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Minimum weight Euclidean t-spanner is NP-hard ☆

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ABSTRACT

Given a set P of points in the plane, an Euclidean t-spanner for P is a geometric graph that preserves the Euclidean distances between every pair of points in P up to a constant factor t. The weight of a geometric graph refers to the total length of its edges. In this paper we show that the problem of deciding whether there exists an Euclidean t-spanner, for a given set of points in the plane, of weight at most w is NP-hard for every real constant $t > 1$, both whether planarity of the t-spanner is required or not.

1. Introduction

Consider a weighted graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}^+$ over the edges. For any two vertices $u, v \in V$, we denote the weight of the shortest path between u and v in G by $\delta_G(u, v)$. Given a spanning subgraph G' of G, we define the *dilation* of G' (with respect to G) to be the value

$$\max_{u,v\in V} \frac{\delta_{G'}(u, v)}{\delta_G(u, v)}.$$

Given a real value $t > 1$, a t-spanner of G is a spanning subgraph G' with dilation at most t with respect to G. Thus, the shortest-path distances in G' approximate the shortest-path distances in the underlying graph G within an approximation ratio t. Typically, G is a dense graph with $\Omega(n^2)$ edges and the t-spanner G' is desired to be sparse, preferably having only a linear number of edges.

Spanners have been studied in many different settings. The various settings differ from one another in the characterization of the underlying graph G, such as different topologies and weight functions over the edges, in the value of the dilation t, and in the required properties of the spanner G', such as planarity. We concentrate on the setting where the underlying graph is geometric. In our context, a graph $G = (P, E)$ is called geometric graph or Euclidean graph if its vertex set P is a set of points in the plane and every edge $\{p, q\} \in E$ is the line segment $\overline{pq}$, weighted by the Euclidean distance $|pq|$ between its endpoints. Moreover, in our setting the underlying graph $G = (P, E)$ is a complete graph and therefore we refer to a t-spanner of G as a t-spanner of the point set P. In other words, we consider networks connecting a given set of points in the plane with bounded dilation. There is a vast body of literature on t-spanners in this geometric setting (see [12] for a comprehensive survey of the area).

The weight of a network $G = (P, E)$ is defined as the sum of the lengths of its edges and denoted $wt(G)$. The weight is a good measure of the cost of building the network; thus, it is often desirable to have spanners with low weight. Since

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any spanner must connect all the given points P, the weight of a t-spanner is bounded from below by the weight of the minimum spanning tree $MST(P)$. Chandra et al. [4] have presented a greedy algorithm for constructing a t-spanner in $O(n^3 \log n)$ time, which has been proved by Das et al. [6,8] to have a weight of size $O(wt(MST(P)))$. The constant factor depends on the value t. A more efficient algorithm that computes the greedy spanner in $O(n^2 \log n)$ was later developed by Bose et al. [1]. A fast implementation of a variant of the greedy algorithm that maintains the $O(wt(MST(P)))$ weight and runs in $O(n \log^2 n)$ time has been developed by Das and Narasimhan [7]. Those spanners obviously approximate the weight of the minimum t-spanner within a constant factor that depends on t, but they are not necessarily optimal. We address the following decision problem, and appropriate optimization problem.

Problem 1. The Low Weight t-Spanner (LWSt) problem:
Input: A set P of points in the plane and a constant $w \geqslant MST(P)$.
Output: Whether there exists an Euclidean t-spanner for P of weight at most w.

Problem 2. The Minimum Weight t-Spanner (MWSt) problem:
Input: A set P of points in the plane.
Output: A minimum weight Euclidean t-spanner for P.

We show that for every real value $t > 1$ the MWSt and the LWSt problems are NP-hard. This is done by a reduction from the Partition problem, which has been proved to be NP-hard [10], and is defined as follows.

Problem 3. The Partition problem:
Input: A set $X = \{x_1, \ldots, x_n\}$ of n positive integers with even $\sum_{x \in X} x = R$.
Output: Whether there exists a subset $X' \subset X$ such that $\sum_{x \in X'} x = R/2$.

Klein and Kutz [11] have proved that the Dilation Graph and the Plane Dilation Graph problems are NP-hard by a reduction from the Partition problem as well. The Dilation Graph problem (resp. the Plane Dilation Graph problem) asks whether there exists a t-spanner (resp. a plane t-spanner) with at most m edges for a given set of points P, an integer m, and a real value $t > 1$. Note that here t is part of the input. Their reduction returns an instance with $t = 7$ for every instance of the Partition problem. This result implies the NP-hardness of minimizing the dilation of a spanner with at most m edges for a given set of points and a value m. This work focuses on the weight parameter rather than the number of edges; therefore, we define a variant of the above minimization problem, where a weight bound is given as an input instead of a bound on the number of edges.

Problem 4. The Minimum Dilation Graph (MDG) problem:
Input: A set P of points in the plane and a constant $w \geqslant MST(P)$.
Output: A minimum dilation Euclidean graph for P of weight at most w.

From the NP-hardness of the LWSt problem proved in this paper, we may deduce the NP-hardness of the MDG problem.

Various minimization problems of different parameters such as weight and number of edges of t-spanners have been proved to be NP-hard. Cai and Corneil have proved in [3] the NP-hardness of determining the existence of a tree t-spanner in a weighted graph for every $t > 1$ and in an unweighted graph for every $t \geqslant 4$. This implies the NP-hardness of a more general problem of determining the existence of a t-spanner with at most m edges for a given weighted or unweighted graph. Regarding geometric graphs, Gudmundsson and Smid have considered in [9] the problem of deciding whether a given geometric graph (not necessarily the complete graph) contains a t-spanner with at most m edges and prove it is NP-hard for every $t > 1$. A variant of the tree t-spanner problem adjusted to our geometric setting, namely, determining if a given set of points in the plane admits a tree t-spanner, has been later proved to be NP-hard by Cheong et al. [5].

Brandes and Handke [2] have introduced the problem of finding a minimum weight planar t-spanner for weighted graphs. They have established it is NP-hard for $t > 1$ by modifying the proof in [3] of the NP-hardness of determining the existence of a tree t-spanner in a weighted graph. An adjustment of the minimum weight planar t-spanner problem to the setting discussed in this paper should restrict the underlying graph to be the complete Euclidean graph. The appropriate decision and optimization problems are defined as follows.

Problem 5. The Low Weight Plane t-Spanner (LWPSt) problem:
Input: A set P of points in the plane and a constant $w \geqslant MST(P)$.
Output: Whether there exists an Euclidean plane t-spanner for P of weight at most w.

Problem 6. The Minimum Weight Plane t-Spanner (MWPSt) problem:
Input: A set P of points in the plane and a constant $w \geqslant MST(P)$.
Output: A minimum weight Euclidean plane t-spanner for P.

The reductions presented in this paper imply that the LWPSt and MWPSt problems are NP-hard for every $t > 1$.

Note that although all the problems presented here refer to the restricted case where the underlying graph is the complete Euclidean graph, our results, obviously, apply to the modified problems where the underlying graph is a general geometric graph (not necessarily the complete graph) and where the underlying graph is a general weighted graph (not necessarily geometric).

The rest of the paper is organized as follows. In Section 2 we define new terms and make some technical observations to be used in the reductions proofs. The reduction functions are presented in Section 3. First the reductions idea is outlined and later described in more details, where the reduction for $t \geqslant 2$ is given in Section 3.1 and the reduction for $1 < t < 2$ is given in Section 3.2. The correctness of both reductions is proved in Section 4.

2. Definitions and technical lemmas

Definition 1. Given a path (p, s, q), a t-*shortcut* refers to the addition of the edge (p, q) and possibly a removal of one of the edges (p, s) or (s, q), as long as the obtained graph is a t-spanner for $\{p, s, q\}$.

Definition 2. Given a path $Q = (p, s, q)$ and a t-*shortcut* that results in a graph G, we define the following terms:

- The *benefit* of the t-*shortcut* is defined as
$$\delta_Q(p, q) - \delta_G(p, q) = |ps| + |sq| - |pq|.$$
- The *cost* of the t-*shortcut* is defined as
$$weight(G) - weight(Q).$$
- The *efficiency* of the t-*shortcut* is defined as the ratio between its *benefit* and its *cost*, i.e.,
$$\frac{\delta_Q(p, q) - \delta_G(p, q)}{weight(G) - weight(Q)}.$$

We say that one t-shortcut is *more efficient* than another t-shortcut if the efficiency of the former is larger than the efficiency of the later.

Lemma 1. *Given two paths $Q = (p, s, q)$ and $Q' = (p', s', q')$, such that $|ps| = |sq| \neq |pq|$, $|p's'| = |s'q'| \neq |p'q'|$, and $\angle(psq) < \angle(p's'q')$ (see Fig. 1(a)), and let e and e' denote the two efficiencies of the most efficient t-shortcuts in Q and Q', respectively, then $e > e'$.*

Proof. By the fact that $\angle(psq) < \angle(p's'q')$, we have
$$\frac{|pq|}{|ps|} < \frac{|p'q'|}{|p's'|}. \tag{1}$$
The most efficient t-shortcut in Q obviously includes the addition of $\{p, q\}$, and if $|ps| + |pq| \leqslant t|qs|$ it also includes the removal of $\{q, s\}$ (or alternatively $\{p, s\}$).

If indeed $|ps| + |pq| \leqslant t|qs|$, then
$$\begin{aligned} e &= \frac{|ps| + |sq| - |pq|}{|pq| - |sq|} = 1\Big/\left(\frac{|pq|}{|ps|} - 1\right) - 1 \\ &>^{(1)} 1\Big/\left(\frac{|p'q'|}{|p's'|} - 1\right) - 1 = \frac{|p's'| + |s'q'| - |p'q'|}{|p'q'| - |s'q'|} \geqslant e', \end{aligned}$$
and $e > e'$ as required.

Otherwise, $|ps| + |pq| > t|qs|$ and therefore
$$1 + \frac{|pq|}{|qs|} > t \quad \overset{(1)}{\Rightarrow} \quad 1 + \frac{|p'q'|}{|q's'|} > t \quad \Rightarrow \quad |p's'| + |p'q'| > t|q's'|$$
and neither $\{q, s\}$ nor $\{q', s'\}$ are removed from the respective most efficient t-shortcuts. Hence,
$$\begin{aligned} e &= \frac{|ps| + |sq| - |pq|}{|pq|} = \frac{2 \cdot |ps|}{|pq|} - 1 \\ &>^{(1)} \frac{2 \cdot |p's'|}{|p'q'|} - 1 = \frac{|p's'| + |s'q'| - |p'q'|}{|p'q'|} = e', \end{aligned}$$
and $e > e'$ as required. $\square$

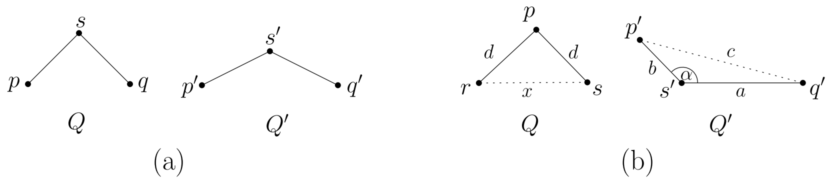

Fig. 1. Illustrated: (a) paths Q and Q' as defined in Lemma 1, and (b) paths Q and Q' as defined in Lemma 2 and Corollary 1.

Lemma 2. *Given a path $Q' = (p', s', q')$, such that $\alpha = \angle(p's'q') > \frac{\pi}{2}$ (see Fig. 1(b), right), then the efficiency of a t-shortcut in Q' is less than $\frac{-1}{\cos\alpha} - 1$.*

Proof. Let e denote the efficiency of the most efficient t-shortcut in Q' and let $a = |s'q'|$, $b = |p's'|$, and $c = |p'q'|$. Assume w.l.o.g. that $a \geqslant b$, then $e \leqslant \frac{a+b-c}{c-a}$.

By the cosines law we have

$$c^2 = a^2 + b^2 - 2ab \cdot \cos\alpha > a^2 + b^2 \cdot (\cos\alpha)^2 - 2ab \cdot \cos\alpha = (a - b \cdot \cos\alpha)^2.$$

Thus, $c > a - b \cdot \cos\alpha$ and we have

$$e \leqslant \frac{a+b-c}{c-a} < \frac{a+b-(a-b\cdot\cos\alpha)}{a-b\cdot\cos\alpha-a} = \frac{1+\cos\alpha}{-\cos\alpha} = \frac{-1}{\cos\alpha} - 1. \qquad \square$$

Corollary 1. *Given two paths $Q = (r, p, s)$ and $Q' = (p', s', q')$, where the points $\{r, s, s', q'\}$ lie on a line, $|rp| = |ps|$, and the two edges (p', s') and (p, s) are parallel (see Fig. 1(b)). Let e and e' denote the two efficiencies of the most efficient t-shortcuts in Q and Q', respectively, then $e > e'$.*

Proof. Let $x = |rs|$ and $d = |rp| = |ps|$, then

$$e \geqslant \frac{2d - x}{x} = \frac{2d}{x} - 1.$$

By the law of sines, $\angle(rps) = 2\arcsin(\frac{x}{2d})$. Since the angle $\angle(p's'q')$, denoted by α, equals to an exterior angle of the isosceles triangle $\triangle(rps)$, we have

$$\alpha = \pi - \left(\frac{\pi}{2} - \arcsin\left(\frac{x}{2d}\right)\right) = \frac{\pi}{2} + \arcsin\left(\frac{x}{2d}\right) = \pi - \arccos\left(\frac{x}{2d}\right)$$

and $\cos(\alpha) = -\frac{x}{2d}$. By Lemma 2, $e' < \frac{2d}{x} - 1 \leqslant e$. $\square$

3. The reduction function

In this section we show that the LWSt decision problem (Problem 1) and the MWSt optimization problem (Problem 2) are NP-hard for every constant $t > 1$. We prove the NP-hardness of the LWSt problem by a reduction from the Partition problem (Problem 3) that is known as NP-hard [10]. The NP-hardness of the MWSt problem follows from an obvious reduction from the appropriate decision problem (the LWSt problem). We propose different reductions for $1 < t < 2$ and for $t \geqslant 2$; however, both follow the same core ideas.

Given an instance $X = \{x_1, \dots, x_n\}$ with $\sum_{x \in X} x = R$ for the Partition problem, both reductions output a weight w and a set P of linear size in n that is composed of points distributed along a path connecting n triples of isosceles triangles' vertices (as illustrated in Fig. 2 and Fig. 4). Each isosceles triangle gadget is associated with a value $x_i \in X$, meaning, the length of its edges are functions of the value x_i (as illustrated in Fig. 3). The distances between adjacent points on the path connecting all triangles are derived from the sum R.

Both reductions outputs are presented below in more details. The reduction for $t \geqslant 2$ is a bit simpler and therefore described first in the next subsection, followed by the reduction for $1 < t < 2$ in Section 3.2.

3.1. The reduction for $t \geqslant 2$

Given a valid input for the Partition problem $X = \{x_1, \dots, x_n\}$ with $\sum_{x \in X} x = R$, the reduction outputs a valid input for the LWSt problem (P, w) as elaborated next.

The points of P are placed roughly along 3 sides of an axis-parallel rectangle: left side, right side, and top side, as illustrated in Fig. 2. The left and right sides are of length $\frac{R}{2}((n+2)(t-1) - \frac{1}{3})$; each is sampled by $m = \lceil (n+2)(t-1) - \frac{1}{3} \rceil + 1$ points with regular spacing of $R/2$ (possibly except for the bottom point). We enumerate the points on the left side from top to bottom by $l_1, \dots, l_m$ and the points on the right side by $r_1, \dots, r_m$. Note that $(n+2)(t-1) - \frac{1}{3} > 1$ and thus $m > 2$.

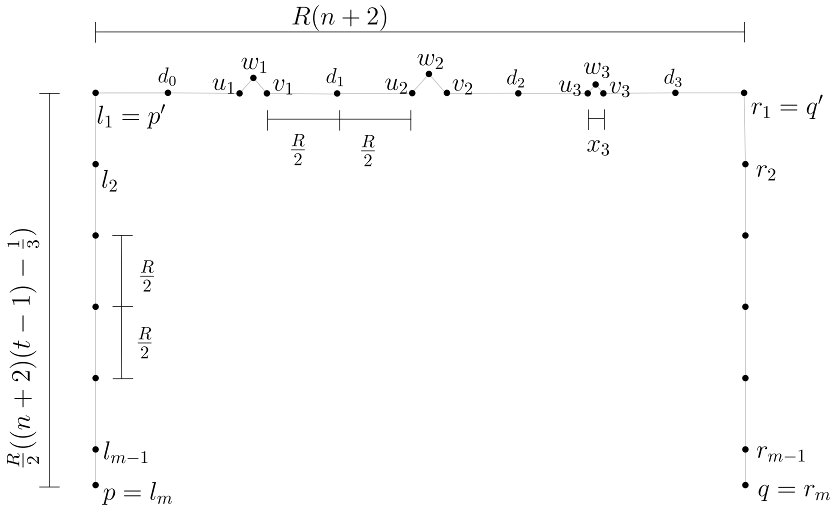

Fig. 2. The set of points P and its minimum weight connected graph as defined in the reduction for $t \geqslant 2$.

The top side is actually a horizontal component of width $R(n+2)$, consists of n isosceles triangle gadgets and connecting points. The $2n$ endpoints of the n isosceles triangles' bases, denoted by u_i and v_i, for $1 \leqslant i \leqslant n$, are located among $n+1$ segments of length R. Each segment is halved by a middle point d_i, for $0 \leqslant i \leqslant n$. The additional vertex of each triangle is located above the rectangle and denoted by w_i, for $1 \leqslant i \leqslant n$. The i-th isosceles triangle gadget has sides of length $\frac{5}{6}x_i$, a base of length x_i, and thus a top angle of $2\arcsin(\frac{3}{5}) < \pi/2$ (see Fig. 3, right).

More formally, P is composed of the following points:

- Left side and right side points:

$$l_i := \left(0, -(i-1)\frac{R}{2}\right),$$
$$r_i := \left((n+2)R, -(i-1)\frac{R}{2}\right),$$

  for $1 \leqslant i < m$, and

$$l_m := \left(0, -\frac{R}{2}\left((n+2)(t-1) - \frac{1}{3}\right)\right),$$
$$r_m := \left((n+2)R, -\frac{R}{2}\left((n+2)(t-1) - \frac{1}{3}\right)\right).$$

- Triangle gadget points:

$$u_i := \left(i \cdot R + \sum_{j=1}^{i-1} x_j, 0\right),$$
$$w_i := \left(i \cdot R + \sum_{j=1}^{i-1} x_j + \frac{x_i}{2}, \frac{2}{3}x_i\right),$$
$$v_i := \left(i \cdot R + \sum_{j=1}^{i} x_j, 0\right),$$

  for $1 \leqslant i \leqslant n$.
- Middle points on the horizontal segments:

$$d_0 := \left(\frac{R}{2}, 0\right),$$

  and

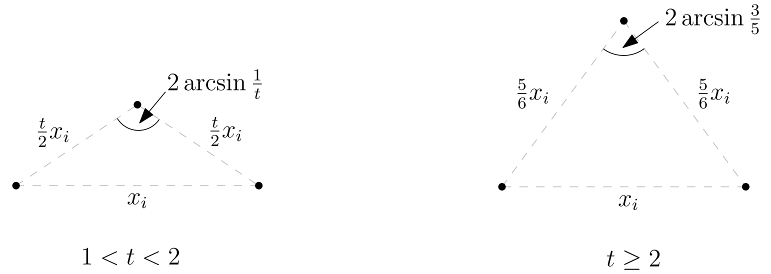

Fig. 3. The triangle gadgets in the constructions of the reductions for $1 < t < 2$ and $t \geqslant 2$.

$$d_i := \left(\sum_{j=1}^{i} x_j + i \cdot R + \frac{R}{2}, 0 \right),$$

for $1 \leqslant i \leqslant n$.

The weight bound w is defined a bit differently for $2 \leqslant t < 2\frac{1}{5}$ and $t \geqslant 2\frac{1}{5}$.

- For $2 \leqslant t < 2\frac{1}{5}$ we define $w = R(t(n+2) + \frac{5}{6})$, and
- for $t \geqslant 2\frac{1}{5}$, $w = R(t(n+2) + \frac{5}{12})$.

3.2. The reduction for $1 < t < 2$

Given a valid input for the Partition problem $X = \{x_1, \dots, x_n\}$ with $\sum_{x \in X} x = R$, the reduction outputs a valid input for the LWSt problem (P, w) as follows. The weight bound is

$$w = R\left(\left(n + \frac{3}{2}\right)\frac{3t}{2} + n + t + \frac{3}{2}\right)$$

and the set of points P resembles the one defined for $t \geqslant 2$. We still have a horizontal component, and left and right sides; however, those are no longer perpendicular to the horizontal component (the x-axis). The angles between the sides and the horizontal component are enlarged by $\alpha_t = \arcsin(\frac{2}{3t^2} + \frac{1}{3t})$ and the rectangle shape turns into an isosceles trapezoid shape (see Fig. 4).

As before, the horizontal component is composed of a horizontal segment of length $R(n+2)$ sampled by $2n$ endpoints of n isosceles triangles' bases, located among $n+1$ segments of length R, halved by a middle point. However, here the i-th isosceles triangle has sides of length $\frac{t}{2}x_i$, a base of length x_i, and thus a top angle of $2\arcsin(1/t)$ (see Fig. 3, left).

The right and left sides are of length $\frac{R}{2}(n+\frac{3}{2})\frac{3t}{2}$; each is sampled by $m = \lceil (n+\frac{3}{2})\frac{3t}{2} \rceil + 1$ points with regular distances of $R/2$ (possibly except for the bottom point). Note that $(n+\frac{3}{2})\frac{3t}{2} > 1$ and thus $m > 2$.

More formally, P is composed of the following points:

- Left side and right side points:

$$l_i := \left(-(i-1)\frac{R}{2}\sin(\alpha_t), -(i-1)\frac{R}{2}\cos(\alpha_t) \right),$$

$$r_i := \left(R(n+2) + (i-1)\frac{R}{2}\sin(\alpha_t), -(i-1)\frac{R}{2}\cos(\alpha_t) \right),$$

for $1 \leqslant i < m$, and

$$l_m := \left(-\frac{R}{2}\left(n+\frac{3}{2}\right)\frac{3t}{2}\sin(\alpha_t), -\frac{R}{2}\left(n+\frac{3}{2}\right)\frac{3t}{2}\cos(\alpha_t) \right),$$

$$r_m := \left(R(n+2) + \frac{R}{2}\left(n+\frac{3}{2}\right)\frac{3t}{2}\sin(\alpha_t), -\frac{R}{2}\left(n+\frac{3}{2}\right)\frac{3t}{2}\cos(\alpha_t) \right).$$

- Triangle gadget points:

$$u_i := \left(i \cdot R + \sum_{j=1}^{i-1} x_j, 0 \right),$$

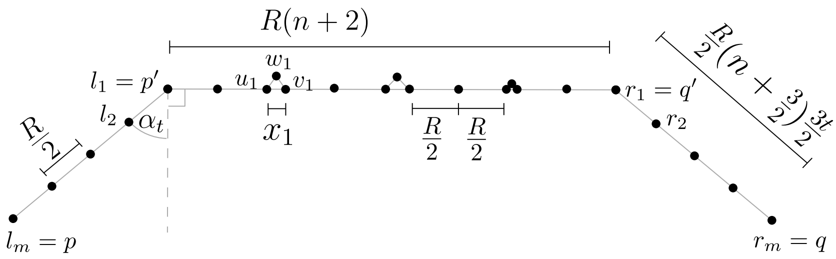

Fig. 4. The set of points P is depicted in black and its minimum weight connected graph is depicted in gray as defined in the reduction for $1 < t < 2$.

$$w_i := \left(i \cdot R + \sum_{j=1}^{i-1} x_j + \frac{x_i}{2}, \frac{\sqrt{t^2-1}}{2} x_i\right),$$

$$v_i := \left(i \cdot R + \sum_{j=1}^{i} x_j, 0\right),$$

for $1 \leqslant i \leqslant n$.

- Middle points on the horizontal segments:

$$d_0 := \left(\frac{R}{2}, 0\right),$$

and

$$d_i := \left(\sum_{j=1}^{i} x_j + i \cdot R + \frac{R}{2}, 0\right),$$

for $1 \leqslant i \leqslant n$.

Overall, in both reductions we have $|P| = 4n + 1 + 2m$. We denote by $p = l_m$ and $q = r_m$ the leftmost and rightmost bottom points in P, respectively, and by $p' = l_1$ and $q' = r_1$ the leftmost and rightmost points on the horizontal component of P.

4. The reduction correctness

In order to conclude the NP-hardness of the LWSt problem from the reductions defined in the previous section, we need to verify the following:

1. The correctness of the reductions; namely, that a subset $X' \subset X$ with $\sum_{x \in X'} x = R/2$ exists iff there exists a t-spanner for P of weight at most w.
2. The reductions can be computed by a Turing machine in polynomial time with respect to the size of the reduction input.

We begin with proving the first, namely, the correctness of the reductions, and at the end of the section we introduce some modifications in order to enable polynomial computation of both reductions. Unless we mention otherwise, the following arguments apply for both reductions.

Consider a reduction input X with output (P, w). In order to prove the correctness of the reductions, we need to show that a subset $X' \subset X$ with $\sum_{x \in X'} x = R/2$ exists iff there exists a t-spanner for P of weight at most w, or in other words, that the minimum weight t-spanner is of weight at most w. We prove the above by the following method.

First, we observe that the minimum weight connected graph over P, $MST(P)$, forms a path between two points, p and q, of weight (length) $w - c = t|pq| + s$. Meaning, a total shortening of s should be performed on the shortest path connecting p and q in order for it to be a proper t-spanning path at the cost of at most c. We show that any shortening can be considered as application of a set of independent t-shortcuts (see Definition 1) and that the most *efficient* (see Definition 2) type of t-shortcut is the one that involves the addition of an isosceles triangle gadget's base and its efficiency is exactly s/c. We refer to such t-shortcuts as *gadget t-shortcuts*. Therefore, no other shortcut may be applied and the distance between p and q can be decreased to $t|pq|$ iff there exists a set of gadget t-shortcuts with a total benefit of s.

Our constructions ensures that a set of gadget t-shortcuts with a total benefit of s exists iff there exists a subset $X' \subset X$ such that $\sum_{x \in X'} x = R/2$ and applying those t-shortcuts yields a t-spanner for P. Moreover, the t-spanner obtained by applying those t-shortcuts is a plane graph. Therefore, our reductions also prove the NP-hardness of the LWPSt problem (Problem 5) and the MWPSt problem (Problem 6).

We now follow the above scheme in more details. Let $G = (P, E)$ be the minimum weight t-spanner for P. If P admits a t-spanner of weight at most w, then, obviously, $wt(G) \leqslant w$. We would like to characterize G under the assumption that $wt(G) \leqslant w$. We do so by observing the minimum weight connected graph over P, $MST(P) = (P, E_{\mathrm{MST}})$ and identifying E^+ and E^-: the sets of edges over P that should be added and removed from $MST(P)$, respectively, in order to obtain the t-spanner G. More formally,

$$E^+ := E \backslash E_{\mathrm{MST}}, \quad \text{and}$$
$$E^- := E_{\mathrm{MST}} \backslash E.$$

We identify E^+ and E^- in the terms of t-shortcuts and thus characterize G as a graph obtained from $MST(P)$ by applying a special type of t-shortcuts on subpaths of $MST(P)$, as stated in Corollary 3.

Consider the minimum weight connected graph over P, $MST(P) = (P, E_{\mathrm{MST}})$. It forms a path between $p = l_m$ and $q = r_m$ as depicted in Fig. 2 and Fig. 4. In the reduction for $t \geqslant 2$, the $MST(P)$ path is of length

$$2 \cdot \frac{R}{2}\left((n+2)(t-1) - \frac{1}{3}\right) + (n+1)R + \frac{5}{3}R = R\left(t(n+2) + \frac{1}{3}\right),$$

and in the reduction for $1 < t < 2$, the $MST(P)$ path is of length

$$2 \cdot \frac{R}{2}\left(n + \frac{3}{2}\right)\frac{3t}{2} + (n+1)R + t \cdot R = R\left(\left(n + \frac{3}{2}\right)\frac{3t}{2} + n + t + 1\right).$$

We would like to examine how much should this path be shortened in order for it to be a proper t-spanning path between p and q. We denote the length of the required shortening in s.

For $1 < t < 2$, a proper t-spanning path between p and q should be of length at most

$$\begin{aligned}
t|pq| &= t\left(2\frac{R}{2}\left(n + \frac{3}{2}\right)\frac{3t}{2}\sin\alpha_t + (n+2)R\right) \\
&= t \cdot R\left(\left(n + \frac{3}{2}\right)\frac{3t}{2}\sin\left(\arcsin\left(\frac{2}{3t^2} + \frac{1}{3t}\right)\right) + n + 2\right) \\
&= t \cdot R\left(\left(n + \frac{3}{2}\right)\frac{3t}{2}\left(\frac{2}{3t^2} + \frac{1}{3t}\right) + n + 2\right) \\
&= R\left(\left(n + \frac{3}{2}\right)\frac{3t}{2}\left(1 - \left(\frac{2}{3} - \frac{2}{3t}\right)\right) + t \cdot n + 2t\right) \\
&= R\left(\left(n + \frac{3}{2}\right)\frac{3t}{2} + n + \frac{3}{2} + \frac{t}{2}\right).
\end{aligned}$$

Therefore, a total shortening of $(t-1)R/2$ should be performed on the path in order for it to be a legal t-spanning path from p to q. Fortunately, $MST(P)$ is $R/2$ units lighter than the given weight w, and this difference can be used in order to shorten the length of the path between p and q. Thus, a total shortening of $(t-1)R/2$ should be performed on the path at the cost of at most $R/2$. We denote the value of the acceptable cost, namely, the remaining weight in c, i.e., $c = w - wt(MST(P))$.

For $t \geqslant 2$, since $|pq| = R(n+2)$, a t-spanning path connecting p and q should be of length at most $t \cdot R(n+2)$. Thus, the path $MST(P)$ is $R/3$ units longer than a legal t-spanning path from p to q. However, it is lighter than the given weight w: for $2 \leqslant t < 2\frac{1}{5}$ the remaining weight is $R/2$ and for $t \geqslant 2\frac{1}{5}$ the remaining weight is $R/12$. Therefore, a total shortening of $R/3$ should be performed on the path at the cost of at most $R/2$ or $R/12$ for $2 \leqslant t < 2\frac{1}{5}$ and $t \geqslant 2\frac{1}{5}$, respectively.

Next we make some observations regarding the edge set E^+.

Claim 1. *If $wt(G) \leqslant w$, then E^+ contains only edges of length at most R.*

Proof. Since $MST(P)$ is the minimum weight connected graph and G must be connected, $|E^-| \leqslant |E^+|$; moreover, there exists an injective function from E^- to E^+ that maps every edge $e^- \in E^-$ to a *longer* edge $e^+ \in E^+$ whose addition creates a cycle in $MST(P)$ and enables the removal of e^-. Assume towards a contradiction that E^+ contains an edge e^+ longer than R. Since all edges in $MST(P)$ are of length at most $R/2$ (under the legitimate assumption that all the elements in X are smaller than $R/2$) and $E^- \subset MST(P)$, we have

$$\sum_{e \in E^+} wt(e) - \sum_{e \in E^-} wt(e) > R/2$$

and thus $wt(G) > wt(MST(P)) + R/2 \geqslant w$. $\square$

There are exactly three types of edges over P of length at most R and that have the potential to be in E^+ (while ignoring edges that overlap subpaths in $MST(P)$, which are useless for shortening the distance between p and q):

1. (u_i, v_i) for $1 \leqslant i \leqslant n$,
2. (w_i, d_i) and (w_i, d_{i-1}) for $1 \leqslant i \leqslant n$, and
3. (l_2, d_0) and (r_2, d_n).

To a t-shortcut that involves an addition of an edge of type 1, 2, or 3 and possibly a removal of an edge in the closed cycle (in case that the required dilation t allows it), we refer as a *relevant* t-shortcut. By Claim 1 we conclude that the addition of E^+ and removal of E^- from $MST(P)$ can be identified as application of a sequence of *relevant* t-shortcuts on subpaths of $MST(P)$. This conclusion is stated in the following corollary.

Corollary 2. *If $wt(G) \leqslant w$, then in order to obtain G from $MST(P)$, a sequence of* relevant *t-shortcuts on subpaths of $MST(P)$ should be applied.*

To the addition of an edge of type 1, namely $\{u_i, v_i\}$ for some $1 \leqslant i \leqslant n$, together with the removal of the left side in the appropriate isosceles triangle, i.e., $\{u_i, w_i\}$, for $t \geqslant 2\frac{1}{5}$, we refer as a *gadget* t-shortcut. Note that the removal of a triangle side in a gadget t-shortcut for $t \geqslant 2\frac{1}{5}$ is indeed possible in the aspect of the dilation between the points u_i and w_i.

Claim 2. *The* gadget *t-shortcuts are more efficient than any other relevant t-shortcut. Moreover, their efficiency is s/c, namely, $t-1$ for $1 < t < 2$, $\frac{2}{3}$ for $2 \leqslant t < 2\frac{1}{5}$, and 4 for $t \geqslant 2\frac{1}{5}$.*

Proof. First, we examine the efficiency of a gadget t-shortcut. In the reduction for $1 < t < 2$, the benefit of the i-th gadget t-shortcut is $tx_i - x_i = (t-1)x_i$, its cost is x_i, and hence its efficiency is indeed $t-1$. In the reduction for $t \geqslant 2$, the benefit of the i-th gadget t-shortcut is

$$\frac{5}{3}x_i - x_i = \frac{2}{3}x_i.$$

The cost is x_i for $2 \leqslant t < 2\frac{1}{5}$ and $1\frac{5}{6}x_i - \frac{5}{3}x_i = \frac{x_i}{6}$ for $t \geqslant 2\frac{1}{5}$; hence, its efficiency is indeed $\frac{2}{3}$ and 4 for $2 \leqslant t < 2\frac{1}{5}$ and $t \geqslant 2\frac{1}{5}$, respectively.

Next, we observe every relevant t-shortcut and show it is less efficient than a gadget t-shortcut.

- Addition of an edge of type 2 and possibly a removal of an edge in the created obtuse triangle: by Corollary 1, this t-shortcut is less efficient than the gadget t-shortcut.
- Addition of an edge of type 3 that forms an isosceles triangle with top angle $\frac{\pi}{2} + \arcsin(\frac{2}{3t^2} + \frac{1}{3t})$ and $\pi/2$ in the reductions for $1 < t < 2$ and $t \geqslant 2$, respectively. Those angles are greater than the top angles of the gadget triangles in the corresponding reductions, which are $2\arcsin(1/t)$ and $2\arcsin(\frac{3}{5})$ in the reductions of $1 < t < 2$ and $t \geqslant 2$, respectively. Thus, by Lemma 1, we conclude that the gadget t-shortcut is more efficient. □

According to Claim 2, only the gadget t-shortcuts have efficiency equal to the ratio between the required shortening of the path between p and q, s, and the acceptable cost, c, while the other t-shortcuts have lower efficiencies. Therefore, if $wt(G) \leqslant w$, then G can be obtained from $MST(P)$ only by applying gadget t-shortcuts. Moreover, in order to achieve the required shortening s of the path connecting p and q, without exceeding the weight bound, a set of gadget t-shortcuts with a total benefit of exactly s should be applied and thus $wt(G) = wt(MST(P)) + c = w$. This conclusion is stated in Corollary 3.

Corollary 3. *Either $wt(G) > w$ or $wt(G) = w$, and then gadget t-shortcuts with a total benefit s should be applied on subpaths of $MST(P)$ in order to obtain G from $MST(P)$.*

Corollary 3 together with the following lemma serve us in the proof of the main lemma of this section, Lemma 4.

Lemma 3. *Let $\triangle(zsy)$ be an isosceles triangle with $|zs| = |sy|$ and let y' be a point on $\overline{sy}$, then $\frac{|zs|+|sy|}{|zy|} \geqslant \frac{|zs|+|sy'|}{|zy'|}$ (see Fig. 5).*

Proof. Let $\gamma = \angle(zsy)$ and $\beta = \angle(zy'y)$, then by the sines law, $|zy| = |zs| 2\sin(\frac{\gamma}{2})$, $|sy'| = |zs|\sin(\beta - \gamma)/\sin(\beta)$, and

$$|zy'| = \frac{|zy|\sin(\frac{\pi}{2} - \frac{\gamma}{2})}{\sin(\beta)} = \frac{|zs| 2\sin(\frac{\gamma}{2})\cos(\frac{\gamma}{2})}{\sin(\beta)}.$$

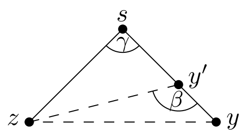

Fig. 5. Illustration of Lemma 3.

Thus,

$$\begin{aligned}
&\frac{|zs|+|sy|}{|zy|} \geqslant \frac{|zs|+|sy'|}{|zy'|} \\
\Leftrightarrow\quad & \frac{2|zs|}{|zs|2\sin(\frac{\gamma}{2})} \geqslant \frac{|zs|+|zs|\sin(\beta-\gamma)/\sin(\beta)}{|zs|2\sin(\frac{\gamma}{2})\cos(\frac{\gamma}{2})/\sin(\beta)} \\
\Leftrightarrow\quad & \frac{2\cos(\frac{\gamma}{2})}{\sin(\beta)} \geqslant 1+\frac{\sin(\beta-\gamma)}{\sin(\beta)} \\
\Leftrightarrow\quad & 2\cos\left(\frac{\gamma}{2}\right)-\sin(\beta)-\sin(\beta-\gamma) \geqslant 0 \\
\Leftrightarrow\quad & 2\cos\left(\frac{\gamma}{2}\right)\left(1-\sin\left(\beta-\frac{\gamma}{2}\right)\right) \geqslant 0.
\end{aligned}$$

The last inequality indeed holds for every $0<\gamma \leqslant \pi$, since $\cos(\frac{\gamma}{2}) \geqslant 0$. □

We are now ready for the main lemma that concludes the correctness of the reductions. Recall that X is the given input for the Partition problem and thus an input for the reduction functions, and P and w are the set of points and weight in the output, i.e., the inputs for the LWSt problem.

Lemma 4. *A subset $X' \subset X$ with $\sum_{x\in X'} x = R/2$ exists iff there exists a t-spanner for P of weight at most w.*

Proof. [$\Leftarrow$] Assume towards a contradiction that there exists a t-spanner for P of weight at most w, i.e., $wt(G) \leqslant w$; however, a subset $X' \subset X$ with $\sum_{x\in X'} x = R/2$ does not exist. According to our construction, the absence of a subset $X' \subset X$ with $\sum_{x\in X'} x = R/2$ implies that a set of gadget t-shortcuts with a total benefit of exactly s does not exist. By Corollary 3, this means that $wt(G) > w$ in contradiction to the existence of a t-spanner for P of weight at most w.

[$\Rightarrow$] Assume that a subset $X' \subset X$ with $\sum_{x\in X'} x = R/2$ exists.
Let $G' = (P, E')$ denote the graph obtained by applying the gadget t-shortcuts on the triangles that correspond to the elements in X'. We show that G' admits a t-spanner for P, i.e., for every two points z and y in P, $\delta_{G'}(z, y) \leqslant t|zy|$. Since G' is obtained by applying gadget t-shortcuts with a total benefit of s, there exists a t-spanning path between p and q in G'. Next we consider all other pairs of points $\{z, y\} \neq \{p, q\}$ in P:

1. $\{z, y\} = \{l_i, r_j\}$ for some $1 \leqslant i, j \leqslant m$.
   In the reduction for $t \geqslant 2$, since $\delta_{G'}(z, y) \leqslant \delta_{G'}(p, q)$ and $|zy| \geqslant |pq|$, we have

   $$\frac{\delta_{G'}(z, y)}{|zy|} \leqslant \frac{\delta_{G'}(p, q)}{|pq|} = t.$$

   In the reduction for $1 < t < 2$, if $|zy| \geqslant |pq|$, then $\delta_{G'}(z, y) \leqslant \delta_{G'}(p, q)$, and we get a proper dilation between z and y as above.
   Otherwise, recall we denote l_1 and r_1 by p' and q', respectively, and let s be the intersection point of the extensions of the segments $\overline{p'p}$ and $\overline{q'q}$ (see Fig. 6) and let $\epsilon = |p's| + |sq'| - \delta_{G'}(p', q')$ (one can verify that $\epsilon > 0$), next we show that the following inequality holds

   $$\big(|zs|+|sy|\big)/|zy| \leqslant \big(|ps|+|sq|\big)/|pq| \qquad (2)$$

   and conclude that

   $$\frac{\delta_{G'}(z, y)}{|zy|} = \frac{|zs|+|sy|-\epsilon}{|zy|} \overset{(2)}{\leqslant} \frac{|ps|+|sq|}{|pq|} - \frac{\epsilon}{|zy|} \leqslant \frac{|ps|+|sq|-\epsilon}{|pq|} = t.$$

   If $\overline{zy}$ is parallel to $\overline{pq}$, then due to the similarity of the triangles $\triangle(zsy)$ and $\triangle(psq)$ we have $(|zs|+|sy|)/|zy| = (|ps|+|sq|)/|pq|$.
   Otherwise, assume w.l.o.g. that $|zp| < |yq|$ and let y' be a point on $\overline{qq'}$ such that $\overline{zy'}$ is parallel to $\overline{pq}$. By Lemma 3 we have

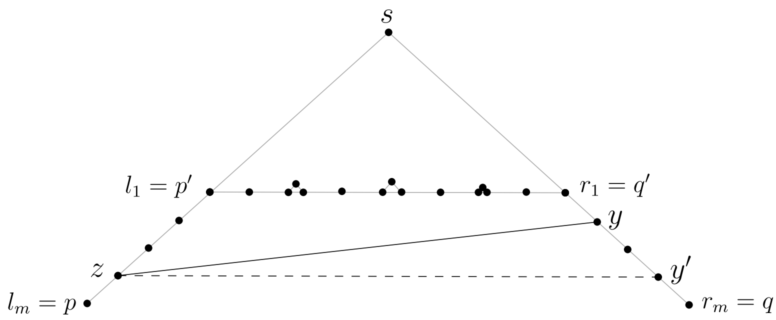

Fig. 6. Illustration of case 1 in the proof of Lemma 4.

$$\frac{|zs|+|sy|}{|zy|} \leqslant \frac{|zs|+|sy'|}{|zy'|} \leqslant \frac{|ps|+|sq|}{|pq|}.$$

2. $\{z, y\}$ where $z \in \{l_i, r_i | 1 \leqslant i \leqslant m\}$ and $y \in \{u_j, w_j, v_j, d_j | 1 \leqslant j \leqslant n\} \cup \{d_0\}$. Namely, z is on the left or the right side and y is on the horizontal component (see Fig. 7(a)).
First consider the reduction for $1 < t < 2$. Assume w.l.o.g. that $z = l_i$ for some $1 \leqslant i \leqslant m$. We would like to examine the dilation between p' and y. The greatest dilation is obtained when $y = v_j$, for some $1 \leqslant j \leqslant n$. Every triangle gadget base is of length at most $R/2$ and is preceded by two edges, each of length $R/2$, to its left, therefore, we have

$$\frac{\delta_{G'}(p', y)}{|p'y|} = \frac{j \cdot 2 \cdot R/2 + \sum_{i=1}^{j} t \cdot x_i}{j \cdot 2 \cdot R/2 + \sum_{i=1}^{j} x_i} \leqslant \frac{j \cdot (R + t \cdot R/2)}{j \cdot (R + R/2)} \leqslant \frac{R + t \cdot R/2}{3 \cdot R/3} = 3/2 + t/3.$$

By the cosines law we have

$$|zy| \geqslant \sqrt{|zp'|^2 + |p'y|^2 - 2|zp'||p'y| \cos\left(\frac{\pi}{2} + \alpha_t\right)}$$

(we do not use equality since for $y = w_l$, $\angle(zp'y) > \pi/2 + \alpha_t$). Hence, we have

$$\begin{aligned}
\delta_{G'}(z, y) &= \delta_{G'}(z, p') + \delta_{G'}(p', y) \\
&\leqslant |zp'| + \left(\frac{2}{3} + \frac{t}{3}\right)|p'y| \\
&= \sqrt{|zp'|^2 + \left(\frac{2}{3} + \frac{t}{3}\right)^2 |p'y|^2 + 2|zp'||p'y|\left(\frac{2}{3} + \frac{t}{3}\right)} \\
&= \sqrt{|zp'|^2 + \left(\frac{2}{3} + \frac{t}{3}\right)^2 |p'y|^2 + 2|zp'||p'y|t^2\left(\frac{2}{3t^2} + \frac{1}{3t}\right)} \\
&\leqslant \sqrt{t^2|zp'|^2 + t^2|p'y|^2 - 2|zp'||p'y|t^2 \cos\left(\frac{\pi}{2} + \alpha_t\right)} \\
&\leqslant t|zy|.
\end{aligned}$$

Now consider the reduction for $t \geqslant 2$. Recall that gadget t-shortcut for $t \geqslant 2\frac{1}{5}$ includes the removal of a triangle's left side. Thus, the case where z is on the left side is not symmetric to the case where z is on the right side with respect to the dilation between z and y. However, it implies that a greater dilation is obtained for $z = l_i$ for some $1 \leqslant i \leqslant m$ (rather than for $z = r_i$).
Consider the dilation between p' and y.
If $y \in \{u_j, v_j, d_j | 1 \leqslant j \leqslant n\} \cup \{d_0\}$, we have

$$\frac{\delta_{G'}(p', y)}{|p'y|} \leqslant \frac{\delta_{G'}(p', v_j)}{|p'v_j|} \leqslant \frac{j \cdot R + \frac{10}{6}\sum_{i=1}^{j} x_i}{j \cdot R + \sum_{i=1}^{j} x_i} \leqslant^{(*)} \frac{R + \frac{10}{6} \cdot R/2}{R + R/2} = 1\frac{2}{9}.$$

Otherwise, $y = w_j$ for $1 \leqslant j < n$, in the worst case $x_i \notin X'$, i.e., $(u_j, w_j) \notin E'$ and we have

$$\frac{\delta_{G'}(p', y)}{|p'y|} \leqslant \frac{j \cdot R + \frac{10}{6}\sum_{i=1}^{j-1} x_i + \frac{11}{6} \cdot x_j}{j \cdot R + \sum_{i=1}^{j-1} x_i + x_j/2} \leqslant^{(*)} \frac{R + \frac{11}{6} \cdot R/2}{R + R/4} = 1\frac{8}{15}.$$

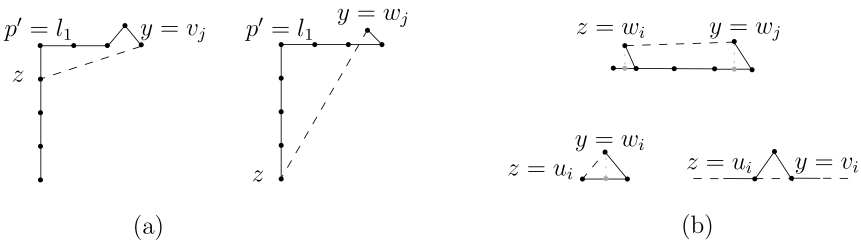

Fig. 7. Illustrated: (a) case 2 in the proof of Lemma 4, and (b) case 3 in the same proof.

Inequalities (*) hold since the left member of the inequality is maximized when j is minimized and x_j is maximized, under the assumption that all the elements in X are smaller than $R/2$. Thus,

$$\begin{aligned}\delta_{G'}(z,y) &= \delta_{G'}\left(z,p'\right)+\delta_{G'}\left(p',y\right)\\ &\leqslant \left|zp'\right| + 1\frac{8}{15}\cdot\left|p'y\right|\\ &<^{(**)} 2\sqrt{\left|zp'\right|^2+\left|p'y\right|^2}\\ &\leqslant t\cdot|zy|.\end{aligned}$$

Inequality (**) holds since the values of the function $(x+1\frac{8}{15}y)/\sqrt{x^2+y^2}$ are smaller than 2 for every x and y.

3. $\{z,y\}\subset\{d_i,u_i,v_i,w_i | 1\leqslant i\leqslant n\}\cup\{d_0\}$. Namely, both z and y are on the horizontal component.
In the reduction for $1<t<2$, the greatest dilation is obtained between $z=u_i$ and $y=v_i$ for some $1\leqslant i\leqslant n$ such that $(u_i,v_i)\notin E'$. Thus,

$$\frac{\delta_{G'}(z,y)}{|zy|}\leqslant\frac{\delta_{G'}(u_i,v_i)}{|u_iv_i|}\leqslant t.$$

In the reduction for $t\geqslant 2$, due to the removal of the triangle's left side in gadget t-shortcuts for $t\geqslant 2\frac{1}{5}$, recognizing the case of the greatest dilation is a bit less straight forward and hence we use case analysis (see Fig. 7(b)).
If $\{z,y\}\subset\{w_i | 1\leqslant i\leqslant n\}$, i.e., $z=w_i$ and $y=w_j$ for $1\leqslant i<j\leqslant n$, let z' and y' be the projections of z and y on the horizontal segment (x-axis) respectively, then we have

$$\begin{aligned}\frac{\delta_{G'}(z,y)}{|zy|}\leqslant\frac{\delta_{G'}(z,y)}{|z'y'|}&\leqslant\frac{\frac{5}{6}\cdot x_i+(j-i)\cdot R+\frac{10}{6}\sum_{k=i+1}^{j-1}x_i+\frac{11}{6}\cdot x_j}{x_i/2+(j-i)\cdot R+\sum_{k=i+1}^{j-1}x_i+x_j/2}\\ &\leqslant\frac{R+\frac{11}{6}\cdot R/2}{\frac{3}{2}\cdot R}=1\frac{5}{18}<t.\end{aligned}$$

Otherwise, the greatest dilation is obtained between two vertices of a triangle gadget. More precisely, for $2\leqslant t<2\frac{1}{5}$ it is obtained for $y=u_i$ and $z=v_i$ for some $1\leqslant i\leqslant n$ such that $(u_i,v_i)\notin E'$, and we have

$$\frac{\delta_{G'}(z,y)}{|zy|}\leqslant\frac{\delta_{G'}(u_i,v_i)}{|u_iv_i|}\leqslant\frac{10}{6}<t,$$

and for $t\geqslant 2\frac{1}{5}$ it is obtained for $y=u_i$ and $z=w_i$ for some $1\leqslant i\leqslant n$ such that $(u_i,w_i)\notin E'$, and then we have

$$\frac{\delta_{G'}(z,y)}{|zy|}\leqslant\frac{\delta_{G'}(u_i,w_i)}{|u_iw_i|}\leqslant 2\frac{1}{5}\leqslant t,$$

4. $\{z,y\}=\{r_i,r_j\}$ or $\{z,y\}=\{l_i,l_j\}$ for $1\leqslant i,j\leqslant n$. Namely, both z and y are on the left or the right side of the rectangle. We have $\frac{\delta_{G'}(z,y)}{|zy|}=1<t$. $\square$

Overall, we proved the correctness of both reductions. The only thing left to prove is that those reductions can be computed by a Turing machine in polynomial time in k, where $k=\max\{n,\log R\}$. Clearly, the number of bits required to represent w is polynomial in k and P is of polynomial size in k. However, while for $t\geqslant 2$ the computation of the points in P can be done in polynomial time in k, in the construction for $1<t<2$ the values of some points cannot be calculated precisely, and sufficient approximation should be computed in polynomial time in k. More precisely, we need to ensure that the values $\sqrt{t^2-1}$ and $\cos(\alpha_t)$, which are factors in the coordinates of the points w_i for $1\leqslant i\leqslant n$ and the points r_i and l_i for $1<i\leqslant m$, can be approximated in a way that does not harm the correctness of the reduction, in polynomial time.

This issue can be easily solved by approximating the values $\sqrt{t^2-1}$ and $\cos(\alpha_t)$ by smaller values, with up to $O(k^2)$ digits after the binary point, with an error of at most $1/2^{k^2}$.

For every parameter a related to the reduction output, we denote by $\hat{a}$ its new modified value after rounding. We have that for every $1 \leqslant i \leqslant n$, the y-coordinate of the modified point $\hat{w}_i$ is smaller than the one of w_i and for every $1 < i \leqslant m$, the y-coordinates of the points $\hat{l}_i$ and $\hat{r}_i$ are greater than the one of l_i and r_i, respectively. We also modify the weight bound to $\hat{w} = w - 2|l_1 l_m| + 2|\hat{l}_1 \hat{l}_m|$.

If a proper partition of X exists, the graph over $\hat{P}$ obtained by applying the gadget t-shortcuts on the triangles that correspond to the elements in X', $\hat{G}$, is of weight at most $\hat{w}$. Moreover, our rounding guarantees that for every $z, y \in P$

$$\frac{\delta_{\hat{G}}(\hat{z}, \hat{y})}{|\hat{z}\hat{y}|} \leqslant \frac{\delta_G(z, y)}{|zy|} = t.$$

Thus, $\hat{G}$ is a proper t-spanner of weight at most $\hat{w}$.

Otherwise (there is no proper partition of X), for every $X' \subset X$ we have either $R/2 + 1 \leqslant \sum_{x \in X'} x$ or $\sum_{x \in X'} x \leqslant R/2 - 1$. Note that our rounding guarantees that the y-coordinates of all points either increased or decreased in at most $O(nR/2^{k^2}) = O(k2^k/2^{k^2}) = O(k/2^{k^2-k})$. Hence, for sufficiently large value k, every graph obtained by applying a set of t-shortcuts either exceed the weight bound $\hat{w}$ or does not admit a t-spanner for P.

Thus, there exists a value k_0, such that the reduction from the Partition problem restricted to instances with $n \geqslant k_0$ that returns $(\hat{P}, \hat{w})$ as described above is correct. Obviously, the Partition problem remains NP-hard after restricting n to be larger than some constant k_0, which implies that the LWSt problem is NP-hard for $1 < t < 2$.

In conclusion, after presenting proper reductions for all values of t, the main theorem follows.

Theorem 1. *The decision problems LWSt and LWPSt, and the optimization problems MWSt, MWPSt, and MDG are NP-hard for every constant $t > 1$.*

Although, we presented reductions to the LWSt and the LWPSt problems in which the underlying graph is restricted to be the complete Euclidean graph, the reductions apply to variants of these problems in which the underlying graph is more general, e.g., general geometric graph (not necessarily a complete graph) and general weighted graph (not necessarily geometric). Therefore, the results presented in Theorem 1 imply the following corollary.

Corollary 4. *The variants of the decision problems LWSt and LWPSt and the optimization problems MWSt, MWPSt, and MDG addressing a geometric or general weighted underlying graph are NP-hard for every constant $t > 1$.*

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