $\mu(G)$ be the size of its smallest maximal independent set. Then G is k-EKR for every $1 \leq k \leq \frac{\mu(G)}{2}$.

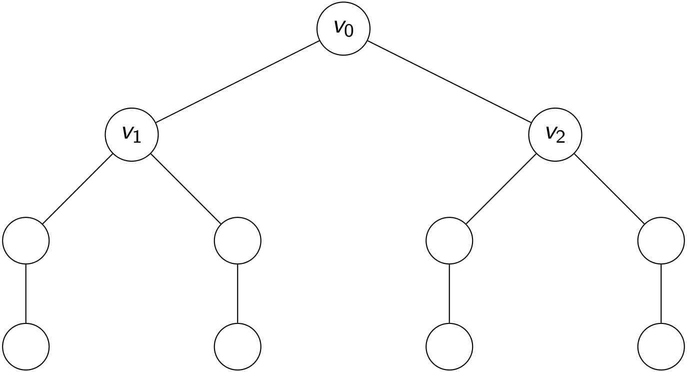
Conjecture (HK-Property)

For any $k \geq 1$ and any tree T, there exists a leaf l of T such that $|\mathcal{I}_T^k(v)| \leq |\mathcal{I}_T^k(l)|$ for each $v \in V(T)$.

[3] **HOLROYD2005165**.

HK-property

- The HK-property was proven for $k \leq 4$, but the conjecture was shown to be false.[4][5][6]



The largest k-star for $k \geq 5$ is centered at v_0

[4] **MR2523796**.
[5] **MR3612439**.
[6] **MR3271819**.

Some graphs that DO satisfy the HK-property

[7] The HK-property holds for spiders, caterpillars, and (partially) lobsters.
¡INSERT IMAGES HERE WHEN YOU HAVE TIME¿

[7] **MR4245360**.

Perfect Binary Tree

Definition (Depth of a vertex)

For a tree $T = (V, E)$ with a root vertex $r \in V$, the **depth** of a vertex $v \in V$ is defined as the length of the path from the r to v.

Definition (Binary Tree)

A **binary tree** is a tree in which each vertex has at most two children, referred to as the left child and the right child.

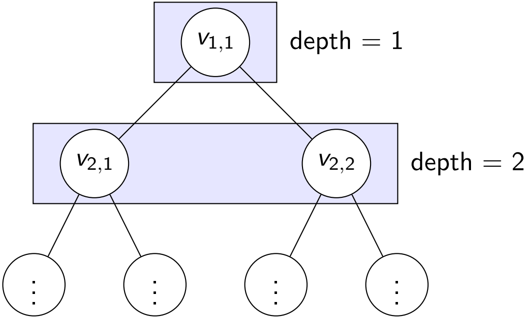
Definition (Perfect Binary Tree)

A **perfect binary tree** is a binary tree in which all the internal nodes have exactly two children and all the leaves are at the same depth.

Perfect Binary Tree

Definition

Let $\mathcal{V}_k \in V(T)$ be the set of vertices of depth k. We call $\mathcal{V}_k$ as the depth vertex set of depth k. Index all vertices in $\mathcal{V}_k$ from left to right as $v_{k,i}$, where k is the depth of the vertex and i is the index of the vertex in $\mathcal{V}_k$ such that $1 \leq i \leq 2^{k-1}$.



Does a perfect binary tree satisfy the HK property?

- Probably.
- The lobster partially satisfies the HK property and the perfect binary tree has a close relation to the lobster.
- In addition, the perfect binary tree is very symmetric and has a lot of structure that we can maniputlate.

Idea 1: Expand the definition of the flip function to relate it to the perfect binary trees

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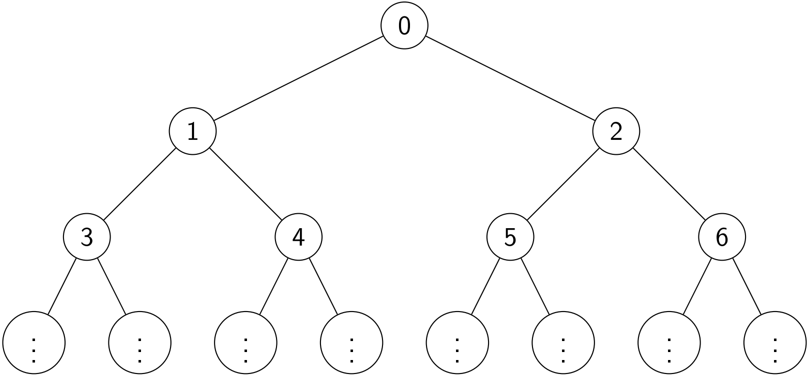
An Algorithm to Generate Pefect Binary Trees

Data: $n \geq 0$, where n is the depth of the perfect binary tree
Result: A perfect binary tree graph's leaves
Function `perfect_binary_tree_generator(`n`)`**:**

$\quad$ $num\_vertices \leftarrow 2^{n+1} - 1$;
$\quad$ $leaves \leftarrow []$;
$\quad$ $last\_row\_start \leftarrow floor(num\_vertices/2)$;
$\quad$ **for** *vertex in range(last_row_start, num_vertices)* **do**
$\quad\quad$ $leaves$.append($vertex$);
$\quad$ **end**
$\quad$ **return** *leaves*

Output of the Perfect Binary Trees

The algorithm will generate a perfect binary tree of this form:



Thank You!

Summary

A slideshow usually ends with a summary slide.