MPC HW #5 Ali Khater

Armon was here :)

1.

$$\hat{x}(k+1|k) = 2\hat{x}(k|k) + u(k)$$
$$\hat{y}(k|k-1) = 3\hat{x}(k|k-1)$$

constraint $-1 \le y(k) \le 2, \quad \forall k$

$\hat{x}(k|k) = 3 \; ; \; u(k-1) = -1.$

$-16/3 \le \Delta u(k) \le -\frac{13}{3} \quad , \quad \Delta u(k) = u(k) - u(k-1)$

Since $\hat{x}(k|k) = 3$;

$$\hat{x}(k+1|k) = 2(3) + u(k)$$
$$\underline{\hat{x}(k+1|k) = 6 + u(k)}$$

$$\hat{y}(k+1|k) = 3\hat{x}(k+1|k)$$
$$= 3(6 + u(k))$$
$$\underline{\hat{y}(k+1|k) = 18 + 3u(k)}$$

we are given the constraint $-1 \le y(k) \le 2$ for $\forall k$

so $-1 \le 18 + 3u(k) \le 2$

$$-19 \le 3u(k) \le -16$$

$$\star \quad \underline{-\frac{19}{3} \le u(k) \le -\frac{16}{3}}$$

Given $\Delta u(k) = u(k) - u(k-1)$;

$\square \quad u(k) = \Delta u(k) + u(k-1)$ ; where $u(k-1) = -1$

$\square \to \star \quad -\frac{19}{3} \le \Delta u(k) - 1 \le -\frac{16}{3}$

$$\boxed{-\frac{16}{3} \le \Delta u(k) \le -\frac{13}{3}}$$

② System dynamics

$$\begin{cases} x_k = x_{k-1} + w_{k-1} \\ z_k = x_k + v_k \end{cases} \quad ; \quad \begin{matrix} w_k \sim N(0,1) \\ v_k \sim N(0,2) \end{matrix}$$

where $\hat{x}_0 = 1,\ z_1 = 2,\ P_0 = 10$

Find $K_1, K_2, P_\infty, \hat{x}_1 = \hat{x}_{1,1}$ $\qquad w_k = (0,Q)\ ;\ v_k = (0,R)$

Kalman

From the given system,

$$Q = 1 \ ; R = 2 \ ; A = 1,\ C = 1$$

k=1

1) $P(k/k-1) = AP(k-1/k-1)A^T + Q$

$$P(1|0) = A P_0 A^T + Q$$
$$= 1(10)1 + 1$$
$$\underline{P(1|0) = 11}$$

2) $K_k = P(k/k-1)C^T\left(CP(k/k-1)C^T + R\right)^{-1}$

$$K_1 = P(1|0)1\left(1P(1/0)1 + 2\right)^{-1}$$
$$= (11)(13)^{-1}$$
$$K_1 = \frac{11}{13}$$

3) $P(k/k) = (I - K_k C)P(k/k-1)$

$$P(1/1) = (1 - K_1 1)P(1/0)$$
$$= \left(1 - \frac{11}{13}\right)11$$
$$\boxed{P(1/1) = \frac{22}{13}}$$

4) $\hat{x}(k/k) = \hat{x}(k/k-1) + K_k\left[y_k - Cx(k/k-1)\right]$

$$\hat{x}(k/k-1) = A\,x(k-1/k-1) + \cancelto{0}{B u(k-1)}$$
$$= A x_0$$
$$\hat{x}(1|0) = \underline{1}$$

$$x(1/1) = x(1/0) + K_1\left[z_1 - 1\,x(1/0)\right]$$
$$x(1/1) = 1 + {}^{11}\!/_{13}\left[2 - 1\right]$$
$$\boxed{x(1/1) = 24/11}$$

$k=2$

1) $P(2/1) = A P(1/1) A + Q$
$= 1(22/13)1 + 1$

$P(2/1) = 35/13$

2) $K_2 = P(2/1) 1 (1 P(2/1) 1^T + R)^{-1}$
$= \frac{35}{13}\left(\frac{35}{13} + 2\right)^{-1} \Rightarrow \frac{35}{13}\left(\frac{61}{13}\right)^{-1}$

$k_2 = \frac{35}{61}$

3) $P(2/2) = (I - k_2 C) P(2/1)$
$(I - \frac{35}{61})(\frac{35}{13})$
$(\frac{26}{61})(\frac{35}{13}) = \frac{70}{61}$

$P(2/2) = \frac{70}{61}$ $\times$ Not needed

4) $P_\infty = P(k/k-1) = A(I - k_k C) P(k/k-1) A^T + Q$

$P(2/1) = 1(1 - \frac{35}{61})(\frac{35}{13}) + 1$

$= (\frac{\cancel{26}\,2}{61})(\frac{35}{13}) + 1 = \frac{131}{61} = P_\infty$

$P(1/0) = (1 - \frac{11}{13}) 11 + 1$

$$P_\infty = A P_\infty A^T - A P_\infty C^T (C P_\infty C^T + R)^{-1} C P_\infty A^T + Q$$

$$A = 1 \, ; \, C = 1 \, ; \, R = 2 \, ; \, Q = 1$$

$$P_\infty = P_\infty - P_\infty (P_\infty + 2)^{-1} P_\infty + 1$$

$$\cancel{P_\infty} = \cancel{P_\infty} - \frac{P_\infty^2}{P_\infty + 2} + 1$$

$$\frac{P_\infty^2}{P_\infty + 2} = 1$$

$$P_\infty^2 = P_\infty + 2$$

$$P_\infty^2 - P_\infty - 2 = 0$$

$$(P_\infty - 2)(P_\infty + 1) = 0$$

$$P_\infty = 2 \, ; \, -1$$