Possible Research Topics USRA 2025

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Geometric, Approximation, and Distributed Algorithms (GADA) lab

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Possible Research Topics

1. Geometric Spanning Trees and Hypergraphs that Minimizes the Wiener Index

2. Inside-Out Dissection

Wiener Index in Graphs

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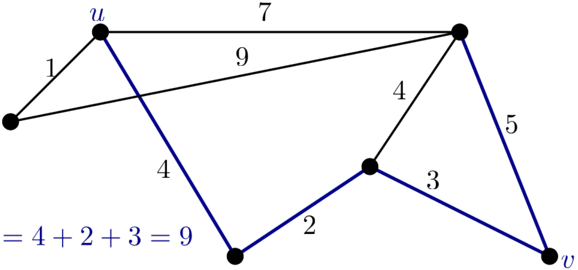

$\delta_G(u, v) = 4 + 2 + 3 = 9$

Wiener Index in Network Design

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The Minimum Routing Cost Spanning Tree (MRCST) problem

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- The MRCST problem is NP-Complete. There exists a PTAS for MRCST.

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2. Goal: Construct a **spanning tree** on P that minimizes the **Wiener index**.
3. The weight function is defined as the Euclidean distance between points.

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Results Summary: Geometric Spanning Trees

- Spanning tree of P that minimizes the Wiener index is planar.
- When P is in convex position, this can be solved in polynomial time.
- The hamiltonian path of P that minimizes the Wiener index is not necessarily planar.
- Computing such a hamiltonian path is NP-Hard.

Hypergraphs and their Wiener Index

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A **hypergraph** $H = (V, E)$ is a pair where V is a set of vertices and E is a set of hyperedges, each of which is a subset of V.

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Distance in Hypergraphs

The **distance** between two vertices u and v in a hypergraph H is defined as the minimum number of hyperedges in a chain connecting u and v.

- A **chain** is a sequence of hyperedges where each consecutive pair shares at least one vertex.
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k-uniform hypergraph

A hypergraph is **k-uniform** if every hyperedge has cardinality k.

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The **Wiener index** of a hypergraph H is defined as:

$$W(H) = \sum_{u,v \in V} \delta_H(u, v)$$

Some facts about HyperGraphs

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- Finding a spanning tree of general hypergraphs is NP-Complete, while spanning tree of 3-uniform hypergraphs has a polynomial time algorithm using matroid matching.
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- Finding a spanning tree of general hypergraphs is NP-Complete, while spanning tree of 3-uniform hypergraphs has a polynomial time algorithm using matroid matching.
[Goodall and de Mier, 2011]
- Finding a spanning tree of a k-uniform 2-regular hypergraph is NP-Complete for any $k \geq 4$. [Demaine and Rudoy, 2018]

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Side note: We could also explore **spatially embedded hypergraphs**, drawing inspiration from [Abu-Affash et al., 2023], and their Wiener index.

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Inside-out dissection

Let P be a polygon (polyhedron). An **inside-out dissection** of P is a decomposition of P into finitely many polygons (polyhedra) $P_1, \ldots, P_k$ such that:

- $P_1, \ldots P_k$ can be rearranged by only applying rotations and translations to form a polygon (polyhedron) P' that is congruent to P.
- The boundary of P' is composed of internal cuts of P.

Visualizing the Problem

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Chalk and Talk

Recent Work Summary

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    - If a polyhedron can be tiled with regular tetrahedra and octahedra, it can be inside-out dissected.

Recent Work Summary

- General n-gon: inside-out dissection possible with at most $2n + 1$ pieces.
- Regular polygons (e.g., equilateral triangle, square, regular pentagon): at most 6 pieces suffice.
- Extension to 3D:
  - If a polyhedron can be tiled with regular tetrahedra and octahedra, it can be inside-out dissected.
- Tools used include symmetry arguments and constructive dissections.

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- Can general n-gons be inside-out dissected with lesser than $2n+1$ pieces? Maybe convex polygons can be done with constant pieces like regular polygons?

Open Work Presented in the Main Reference

- Can a triangle be inside-out dissected with 3 pieces?
- Can general n-gons be inside-out dissected with lesser than $2n + 1$ pieces? Maybe convex polygons can be done with constant pieces like regular polygons?
- They have a method of general polyhedra, but the number of pieces required is quite large. Therefore, an efficient method for inside-out dissections of polyhedra is still open.

References I

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The end.

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