STATISTICAL PHYSICS AND GRAPH LIMITS LECTURE NOTE

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1. Introduction

This lecture note concerns with some very basic problems in statistical physics. These particular problems are coming from lattice gas models, and have some combinatorial flavor. To describe an exemplary problem let us consider the d-dimensional grid, $\mathbb{Z}^d$, and let us consider some configuration on it, for instance, an independent set of it. To understand the behavior of this configuration, it is worth first considering a large box of $\mathbb{Z}^d$, say box B_n of size $n \times \cdots \times n$, and understand the behavior of the configuration there. For instance, we may wish to understand what is the expected fraction of the box that is covered by a random independent set chosen uniformly at random from all independent sets of the box B_n. As n goes to infinity we might (or might not) expect that these fractions will converge to some number. If we try to translate this problem to a more general setting, then it is natural to look at the sequence of boxes as a converging graph sequence and $\mathbb{Z}^d$ as a limit of this graph sequence. The proper understanding of this convergence will lead to the definition of the Benjamini–Schramm convergence. The problem of the convergence of the independence ratio, the fraction of vertices covered by a random independent set, is a special case of the more general question of describing those graph parameters that are continuous with respect to the topology determined by the Benjamini–Schramm convergence.

At this point it might look that the only goal of this graph convergence is to extend some graph parameter from finite graphs to infinite ones. Surprisingly, sometimes the other direction is also fruitful. Here the understanding of the infinite object is easy, and then this leads to a better understanding of the finite version of the problem.

This lecture note is organized as follows. In the next section we describe some statistical physical models. In the third section we introduce the Benjamini–Schramm convergence. In Section 4 we give various results about dimer and monomer-dimer models. In particular, we study the number of dimer configurations of grid graphs, and we introduce the so-called matching polynomial. In Section 5 we apply graph limit theory to the monomer-dimer model.

2. Statistical physical models

In this section we collected some very basic statistical physical models.

2.1. **Ising-model.** Ising-model was introduced to model the phenomenon that if we heat up a magnetic metal, then it eventually lose its magnetism. In this model the vertices of the graph G represent particles. These particles have a spin that can be up $(+1)$ or down (-1). Two adjacent particles have an interaction e^{β} if they have the same spins, and $e^{-\beta}$ if they have different spins (explanation of this sentence comes soon). In the statistical physics literature β is called the inverse temperature. If the temperature is very small, then β is very large and adjacent vertices have the same spin with high probability as we will see soon. On the other hand, if the temperature is very large, then β is very small so the spins of adjacent vertices will be more or less uncorrelated. Suppose also that there is an external magnetic field that breaks the symmetry between $+1$ and -1. This defines a probability distribution on the possible configurations as follows: for a random spin configuration $\mathbf{S}$:

$$\mathbb{P}(\mathbf{S}=\sigma)=\frac{1}{Z}\exp\left(\sum_{(u,v)\in E(G)}\beta\sigma(u)\sigma(v)+B\sum_{u\in V(G)}\sigma(u)\right),$$

where Z is the normalizing constant:

$$Z_{\mathrm{Is}}(G,B,\beta)=\sum_{\sigma:V(G)\to\{-1,1\}}\exp\left(\sum_{(u,v)\in E(G)}\beta\sigma(u)\sigma(v)+B\sum_{u\in V(G)}\sigma(u)\right).$$

Here Z is the so-called partition function of the Ising-model. Now we can see that if β is large, then according to this distribution it is much more likely that adjacent vertices will have the same spin, and so the sum of the spins will have large absolute value which measures the magnetism of the metal.

A priori we assumed so far that $\beta>0$, because this case provides the proper physical intuition. On the other hand, the model makes perfect sense mathematically even if $\beta<0$. When β is a negative number with large absolute value, then the system favors configurations where most pairs of adjacent vertices have different spins. When $\beta>0$ we say that it is a ferromagnetic Ising-model, and when $\beta<0$, then we say that it is an antiferromagnetic model. As one might expect it, the model behaves very differently in the two regimes.

2.2. **Monomer-dimer and dimer model.** For a graph G let $\mathcal{M}(G)$ be the set of all matchings. Recall that a matching M of G is simply a set of edges such that no two edges in the set intersect each other. When this set has k edges, then we say that it is a k-matching or alternatively, the matching M is of size k (and it covers $2k$ vertices). For a $\lambda>0$ we can associate a probability space on $\mathcal{M}(G)$ by choosing a random matching $\mathbf{M}$ as follows:

$$\mathbb{P}(\mathbf{M}=M)=\frac{\lambda^{|M|}}{M(G,\lambda)},$$

λ : same parameter

where $M(G,\lambda)$ is the normalizing constant:

$$M(G,\lambda)=\sum_{M\in\mathcal{M}(G)}\lambda^{|M|}.$$

This model is the monomer-dimer model. The name has the following origin. In statistical physics the vertices of the graph represent particles, and edges represent some sort of interaction between certain pair of particles. A dimer is a pair of particles where the interaction is active. Supposing that one particle can be active with at most one other particle, we get that the dimers form a matching. The uncovered vertices are called monomers. We say that $M(G,\lambda)$ is the partition function of the monomer-dimer model. In mathematics it is called the matching generating function. Let $m_k(G)$ denote the number of k-matchings. Then

$$M(G,\lambda)=\sum_k m_k(G)\lambda^k.$$

Note that the sum runs from $k=0$ as the empty set is a matching by definition. Naturally, once we introduced a probability distribution we can ask various natural questions like what is $\mathbb{E}|\mathbf{M}|$. It is not hard to see that

$$\boxed{\mathbb{E}|\mathbf{M}| = \sum_{M \in \mathcal{M}(G)} \frac{|M|\lambda^{|M|}}{M(G,\lambda)} = \frac{\lambda M'(G,\lambda)}{M(G,\lambda)}.}$$

If G has $2n$ vertices, then we call an n-matching a perfect matching as it covers all vertices. The dimer model is the model where we consider a uniform distribution on the perfect matchings. Clearly, a dimer model is a monomer-dimer model without monomers. The number of perfect matchings is denoted by $\mathrm{pm}(G)$. With our previous notation we have $\mathrm{pm}(G) = m_n(G)$.

2.3. **Hard-core model.** For a graph G let $\mathcal{I}(G)$ be the set of all independent sets. Recall that an independent set is a subset I of the vertices such that no two elements of I is adjacent in G. Here the vertices of G represent possible places for particles that repulse each other so that no two adjacent vertex can be occupied by particles. For a $\lambda > 0$ we can associate a probability space on $\mathcal{I}(G)$ by choosing a random independent set $\mathbf{I}$ as follows:

$$\mathbb{P}(\mathbf{I} = I) = \frac{\lambda^{|I|}}{I(G,\lambda)},$$

where $I(G,\lambda)$ is the normalizing constant:

$$I(G,\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|M|}.$$

Then $I(G,\lambda)$ is the partition function of the hard-core model. In mathematics it is called the independence polynomial of the graph G. Let $i_k(G)$ denote the number of independent sets of size k. Then

$$I(G,\lambda) = \sum_{k=0}^{n} i_k(G)\lambda^k.$$

Note that the sum runs from $k = 0$ as the empty set is an independent set by definition. Similarly to the case of the matchings we have

$$\mathbb{E}|\mathbf{I}| = \sum_{I \in \mathcal{I}(G)} \frac{|I|\lambda^{|I|}}{I(G,\lambda)} = \frac{\lambda I'(G,\lambda)}{I(G,\lambda)}.$$

Hard-core model describes models with repulsive interactions. For instance, if we pack spheres of radius 1 into some space, then it is a hard-core model where the vertices of the graph are all possible points that can be a center of a sphere and two vertices are adjacent if they have distance at least 2. If we discretize this problem, then maybe we can allow only the points of a lattice to be the center of a sphere.

3. Graph limits and examples

In the introduction we have seen that it would be useful if we can consider a lattice as a limit object of its large boxes. This establishes a claim to handle infinite graphs and connect them to the theory of finite graphs. This is exactly the goal of this section. In what follows we introduce the concept of Benjamini–Schramm convergence with some examples. We will see that this concept will be much more flexible than just considering lattices and its subgraphs.

Before we define this concept one more remark is in order: in this lecture note we will always assume that there is some Δ such that the largest degree of any graph G_i in a given sequence of graphs is at most Δ. In such a case we say that the graph sequence (G_i) is a *bounded-degree* graph sequence. This assumption simplifies our task significantly.

Definition 3.1. For a finite graph G, a finite connected rooted graph α and a positive integer r, let $\mathbb{P}(G,\alpha,r)$ be the probability that the r-ball centered at a uniform random vertex of G is isomorphic to α.

Let L be a probability distribution on (infinite) connected rooted graphs; we will call L a *random rooted graph*. For a finite connected rooted graph α and a positive integer r, let $\mathbb{P}(L,\alpha,r)$ be the probability that the r-ball centered at the root vertex is isomorphic to α, where the root is chosen from the distribution L.

We say that a bounded-degree graph sequence (G_i) is *Benjamini–Schramm convergent* if for all finite rooted graphs α and $r > 0$, the probabilities $\mathbb{P}(G_i,\alpha,r)$ converge. Furthermore, we say that (G_i) *Benjamini-Schramm converges to* L, if for all positive integers r and finite rooted graphs α, $\mathbb{P}(G_i,\alpha,r) \to \mathbb{P}(L,\alpha,r)$. Def^n

The Benjamini–Schramm convergence is also called *local convergence* as it primarily grasps the local structure of the graphs (G_i).

If we take larger and larger boxes in the d-dimensional grid $\mathbb{Z}^d$, then it will converge to the rooted $\mathbb{Z}^d$, that is, the corresponding random rooted graph L is simply the distribution which takes a rooted $\mathbb{Z}^d$ with probability 1. When L is a certain rooted infinite graph with probability 1 then we simply say that this rooted infinite graph is the limit without any further reference on the distribution.

There are other very natural graph sequences which are Benjamini–Schramm convergent, for instance, if (G_i) is a sequence of d–regular graphs such that the girth $g(G_i) \to \infty$ (length of the shortest cycle), then it is Benjamini–Schramm convergent and we can even see its limit object: the rooted infinite d-regular tree $\mathbb{T}_d$

The following problem is one of the main problems in the area, and will be especially crucial for us.

Problem. For which graph parameters $p(G)$ is it true that the sequence $(p(G_i))_{i=1}^{\infty}$ converges whenever the graph sequence $(G_i)_{i=1}^{\infty}$ is Benjamini–Schramm convergent?

The problem in such a generality is intractable, but there are various tools to attack it in special cases. One of the most popular tools is the so-called belief propagation. For matchings we will use another way to attack this problem using certain empirical measures called matching measures.

Concerning the general problem the reader might wish to consult with the papers [4, 6, 7, 8, 21, 22], the book [16] and the references therein.

4. Monomer-dimer and dimer model

In this section we study the monomer-dimer and dimer models. In the first part we prove a result of Kasteleyn [13], and independently Fisher and Temperley [23], about the number of perfect matchings of grid graphs. Then we give a lower bound on the number of perfect matchings of regular bipartite graphs (though the proof will be given in Section 5). Finally, we introduce the concept of matching polynomial and review its basic properties.

4.1. **Dimer model on the lattice.** In this section we study the number of perfect matchings of grid graphs. It turns out that if G is a bipartite graph with classes of size n, then the problem of counting the number of perfect matchings of G is equivalent to computing the permanent of a $0-1$ matrix of size n by n. Recall that the permanent of a matrix A is defined as follows:

$$\mathrm{per}(A)=\sum_{\pi\in S_n} a_{1,\pi(1)}a_{2,\pi(2)}\dots a_{n,\pi(n)}.$$

Let us suppose for a moment that all $a_{ij}\in\{0,1\}$, and define a graph G on the vertex set $R\cup C$, where $R=\{r_1,r_2,\dots,r_n\}$ and $C=\{c_1,c_2,\dots,c_n\}$ correspond to the rows and columns of the matrix, respectively. If $a_{ij}=1$, then put an edge between the vertices r_i and c_j. Now it is clear that $\mathrm{per}(A)=\mathrm{pm}(G)$, the number of perfect matchings of G. Unfortunately, permanents are hard to compute in spite of the fact that their siblings, determinants can be computed in polynomial time. Still we can use their similarity as the proof of the following theorem shows.

Theorem 4.1 (Kasteleyn [13] and independently Fisher and Temperley [23])**.** *Let $Z_{m,n}$ be the number of perfect matchings of the grid of size $m\times n$. Then*

For grid graph

$$Z_{m,n}=\left(\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\left(\frac{\pi j}{m+1}\right)+4\cos^2\left(\frac{\pi k}{n+1}\right)\right)\right)^{1/4}.$$

Proof. Note that the grid is a bipartite graph, so we can color the vertices of the grid by black and white such that only vertices of different colors are adjacent. Let S be the incidence matrix (also called bipartitie adjacency matrix) of the bipartite graph: $S_{ij}=1$ if black and white vertices b_i and w_j are adjacent, and 0 otherwise. Then the number of perfect matchings is exactly $\mathrm{per}(S)$, the permanent of S. We will give a "signing" σ of S such that $\mathrm{per}(S)=|\det(S^\sigma)|$.

Let $S^\sigma_{(x,y),(x,y\pm1)}=i$ and $S^\sigma_{(x,y),(x\pm1,y)}=1$, and 0 otherwise. We claim that $\mathrm{per}(S)=|\det(S^\sigma)|$. One way to see it is the following: from any perfect matching M_1 we can arrive to any other perfect matching M_2 by a sequence of moves of the following type: choose two edges of the form $e=((x,y),(x+1,y)), f=((x,y+1),(x+1,y+1)$ and replace them by $e'=((x,y),(x,y+1)), f'=((x+1,y),(x+1,y+1)$, or do the reverse of this operation (why?). In $\det(S^\sigma)$ this operation does the following thing: the sign of the corresponding permutation changes because we did a transposition, but also the weight of the perfect matching changes since we changed two edges of weight 1 to two edges of weight i or vice versa. So every expansion term of $\det(S^\sigma)$ corresponding to a perfect matching will give the same quantity, hence $\mathrm{per}(S)=|\det(S^\sigma)|$.

Next we will compute $\det(S^\sigma)$. It will be more convenient to work with the matrix

$$A=\begin{pmatrix}0 & S^\sigma\\ (S^\sigma)^T & 0\end{pmatrix}.$$

Clearly, $\det(A)=\det(S^\sigma)^2$. It turns out that we can give all eigenvectors and eigenvalues explicitly. Note that the vector consisting of the values $f(x,y)$ is an eigenvector of A belonging to the eigenvalue λ if

$$\lambda f(x,y)=f(x+1,y)+f(x-1,y)+if(x,y+1)+if(x,y-1),$$

where $f(r,t)=0$ if $r\in\{0,m+1\}$ or $t\in\{0,n+1\}$. Let $1\le j\le m$, $1\le k\le n$, and $z=e^{2\pi i\frac{j}{m+1}}$ and $w=e^{2\pi i\frac{k}{n+1}}$. Let us consider the vector $f_{j,k}$ defined as follows:

$$f_{j,k}(x,y)=(z^x-z^{-x})(w^y-w^{-y})=-4\sin\left(\frac{\pi jx}{m+1}\right)\sin\left(\frac{\pi ky}{n+1}\right).$$

Then with $\lambda_{j,k} = z + \frac{1}{z} + i\left(w + \frac{1}{w}\right)$ we have

$$\lambda_{j,k} f_{j,k}(x,y) = f_{j,k}(x+1,y) + f_{j,k}(x-1,y) + i f_{j,k}(x,y+1) + i f_{j,k}(x,y-1).$$

Indeed,

$$f_{j,k}(x+1,y) + f_{j,k}(x-1,y) = (z^{x+1} - z^{-x-1})(w^y - w^{-y}) + (z^{x-1} - z^{-x+1})(w^y - w^{-y}) =$$

$$= (z + z^{-1})(z^x - z^{-x})(w^y - w^{-y}) = (z + z^{-1}) f_{j,k}(x,y),$$

and

$$i f_{j,k}(x,y+1) + i f_{j,k}(x,y-1) = i((z^x - z^{-x})(w^{y+1} - w^{-y-1}) + (z^x - z^{-x})(w^{y-1} - w^{-y+1}) =$$

$$= i(w + w^{-1})(z^x - z^{-x})(w^y - w^{-y}) = i(w + w^{-1}) f_{j,k}(x,y),$$

It is easy to see that the vectors $f_{j,k}$ are pairwise orthogonal to each other, consequently they are linearly independent. Since there are nm eigenvalues of A, we have found all of them. Note that

$$\lambda_{j,k} = 2\cos\left(\frac{\pi j}{m+1}\right) + 2\cos\left(\frac{\pi k}{n+1}\right) i.$$

Hence

$$Z_{m,n} = \left(\prod_{j=1}^{m}\prod_{j=1}^{n} \lambda_{j,k}\right)^{1/2} = \left(\prod_{j=1}^{m}\prod_{j=1}^{n} |\lambda_{j,k}|^2\right)^{1/4} =$$

$$= \left(\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right)\right)\right)^{1/4}.$$

$\square$

Corollary 4.2. *We have*

$$\boxed{\lim_{\substack{m,n\to\infty \\ 2 \mid mn}} \frac{1}{mn} \log Z_{m,n}} = \frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \log(4\cos^2(x) + 4\cos^2(y))\,dx\,dy.$$

Grid graph in limit

Remark 4.3. Surprisingly, there is a nice expression for the above integral:

$$\frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \log(4\cos^2(x) + 4\cos^2(y))\,dx\,dy = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}.$$

4.2. **Lower bounds on the number of perfect matchings.** In the previous section we considered the number of perfect matchings of large grids. These graphs were almost 4–regular bipartite graphs, that is, almost all vertices had exactly 4 neighbors. It is a natural question to ask for a lower bound for the number of perfect matchings of d–regular bipartite graphs. This problem was solved in some approximate sense by Voorhoeve and Schrijver.

Theorem 4.4 (A. Schrijver [19], for $d = 3$ M. Voorhoeve [25])**.** *Let G be a d–regular bipartite graph on $v(G) = 2n$ vertices, and let $\mathrm{pm}(G)$ denote the number of perfect matchings of G. Then*

$$\mathrm{pm}(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n.$$

In other words, for every d–regular bipartite graph G we have

d-regular bipartite

$$\frac{\ln \mathrm{pm}(G)}{v(G)} \geq \frac{1}{2} \ln \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right).$$

It turns out that the constant $\frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)$ is the best possible constant as it was shown by Wilf [26], see also [3, 20]. They showed it by computing the expected value of $\mathrm{pm}(G)$ for d–regular random bipartite graphs. There was no explicit construction for regular bipartite graphs with small number of perfect matchings for a long time. Very recently it turned out that if a d–regular bipartite graph has small number of short cycles, then it has asymptotically the same number of perfect matchings as a random d–regular graph, the more precise formulation is the following.

Theorem 4.5 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1])**.** *Let (G_i) be a sequence of d–regular graphs such that $g(G_i) \to \infty$, where g denotes the girth, that is, the length of the shortest cycle.*

(a) For the number of perfect matchings $\mathrm{pm}(G_i)$, *we have*

$$\limsup_{i\to\infty} \frac{\ln \mathrm{pm}(G_i)}{v(G_i)} \leq \frac{1}{2} \ln \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right).$$

(b) If, in addition, the graphs (G_i) are bipartite, then

$$\lim_{i\to\infty} \frac{\ln \mathrm{pm}(G_i)}{v(G_i)} = \frac{1}{2} \ln \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right).$$

d-reg bipartite
in lim as girth $\to \infty$

This theorem, more precisely, the condition $g(G_i) \to \infty$ together with our knowledge on Benjamini–Schramm convergence suggests that the quantity $\frac{\ln \mathrm{pm}(G)}{v(G)}$ is minimized by the infinite d–regular tree $\mathbb{T}_d$. Indeed, we will give a proof of Theorem 4.4 based on this intuition in the next section. In fact, we will prove the following more general theorem.

Theorem 4.6 ([5])**.** *Let G be a d–regular bipartite graph on $v(G) = 2n$ vertices, and let $m_k(G)$ denote the number of matchings of size k. Let $0 \leq p \leq 1$, then*

$$\sum_{k=0}^{n} m_k(G) \left(\frac{p}{d} \left(1 - \frac{p}{d} \right) \right)^k (1-p)^{2(n-k)} \geq \left(1 - \frac{p}{d} \right)^{nd}.$$

Observe that in case of $p = 1$ this theorem directly reduces to Theorem 4.4. To prove this theorem, we will need some preparation, and so the rest of this section is devoted to the study of the so-called matching polynomial.

4.3. **Matching polynomial.** Recall that if $G = (V, E)$ is a finite graph, then $v(G)$ denotes the number of vertices, and $m_k(G)$ denotes the number of k-matchings ($m_0(G) = 1$). Let

$$\boxed{\mu(G, x) = \sum_{k=0}^{\lfloor v(G)/2 \rfloor} (-1)^k m_k(G) x^{v(G)-2k}.}$$

We call $\mu(G, x)$ the matching polynomial. Clearly, the matching generating function $M(G, \lambda)$ introduced in Section 2 and the matching polynomial encode the same information.

Proposition 4.7 ([12, 10])**.** *(a) Let $u \in V(G)$. Then*

$$\mu(G,x) = x\mu(G-u,x) + \sum_{v \in N(u)} \mu(G - \{u,v\}, x).$$

(b) For $e=(u,v) \in E(G)$ we have

$$\mu(G,x) = \mu(G-e,x) + \mu(G-\{u,v\},x).$$

(c) For $G = G_1 \cup G_2 \cup \cdots \cup G_k$ we have

$$\mu(G,x) = \prod_{i=1}^{k} \mu(G_i, x).$$

(d) We have

$$\mu'(G,x) = \sum_{u \in V(G)} \mu(G-u,x).$$

Proof. (a) By comparing the coefficients of x^{n-2k} we need to prove that

$$m_k(G) = m_k(G-u) + \sum_{v \in N(u)} m_{k-1}(G - \{u,v\}).$$

This is indeed true since we can count the number of k-matchings of G as follows: there are $m_k(G-u)$ k-matchings which do not contain u, and if a k-matching contains u, then there is a unique $v \in N(u)$ such that the edge (u,v) is in the matching, and the remaining $k-1$ edges are chosen from $G - \{u,v\}$.

(b) By comparing the coefficient of x^{n-2k} we need to prove that

$$m_k(G) = m_k(G-e) + m_{k-1}(G - \{u,v\}).$$

This is indeed true since the number of k-matchings not containing e is $m_k(G-e)$, and the number of k-matchings containing $e=(u,v)$ is $m_{k-1}(G-\{u,v\})$.

(c) It is enough to prove the claim when $G = G_1 \cup G_2$, for more components the claim follows by induction. By comparing the coefficient of x^{n-2k} we need to prove that

$$m_k(G) = \sum_{r=0}^{k} m_r(G_1) m_{k-r}(G_2).$$

This is indeed true since a k-matching fof G uniquely determine an r-matching of G_1 and a $(k-r)$-matching of G_2 for some $0 \leq r \leq k$.

(d) This follows from the fact that

$$(m_k(G)x^{n-2k})' = (n-2k)m_k(G)x^{n-1-2k} = \sum_{u \in V(G)} m_k(G-u)x^{n-1-2k}$$

since we can compute the cardinality of the set

$$\{(M,u) \mid u \notin V(M), |M| = k\}$$

in two different ways. □

Theorem 4.8 (Heilmann and Lieb [12])**.** *All zeros of the matching polynomial $\mu(G,x)$ are real.*

Proof. We will prove the following two statements by induction on the number of vertices.

(i) All zeros of $\mu(G,x)$ are real.

(ii) For an x with $\mathrm{Im}(x) > 0$ we have

$$\mathrm{Im}\frac{\mu(G,x)}{\mu(G-u,x)} > 0$$

for all $u \in V(G)$.

Note that in (ii) we already use the claim (i) inductively, namely that $\mu(G-u,x)$ does not vanish for an x with $\mathrm{Im}(x) > 0$. On the other hand, claim (ii) for G implies claim (i). So we need to check claim (i).

By the recursion formula we have

$$\frac{\mu(G,x)}{\mu(G-u,x)} = \frac{x\mu(G-u,x) - \sum_{v\in N(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)} = x - \sum_{v\in N(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)}.$$

By induction we have

$$\mathrm{Im}\frac{\mu(G-u,x)}{\mu(G-\{u,v\},x)} > 0$$

for $\mathrm{Im}(x) > 0$. Hence

$$-\mathrm{Im}\frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} > 0$$

which gives that

$$\mathrm{Im}\frac{\mu(G,x)}{\mu(G-u,x)} > 0.$$

□

Remark 4.9. One can also show that the zeros of $\mu(G,x)$ and $\mu(G-u,x)$ interlace each other just like the zeros of a real-rooted polynomial and its derivative.

Theorem 4.10 (Heilmann and Lieb [12])**.** *If the largest degree Δ is at least 2, then all zeros of the matching polynomial lie in the interval $(-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1})$.*

Proof. First we show that if u is a vertex of degree at most $\Delta - 1$, then for any $x \geq 2\sqrt{\Delta-1}$ we have

$$\frac{\mu(G,x)}{\mu(G-u,x)} \geq \sqrt{\Delta-1}.$$

We prove this statement by induction on the number of vertices. This is true if $G = K_1$, so we can assume that $v(G) \geq 2$. Then

$$\begin{aligned}\frac{\mu(G,x)}{\mu(G-u,x)} &= \frac{x\mu(G-u,x) - \sum_{v\in N_G(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)} \\ &= x - \sum_{v\in N_G(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} \geq x - (\Delta-1)\frac{1}{\sqrt{\Delta-1}} \geq \sqrt{\Delta-1}.\end{aligned}$$

We used the fact that $v \in N_G(u)$ has degree at most $\Delta - 1$ in the graph $G - u$.

Then for any vertex u we have

$$\begin{aligned}\frac{\mu(G,x)}{\mu(G-u,x)} &= \frac{x\mu(G-u,x) - \sum_{v\in N_G(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)} \\ &= x - \sum_{v\in N_G(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} \geq x - \Delta\frac{1}{\sqrt{\Delta-1}} > 0\end{aligned}$$

since $v \in N_G(u)$ has degree at most $\Delta - 1$ in the graph $G - u$. This shows $\mu(G,x) \neq 0$ if $x \geq 2\sqrt{\Delta-1}$. Since the zeros of the matching polynomial are symmetric to 0 we get that all zeros lie in the interval $(-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1})$. □

Suppose that $\mu(G,x)=\prod_{i=1}^{v(G)}(x-\alpha_i)$. Our next goal is to understand the quantity $\sum_{i=1}^{v(G)}\alpha_i^k$ for some fixed k. It turns out that this quantity is a non-negative integer, and it has some combinatorial interpretation counting certain special walks. Next we will make it more explicit by the use of the so-called path-tree.

Before we proceed it is worth motivate and compare our results with the corresponding result for the characteristic polynomial of a graph G. For a graph G, the adjacency matrix of G denoted by A_G is defined as follows: A_G has size $v(G)\times v(G)$ and its rows and columns are labeled by the vertices of G, the element $(A_G)_{uv}=1$ if the vertices u and v are adjacent, and 0 otherwise. The characteristic polynomial of G is the characteristic polynomial of the adjacency matrix, $\phi(G,x)=\det(xI-A_G)$. Then $\phi(G,x)=\prod_{i=1}^{v(G)}(x-\lambda_i)$. The numbers λ_i are the eigenvalues of the matrix A_G, they are real since A_G is symmetric. Then

$$\sum_{i=1}^{v(G)}\lambda_i^k=\mathrm{Tr}(A_G^k)$$

since λ_i^k are the eigenvalues of A_G^k and the sum of the eigenvalues is the trace of the matrix, that is, the some of the diagonal elements. The quantity

$$\mathrm{Tr}(A_G^k)=\sum_{v_0\in V(G)}(A_G)_{v_0v_1}(A_G)_{v_1v_2}\dots(A_G)_{v_{k-1}v_0}$$

by the definition of matrix multiplication. The quantity $(A_G)_{v_0v_1}(A_G)_{v_1v_2}\dots(A_G)_{v_{k-1}v_0}$ is 1 if and only if $(v_j,v_{j+1})\in E(G)$ for $j=0,\dots,k-1$ and $v_k=v_0$, this is called a closed walk. So $\mathrm{Tr}(A_G^k)$ counts the number of closed walks of length k. This is determined by the statistics of the k-neighborhoods.

Definition 4.11. Let G be graph with a given vertex u. The *path-tree* $T(G,u)$ is defined as follows. The vertices of $T(G,u)$ are the paths[1] in G which start at the vertex u and two paths joined by an edge if one of them is a one-step extension of the other.

each node $v_1v_2v_3\dots u'$ in T corresponds to the last node u' in G along with the information of the path taken to reach u' from root of G

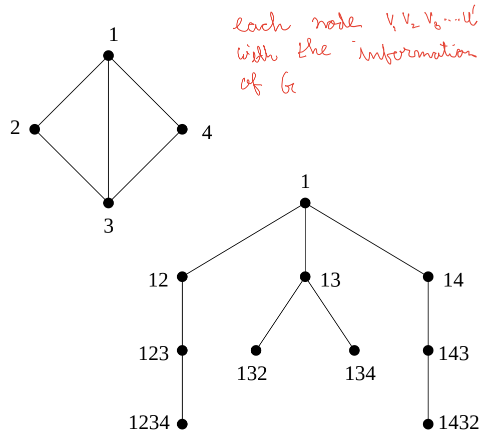

FIGURE 1. A path-tree from the vertex 1.

Proposition 4.12. *Let G be a graph with a root vertex u. Let $T(G,u)$ be the corresponding path-tree in which the root is again denoted by u for sake of convenience. Then*

$$\frac{\mu(G-u,x)}{\mu(G,x)}=\frac{\mu(T(G,u)-u,x)}{\mu(T(G,u),x)},$$

[1]In statistical physics, paths are called self-avoiding walks.

and $\mu(G,x)$ divides $\mu(T(G,u),x)$.

Proof. The proof of this proposition is again by induction using part (a) of Proposition 4.7. Indeed,

$$\frac{\mu(G,x)}{\mu(G-u,x)} = \frac{x\mu(G-u,x) - \sum_{v\in N(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)} =$$

$$= x - \sum_{v\in N(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} = x - \sum_{v\in N(u)} \frac{\mu(T(G-u,v)-v,x)}{\mu(T(G-u,v),x)}$$

$$= x\frac{\prod_{v\in N(u)} \mu(T(G-u,v),x) - \sum_{v\in N(u)} \mu(T(G-u,v)-v,x)\prod_{v'\in N(u)\setminus\{v\}} \mu(T(G-u,v'),x)}{\prod_{v\in N(u)} \mu(T(G-u,v),x)}$$

$$= \frac{x\mu(T(G,u)-u,x) - \sum_{v\in N(u)} \mu(T(G,u)-\{u,v\},x)}{\mu(T(G,u)-u,x)} = \frac{\mu(T(G,u),x)}{\mu(T(G,u)-u,x)}.$$

In the first step we used the recursion formula, and in the third step we used the induction step to the graph $G-u$ and root vertex v. Here it is an important observation that $T(G-u,v)$ is exactly the branch of the tree $T(G,u)$ that we get if we delete the vertex u from $T(G,u)$ and consider the subtree rooted at the path uv. □

Proposition 4.13 ([12, 10])**.** *For a forest T, the matching polynomial $\mu(T,x)$ coincides with the characteristic polynomial $\phi(T,x) = \det(xI - A_T)$.*

Proof. Indeed, when we expand the $\det(xI-A)$ we only get non-zero terms when the cycle decomposition of the permutation consists of cycles of length at most 2. These terms correspond to the terms of the matching polynomial.

□

Remark 4.14. Clearly, Propositions 4.12 and 4.13 together give a new proof of the Heilmann-Lieb theorem since $\mu(G,x)$ divides $\mu(T(G,u),x) = \phi(T(G,u),x)$ whose zeros are real since they are the eigenvalues of a symmetric matrix.

Proposition 4.15. *Let*

$$\frac{\mu(G-u,x)}{\mu(G,x)} = \sum_k a_k(G,u)x^{-(k+1)}.$$

Then $a_k(G,u)$ counts the number of closed walks of length k in the tree $T(G,u)$ from u to u.

Proof. This proposition follows from Proposition 4.12 and 4.13 and the fact that

$$\boxed{\frac{\phi(H-u,x)}{\phi(H,x)} = \sum_k W_k(H,u)x^{-(k+1)},}$$

its magic

where $W_k(H,u)$ counts the number of closed walks of length k from u to u in a graph H. Indeed,

$$\frac{\mu(G-u,x)}{\mu(G,x)} = \frac{\mu(T(G,u)-u,x)}{\mu(T(G,u),x)} = \frac{\phi(T(G,u)-u,x)}{\phi(T(G,u),x)} = \sum_k W_k(T(G,u),u)x^{-k}.$$

Here $W_k(T(G,u),u) = a_k(G,u)$ by definition. □

Remark 4.16. A walk in the tree $T(G,u)$ from u can be imagined as follows. Suppose that in the graph G a worm is sitting at the vertex u at the beginning. Then at each step the worm can either grow or pull back its head. When it grows it can move its head to a neighboring unoccupied vertex while keeping its tail at vertex u. At each step the worm occupies a path in the graph G. A closed walk in the tree $T(G,u)$ from u to u corresponds to the case when at the final step the worm occupies only vertex u. C. Godsil calls these walks tree-like walks in the graph G.

Proposition 4.17. *(a) Let*

$$\frac{\mu'(G,x)}{\mu(G,x)} = \sum_k a_k(G) x^{-(k+1)}.$$

Then $a_k(G)$ counts the number of closed tree-like walks of length k.

(b) If $\mu(G,x) = \prod_{i=1}^{v(G)}(x-\alpha_i)$, then for all $k \geq 1$ we have

$$a_k(G) = \sum_{i=1}^{v(G)} \alpha_i^k.$$

Remark 4.18. The quantity $a_k(G,u)$ makes perfect sense for a random rooted graph L as it only depends on the k neighborhood of the root vertex. So we can define:

$$a_k(L) = \sum_{\alpha} \mathbb{P}(L,\alpha,k)\mathrm{TW}_k(\alpha),$$

where $\mathrm{TW}_k(\alpha)$ is the number of closed tree-like walks of length k from the root vertex u of α in the rooted graph α.

5. A proof strategy

The goal of this section is to present an almost complete proof of Theorem 4.4 and 4.6. As we have discussed in section 4.2 certain intuition suggests that the extremal graph for (perfect) matchings is not finite, but the infinite d–regular tree $\mathbb{T}_d$. This raises the question: how can we attack a problem if we conjecture that the d–regular tree is the extremal graph for a given graph parameter:

Problem: Given a graph parameter $p(G)$. We would like to prove that among d–regular graphs we have

$$p(G) \geq p(\mathbb{T}_d).$$

A possible two-step solution.

- Find a graph transformation φ for which $p(G) \geq p(\varphi(G))$, and for every graph G there exists a sequence of graphs (G_i) such that $G = G_0$ and $G_i = \varphi(G_{i-1})$, and $G_i \to \mathbb{T}_d$.
- Show that if (G_i) converges in Benjamini–Schramm sense, then $p(G_i)$ is convergent, and compute $p(\mathbb{T}_d)$. (Or at least, show it in the case of $G_i \to \mathbb{T}_d$.) Then

$$p(G) = p(G_0) \geq p(G_1) \geq p(G_2) \geq \cdots \geq p(\mathbb{T}_d).$$

$\square$

Concerning the first step we will be more explicit: it seems that the 2-lift transformation can be used in a wide range of problems. Experience shows that the second step can be the most difficult, but the first step can also be tricky. Nevertheless, in the special case when we only consider a graph sequence converging to $\mathbb{T}_d$, there are many available tools: see for instance the paper of D. Gamarnik and D. Katz [9]. If $p(G) = \ln I(G, \lambda)/v(G)$, where $I(G, \lambda)$ denotes the partition function of the hard-core model, then the first step is very easy for d–regular bipartite graphs, while the second step concerning the limit theorem was established by A. Sly and N. Sun [21]. If $p(G) = \ln Z_{\mathrm{Is}}(G, B, \beta)/v(G)$, then both steps are somewhat tricky, but the first step is still easier.

In this section we demonstrate this plan by sketching the proof of Theorem 4.6. In the following sections we study each step separately.

5.1. **First step: graph transformation.** In this section we introduce the concept of 2-lift.

Definition 5.1. A k-cover (or k-lift) H of a graph G is defined as follows. The vertex set of H is $V(H) = V(G) \times \{0, 1, \ldots, k-1\}$, and if $(u, v) \in E(G)$, then we choose a perfect matching between the vertices (u, i) and (v, j) for $0 \le i, j \le k-1$. If $(u, v) \notin E(G)$, then there are no edges between (u, i) and (v, j) for $0 \le i, j \le k-1$.

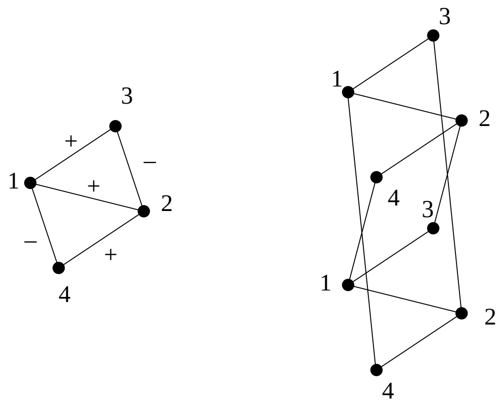

FIGURE 2. A 2-lift.

When $k = 2$ one can encode the 2-lift H by putting signs on the edges of graph G: the $+$ sign means that we use the matching $((u, 0), (v, 0)), ((u, 1), (v, 1))$ at the edge (u, v), the $-$ sign means that we use the matching $((u, 0), (v, 1)), ((u, 1), (v, 0))$ at the edge (u, v). For instance, if we put $+$ signs to every edge, then we simply get $G \cup G$ as H, and if we put $-$ signs everywhere, then the obtained 2-cover H is simply $G \times K_2$.

The following result will be crucial for our argument.

Lemma 5.2 (N. Linial [15]). *For any graph G, there exists a graph sequence $(G_i)_{i=0}^{\infty}$ such that $G_0 = G$, G_i is a 2-lift of G_{i-1} for $i \ge 1$, and $g(G_i) \to \infty$, where $g(H)$ is the girth of the graph H, that is, the length of the shortest cycle. In particular, if G_0 is d–regular, then $G_i \to \mathbb{T}_d$.*

Proof. It is clear that if H' is a 2-lift of H, then $g(H') \ge g(H)$. Hence it is enough to show that for every H there exists an H'' obtained from H by a sequence of 2-lifts such that $g(H'') > g(H)$. We show that if the girth $g(H) = k$, then there exists a lift of H with fewer k-cycles than H. Let X be the random variable counting the number of k-cycles in a random 2-lift of H. Every k-cycle of H lifts to two k-cycles or a $2k$-cycle with probability $1/2$ each, so $\mathbb{E}X$ is exactly the number of k-cycles of H. But $H \cup H$ has two times as many k-cycles than H, so there must be a lift with strictly fewer k-cycles than H has. Choose this 2-lift and iterate this step to obtain an H'' with girth at least $k+1$. $\square$

Note that if G is a bipartite d–regular graph, and H is a 2-lift of G, then H is again a d–regular bipartite graph.

The following theorem shows that the first step of the plan works for matchings of bipartite graphs.

Theorem 5.3. *Let G be a graph, and let H be an arbitrary 2-lift of G. Then*

$$m_k(H) \leq m_k(G \times K_2),$$

where $m_k(\cdot)$ denotes the number of matchings of size k.

In particular, if $H = G \cup G$, then $m_k(G \cup G) \leq m_k(G \times K_2)$ for every k. It follows that we have $\mathrm{pm}(G)^2 \leq \mathrm{pm}(G \times K_2)$.

Furthermore, if G is a bipartite graph and H is a 2-lift of G, then

$$\frac{\ln M(G,\lambda)}{v(G)} = \frac{\ln M(G \cup G, t)}{v(G \cup G)} \geq \frac{\ln M(H,\lambda)}{v(H)},$$

where $M(G,\lambda) = \sum_k m_k(G)\lambda^k$. (Note that $M(G \cup G, \lambda) = M(G,\lambda)^2$.)

Proof. Let M be any matching of a 2-lift of G. Let us consider the projection of M to G, then it will consist of cycles, paths and "double-edges" (i.e, when two edges project to the same edge). Let $\mathcal{R}$ be the set of these configurations. Then

$$m_k(H) = \sum_{R \in \mathcal{R}} |\phi_H^{-1}(R)|$$

and

$$m_k(G \times K_2) = \sum_{R \in \mathcal{R}} |\phi_{G \times K_2}^{-1}(R)|,$$

where ϕ_H and $\phi_{G \times K_2}$ are the projections from H and $G \times K_2$ to G. Note that

$$|\phi_{G \times K_2}^{-1}(R)| = 2^{k(R)},$$

where $k(R)$ is the number of cycles and paths of R. Indeed, in each cycle or path we can lift the edges in two different ways. The projection of a double-edge is naturally unique. On the other hand,

$$|\phi_H^{-1}(R)| \leq 2^{k(R)},$$

since in each cycle or path if we know the inverse image of one edge, then we immediately know the inverse images of all other edges. Clearly, there is no equality in general for cycles. Hence

$$|\phi_H^{-1}(R)| \leq |\phi_{G \times K_2}^{-1}(R)|$$

and consequently,

$$m_k(H) \leq m_k(G \times K_2).$$

Note that if G is bipartite, then $G \times K_2 = G \cup G$, and so

$$\frac{1}{v(H)} \ln M(H,\lambda) \leq \frac{1}{v(G \cup G)} \ln M(G \cup G, \lambda) = \frac{1}{2v(G)} \ln M(G,\lambda)^2 = \frac{1}{v(G)} \ln M(G,\lambda)$$

$\square$

Remark 5.4. Sometimes it is also possible to prove that for a certain graph parameter $p(\cdot)$ one has $p(G) \geq p(H)$ for all k-cover H of G. Such a result was given by N. Ruozzi in [18] for attractive graphical models. The advantage of using k-covers is that one can spare the graph limit step in the above plan, and replace it with a much simpler averaging argument over all k-covers of G with k converging to infinity. For homomorphisms this averaging argument was given by P. Vontobel [24]. For matchings such a result was established by C. Greenhill, S. Janson and A. Ruciński [11].

Exercise 5.5. Let G be a graph, and let H be an arbitrary 2-lift of G. Show that

$$i_k(H) \leq i_k(G \times K_2),$$

where $i_k(\cdot)$ denotes the number of independent sets of size k. Furthermore, show that if G is a bipartite graph and H is a 2-lift of G, then

$$\frac{\ln I(G,\lambda)}{v(G)} = \frac{\ln I(G \cup G,\lambda)}{v(G \cup G)} \geq \frac{\ln I(H,\lambda)}{v(H)},$$

where $I(G,\lambda) = \sum_k i_k(G)\lambda^k$, that is, the partition function of the hard-core model. (Note that $I(G \cup G,\lambda) = I(G,\lambda)^2$.)

Exercise 5.6. (hard) Let G be a graph, and let H be an arbitrary 2-lift of G. Show that if $\beta > 0$ then

$$\frac{\ln Z_{\mathrm{Is}}(G,B,\beta)}{v(G)} \geq \frac{\ln Z_{\mathrm{Is}}(H,B,\beta)}{v(H)},$$

where $Z_{\mathrm{Is}}(G,B,\beta)$ is the partition function of the Ising-model.

5.2. **Second step: graph limit theory.** In this subsection we carry out the second step of our plan. First we develop the necessary terminology.

Definition 5.7 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1])**.** The matching measure of a finite graph G is defined as

$$\boxed{\rho_G = \frac{1}{v(G)} \sum_{z_i:\ \mu(G,z_i)=0} \delta(z_i),}$$

where $\delta(s)$ is the Dirac-delta measure on s, and we take every z_i into account with its multiplicity. In other words, it is the uniform distribution on the zeros of $\mu(G,x)$.

Example: Let us consider the matching measure of C_6.

$$\mu(C_6,x) = x^6 - 6x^4 + 9x^2 - 2 =$$

$$= (x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{2+\sqrt{3}})(x+\sqrt{2+\sqrt{3}})(x-\sqrt{2-\sqrt{3}})(x+\sqrt{2-\sqrt{3}})$$

Hence

$$\int f(z)\, d\rho_{C_6}(z) =$$

$$= \frac{1}{6}\left(f(\sqrt{2}) + f(-\sqrt{2}) + f(\sqrt{2+\sqrt{3}}) + f(-\sqrt{2+\sqrt{3}}) + f(\sqrt{2-\sqrt{3}}) + f(-\sqrt{2-\sqrt{3}})\right).$$

The following theorem enables us to consider the matching measure of a unimodular random graph which can be obtained as a Benjamini–Schramm limit of finite graphs. In particular, it provides an important tool to establish the second step of our plan.

Theorem 5.8 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1])**.** *Let (G_i) be a Benjamini–Schramm convergent bounded degree graph sequence. Let ρ_{G_i} be the matching measure of the graph G_i. Then the sequence (ρ_{G_i}) is weakly convergent, i. e., there exists some measure ρ_L such that for every bounded continuous function f, we have*

$$\lim_{i\to\infty}\int f(z)\,d\rho_{G_i}(z)=\int f(z)\,d\rho_L(z).$$

Proof. For $k\geq 0$ let

$$\mu_k(G)=\int z^k\,d\rho_G(z)$$

be the k-th moment of ρ_G. By Proposition 4.17 we have

$$\mu_k(G)=\mathbb{E}_v a_k(G,v)$$

where $a_k(G,v)$ denotes the number of closed walks of length k of the tree $T(G,v)$ starting and ending at the vertex v.

Clearly, the value of $a_k(G,v)$ only depends on the k-ball centered at the vertex v. Let $\mathrm{TW}_k(\alpha)=a_k(G,v)$ where the k-ball centered at v is isomorphic to α. Note that the value of $\mathrm{TW}_k(\alpha)$ depends only on the rooted graph α and k and does not depend on G.

Let $\mathcal{N}_k$ denote the set of possible k-balls in G. The size of $\mathcal{N}_k$ and $\mathrm{TW}_k(\alpha)$ are bounded by a function of k and the largest degree of G. By the above, we have

$$\mu_k(G)=\mathbb{E}_v a_k(G,v)=\sum_{\alpha\in\mathcal{N}_k}\mathbb{P}(G,\alpha,k)\cdot\mathrm{TW}_k(\alpha).$$

Since (G_i) is Benjamini–Schramm convergent, we get that for every fixed k, the sequence of k-th moments $\mu_k(G_i)$ converges. The same holds for $\int q(z)\,d\rho_{G_i}(z)$ where q is any polynomial. By the Heilmann–Lieb theorem, ρ_{G_i} is supported on $[-2\sqrt{\Delta-1},2\sqrt{\Delta-1}]$ where Δ is the absolute degree bound for G_i. Since every continuous function can be uniformly approximated by a polynomial on $[-2\sqrt{\Delta-1},2\sqrt{\Delta-1}]$, we get that the sequence (ρ_{G_i}) is weakly convergent.

Assume that (G_i) Benjamini–Schramm converges to L. Then for all $k\geq 0$ we have $\mathbb{P}(G_n,\alpha_k,k)\to 1$ where α_k is the k-ball in L, which implies

$$\lim_{n\to\infty}\mu_k(G_n)=\lim_{n\to\infty}\sum_{\alpha\in\mathcal{N}_k}\mathbb{P}(G_n,\alpha,k)\cdot\mathrm{TW}_k(\alpha)=\sum_{\alpha}\mathbb{P}(L,\alpha,k)\cdot\mathrm{TW}_k(\alpha)=a_k(L)$$

where v is any vertex in L. This means that all the moments of ρ_L and $\lim\rho_{G_n}$ are equal, so $\lim\rho_{G_n}=\rho_L$. □

Remark 5.9. It is clear from the proof that the crucial point of the proof is that $\int z^k\,d\rho_G(z)$ can be determined by knowing the statistics of the k-balls. This phenomenon is not restricted to the matching polynomial. In fact, it is a very general phenomenon. A better-known example is that for the spectral measure, that is, for the uniform distribution on the eigenvalues of the adjacency matrix of the graph, this integral is determined by the number of closed walks of length k as we discussed before Definition 4.11.

To illustrate the power of Theorem 5.8, let us consider an application that also provides us the second step of our plan.

Theorem 5.10 (M. Abért, P. Csikvári, T. Hubai [2])**.** *Let (G_i) be a Benjamini–Schramm convergent graph sequence of bounded degree graphs. Then the sequences of functions*

$$\frac{\ln M(G_i,\lambda)}{v(G_i)}$$

is pointwise convergent.

Proof. If G is a graph on $v(G) = 2n$ vertices and

$$M(G,\lambda) = \prod_{i=1}^{v(G)/2} (1+\gamma_i\lambda),$$

then

$$\mu(G,x) = \prod_{i=1}^{v(G)/2} (x-\sqrt{\gamma_i})(x+\sqrt{\gamma_i}),$$

and therefore

$$\frac{\ln M(G,\lambda)}{v(G)} = \frac{1}{v(G)}\sum_{i=1}^{v(G)/2} \ln(1+\gamma_i\lambda) = \int \frac{1}{2}\ln(1+\lambda z^2)\, d\rho_G(z).$$

Since $\frac{1}{2}\ln(1+\lambda z^2)$ is a continuous function for every fixed positive λ, the theorem immediately follows from Theorem 5.8. □

Exercise 5.11. When we introduced the monomer-dimer model we have seen that the expected size of a random matching is

$$\mathbb{E}_G|\mathbf{M}| = \frac{\lambda M'(G,\lambda)}{M(G,\lambda)}.$$

Show that if (G_i) is a Benjamini–Schramm convergent graph sequence of bounded degree graphs, then $\mathbb{E}_{G_i}|\mathbf{M}|/v(G_i)$ is convergent.

It is worth introducing the notation

$$p_\lambda(G) = \frac{\ln M(G,\lambda)}{v(G)}.$$

By Theorem 5.10 we can also introduce $p_\lambda(L)$ if L is a Benjamini–Schramm-limit of a sequence of finite graphs (G_i). (In fact, from the proof it is clear that it is possible to define the function $p_\lambda(L)$ if L is not the Benjamini–Schramm-limit of finite graphs.) In particular, we can speak about $p_\lambda(\mathbb{T}_d)$.

If we know the matching measure of a random unimodular graph, then it is just a matter of computation to derive various results on matchings.

In the particular case when the sequence (G_i) converges to the infinite d–regular tree $\mathbb{T}_d$, the limit measure $\rho_{\mathbb{T}_d}$ turns out to be the so-called Kesten-McKay measure. It is true in general that for any finite tree or infinite random rooted tree the matching measure coincides with the so-called spectral measure, for details see [1]. In particular, this is true for the infinite d–regular tree $\mathbb{T}_d$. Its spectral measure is computed explicitly in the papers [14] and [17]. The Kesten-McKay measure is given by the density function

$$f_d(x) = \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}\chi_{[-\omega,\omega]},$$

where $\omega = 2\sqrt{d-1}$. Hence for any continuous function $h(z)$ we have

$$\int h(z)\, d\rho_{\mathbb{T}_d}(z) = \int_{-\omega}^{\omega} h(z) f_d(z)\, dz.$$

In particular,

$$p_\lambda(\mathbb{T}_d) = \int \frac{1}{2}\ln(1+\lambda z^2)\, d\rho_{\mathbb{T}_d}(z) = \frac{1}{2}\ln S_d(\lambda),$$

where

$$S_d(\lambda) = \frac{1}{\eta_\lambda^2}\left(\frac{d-1}{d-\eta_\lambda}\right)^{d-2} \quad \text{and} \quad \eta_\lambda = \frac{\sqrt{1+4(d-1)\lambda}-1}{2(d-1)\lambda}.$$

It is worth introducing the following substitution:

$$\lambda = \frac{\frac{p}{d}\left(1-\frac{p}{d}\right)}{(1-p)^2}.$$

As p runs through the interval $[0,1)$, λ runs through the interval $[0,\infty)$ and we have

$$\eta_\lambda = \frac{1-p}{1-\frac{p}{d}} \quad \text{and} \quad S_d(\lambda) = \frac{\left(1-\frac{p}{d}\right)^d}{(1-p)^2}.$$

5.3. **The end of the proof of Theorem 4.6.** For every sequence of 2-covers we know from Theorem 5.3 that

$$p_\lambda(G_0) \geq p_\lambda(G_1) \geq p_\lambda(G_2) \geq p_\lambda(G_3) \geq \dots$$

Furthermore, from Theorem 5.2 and 5.10 we know that we can choose the sequence of 2-covers such that the sequence $p_\lambda(G_i)$ converges to $p_\lambda(\mathbb{T}_d)$, hence $p_\lambda(G) \geq p_\lambda(\mathbb{T}_d)$ for any d–regular bipartite graph G. In other words,

$$\frac{1}{2n}\ln M(G,\lambda) \geq \frac{1}{2}\ln S_d(\lambda).$$

With the substitution $\lambda = \frac{\frac{p}{d}\left(1-\frac{p}{d}\right)}{(1-p)^2}$ we arrive to the inequality

$$M\left(G, \frac{\frac{p}{d}\left(1-\frac{p}{d}\right)}{(1-p)^2}\right) \geq \frac{1}{(1-p)^{2n}}\left(1-\frac{p}{d}\right)^n.$$

After multiplying by $(1-p)^{2n}$, we get that

$$\sum_{k=0}^{n} m_k(G)\left(\frac{p}{d}\left(1-\frac{p}{d}\right)\right)^k (1-p)^{2(n-k)} \geq \left(1-\frac{p}{d}\right)^{nd}.$$

This is true for all $p \in [0,1)$ and so by continuity it is also true for $p=1$, where it directly reduces to Schrijver's theorem since all but the last term vanish on the left hand side.

References

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