Geometric Spanning Trees and Hypergraphs that Minimizes the Wiener Index

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Wiener Index in Graphs

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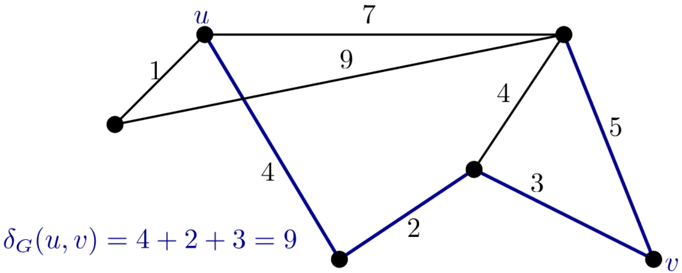

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2. Goal: Construct a **spanning tree** on P that minimizes the **Wiener index**.
3. The weight function is defined as the Euclidean distance between points.

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- Spanning tree of P that minimizes the Wiener index is planar.
- When P is in convex position, this can be solved in polynomial time.
- The hamiltonian path of P that minimizes the Wiener index is not necessarily planar.
- Computing such a hamiltonian path is NP-Hard.

Minimum Spanning Tree (MST)

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A **minimum spanning tree** of a weighted undirected graph is a spanning tree that has the minimum total edge weight.

- The MST connects all vertices with the minimum total edge weight.
- It is unique if all edge weights are distinct.

Euclidean Minimum Spanning Tree

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The **Euclidean minimum spanning tree** (EMST) of a set of points in the Euclidean space is the minimum spanning tree where the edge weights are the Euclidean distances between points.

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Note: It can be shown that the EMST is a subgraph of special geometric graphs called **Delaunay triangulations** which are a dual of **Voronoi Diagrams** [de Berg et al., 2000].

Euclidean Minimum Spanning Tree VS Wiener Index

Question: Is the EMST a good approximation for the spanning tree that minimizes the Wiener index?

- **Answer:** No, see the code examples.

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Aside: The EMST **could** be a good approximation or a heuristic for the spanning tree that minimizes the Wiener index. This may be worth exploring if we go the route of finding an approximation algorithm for the problem.

Hypergraphs and their Wiener Index

Hypergraph

A **hypergraph** $H = (V, E)$ is a pair where V is a set of vertices and E is a set of hyperedges, each of which is a subset of V.

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Distance in Hypergraphs

The **distance** between two vertices u and v in a hypergraph H is defined as the minimum number of hyperedges in a chain connecting u and v.

- A **chain** is a sequence of hyperedges where each consecutive pair shares at least one vertex.
- The distance is denoted as $\delta_H(u, v)$.

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k-uniform hypergraph

A hypergraph is **k-uniform** if every hyperedge has cardinality k.

Wiener Index in Hypergraphs

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The **Wiener index** of a hypergraph H is defined as:

$$W(H) = \sum_{u,v \in V} \delta_H(u, v)$$

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- Finding a spanning tree of general hypergraphs is NP-Complete, while spanning tree of 3-uniform hypergraphs has a polynomial time algorithm using matroid matching. [Goodall and de Mier, 2011]
- Finding a spanning tree of a k-uniform 2-regular hypergraph is NP-Complete for any $k \geq 4$. [Demaine and Rudoy, 2018]

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Side note: We could also explore **spatially embedded hypergraphs**, drawing inspiration from [Abu-Affash et al., 2023], and their Wiener index.

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The end.
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