MPC HW2

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① $x(k+1) = \overset{A}{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} x(k) + \overset{B}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} u(k)$

$y(k) = \underset{C}{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}} x(k)$

Is this system observable? Is the system controllable?

Observability

$C = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$\begin{pmatrix} C \\ CA \end{pmatrix}$

$CA = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$

$$O = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} \underset{R_2+R_3=R_2}{\Rightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} \underset{R_3-R_1=R_3}{\Rightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 2 \end{pmatrix} \underset{2R_3+R_4=R_4}{\Rightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$\boxed{\text{Rank}(O) = 2 \quad \text{Observable}}$

Controllability

$(B \ \ AB)$

$B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\mathcal{C} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ $\boxed{\text{Rank}(\mathcal{C}) = 2 \quad \text{Controllable}}$

(2)

$$L(x,u) = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right)$$

Linear constraint $\Rightarrow$ $f(x,u) = x + mu + c$

a) Hamiltonian Function N fixed

$$\boxed{H(x_k, u_k) = \frac{1}{2}\left(\frac{x_k^2}{a^2} + \frac{u_k^2}{b^2}\right) + \lambda_{k+1}^T (x_k + m u_k + c)}$$

b) Find optimal u and x

state eqn: $x_{k+1} = \frac{\partial H_k}{\partial \lambda_{k+1}} = x_k + m u_k + c$

costate eqn: $\lambda_k = \frac{\partial H_k}{\partial x_k} = \frac{x_k}{a^2} + \lambda_{k+1}$

stationarity: $0 = \frac{\partial H_k}{\partial u_k} = \frac{u_k}{b^2} + \lambda_{k+1} m \Rightarrow u_k = -mb^2 \lambda_{k+1}$

* Plug in u_k into x_{k+1}

$$x_{k+1} = x_k - m^2 b^2 \lambda_{k+1} + c$$

$$x_k = x_{k+1} + m^2 b^2 \lambda_{k+1} - c$$

* Get λ_N terms for λ_k and λ_{k+1}

$$\lambda_k = \frac{x_k}{a^2} + \lambda_{k+1} = \frac{x_k + x_{k+1}}{a^2} + \lambda_{k+2} = \frac{x_k + x_{k+1} + x_{k+2}}{a^2} + \lambda_{k+3} = \frac{1}{a^2}\sum_{i=k}^{N-1} x_i + \lambda_N$$

$$\lambda_{k+1} = \frac{x_{k+1}}{a^2} + \lambda_{k+2}$$

$$\lambda_{k+2} = \frac{x_{k+2}}{a^2} + \lambda_{k+3}$$

$$\lambda_k = \frac{1}{a^2}\sum_{i=k}^{N-1} x_i + \lambda_N \quad ; \quad \lambda_{k+1} = \frac{1}{a^2}\sum_{i=k+1}^{N-1} x_i + \lambda_N$$

(2.1)

$$x_{k+1} = x_k + \underbrace{m^2 b^2}_{\gamma} \lambda_{k+1} + C \quad ; \quad \text{let } \gamma = m^2 b^2$$

★ Assume initial state is fixed, x_0

$$x_k = x_0 + \gamma \lambda_{k+1} + C$$

$$x_k = x_0 + \frac{\gamma}{a^2} \sum_{i=k}^{N-1} x_i + \lambda_N$$

★ Assume final state is fixed; $x_N = R_N$

$$x_N = x_0 + \frac{\gamma}{a^2}\left(0 \right) + \lambda_N = R_N$$

Note; When $\sum_{i=N}^{N-1} x_i$, the index is out of bounds. Boundary conditions sets this to 0.

$$\lambda_N = R_N - x_0$$

$$u_k^* = -mb^2\left(\frac{1}{a^2} \sum_{i=k+1}^{N-1}(x_i) + R_N - x_0\right)$$

$$x_{k+1}^* = x_k^* + m u_k^* + C$$

$$x_{k+1}^* = x_k^* - \gamma \lambda_{k+1} + C$$

$$x_k^* = x_0 - \gamma \lambda_k + C$$

$$= x_0 - \gamma\left(\frac{1}{a^2} \sum_{i=k}^{N-1}(x_i) + R_N - x_0\right) + C$$

$$x_k^* = x_0 - \gamma\left(\frac{1}{a^2} \sum_{i=k}^{N-1}(x_i) + R_N - x_0\right) + C$$

2.2

c) Find performance index L optimal value

$$L(x,u) = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right)$$

$$L(x_k^*, u_k^*) = \frac{1}{2}\left(\frac{x_k^{*2}}{a^2} + \frac{u_k^{*2}}{b^2}\right)$$

$$x_k^* = x_0 - \gamma\alpha + c \; ; \text{ where } \alpha = \left(\frac{1}{a^2}\sum_{i=k}^{N-1}(x_i) + R_N - x_0\right)$$

$$u_k^* = -mb^2\beta \qquad ; \text{ where } \beta = \left(\frac{1}{a^2}\sum_{i=k+1}^{N-1}(x_i) + R_N - x_0\right)$$

$$L(x_k^*, u_k^*) = \frac{1}{2}\left[\frac{(x_0 - \gamma\alpha + c)^2}{a^2} + \frac{(-mb^2\beta)^2}{b^2}\right)$$

③ $L(x,u) = \frac{1}{2}(x^T Q x + u^T R u)$

linear constraint;

$$f(x,u) = x + Bu + c = 0$$

$$Q \geq 0, R > 0$$

$x \in \mathbb{R}^n, u \in \mathbb{R}^m, f \in \mathbb{R}^n, \lambda \in \mathbb{R}^n$. Q and R are symmetric

a) Hamiltonian Function $\quad J = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (L(x,u))$; $S_N \geq 0$

$$\boxed{H(x_k, u_k) = \frac{1}{2}(x_k^T Q x_k + u_k^T R u_k) + \lambda_{k+1}^T (x_k + Bu_k + c)}$$

b) Find optimal u and x

state eqn: $\quad x_{k+1} = \frac{\delta H_k}{\delta \lambda_{k+1}} = x_k + Bu_k + c$

costate eqn: $\quad \lambda_k = \frac{\delta H_k}{\delta x_k} = Qx_k + \lambda_{k+1}$

stationarity: $\quad 0 = \frac{\delta H_k}{\delta u_k} = Ru_k + B^T \lambda_{k+1} \Rightarrow u_k = -R^{-1}B^T\lambda_{k+1}$

* Plug in u_k into x_{k+1}

$$x_{k+1} = x_k - BR^{-1}B^T\lambda_{k+1}$$

$$\{ x_k = x_{k+1} + BR^{-1}B^T\lambda_{k+1}$$

* Plug x_k into λ_k

$$\lambda_k = Q[x_{k+1} + BR^{-1}B^T\lambda_{k+1}] + \lambda_{k+1}$$

$$\{ \lambda_k = Qx_{k+1} + (QBR^{-1}B^T + 1)\lambda_{k+1}$$

$$\begin{pmatrix} x_k \\ \lambda_k \end{pmatrix} = \begin{pmatrix} 1 & BR^{-1}B^T \\ Q & QBR^{-1}B^T + 1 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix}$$

(3.1) - Fixed initial state, Free Final state $dx_N \neq 0,\ Q \neq 0.$

$$x_{k+1} = x_k - BR^{-1}B^T\lambda_{k+1}$$

$$\lambda_N = \frac{\delta\Phi}{\delta x_N} = S_N x_N$$

$$\lambda_k = Qx_k + \lambda_{k+1}$$

$$\Phi = \frac{1}{2} x_N^T S_N x_N$$

$\lambda_N = S_N x_N$ ☆ Sweep Method

$\lambda_k = S_k x_k \Rightarrow \lambda_{k+1} = S_{k+1}x_{k+1}$

$$x_{k+1} = x_k - BR^{-1}B^T S_{k+1} x_{k+1}$$

$$x_k = x_{k+1} + BR^{-1}B^T S_{k+1} x_{k+1} = x_{k+1}(I + BR^{-1}B^T S_{k+1}),$$

$$\underline{x_{k+1} = (I + BR^{-1}B^T S_{k+1})^{-1} x_k}$$

$$S_k x_k = \lambda_k = Qx_k + \lambda_{k+1}$$

$$= Qx_k + S_{k+1}x_{k+1}$$

$$S_k \cancel{x_k} = Q\cancel{x_k} + S_{k+1}(I + BR^{-1}B^T S_{k+1})^{-1}\cancel{x_k}$$

$$= Q + S_{k+1}(\underbrace{I}_{A} + \underbrace{B}_{B}\underbrace{R^{-1}}_{C}\underbrace{B^T S_{k+1}}_{D})^{-1}$$

☆ Matrix Inversion Lemma

$$(A^{-1} + BCD)^{-1} = A - AB(DAB + C^{-1})^{-1}DA$$

$$S_k = Q + S_{k+1}[I - B(B^T S_{k+1}B + R)^{-1}B^T S_{k+1}]$$

$$\underline{S_k = Q + S_{k+1} - S_{k+1}B(B^T S_{k+1}B + R)^{-1}B^T S_{k+1}}$$

Ricatti Equation

General form: $P_{t-1} = Q + A^T P_t A - A^T P_t B(B^T P_t B + R)^{-1}B^T P_t A$

$$\underline{S_k = Q + (S_{k+1}^{-1} + BR^{-1}B^T)^{-1}}$$

(3.2)

$$u_k = -R^{-1}B^T\lambda_{k+1} = -R^{-1}B^T S_{k+1}x_{k+1}$$
$$= -R^{-1}B^T S_{k+1}(x_k + Bu_k + c)$$
$$(I + R^{-1}B^T S_{k+1}B)u_k = -R^{-1}B^T S_{k+1}(x_k + c)$$

*Multiply R on bothsides

$$(R + B^T S_{k+1}B)u_k = B^T S_{k+1}(x_k + c)$$
$$= \underbrace{(R + B^T S_{k+1}B)^{-1}B^T S_{k+1}}_{K_k}(x_k + c)$$

$$u_k = K_k(x_k + c)$$

kalman gain: $K_k = \underline{(R + B^T S_{k+1}B)^{-1}B^T S_{k+1}}$

$$\boxed{u_k^* = K_k(x_k + c)}$$

$$x_{k+1}^* = x_k^* + Bu_k^* + c$$
$$= x_k^* + BK_k(x_k + c) + c$$

$$\boxed{x_k^* = x_0 + BK_k(x_0 + c) + c}$$

c) Performance index L optimal value

$$L(x,u) = \frac{1}{2}(x^TQx + u^TRu)$$

$$L(x^*,u^*) = \frac{1}{2}(x_k^{*T}Qx^* + u^{*T}Ru^*)$$

$$L(x^*,u^*) = \frac{1}{2}\Big[(x_0 + BK_k(x_0+c)+c)^TQ(x_0 + BK_k(x_0+c)+c)$$
$$+ (K_k(x_k+c))^TR(K_k(x_k+c))\Big]$$