Journal of Discrete Algorithms 15 (2012) 16–31

Contents lists available at SciVerse ScienceDirect

Journal of Discrete Algorithms

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On bounded degree plane strong geometric spanners

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ARTICLE INFO

Article history:
Received 13 October 2010
Received in revised form 18 March 2012
Accepted 19 March 2012
Available online 21 March 2012

Keywords:
Computational geometry
Geometric spanners

ABSTRACT

Given a set P of n points in the plane, we show how to compute in $O(n\log n)$ time a spanning subgraph of their Delaunay triangulation that has maximum degree 7 and is a strong plane t-spanner of P with $t = (1+\sqrt{2})^2 * \delta$, where δ is the spanning ratio of the Delaunay triangulation. Furthermore, the maximum degree bound can be reduced slightly to 6 while remaining a strong plane constant spanner at the cost of an increase in the spanning ratio and no longer being a subgraph of the Delaunay triangulation.

1. Introduction

Given a weighted graph $G = (V, E)$ and a real number $t \geqslant 1$, a *t-spanner* of G is a spanning subgraph G^* with the property that for every edge $\{p, q\} \in G$, there exists a path between p and q in G^* whose weight is no more than t times the weight of the edge $\{p, q\}$. Such a path is referred to as a *spanning path*. Typically, G is a dense graph with $\Omega(n^2)$ edges. It is desirable for the t-spanner G^* to be sparse, preferably having only a linear number of edges. Note that the shortest-path distances in G^* approximate shortest-path distances in the underlying graph G and the parameter t represents the approximation ratio. The smallest t, for which G^* is a t-spanner of G, is the spanning ratio. A path between p and q is *strong* when every edge in the path has weight at most the weight of the edge $\{p, q\}$. A spanner is *strong* when there exists a strong spanning path in G^* for every edge in G. For example, the Delaunay triangulation of a point set is known to be a strong constant spanner of the complete geometric graph on the same point set [5].

Spanners have been studied in many different settings. The various settings depend on the type of underlying graph G, on the way weights are assigned to edges in G, on the specific value of the spanning ratio t, and on the function used to measure the weight of a shortest path. We concentrate on the setting where the underlying graph is geometric. In our context, a geometric graph is a weighted graph whose vertex set is a set of points in $\Re^2$ and whose edge set consists of line segments connecting pairs of vertices. The edges are weighted by the Euclidean distance between their endpoints.

There is a vast body of literature on different methods for constructing t-spanners with various properties in this geometric setting (see [13] and [6] for a survey of the area). Aside from trying to build a spanner that has a small spanning ratio, additional properties of the spanners are desirable, e.g., planarity and bounded degree.

In this paper we consider the following problem. Given a set of points in the plane, compute a bounded degree plane spanner of this set of points. Bose et al. [4] were the first to show the existence of a plane t-spanner (for some constant t) whose maximum vertex degree is bounded by 27. Subsequently, Li and Wang [12] reduced the degree bound to 23. In [7], Bose et al. improved the degree bound to 17. Kanj et al. [9] further reduced the degree bound to 14. This was then improved

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1 Research is partially supported by grant 2240-2100.6/2009 from the German–Israeli Foundation for scientific research and development (GIF).
2 Research is partially supported by Lynn and William Frankel Center for Computer Science.

1570-8667/$ – see front matter
http://dx.doi.org/10.1016/j.jda.2012.03.004

to 11 by Kanj and Xia [10]. Recently, Bonichon et al. [1] uncovered a beautiful yet surprising connection between θ-graphs and $\triangle$-Delaunay graphs (where the empty circle is an equilateral triangle). They showed that the θ-graph where $\theta = \pi/3$ is the overlay of two $\triangle$-Delaunay graphs. This connection was exploited by Bonichon et al. [2] who showed how to construct a subgraph of the $\triangle$-Delaunay graph that is a plane 6-spanner with maximum degree 9 and how to construct a plane 6-spanner with maximum degree 6, based on the previous one, which is no longer a subgraph of the $\triangle$-Delaunay graph. The use of the $\triangle$-Delaunay graph comes at a cost since these spanners are not necessarily strong. All of the above algorithms, including ours, use the same approach: start with a Delaunay graph (either Euclidean or $\triangle$), carefully prune edges from this graph to obtain a bounded degree subgraph that retains the desired properties.

Our results obtained independently and in parallel are different from those of [2] in the following way. Given a set P of n points in the plane, we show how to compute in $O(n \log n)$ time a subgraph of the Delaunay triangulation $\mathrm{DT}(P)$ that has maximum degree 7 and is a strong t-spanner of P with $t = (1+\sqrt{2})^2 * \delta$, where δ is the spanning ratio of the Delaunay triangulation. We denote this subgraph as $\mathrm{BDDT}(P)$. Our results are different from those of [2] in the following way. $\mathrm{BDDT}(P)$ is a subgraph of the Euclidean Delaunay triangulation as opposed to the $\triangle$-Delaunay graph. Since the Euclidean Delaunay triangulation is strong, we are able to build a strong spanner. It is important to note that just because the Delaunay triangulation is a strong spanner does not imply that every subgraph of the Delaunay triangulation is a strong spanner. As such, we prove that $\mathrm{BDDT}(P)$ is indeed a strong spanner. The fact that $\mathrm{BDDT}(P)$ is a strong spanner means that $\mathrm{BDDT}(P) \cap \mathrm{UDG}(P)$ is a spanner of $\mathrm{UDG}(P)$, where $\mathrm{UDG}(P)$ is the unit disk graph of P. This is of importance in the context of ad hoc wireless networks which are often modeled as unit disk graphs. Finally, at the cost of increasing the spanning ratio and no longer being a subgraph of the Delaunay triangulation, we are able to build a strong plane constant spanner with maximum degree 6.

The actual spanning ratio of the Delaunay triangulation remains an open problem. Although long conjectured to be $\pi/2$, it was shown in [3] that the actual spanning ratio of the Delaunay triangulation is strictly larger than $1.5846 > \pi/2$ and recently, in [15], it has been proved to be at least 1.5907. In [11] an upper bound of $4\pi\sqrt{3}/9$ was shown on the spanning ratio of the Delaunay triangulation and lately Xia has claimed a spanning ratio of 1.998 [14]. En route to proving our main result, we uncover some structural properties of Delaunay triangulations that allow us to provide tighter bounds on the spanning ratio in some restricted settings. We hope that these results shed some light to help resolve this longstanding open problem.

2. Algorithm for building a bounded degree 7 strong plane spanner

In this section we describe an algorithm that computes a bounded degree strong plane spanner. The approach we take to build such a spanner is to start with the Delaunay triangulation and prune some edges to achieve the degree bound of 7 while maintaining a constant spanning ratio. We ensure that for every edge of the Delaunay triangulation that we do not add to our resulting spanner, there is a strong spanning path approximating this edge.

The algorithm consists of two main components, *BoundSpanner*() and *Wedge*() outlined below, that work together to compute a plane degree 7 t-spanner P. The first step is to compute the Delaunay triangulation ($\mathrm{DT}(P)$) and sort the edges in nondecreasing length order. For each $p \in P$, let $C_p = \{C_p^0, C_p^1, \ldots, C_p^6, C_p^7\}$ denote a set of 8 closed cones labeled in clockwise order, with apex p and angle $\frac{\pi}{4}$. For each point p, let q_{min} be its nearest point in P and orient the closed cones C_p such that edge $\{p, q_{min}\}$ is shared by cones C_p^0 and C_p^7. An edge $\{p,q\} \in \mathrm{DT}(P)$ is added to G if both p and q *agree* on it. A point p *agrees* on an edge $\{p,q\}$ if the set of edges added by the algorithm so far **excluding** the edges that have been added by subroutine *Wedge*() is empty of edges in one of the cones $C_p^i \in C_p$ containing $\{p,q\}$. (Note that throughout this paper we intersect the interior of an edge with a cone, otherwise every cone of p would contain the edge $\{p,q\}$.) By construction, an edge can be contained in at most two closed cones, and in exactly two cones if this edge is on the common boundary of both cones. After adding an edge $\{p,q\}$ in *BoundSpanner*(), we call the second subroutine *Wedge*() twice, once for point p and then once for point q. In the second subroutine, we add more edges to the spanner that do not affect the degree of the spanner but help to bound the spanning ratio of the resulting graph. We refer to the edges added during Algorithm 1 **excluding** the edges that have been added in the subroutine *Wedge*() as *short* edges (see Fig. 1).

Remark. Note that the output t-spanner G of P obtained by Algorithm 1 is a subgraph of $\mathrm{DT}(P)$, therefore, it is plane and has a linear number of edges.

3. Bounded degree

In this section we show that the degree of a vertex in the resulting spanner is at most 7. We begin by making a few basic observations, and then conclude with Lemma 3.1, where we prove that the maximum degree is at most 7.

Claim 1. *Let C_p^i be a cone of a point p with angle $\alpha < \pi/2$. Let $\{p, q_j\}$ (resp., $\{p, q_k\}$) be the first (resp., last) edge in C_p^i. Then, only the edges $\{p, q_j\}$, $\{p, q_{j+1}\}$, $\{p, q_{k-1}\}$ and $\{p, q_k\}$ can be added to the spanner by a call to Wedge() in Algorithm 2.*

Proof. A call to subroutine *Wedge*() with first parameter x adds an edge e only if x is adjacent to both endpoints of e in $\mathrm{DT}(P)$. Thus, the edge $\{p, q_m\}$ is added during a call to *Wedge*() with the first parameter q_{m-1} or q_{m+1}. Assume that the

Algorithm 1 BoundSpanner(P)

Input: A set P of points in the plane
Output: A plane t-spanner $G = (P, E)$ with maximum degree 7

```
1: Compute DT(P) → (P, E_DT)
2: Let L be a list of the edges of DT sorted in nondecreasing length
3: E ← ∅ /* Edges of the resulting spanner */
4: E* ← ∅ /* Edges added during calls to Wedge() */
5: Initialize C_p for each p ∈ P        /* with respect to edge {p, q_min} */
6: for each edge {p, q} ∈ L (* in the sorted order *) do
7:    if (∀C_p^i containing {p, q}, C_p^i ∩ E = ∅) and (∀C_q^j containing {p, q}, C_q^j ∩ E = ∅)
      /* Note: every edge can be contained in at most two adjacent closed cones */
      then
8:       E ← E ∪ {{p, q}}
9:       Wedge(p, q)                    /* calling subroutine Wedge() to check if some
10:      Wedge(q, p)                       edges (E*) need to be added to E   */
11: E ← E ∪ E*
```

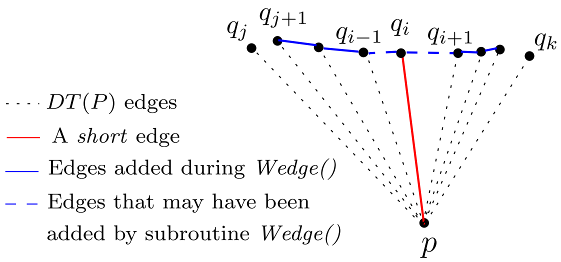

Fig. 1. Illustration of a cone C_p^i after applying Algorithm 1.

Algorithm 2 *Wedge*(p, q_i)

Input: Two points p and q_i such that the edge $\{p, q_i\} \in \mathrm{DT}(P)$
Output: A set of edges E^* to be added to the spanner $G = (P, E)$

```
for every C_p^z containing {p, q_i} do
   Let {p, q_j} and {p, q_k} be the first and the last edges of DT(P), in cone C_p^z (clockwise).
   E* ← E* ∪ {{q_m, q_{m+1}}} for each j < m < i − 1
   E* ← E* ∪ {{q_m, q_{m+1}}} for each i < m < k − 1
   if (edge {p, q_{i+1}} ∈ C_p^z) and (q_{i+1} ≠ q_k) and (angle ∠(p q_i q_{i+1}) > π/2) then
      E* ← E* ∪ {{q_i, q_{i+1}}}
   if (edge {p, q_{i−1}} ∈ C_p^z) and (q_{i−1} ≠ q_j) and (angle ∠(p q_i q_{i−1}) > π/2) then
      E* ← E* ∪ {{q_i, q_{i−1}}}
```

first parameter is q_{m-1}. Assume for the sake of a contradiction that $m \notin \{j, j+1, k-1, k\}$. Notice that a call to *Wedge*() with first parameter p does not add edges $\{q_j, q_{j+1}\}$ and $\{q_{k-1}, q_k\}$, where, $\{p, q_j\}$ (resp., $\{p, q_k\}$) is the first (resp., last) edge in the relevant cone. Thus, edges $\{q_{m-1}, q_{m-2}\}$, $\{q_{m-1}, p\}$ and $\{q_{m-1}, q_m\}$ are in the same cone with apex q_{m-1}. Otherwise $\{p, q_m\}$ would be the first or the last edge in the cone.

Note that $\angle(q_{m-2} p q_m) < \alpha$ since q_{m-2} and q_m are in C_p^i. By the empty circle property of Delaunay triangulations, we have that $\angle(q_{m-2} p q_m) + \angle(q_{m-2} q_{m-1} q_m) > \pi$. This implies that $\angle(q_{m-2} q_{m-1}, q_m) > \alpha$ where α is the angle of the cone, (for $\alpha < \pi/2$). Therefore, we contradict the fact that edges $\{q_{m-1}, q_{m-2}\}$, $\{q_{m-1}, p\}$ and $\{q_{m-1}, q_m\}$ are in the same cone with apex q_{m-1}. □

Observation 1. *When an edge $\{p, q\} \in C_p^i$ is added to E in step 8 of Algorithm 1, $C_p^i \cap E \setminus E^* = \emptyset$ where intersections are only with interiors of edges.*

Observation 2. *When the cone angle α is less than $\pi/3$ then the first edge incident to a point p added to $E \setminus E^*$ during Algorithm 1 is $\{p, q_{min}\}$. Thus, the edge $\{p, q_{min}\}$ belongs to two closed cones C_p^0 and C_p^7 in C_p.*

Proof. Consider the rank of edge $\{p, q_{min}\}$. Since it is the shortest edge incident to p in $\mathrm{DT}(P)$ it implies that all the cones in C_p are empty; thus, p "agrees" to add edge $\{p, q_{min}\}$ to E. Let C_q^j be a cone in C_q that contains the edge $\{p, q_{min}\}$. Since the disk centered at p with radius $|p, q_{min}|$ is empty of points, it implies that cone C_q^j that contains $\{p, q_{min}\}$ is empty. Thus, q "agrees" on adding edge $\{p, q_{min}\}$ as well, and the edge is added to E. □

The empty circle property of Delaunay triangulations allows us to make two more basic but crucial observations. Let $S_p = \{q_0, \ldots, q_k\}$ be the set of neighbors of p in DT(P) labeled in clockwise order. For angle $\angle(q_i p q_j) < \pi$, let $S_{p,q_i,q_j} = \{q_i, q_j\} \cup \{q_k \in S_p\colon q_k$ is between q_i and q_j in clockwise order$\}$. For ease of presentation, in the rest of the paper we assume that in S_{p,q_i,q_j} the index i is less than j in clockwise order, and all indices are manipulated modulo the number of neighbors of p in DT(P). Let $D_{p,a,z}$ denote the disk having p, a and z on its boundary.

Observation 3. *From the empty cycle property of Delaunay triangulations it follows that each $x \in S_{s,q_i,q_j}$ lies inside D_{s,q_i,q_j}.*

Proof. Let q_k be a point in S_{s,q_i,q_j}. Since $\{s, q_k\}$ is an edge in DT(P), necessarily there is a disk containing s and q_k on its boundary and empty of points from P, especially q_i and q_j. Hence, the sum of the angles $\angle s q_i q_k$ and $\angle s q_j q_k$ which lie on opposite sides of the same chord is smaller than π and the sum of the other two angles in the quadrilateral $(s q_i q_k q_j)$, $\angle q_i q_k q_j$ and $\angle q_i s q_j$ is grater than π. That implies q_k is inside D_{s,q_i,q_j}. $\square$

Observation 4. *For $q_j, q_i, q_k \in S_{p,q_j,q_k}$ such that q_i is between q_j and q_k in clockwise order, the angle $\angle q_j q_i q_k$ (referring to the angle containing p in its wedge) $\geqslant \pi - \angle q_j p q_k$.*

Proof. Due to the empty cycle property of Delaunay triangulation, the point q_i lies inside the disk D_{p,q_j,q_k} having p, q_j, q_k on its boundary (Observation 3). The angle $\angle q_j q_i q_k$ is minimized when q_i is on the boundary of D_{p,q_j,q_k}. In that case $\angle q_j q_i q_k = \pi - \angle q_j p q_k$ since the two angles lie on the same chord (q_j, q_k). Therefore, $\angle q_j q_i q_k \geqslant \pi - \angle q_j p q_k$. $\square$

Throughout this paper by $\angle abc$ we refer to the smaller angle created by the three points a, b, c, unless we specify otherwise.

Lemma 3.1. *The maximum degree of graph G output by Algorithm 1 is bounded by 7.*

Proof. Eight closed cones C_p are defined for each point $p \in P$ during Algorithm 1. By Observation 2, there are two cones C_p^0 and C_p^7 in C_p sharing a common edge. Consider the short edges E'_p incident to p, i.e., these edges are *not* added to p from a call to *Wedge*(). Then each edge $e \in E'_p$ is added to E only if the cone in C_p containing e is empty at the time that e is considered. Moreover, the first edge in E'_p added to E shares two cones, thus $|E'_p| \leqslant 7$ since there are 8 cones. Next, we show that the edges added during calls to *Wedge*() can be charged uniquely to empty cones, and thus do not increase the degree bound of 7. Let $\{p, q\}$ be an edge added to E^* during a call to *Wedge*(); thus, there exists a point z such that the edge $\{p, q\}$ has been added to E^* during the call *Wedge*(s, r). Moreover, this edge has been added in steps 3, 4 or during steps 6, 8 of the call to *Wedge*(s, r).

- **Case 1:** The edge has been added during step 3 or 4.
  Let $\{p, z\}$ be the edge consecutive to $\{p, q\}$ in the neighborhood of s, such that $q \neq z$. Since the edge $\{p, q\}$ has been added to E^* during the call *Wedge*(s, r), it follows that the edge $\{s, z\}$ is in the same cone (of C_s) as edges $\{s, p\}$ and $\{s, q\}$. Thus, the angle $\angle(zsq) \leqslant \pi/4$ and by Observation 4 angle $\angle(qpz)$ (referring to the angle containing p in its wedge) $\geqslant 3\pi/4$. The addition of the edge $\{p, q\}$ by a call to *Wedge*(s, r) implies that $\{s, p\}$ cannot be a *short* edge (i.e. it is not added in step 8 of Algorithm 1). By Claim 1 it also cannot be added during a call to *Wedge*(). Therefore, there are at least two empty cones of C_p located between $\{p, q\}$ and $\{p, z\}$. One of them is charged for the edge $\{p, q\}$ and the second pays for the edge $\{p, z\}$ if needed.
- **Case 2:** The edge has been added during step 6 or 8.
  In this case $r = p$. We know that the angle $\angle(qps)$ (referring to the angle containing p in its wedge) $\geqslant \pi/2$, thus there is at least one empty cone c' of C_p located between $\{p, q\}$ and $\{p, s\}$. Therefore, this empty cone c' is charged for the edge $\{p, q\}$.

Therefore, the degree of every point $p \in P$ is bounded by 7. $\square$

4. Spanning ratio

In this section we show that the graph output by Algorithm 1 is a strong spanner and has bounded spanning ratio. The approach of the proof is the following. If an edge $\{p, q\}$ of the Delaunay triangulation is not in the resulting spanner, then either p or q did not *agree* to this edge. Suppose, without loss of generality, that p did not agree to the edge $\{p, q\}$. This means that in the cone with apex p that contains q, there must be a smaller edge adjacent to p. By using this smaller edge, we are able to construct a strong spanning path from p to q whose length is at most $(1 + \sqrt{2})^2$ times $|pq|$. The main difficulty is in showing that the edges of such a spanning path were not pruned. We begin with bounding the length of paths in wedges before proving our main theorem.

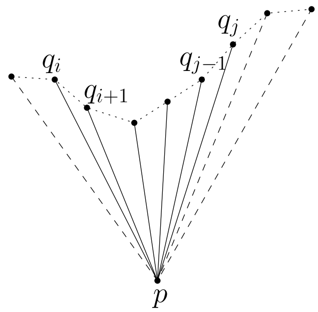

Fig. 2. Wedge W_{p,q_i,q_j} and its edges depicted in solid lines.

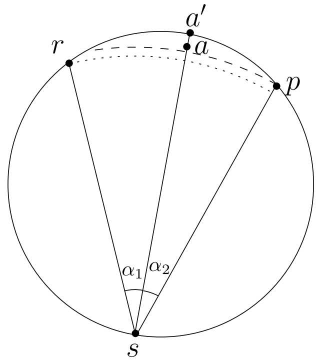

Fig. 3. Illustrating the proof of Lemma 4.1.

4.1. Bounding the length of wedges paths

For a graph $G = (V, E)$ and two points $p, q \in V$, let $\delta_G(p, q)$ denote the length of the shortest path between p and q in G. Let wedge $W_{p,q_i,q_j} = \{\{p, q_k\}\colon\ q_k \in S_{p,q_i,q_j}\}$ (see Fig. 2). Let $P_{S_{p,q_i,q_j}}(q_i, q_j)$ denote the path in DT(P) from q_i to q_j restricted to points in S_{p,q_i,q_j}, and let $\delta_{S_{p,q_i,q_j}}(q_i, q_j)$ denote the length of this path.

The following observations and claims leads to Corollary 4.3 which bounds the length $\delta_{S_{p,q_i,q_j}}(q_i, q_j)$.

Observation 5. *Let $D_{p,a,z}$ be a disk having p, a, and z on its boundary in clockwise order and let β denote the angle $\angle(pza)$. Then, $\frac{\beta}{\sin(\beta)}|\{p, a\}|$ is the length of the arc from p to a on the boundary of $D_{p,a,z}$ ($\hat{pa}$).*

Proof. Let o be the center of $D_{p,a,z}$ and let r be the length of its radius, thus, angle $\angle(poa) = 2\beta$. By Pythagoras, $\frac{|pa|}{\sin(\beta)} = 2r$. Therefore, the length of the arc $\hat{pa}$ is

$$2\pi r \Big/ \frac{2\pi}{2\beta} = 2r\beta = \frac{\beta}{\sin(\beta)}|pa|. \qquad \square$$

Lemma 4.1. *Consider a wedge $W_{s,r,p}$ in DT(P) where $\alpha = \angle(rsp) < \pi$ and assume that $\{s, r\}$, $\{s, p\}$ are the shortest edges in $W_{s,r,p}$ incident to s (i.e., $|sr|, |sp| \leqslant |sx|$ for all $x \in S_{s,r,p} \backslash \{r, p\}$). Then, $\delta_{S_{s,r,p}}(r, p) \leqslant |rp| \frac{\alpha}{\sin(\alpha)}$.*

Proof. We prove the claim by induction on the rank of the angle α, i.e., the place of α in a nondecreasing order of the angles of wedges in DT(P), with ties broken arbitrarily.

Base case: Angle α is the smallest angle in DT(P), thus, $\{r, p\} \in$ DT(P) and clearly $\delta_{S_{s,r,p}}(r, p) \leqslant |rp|$.

The induction hypothesis: Assume that for every $\alpha' < \alpha$ the claim holds.

The inductive step: If $W_{s,r,p} \backslash \{\{s, r\}, \{s, p\}\} = \emptyset$, then $\{r, p\} \in$ DT(P) and we are done. Otherwise, let $\{s, a\}$ be the shortest edge in $W_{s,r,p} \backslash \{\{s, r\}, \{s, p\}\}$, i.e., $\{s, a\} = \min_{x \in S_{s,p,r} \backslash \{p,r\}}\{|sx|\}$ (see Fig. 3).

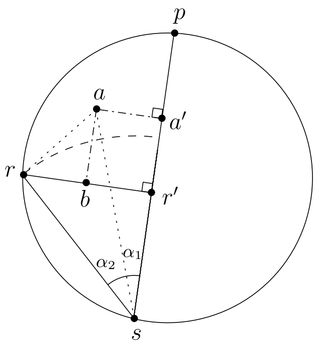

Fig. 4. Illustrating the proof of Lemma 4.2.

Let $\alpha_1 = \angle(rsa)$ and $\alpha_2 = \angle(psa)$. Since $\alpha_1, \alpha_2 < \alpha$, by the induction hypothesis, $\delta_{S_{s,r,a}}(r,a) \leqslant |ra|\frac{\alpha_1}{\sin(\alpha_1)}$, and $\delta_{S_{s,a,p}}(a,p) \leqslant |ap|\frac{\alpha_2}{\sin(\alpha_2)}$. Notice that $S_{s,r,a} \subseteq S_{s,r,p}$ and $S_{s,a,p} \subseteq S_{s,r,p}$. Thus,

$$\delta_{S_{s,r,p}}(r,p) = \delta_{S_{s,r,a}}(r,a) + \delta_{S_{s,a,p}}(a,p) \leqslant |ra|\frac{\alpha_1}{\sin(\alpha_1)} + |ap|\frac{\alpha_2}{\sin(\alpha_2)}.$$

Note that by Observation 3 we have that a is located inside $D_{s,p,r}$. Let a' be the intersection point of $D_{s,p,r}$ and the extension of $\{s,a\}$. Since $|sp| \leqslant |sa|$ and $\angle(sap) \leqslant \frac{\pi}{2}$, we have that $\angle(a'ap) \geqslant \frac{\pi}{2}$ and $|a'p| \geqslant |ap|$. Symmetrically, we get $|ra'| \geqslant |ra|$ and therefore,

$$\delta_{S_{s,r,p}}(r,p) \leqslant |ra'|\frac{\alpha_1}{\sin(\alpha_1)} + |a'p|\frac{\alpha_2}{\sin(\alpha_2)}.$$

According to Observation 5, $|ra'|\frac{\alpha_1}{\sin(\alpha_1)}$ and $|a'p|\frac{\alpha_2}{\sin(\alpha_2)}$ are the lengths of the arcs from r to a' ($\widehat{ra'}$) and from a' to p ($\widehat{a'p}$) on the boundary of $D_{s,p,r}$, respectively. Moreover, $|rp|\frac{\alpha}{\sin(\alpha)}$ is the length of the arc from r to p on the boundary of $D_{s,p,r}$, which is the sum of $\widehat{ra'}$ and $\widehat{a'p}$. Therefore,

$$\delta_{S_{s,r,p}}(r,p) \leqslant |ra'|\frac{\alpha_1}{\sin(\alpha_1)} + |a'p|\frac{\alpha_2}{\sin(\alpha_2)} = |rp|\frac{\alpha}{\sin(\alpha)}. \qquad \square$$

Lemma 4.2. *Let* DT(P) *be the Delaunay triangulation of the set of points P and let $W_{s,r,p}$ be a wedge in* DT(P)*, such that $\{s,r\}$ is the shortest edge in $W_{s,r,p}$ ($|sr| \leqslant |sx|\ \forall x \in S_{s,r,p}$) and $\alpha = \angle(rsp) < \pi/2$. Let r' be the orthogonal projection of r on $\{s,p\}$. Then,*

$$\delta_{S_{s,r,p}}(r,p) \leqslant \frac{\alpha}{\sin\alpha}\left(|pr'| + |r'r|\right),$$

(*see Fig. 4*).

Proof. We prove the lemma by induction on the rank of the angle α.

Base case: Angle α is the smallest angle in DT(P); therefore, $\{r,p\} \in$ DT(P) and $\delta_{S_{s,r,p}}(r,p) = |rp|$. Since $|rp| < |pr'| + |r'r|$ and $\alpha/\sin(\alpha) > 1$ for $0 < \alpha < \pi/2$, we get $\delta_{S_{s,r,p}}(r,p) \leqslant \frac{\alpha}{\sin\alpha}(|pr'| + |r'r|)$.

The induction hypothesis: Assume the claim holds for every angle $\alpha' < \alpha$.

The inductive step: If $S_{s,r,p} \backslash \{r,p\} = \emptyset$, then $\{r,p\} \in$ DT(P) and we are done by the same argument of the base case. Otherwise, recall that from the empty cycle property of Delaunay triangulation it follows that each $x \in S_{s,r,p} \backslash \{r,p\}$ is inside $D_{s,p,r}$. Let $a \in S_{s,r,p} \backslash \{r,p\}$ be a point such that for every $x \in S_{s,r,p} \backslash \{r,p\}$, $|sa| \leqslant |sx|$. If $|sa| \geqslant |sp|$ by Lemma 4.1

$$\delta_{S_{s,r,p}}(r,p) \leqslant \frac{\alpha}{\sin(\alpha)}|rp| \leqslant \frac{\alpha}{\sin\alpha}\left(|pr'| + |r'r|\right)$$

and we are done. Otherwise, ($|sa| < |sp|$), let a' be the orthogonal projection of a on $\{s,p\}$. Denote $\alpha_1 = \angle(asp)$ and $\alpha_2 = \angle(asr)$. Since $\alpha_1 < \alpha$, we can apply the induction hypothesis and get

$$\delta_{S_{s,a,p}}(a,p) \leqslant \frac{\alpha_1}{\sin\alpha_1}\left(|pa'| + |a'a|\right). \tag{1}$$

Moreover, by Lemma 4.1

$$\delta_{S_{s,r,a}}(r,a) \leqslant \frac{\alpha_2}{\sin(\alpha_2)}|ra|. \tag{2}$$

Therefore,

$$\begin{aligned}
\delta_{S_{s,r,p}}(r,p) &\leqslant \delta_{S_{s,r,a}}(r,a) + \delta_{S_{s,a,p}}(a,p) \\
&\leqslant^{(2)} \frac{\alpha_2}{\sin\alpha_2}|ra| + \delta_{S_{s,a,p}}(a,p) \\
&\leqslant^{(1)} \frac{\alpha_2}{\sin\alpha_2}|ra| + \frac{\alpha_1}{\sin\alpha_1}\left(\left|pa'\right| + \left|a'a\right|\right) \\
&\leqslant \frac{\alpha}{\sin\alpha}\left(|ra| + \left|pa'\right| + \left|a'a\right|\right) \\
&\leqslant^{(*)} \frac{\alpha}{\sin\alpha}\left(\left|pr'\right| + \left|rr'\right|\right).
\end{aligned}$$

The last inequality $(*)$ is obtained by the following. Let b denote the orthogonal projection of a on (r', r). Thus, $|ba| = |a'r'|$ and $|br'| = |a'a|$. Therefore,

$$\begin{aligned}
|ra| + \left|pa'\right| + \left|a'a\right| &\leqslant^{(**)} |rb| + |ba| + \left|pa'\right| + \left|aa'\right| \\
&= |rb| + \left|a'r'\right| + \left|pa'\right| + \left|br'\right| \\
&= \left|pr'\right| + \left|rr'\right|.
\end{aligned}$$

Inequality $(**)$ follows by triangle inequality, $|ra| \leqslant |rb| + |ba|$. □

Lemma 4.2 leads to the following corollary.

Corollary 4.3. *Let $\mathrm{DT}(P)$ be the Delaunay triangulation of a set of points P. Consider an arbitrary wedge $W_{s,r,p}$. If $\{s,r\}$ is the shortest edge in $W_{s,r,p}$, then*

$$\delta_{\mathrm{DT}(P)}(r,p) \leqslant \frac{\alpha}{\sin\alpha}\sqrt{2}|rp|,$$

where $\alpha = \angle(rsp)$.

4.2. Bounding the spanning ratio

We begin with a few observations and geometric lemmas before proving the upper bound on the spanning ratio.

Lemma 4.4. *Let $\{s,r\}$, $\{s,p\}$ be two edges in $C_s^i \cap \mathrm{DT}(P)$ for C_s^i with angle $\pi/4$, such that $\{s,r\}$ has been chosen by Algorithm 1 in step 8 to be added to E. Then for every $\{s,x\} \in W_{s,r,p}$, $|sx| \geqslant \min\{|sr|,|sp|\}$. **Note**: it is not necessarily true that $|sr| < |sp|$.*

Proof. Assume, to the contrary, there is a point $x \in S_{s,r,p}$ such that $|sx| < \min\{|sr|,|sp|\}$. Let $\{s,w\}$ be the shortest edge among all these edges (in $W_{s,r,p}$). Since $|sw| < |sr|$, when $\{s,w\}$ was examined by Algorithm 1, the cone C_s^i was empty of edges in $E \backslash E^*$. Therefore, the only possible reason that could cause $\{s,w\}$ not to be added to $E \backslash E^*$ at that time is that $E \backslash E^*$ already contained an edge in the cone with apex w that $\{s,w\}$ belongs to (w.l.o.g. to C_w^j). Let $\{w,t\}$ be an adjacent edge to $\{w,s\}$ in C_w^j. Necessarily $\{t,s\} \in \mathrm{DT}(P)$, and it is also in $W_{s,r,p}$. Consider the triangle $\triangle(swt)$ and its internal angles. Since $\{w,t\}$ and $\{w,s\}$ are both in C_w^j, $\angle(swt) \leqslant \frac{\pi}{4}$. Thus,

$$\begin{aligned}
\angle(swt) &= \pi - \angle(wts) - \angle(wst) \\
&\geqslant^{(*)} \pi - \angle(swt) - \angle(wst) \\
&\geqslant \pi - \angle(swt) - \frac{\pi}{4} \\
&= \frac{3\pi}{4} - \angle(swt) \\
&\geqslant \frac{\pi}{2}
\end{aligned}$$

in contradiction to the assumption that $\{w,t\}$ and $\{w,s\}$ are both in C_w^j.

Inequality $(*)$ follows from the fact that $\{s,w\}$ is the shortest edge among all the edges in $W_{s,r,p}$, and therefore $\{s,w\} \leqslant \{s,t\}$. □

Claim 2. *Let $\triangle(rqp)$ be a triangle with $\angle(rqp) \geqslant \pi - \alpha$, then for $k \geqslant \frac{1}{\cos\alpha}d$, and $\alpha < \frac{\pi}{2}$, we have $k|qp| + d|rq| \leqslant k|rp|$.*

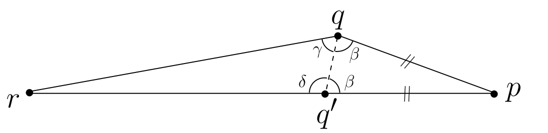

Fig. 5. Illustrating the proof of Claim 2.

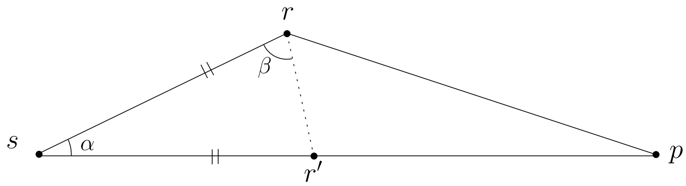

Fig. 6. Illustrating the proof of Claim 3.

Proof. Let q' be the point on $\{r, p\}$, such that $|qp| = |q'p|$ (see Fig. 5). Since $\angle(rqp) \geqslant \pi - \alpha > \pi/2$, then $|rp| > |qp|$, therefore such a point exists.

Since $|rp| = |rq'| + |q'p| = |rq'| + |qp|$, we need to show

$$k|qp| + d|rq| \leqslant k\big(\big|rq'\big| + |qp|\big),$$

which is equivalent to showing

$$d|rq| \leqslant k\big|rq'\big|.$$

Denote $\angle(pqq') = \angle(pq'q) = \beta$, $\angle(qq'r) = \delta$, and $\angle(q'qr) = \gamma$. Notice that $\beta < \frac{\pi}{2}$, therefore, $\gamma > \frac{\pi}{2} - \alpha$.

By the law of sines and since sin is increasing function in the interval $(0, \pi/2]$,

$$\frac{|rq|}{|rq'|} = \frac{\sin(\delta)}{\sin(\gamma)|} \leqslant \frac{1}{\sin(\frac{\pi}{2} - \alpha)} = \frac{1}{\cos\alpha}.$$

Thus, $|rq| \leqslant |rq'|\frac{1}{\cos\alpha}$, which finishes the proof. $\square$

Claim 3. *Let $\{s, r\}$ and $\{s, p\}$ be two edges in $\mathrm{DT}(P)$, such that $|sr| \leqslant |sp|$ and the angle between $\{s, r\}$ and $\{s, p\}$, $\angle(rsp) = \alpha < \pi/3$. Then,*

$$|sr| + K\big(d\big|rr'\big| + \big|r'p\big|\big) \leqslant K|sp|$$

for $K \geqslant \frac{1}{1-2\sin(\frac{\alpha}{2})}$, where r' is a point on $\{s, p\}$ such that $|sr'| = |sr|$.

Proof. Let $\angle(r'sr) = \alpha$ and $\angle(srr') = \beta = \frac{\pi-\alpha}{2}$ (see Fig. 6); then by the law of sines, $|rr'| = \frac{\sin(\alpha)}{\sin(\beta)}|sr'|$. Therefore,

$$\begin{aligned}
|sr| + K\big(\big|rr'\big| + \big|r'p\big|\big) &= |sr| + K\left(d\frac{\sin(\alpha)}{\sin(\beta)}\big|sr'\big| + \big|r'p\big|\right) \\
&= |sr| + K\left(d\frac{\sin(\alpha)}{\sin(\beta)}\big|sr'\big| + |sp| - \big|sr'\big|\right) \\
&= |sr| + K\left(\left(d\frac{\sin(\alpha)}{\sin(\beta)} - 1\right)\big|sr'\big| + |sp|\right) \\
&= |sr| + K\left(\left(d\frac{\sin(\alpha)}{\sin(\frac{\pi-\alpha}{2})} - 1\right)\big|sr'\big| + |sp|\right) \\
&= |sr| + K\left(\left(d\frac{\sin(\alpha)}{\cos(\alpha/2)} - 1\right)\big|sr'\big| + |sp|\right) \\
&= |sr| + K\big(2\sin(\alpha/2) - 1\big)\big|sr'\big| + |sp|) \\
&= |sr|\big(1 + K\big(2d\sin(\alpha/2) - 1\big)\big) + K|sp|) \\
&\leqslant^{(*)} K|sp|.
\end{aligned}$$

The last inequality $(*)$ follows from the fact that $(1 + K(2d\sin(\alpha/2) - 1))$ is less than zero for $K \geqslant \frac{1}{1-2\sin(\frac{\alpha}{2})}$. $\square$

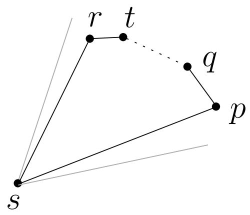

Fig. 7. Illustrating the proof of Lemma 4.5.

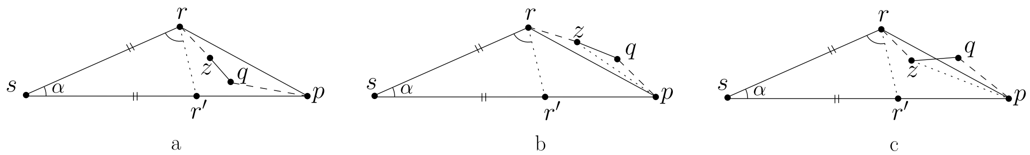

Fig. 8. Illustrating the proof of Claim 4.

The following lemma bounds the stretch factor of the resulting spanner of Algorithm 1 from above.

Lemma 4.5. *Let $k = (1+\sqrt{2})^2$, the resulting graph of Algorithm 1 is a t-spanner with $t = k \cdot \delta$, where δ is the stretch factor of Delaunay triangulation.*

Proof. Let $G = (P, E)$ be the output graph of Algorithm 1. To prove the lemma we show that for every edge $\{s, p\} \in \mathrm{DT}(P)$, $\delta_G(s, p) \leqslant (1+\sqrt{2})^2|sp|$. We prove the above by induction on the rank of the edge $\{s, p\}$, i.e., the place of the edge $\{s, p\}$ in a nondecreasing length order of the edges in $\mathrm{DT}(P)$.

Base case: Let $\{s, p\}$ be the shortest edge in $\mathrm{DT}(P)$. Then, edge $\{s, p\}$ has been added to E during the first iteration of the loop in step 6, and therefore $\delta_G(s, p) = |sp|$.

Induction hypothesis: Assume for every edge $\{r, q\} \in \mathrm{DT}(P)$ shorter than $\{s, p\}$, the lemma holds, i.e., $\delta_G(r, q) \leqslant (1+\sqrt{2})^2|rq|$.

The inductive step: If $\{s, p\} \in E$, we are done. Otherwise, w.l.o.g. assume $\{s, p\} \in C_s^i$ and $\{s, p\} \in C_p^j$; then, there exists either an edge $\{s, r\} \in C_s^i \cap E \backslash E^*$, such that $|sr| \leqslant |sp|$, or an edge $\{p, r\} \in C_p^j \cap E \backslash E^*$, such that $|pr| \leqslant |sp|$. Assume w.l.o.g. there exists an edge $\{s, r\} \in C_s^i \cap E \backslash E^*$, such that $|sr| \leqslant |sp|$. By Lemma 4.4, for every $x \in S_{s,r,p}$, $|sx| \geqslant \min\{|sr|, |sp|\} = |sr|$. Let $\{r, t\}$ be the first edge in $P_{S_{s,r,p}}(r, p)$, and $\{q, p\}$ the last. Note that all edges of $P_{S_{s,t,q}}$ have been added to E during Algorithm 2 (see Fig. 7).

Claim 4. *The edges in the path $P_{S_{s,r,p}}(r, p)$ are shorter than $\{s, p\}$.*

Proof. Let $\{z, q\}$ be an edge in $P_{S_{s,r,p}}(r, p)$. Note that since $\angle(rsp) \leqslant \frac{\pi}{4}$ and $|sr| \leqslant |sp|$, we get $|rp| < |sp|$ and $\{s, p\}$ is the longest edge in the triangle $\triangle(srp)$. If $\{z, q\}$ is bounded inside the triangle $\triangle(srp)$ (see Fig. 8a), then it is necessarily inside the $\triangle(prr')$, where r' is a point on $\{s, p\}$, such that $|sr'| = |sr|$ (since $|sr| \leqslant |sp|$, such a point exists), and therefore $\{z, q\}$ is shorter than $\{s, p\}$.

If $\{z, q\}$ is outside the triangle $\triangle(srp)$, assume q is closer to p than z (see Fig. 8b). Either $\{r, q\}$ or $\{z, p\}$ is inside the polygon $(rzqp)$. Assume w.l.o.g. that $\{z, p\}$ is inside the polygon $(rzqp)$. By Observation 4 the angle $\angle(zqp) \geqslant \pi - \angle(zsp) \geqslant \frac{3\pi}{4}$, and therefore, $\{z, q\}$ is shorter than $\{z, p\}$ and by the same argument $\{z, p\}$ is shorter than $\{r, p\}$. Otherwise, assume w.l.o.g. that z is inside the triangle $\triangle(prr')$ and q is outside the triangle $\triangle(srp)$ (see Fig. 8c). By Observation 4 we get $|zq| < |zp|$ as before. Since z is inside the triangle $\triangle(prr')$ the angle $\angle(rzp)$ facing towards q is greater than $\angle(rr'p)$ that is greater than $\pi/2$ and therefore $|zp| < |rp|$. Since $|rp| < |sp|$ we get $|zq| < |sp|$. $\square$

Applying the induction hypothesis on $\{r, t\}$ and $\{q, p\}$ results in:

$$\delta_G(r, t) \leqslant (1+\sqrt{2})^2|rt|, \tag{3}$$

$$\delta_G(p, q) \leqslant (1+\sqrt{2})^2|pq|. \tag{4}$$

Let $\{s, a\}$ be the shortest edge in the wedge $W_{s,t,q}$ (i.e., the closest point to s in $S_{s,r,p} \backslash \{r, p\}$). By Corollary 4.3,

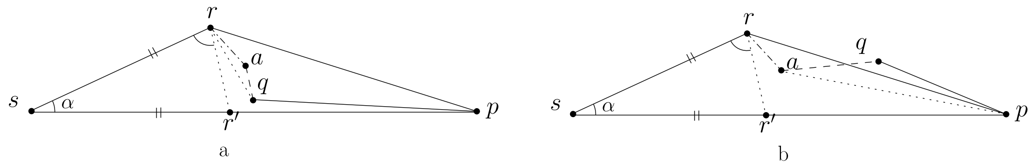

Fig. 9. Illustrating the proof of Lemma 4.5, Case 1.2. In figure (a) point a violates the convexity of polygon $(raqp)$, while in figure (b) point q violates the convexity of the polygon.

$$\delta_{S_{s,t,a}}(t,a) \leqslant \frac{\pi}{2}|ta|, \tag{5}$$

$$\delta_{S_{s,a,q}}(a,q) \leqslant \frac{\pi}{2}|aq|. \tag{6}$$

Since $|sr| \leqslant |st|$ and $|sa| \leqslant |st|$, the angle $\angle(rta)$ facing towards s is less than π. By Observation 4 we get that $\angle(rta) \geqslant \frac{3\pi}{4}$. Applying Claim 2 on the triangle $\triangle(rta)$, with $d = \frac{\pi}{2}$ gives us

$$(1+\sqrt{2})^2|rt| + \frac{\pi}{2}|ta| \leqslant (1+\sqrt{2})^2|ra|. \tag{7}$$

Therefore,

$$\begin{aligned}
\delta_G(s,p) &\leqslant |sr| + \delta_G(r,p) \\
&\leqslant |sr| + \delta_G(r,t) + \delta_G(t,q) + \delta_G(q,p) \\
&\leqslant^{(3),(4)} |sr| + (1+\sqrt{2})^2\big(|rt| + |pq|\big) + \delta_G(t,q) \\
&\leqslant |sr| + (1+\sqrt{2})^2\big(|rt| + |pq|\big) + \delta_{S_{s,t,a}}(t,a) + \delta_{S_{s,a,q}}(a,q) \\
&\leqslant^{(5),(6)} |sr| + (1+\sqrt{2})^2\big(|rt| + |pq|\big) + \frac{\pi}{2}\big(|ta| + |qa|\big) \\
&\leqslant^{(7)} |sr| + (1+\sqrt{2})^2\big(|ra| + |pq|\big) + \frac{\pi}{2}\big(|aq|\big).
\end{aligned}$$

There are two cases regarding the location of points q and a:

- **Case 1:** Either point q or a lies inside the triangle $\triangle(srp)$.
  Let r' be a point on $\{s,p\}$ such that $|sr| = |sr'|$. Notice $|sr| \leqslant |sp|$, and therefore, such a point exists. Since $|sr| \leqslant |sa|$ and $|sr| \leqslant |sq|$, q and t lie outside the disk centered at s and with radius $|sr|$. Therefore, either point q or point a is located inside the triangle $\triangle(rr'p)$.
  Since $(1+\sqrt{2})^2 > \frac{1}{1-2\sin(\pi/8)}$, by Claim 3 with $d=1$ we get,

  $$|sr| + (1+\sqrt{2})^2\big(\left|rr'\right| + \left|r'p\right|\big) \leqslant (1+\sqrt{2})^2|sp|.$$

  Therefore, it is enough to show that

  $$(1+\sqrt{2})^2\big(|ra| + |pq|\big) + \frac{\pi}{2}\big(|aq|\big) \leqslant (1+\sqrt{2})^2\big(\left|rr'\right| + \left|r'p\right|\big).$$

  Observe the following two cases regarding the convexity of the polygon $(raqp)$:
  - **Case 1.1:** The polygon $(raqp)$ is convex.
    Since $\frac{\pi}{2} < (1+\sqrt{2})^2$, we get

    $$\begin{aligned}
    (1+\sqrt{2})^2\big(|ra| + |qp|\big) + \frac{\pi}{2}\big(|aq|\big) &< (1+\sqrt{2})^2\big(|ra| + |qp| + |aq|\big) \\
    &\leqslant (1+\sqrt{2})^2\big(\left|rr'\right| + \left|r'p\right|\big).
    \end{aligned}$$

    The last inequality follows from the convexity of the polygon $(raqp)$.
  - **Case 1.2:** The polygon $(raqp)$ is not convex.
    The vertex that violates the convexity is either a or q, and the other vertex is inside triangle $\triangle(srp)$ (see Fig. 9). Assume w.l.o.g. that a is the vertex that violates the convexity; then the angle $\angle(raq)$ facing towards s is less than π. By Observation 4 $\angle(raq) \geqslant \frac{3\pi}{4}$; therefore, applying Claim 2 on the triangle $\triangle(raq)$ with $d = \frac{\pi}{2}$ gives us

    $$(1+\sqrt{2})^2|ra| + \frac{\pi}{2}|aq| \leqslant (1+\sqrt{2})^2|rq|. \tag{8}$$

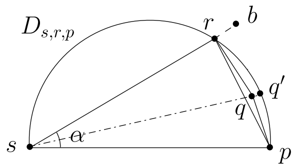

Fig. 10. Illustrating the proof of Lemma 4.5, Case 2, when $\angle(srp) < \frac{\pi}{2}$.

Thus, we get

$$(1+\sqrt{2})^2\big(|ra|+|qp|\big)+\frac{\pi}{2}(|aq|) \leqslant^{(8)} (1+\sqrt{2})^2\big(|rq|+|qp|\big) \leqslant^{(*)} (1+\sqrt{2})^2\big(|rr'|+|r'p|\big).$$

The last inequality $(*)$ follows from the convexity of triangle $\triangle(rqp)$.

- **Case 2:** Both points q and a lie outside the triangle $\triangle(srp)$.
  In this case, the edges added to E depend on the angle $\angle(srt)$. There are two cases:
  - **Case 2.1:** Angle $\angle(srp) \geqslant \frac{\pi}{2}$.
    In this case, $\angle(srt) \geqslant \angle(srp) \geqslant \frac{\pi}{2}$, and therefore, the algorithm adds the edge $\{r,t\}$ to E. Thus, instead of showing

    $$|sr|+(1+\sqrt{2})^2\big(|ra|+|qp|\big)+\delta_{S_{s,a,q}}(a,q) \leqslant (1+\sqrt{2})^2|sp|,$$

    it is enough to show

    $$|sr|+(1+\sqrt{2})^2|qp|+\delta_{S_{s,r,q}}(r,q) \leqslant (1+\sqrt{2})^2|sp|.$$

    By Corollary 4.3,

    $$\delta_{S_{s,r,q}}(r,q) \leqslant \frac{\pi}{2}|rq|. \tag{9}$$

    By Observation 4, $\angle(rqp) \geqslant \frac{3\pi}{4}$, and by applying Claim 2 on the triangle $\triangle(rqp)$ with $d=\frac{\pi}{2}$ we get

    $$(1+\sqrt{2})^2|qp|+\frac{\pi}{2}|rq| \leqslant (1+\sqrt{2})^2|rp|. \tag{10}$$

    Thus,

    $$\begin{aligned}|sr|+(1+\sqrt{2})^2|qp|+\delta_{S_{s,r,q}}(r,q) &\leqslant^{(9)} |sr|+(1+\sqrt{2})^2|qp|+\frac{\pi}{2}|rq| \\ &\leqslant^{(10)} |sr|+(1+\sqrt{2})^2|rp| \\ &\leqslant^{(**)} |sr|+(1+\sqrt{2})^2\big(|rr'|+|pr'|\big) \\ &\leqslant^{(***)} (1+\sqrt{2})^2|sp|.\end{aligned}$$

    Inequality $(**)$ follows from triangle inequality for any point r', and thus it also holds for a point r' on $\{s,p\}$, such that $|sr|=|sr'|$. Therefore, by Claim 3 with $d=1$ and since $(1+\sqrt{2})^2 > \frac{1}{1-2\sin(\pi/8)}$, inequality $(***)$ follows.
  - **Case 2.2:** Angle $\angle(srp) < \frac{\pi}{2}$.
    Since a and q are outside $\triangle(rps)$, either angle $\angle(raq)$ or angle $\angle(aqp)$ (facing towards s) is less than π. Assume w.l.o.g. that $\angle(raq) < \pi$, thus, by Observation 4, $\angle(raq) \geqslant \frac{3\pi}{4}$. Applying Claim 2 on triangle $\triangle(raq)$ with $d=\frac{\pi}{2}$ gives us

    $$(1+\sqrt{2})^2|ra|+\frac{\pi}{2}|aq| \leqslant (1+\sqrt{2})^2|rq|. \tag{11}$$

    Let q' be a point on the intersection of disk $D_{s,r,p}$ and the extension of $\{s,q\}$ (see Fig. 10). Then, by convexity we get

    $$|rq|+|pq| \leqslant |rq'|+|pq'| \leqslant^{(*)} \frac{|pr|}{\cos(\alpha/2)}. \tag{12}$$

    The last inequality $(*)$ is obtained by Claim 5.
    Let b be a point on the extension of $\{s,r\}$, such that $|sb|=|sp|$; therefore, $\angle(sbp)=\angle(spb)=\frac{\pi}{2}-\frac{\alpha}{2}$.
    By the law of sines,

    $$|pb| = |sp|\frac{\sin(\alpha)}{\sin(\frac{\pi}{2}-\frac{\alpha}{2})} = |sp|\frac{\sin(\alpha)}{\cos(\frac{\alpha}{2})} = 2|sp|\sin(\alpha/2). \tag{13}$$

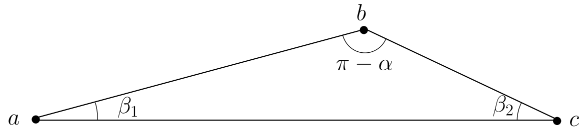

Fig. 11. Illustrating the proof of Claim 5.

Since angle $\angle(srp) < \pi/2$, it follows that angle $\angle(brp) > \pi/2$, thus,

$$|pb| > |pr|. \tag{14}$$

Now we are ready to bound the length of the path:

$$\begin{aligned}
|sr| + \delta_G(r,p) &\leqslant |sr| + (1+\sqrt{2})^2\big(|ra| + |pq|\big) + \frac{\pi}{2}|qa| \\
&\leqslant^{(11)} |sr| + (1+\sqrt{2})^2\big(|rq| + |pq|\big) \\
&\leqslant^{(12)} |sr| + (1+\sqrt{2})^2 \frac{|pr|}{\cos(\alpha/2)} \\
&\leqslant^{(14)} |sr| + (1+\sqrt{2})^2 \frac{|pb|}{\cos(\alpha/2)} \\
&\leqslant^{(13)} |sb| + (1+\sqrt{2})^2 \frac{2|sp|\sin(\alpha/2)}{\cos(\alpha/2)} \\
&= |sp|\big(1 + (1+\sqrt{2})^2\big(2\tan(\alpha/2)\big)\big) \\
&\leqslant^{(**)} |sp|\big(1 + (1+\sqrt{2})^2\big(2\tan(\pi/8)\big)\big) \\
&= |sp|\big(1 + 2(1+\sqrt{2})(1+\sqrt{2})(\sqrt{2}-1)\big) \\
&= |sp|\big(1 + 2(1+\sqrt{2})\big) \\
&= |sp|(1+\sqrt{2})^2.
\end{aligned}$$

The last inequality $(**)$ follows from the fact that *tangent* is a monotone increasing function in the range $(0, \pi/4]$. □

Claim 5. *Let a, b, and c be three points on a circle, such that $\angle(abc) = \pi - \alpha$. Then, $|ab| + |bc| \leqslant \frac{|ac|}{\cos(\frac{\alpha}{2})}$.*

Proof. Let β_1 be the angle between ba and ca, and let β_2 be the angle between bc and ca, as depicted in Fig. 11. By the law of Sines we have

$$\frac{|ac|}{\sin(\pi - \alpha)} = \frac{|bc|}{\sin(\beta_1)} = \frac{|ab|}{\sin(\beta_2)}.$$

Therefore,

$$|ab| + |bc| = |ac|\frac{\sin(\beta_2)}{\sin(\pi - \alpha)} + |ac|\frac{\sin(\beta_1)}{\sin(\pi - \alpha)} = \frac{|ac|}{\sin(\pi - \alpha)}\big(\sin(\beta_2) + \sin(\beta_1)\big).$$

For $0 \leqslant \alpha \leqslant \pi/2$, this function is maximized when $\beta_1 = \beta_2 = \frac{\alpha}{2}$. Thus,

$$\begin{aligned}
|ab| + |bc| &\leqslant 2|ac|\frac{\sin(\frac{\alpha}{2})}{\sin(\pi - \alpha)} \\
&= 2|ac| \Big/ \frac{\sin(\alpha)}{\sin(\frac{\alpha}{2})} \\
&= \frac{2|ac|}{2\cos(\frac{\alpha}{2})} \\
&= \frac{|ac|}{\cos(\frac{\alpha}{2})}. \qquad \square
\end{aligned}$$

Claim 6. *The resulting t-spanner of Algorithm 1, $G = (P, E)$ is a strong t-spanner.*

Algorithm 3 *Wedge*(p, q_i)

Input: Two points p and q_i such that the edge $\{p, q_i\} \in \mathrm{DT}(P)$
Output: A set of edges E^* to be added to the spanner $G = (P, E)$

for every C_p^z that contains $\{p, q_i\}$ **do**
  Let $Q = \{q_n\colon \{p, q_n\} \in C_p^z \cap DT(P)\} = \{q_j, \dots, q_k\}$
  Let $Q' = \{q_n\colon \angle(q_{n-1}q_nq_{n+1}) < \frac{6\pi}{7}, q_n \in Q \setminus \{q_j, q_i, q_k\}\}$
  $E^* \leftarrow E^* \cup \{\{q_n, q_{n+1}\} \mid q_n, q_{n+1} \notin Q'$ and $n \in [j+1, i-2] \cup [i+1, k-2]\}$
  W.l.o.g. the points of Q' lie between q_i and q_k
  (* the symmetric case is handled analogously *)
  if $(\angle(pq_iq_{i-1}) > 4\pi/7)$ and $(i, i-1 \neq j)$ **then**
    $E^* \leftarrow E^* \cup \{\{q_i, q_{i-1}\}\}$
  Let q_f be the first point in Q'
  Let $a = \min\{n \mid n > f$ and $q_n \in Q \setminus Q'\}$
  if $f = i+1$ **then**
    if $(\angle(pq_iq_{i+1}) \leqslant 4\pi/7)$ and $(a \neq k)$ **then**
      $E^* \leftarrow E^* \cup \{\{q_f, q_a\}\}$
    if $(\angle(pq_iq_{i+1}) > 4\pi/7)$ and $(f+1 \neq k)$ **then**
      $E^* \leftarrow E^* \cup \{\{q_i, q_{f+1}\}\}$
  else
    Let q_l be the last point in Q'
    Let $b = \max\{n \mid n < l$ and $q_n \in Q \setminus Q'\}$
    if $l = k-1$ **then**
      $E^* \leftarrow E^* \cup \{\{q_l, q_b\}\}$
    else
      $E^* \leftarrow E^* \cup \{\{q_b, q_{l+1}\}\}$
      if $q_{l-1} \in Q'$ **then**
        $E^* \leftarrow E^* \cup \{\{q_l, q_{l-1}\}\}$

Proof. Let $\{r, q\}$ be an edge in G. Since $\mathrm{DT}(P)$ is a strong t-spanner of P, it is enough to show that for each $\{s, p\} \in \mathrm{DT}(P)$ there is a spanning path consisting of edges shorter than $|sp|$ and the rest follows inductively.

For $\{s, p\} \in \mathrm{DT}(P)$ consider the path $\{s, r\} \cdot P_{S_{s,r,p}}(r, p)$ from s to p as presented in the proof of Lemma 4.5. Except for the first and last edges in $P_{S_{s,r,p}}(r, p)$ all of the rest are in E. Note $|sr| \leqslant |sp|$ by definition. By Claim 4 all the edges in $P_{S_{s,r,p}}(r, p)$ are shorter than $\{s, p\}$, and by induction on the edges lengths, the path in G connecting the endpoints of the first and last edges is shorter than $\{s, p\}$. □

Theorem 4.6. *For every set of points P, there is a strong planar t-spanner which is subgraph of* $\mathrm{DT}(P)$ *with $t = (1 + \sqrt{2})^2 \cdot \delta$, where δ is the stretch factor of Delaunay triangulation with bounded degree* 7.

5. Reducing to degree 6

In this section we show how to further reduce the degree bound to 6 at the expense of increasing the spanning ratio and no longer being a subgraph of the Delaunay triangulation. The idea is to use 7 cones of degree $2\pi/7$ as opposed to 8 cones of angle $\pi/4$. One can see that all the claims and lemmas concerning the spanning ratio are easily adjusted to this new division. However, there is a problem in the analysis of the degree bound, since for edges added in Algorithm 2, we charge the addition of these edges to empty cones. For example, consider a call to *wedge*(p, q_i) and assume we add edges $\{q_{j-1}, q_j\}$ and $\{q_j, q_{j+1}\}$ to E^*. For our charging argument to go through, we need to show the existence of two empty cones of angle $2\pi/7$. This requires that angle $\angle(q_{j-1}q_jq_{j+1}) \geqslant 6\pi/7$, but from the empty circle property we can only show that this angle is at least $5\pi/7$. Fortunately, there can be at most two such points (in a cone of angle $2\pi/7$) that have angle less than $6\pi/7$, moreover, these two points are consecutive (Observation 6). Therefore, we change Algorithm 2 to handle this case by adding only one edge to these points. However, adding such edges affects the stretch factor, thus, we add an extra edge (as a bridge) to bypass the gaps. However, we now need to show that the resulting graph is planar. Let Algorithm BD6 denote the algorithm based on Algorithm 1 obtained by using cones of degree $2\pi/7$ instead of $\pi/4$ and replacing Algorithm 2 with Algorithm 3. See Algorithm 3, the new subroutine of *wedge*(), and Fig. 12 for illustration.

Observation 6. *Let $Q = \{q_n\colon \{p, q_n\} \in C_p^z \cap \mathrm{DT}(P)\}$ and let $Q' = \{q_n\colon \angle(q_{n-1}q_nq_{n+1}) < \frac{6\pi}{7}, q_n \in Q \setminus \{q_j, q_i, q_k\}\}$ then:*

1. $|Q'| \leqslant 2$,
2. *if $|Q'| = 2$ and $Q' = \{x, y\}$, then x and y are consecutive, and*
3. *the points of Q' lie on the same side of the short edge in the cone.*

Proof. Assume to the contrary that $|Q'| > 2$ or $Q' = \{x, y\}$ and x and y are not consecutive. By Observation 4 the cone C_p^z is of degree greater than $2(\pi - \frac{6\pi}{7}) = \frac{2\pi}{7}$, in contradiction to the definition of $\frac{2\pi}{7}$ degree cones. Statement 3 follows from statement 2. □

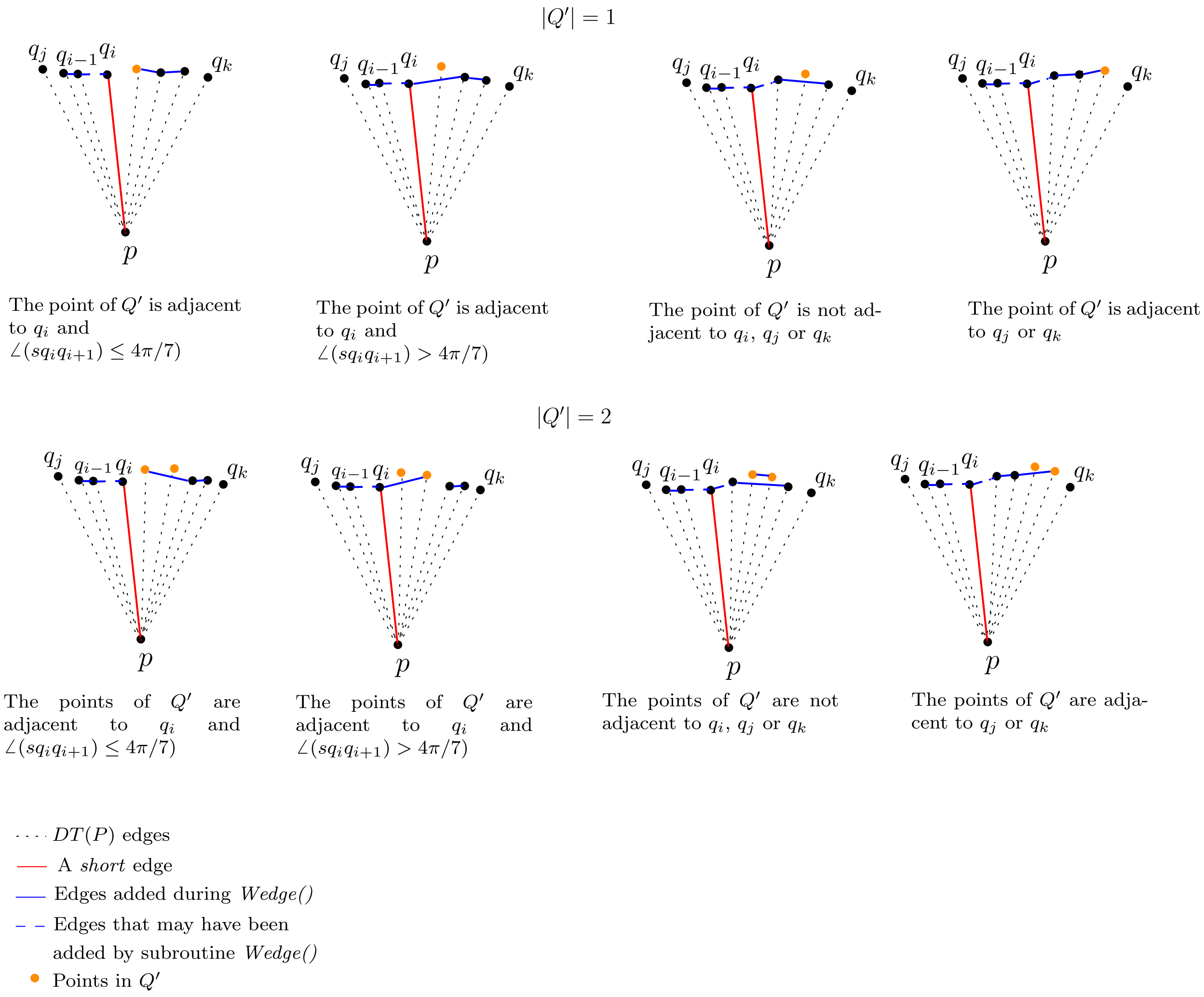

Fig. 12. Examples of Algorithm BD6 executions.

Notation 5.1. Let *bypass* edge denote an edge added during Algorithm 3 that is not in DT(P).

In the following subsections we prove the degree bound, spanning ratio, and planarity of the resulting graph.

5.1. Bounded degree

Observation 7. *The property in Claim 1 holds for the edges in* DT(P) *chosen by Algorithm 3.*

Lemma 5.2. *The degree of spanner G constructed by Algorithm* BD6 *is bounded by* 6.

Proof. Observe Lemma 3.1 regarding the bounded degree 7. Applying the same arguments of Algorithm 1 on cones of degree $2\pi/7$ we get that the edges added to $E \backslash E^*$ (during Algorithm 1 **not** including the edges added during Algorithm 3) contribute at most 6 to the degree of every $p \in P$. What remains to be shown is that the edges added during Algorithm 3 can be charged uniquely to empty cones, thus do not increase the degree bound of 6. Let p be a point whose degree has been increased during Algorithm 3. Note that p is neither the first nor the last point in the wedge.

- **Case 1:** $p = q_i$.
  In this case the edges $\{q_i, q_{i+1}\}$ is added during Algorithm 3 only if $\angle(sqq_{i+1}) > 4\pi/7$ and $\{q_i, q_{i-1}\}$ is added only if $\angle(sqq_{i-1}) > 4\pi/7$. Meaning, both edges can be charged to empty cones in the corresponding wedges.

- **Case 2:** $p = q_m \in Q'$.
  In this case the degree of p is increased only by one during Algorithm 3. By Observation 4, $\angle(q_{m-1}, q_m, q_{m+1}) \geqslant 5\pi/7 \geqslant 4\pi/7$ and therefore there is an empty cone in the wedge between $\{q_{m-1}, q_m\}$ and $\{q_m, q_{m+1}\}$ to whom the new edge can be charged.
- **Case 3:** $p = q_n \notin Q'$.

Therefore, the degree of every point $p \in P$ is bounded by 6. □

5.2. Spanning ratio

Lemma 5.3. *Let $k = \frac{1}{(1-\tan(\pi/7)(1+1/\cos(\pi/14)))}$, then resulting graph of Algorithm* BD6 *is a t-spanner with $t = k \cdot \delta$, where δ is the stretch factor of Delaunay triangulation.*

Proof. To prove the lemma we show that for every edge $\{s, p\} \in DT(P)$, $\delta_G(s, p) \leqslant k|sp|$, where G is the resulting graph of Algorithm BD6. Observe the proof of Lemma 4.5. We use the same proof method; the only change is in the inductive step. If $\{s, p\} \in E$, we are done. Otherwise, w.l.o.g. assume $\{s, p\} \in C_s^i$ and $\{s, p\} \in C_p^j$; then, there exists either an edge $\{s, r\} \in C_s^i \cap E \backslash E^*$, such that $|sr| \leqslant |sp|$, or an edge $\{p, r\} \in C_p^j \cap E \backslash E^*$, such that $|pr| \leqslant |sp|$. Assume w.l.o.g. there exists an edge $\{s, r\} \in C_s^i \cap E \backslash E^*$, such that $|sr| \leqslant |sp|$. By Lemma 4.4, for every $x \in S_{s,r,p}$, $|sx| \geqslant \min\{|sr|, |sp|\} = |sr|$. Let $\{r, t\}$ be the first edge in $P_{S_{s,r,p}}(r, p)$, and $\{q, p\}$ the last. Note that except for one case, all edges of $P_{S_{s,t,q}}$ have been added or bypassed by a *bypass* edge during Algorithm 2. Let $\alpha \leqslant 2\pi/7$, by applying the induction hypothesis on $\{r, t\}$ and $\{q, p\}$ and Corollary 4.3 on $P_{S_{s,t,q}}$, we get

$$\delta_G(s, p) \leqslant |sr| + k\big(|rt| + |qp|\big) + \frac{\alpha}{\sin(\alpha)}\sqrt{2}|tq|.$$

The exception case occurs when $p \notin Q'$, $\angle(srt) > 4\pi/7$ and $Q' = \{t, z\}$ where t, z are the consecutive points to r in the neighborhood of s in $DT(P)$ (see Fig. 12, bottom row, the second Wedge on the left). Let u be the consecutive point to z, in this case $\{r, z\} \in E^*$, but $\{z, u\} \notin E^*$. By the induction hypothesis we get, $\delta_G(z, u) \leqslant k|zu|$. Since $\angle(rtz) < 6\pi/7$ and $\angle(tzu) < 6\pi/7$, $\angle(rzu)$ facing towards s is less than π and by Observation 4, $\angle(rzu) \geqslant 5\pi/7$. Thus, by Claim 2, $|rz| + k|zu| \leqslant k|ru|$. Similarly to the other cases, we get

$$\delta_G(s, p) \leqslant |sr| + k\big(|ru| + |qp|\big) + \frac{\alpha}{\sin(\alpha)}\sqrt{2}|uq|.$$

Let $\{s, a\}$ be the shortest edge in the wedge $W_{s,t,q}$ (alternatively the wedge $W_{s,u,q}$ in the exception case). By the same ideas as in the proof of Lemma 4.5, we get similarly,

$$\delta_G(s, p) \leqslant |sr| + k\big(|ra| + |pq|\big) + \frac{\alpha}{\sin(\alpha)}\sqrt{2}|aq|.$$

Observe the cases presented in the proof of Lemma 4.5. The arguments of Case 1 in the proof of Lemma 4.5 are still valid for Algorithm BD6 and $k = \frac{1}{1-\tan(\pi/7)(1+1/\cos(\pi/14))}$ instead of $(1+\sqrt{2})^2$. In Case 2 of the proof, if angle $\angle(srp) > \frac{4\pi}{7}$ then the arguments of Case 2.1 hold and if $\angle(srp) < \frac{\pi}{2}$ the arguments of Case 2.2 hold. Otherwise, $\frac{\pi}{2} \leqslant \angle(srp) \leqslant \frac{4\pi}{7}$ and the arguments of Case 2.2 hold except for Eqs. (13) and (14) that can be replaced with the following one. For $\frac{\pi}{2} \leqslant \angle(srp) \leqslant \frac{4\pi}{7}$, $\sin(\angle(srp)) \geqslant \sin(3\pi/7) \geqslant \sin(2.5\pi/7)$ and thus,

$$|pr| = \frac{\sin(\alpha)}{\sin(\angle(srp))}|sp| \leqslant \frac{\sin(2\pi/7)}{\sin(2.5\pi/7)}|sp| = 2\sin(\pi/7)|sp|. \qquad \square \tag{15}$$

5.3. Planarity

Observation 8. *Let $\{p, q\}$ be a* bypass *edge added during the call wedge(z, r), such that $\{z, r\} \in C_z^i$. Then:*

1. *At most one* bypass *edge is added in each cone C_p^z during Algorithm 3.*
2. *The triangle $\triangle(pzq)$ is empty of points in P.*
3. *$\{z, q\}$ and $\{z, p\}$ are edges in C_z^i, but neither the first nor the last edges is in the cone.*

Lemma 5.4. *The resulting t-spanner of Algorithm* BD6 *is planar.*

Proof. Note that the only edges that may harm the planarity of the resulting t-spanner are the *bypass* edges. Let $\{p, q\}$ be such an edge, chosen during a call to *wedge*(z, r), such that $\{z, r\} \in C_z^i$. Note there are two types of edges edge $\{p, q\}$ may cross:

1. Edges of DT(P).
   An edge in DT(P) is chosen by Algorithm BD6 either as a *short* edge or during Algorithm 3. According to Algorithm 3 the edges of DT(P) that cross $\{p, q\}$ are not the edges that have been chosen as *short* edges. Moreover, since $\{z, q\}$ and $\{z, p\}$ are neither the first nor the last edges in C_z^i, the edges of DT(P) that cross $\{p, q\}$ are neither the external nor adjacent to the external edges in C_z^i. By Observation 7 those edges are not chosen among the edges of DT(P) in Algorithm 3.
2. *Bypass* edges.
   By Observation 8, each *bypass* edge should have all the three properties. However, two such edges cannot cross each other. □

Observation 9. *The resulting t-spanner of Algorithm* BD6*, is a strong t-spanner.*

Theorem 5.5. *For every set of points P, there is a strong degree* 6 *planar t-spanner with $t = \frac{1}{(1-\tan(\pi/7)(1+1/\cos(\pi/14)))} \cdot \delta$, where δ is the stretch factor of Delaunay triangulation.*

6. Conclusion

We have shown how to construct a spanning subgraph of the Euclidean Delaunay triangulation that is a strong plane constant spanner with maximum degree 7. In addition, we have shown that a similar technique can yield a strong plane constant spanner that has maximum degree 6 but that is no longer a subgraph of the Euclidean Delaunay triangulation. This investigation naturally leads to the following two open problems: What is the smallest maximum degree that can be achieved for plane spanners that are subgraphs of the Delaunay triangulation? What is the smallest maximum degree that can be achieved for plane spanners?

Finally, we conclude with an intriguing open question. Does planarity actually affect the degree bound? Specifically, it is known that one can always build a constant spanner with maximum degree 3 that is not necessarily planar [8]. It is easy to see that there exist point sets such that every graph of maximum degree 2 defined on that point set has unbounded spanning ratio. As such, the question is whether the planarity constraint actually imposes a higher lower bound on the maximum degree. Thus, we have the following open problem:

Is there a lower bound on the maximum degree that is greater than 3 when one requires the spanner be planar? That is, can we show the following: For every real number $t > 1$, there exists a set P of points, such that every plane degree-3 spanning graph of P has spanning ratio greater than t.

References

[1] N. Bonichon, C. Gavoille, N. Hanusse, D. Ilcinkas, Connections between theta-graphs, Delaunay triangulations, and orthogonal surfaces, in: WG, 2010.
[2] N. Bonichon, C. Gavoille, N. Hanusse, L. Perkovic, Plane spanners of maximum degree 6, in: ICALP, 2010.
[3] P. Bose, L. Devroye, M. Loffler, J. Snoeyink, V. Verma, The spanning ratio of the Delaunay triangulation is greater than $\pi/2$, Comp. Geom.: Theory Appl. 44 (2) (2011) 121–127.
[4] P. Bose, J. Gudmundsson, M.H.M. Smid, Constructing plane spanners of bounded degree and low weight, in: ESA, 2002, pp. 234–246.
[5] P. Bose, A. Maheshwari, G. Narasimhan, M.H.M. Smid, N. Zeh, Approximating geometric bottleneck shortest paths, Comp. Geom.: Theory Appl. 29 (3) (2004) 233–249.
[6] P. Bose, M.H.M. Smid, On plane geometric spanners: A survey and open problems, 2009, submitted for publication.
[7] P. Bose, M.H.M. Smid, D. Xu, Delaunay and diamond triangulations contain spanners of bounded degree, Internat. J. Comput. Geom. Appl. 19 (2) (2009) 119–140.
[8] G. Das, P.J. Heffernan, Constructing degree-3 spanners with other sparseness properties, Internat. J. Found. Comput. Sci. 7 (2) (1996) 121–136.
[9] I.A. Kanj, L. Perkovic, G. Xia, On spanners and lightweight spanners of geometric graphs, SIAM J. Comput. 39 (6) (2010), pp. 2132–2161.
[10] I.A. Kanj, G. Xia, Improved local algorithms for spanner construction, in: ALGOSENSORS, 2010, pp. 1–15.
[11] J.M. Keil, C.A. Gutwin, Classes of graphs which approximate the complete Euclidean graph, Discrete Comput. Geom. (1992) 13–28.
[12] X.-Y. Li, Y. Wang, Efficient construction of low weighted bounded degree planar spanner, Internat. J. Comput. Geom. Appl. 14 (1–2) (2004) 69–84.
[13] G. Narasimhan, M. Smid, Geometric Spanner Networks, Cambridge University Press, New York, NY, USA, 2007.
[14] G. Xia, Improved upper bound on the stretch factor of Delaunay triangulations, in: SOCG, 2011.
[15] G. Xia, L. Zhang, L. College, Improved lower bound for the stretch factor of Delaunay triangulations, in: FWCG, 2010.