HW #4

Aly Khater

1)

a) F
b) F
c) T
d) T

e) T
f) T
g) F
h) F

2) $\begin{pmatrix} c_1c_{23} & -c_1s_{23} & s_1 & l_1c_1+l_2c_1c_2 \\ s_1c_{23} & -s_1s_{23} & -c_1 & l_1s_1+l_2s_1c_2 \\ s_{23} & c_{23} & 0 & l_2s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$x = l_1c_1 + l_2c_1c_2$
$y = l_1s_1 + l_2s_1c_2$
$z = l_2s_2$

$$J = \begin{pmatrix} \frac{dx}{d\theta_1} & \frac{dx}{d\theta_2} & \frac{dx}{d\theta_3} \\ \frac{dy}{d\theta_1} & \frac{dy}{d\theta_2} & \frac{dy}{d\theta_3} \\ \frac{dz}{d\theta_1} & \frac{dz}{d\theta_2} & \frac{dz}{d\theta_3} \end{pmatrix}$$

MATLAB

$$J = \begin{pmatrix} -l_1s_1 - l_2c_2s_1 & -l_2c_1s_2 & 0 \\ l_1c_1 + l_2c_1c_2 & -l_2s_1s_2 & 0 \\ 0 & l_2c_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix}$$

③ SEE MATLAB.

$$\cancel{L_2L_3^2C_{23}^2C_1^2S_2 - 2L_1L_3^2C_{23}S_{23}S_1^2 - 2L_3^3C_{23}^2S_{23}S_1^2 - L_2^2L_3S_{23}C_1^2C_2^2}$$

$$\cancel{-L_2L_3^2C_{23}^2S_1^2S_2 - L_2^2L_3S_{23}C_2^2S_1^2 - L_2L_3^2C_{23}S_{23}C_1^2C_2 - 3L_2L_3^2C_{23}S_{23}C_2S_1^2}$$

$$\cancel{+L_2^2L_3C_{23}C_1^2C_2S_2 - L_2^2L_3C_{23}C_2S_1^2S_2 + L_1L_2L_3C_{23}C_1^2S_2 - L_1L_2L_3S_{23}C_1^2}$$

$$\cancel{-L_1L_2L_3C_{23}S_1^2S_2 - L_1L_2L_3S_{23}C_2S_1^2 = 0}$$

$$-\cancel{L_2L_3(L_1+L_3C_{23}+L_2C_2)})(S_{23}C_1^2C_2 - C_{23}C_1^2S_2 + C_{23}S_1^2S_2 + S_{23}C_2S_1^2 -$$

$$S_{23}C_1^2C_2 - C_{23}C_1^2S_2 + C_{23}S_1^2S_2 + S_{23}C_2S_1^2 = 0$$

$\underline{\theta_1 = \frac{\pi}{2}}$

$$0 - 0 + C_{23}S_2 + S_{23}C_2$$

$\underline{\theta_2 = 0}$

$$0 - 0 + 0 + S_{23}$$

$\underline{\theta_3 = 0}$

$$0 - 0 + 0 + 0 = 0$$

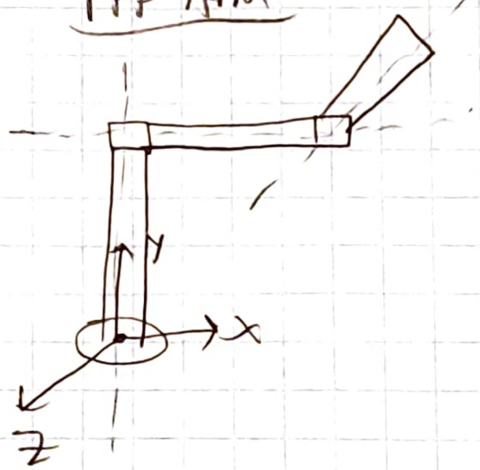
(4) $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = {}^nJ_i$

$${}^nJ = {}^n_mR\,{}^mJ \Rightarrow {}^nJ = {}^m_nR^{-1}\,{}^mJ$$

Rotation matrix has to be the identity to keep Jacobians in the identity across all configurations.

This will be a PPP arm, moving in x, y, z direction respectively.

PPP Arm



$$R = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

5 Law of cosines $[\theta_1, \theta_2, \theta_3]$

Has to put ee at origin

$$\underbrace{x^2+y^2}_{=0} = L_1^2+L_2^2 - 2L_1L_2\cos(180-\theta_2)$$

$$\cos(180-\theta_2) = \frac{L_1^2+L_2^2}{2L_1L_2}$$

$$180-\theta_2 = \cos^{-1}(\sim)$$

$$\theta_2 = 180 - \cos^{-1}\left(\frac{L_1^2+L_2^2}{2L_1L_2}\right)$$

θ_1 position doesn't matter, 0 for the purpose of this problem.

$$\theta_3 = 180 - \cos^{-1}\left(\frac{L_1^2+L_2^2}{2L_1L_2}\right)$$

$[\theta_1, \theta_2, \theta_3]$

boundary $= [0, 0, 0]$

interior $= \left[0, 180-\cos^{-1}\left(\frac{L_1^2+L_2^2}{2L_1L_2}\right), 180-\cos^{-1}\left(\frac{L_1^2+L_2^2}{2L_1L_2}\right)\right]$

6) a)

$$ {}^{0}_{ee}T = \begin{pmatrix} c_{13} & -s_{13} & 0 & L_3 c_{13} + L_1 c_1 + d_2 s_1 \\ s_{13} & c_{13} & 0 & L_3 s_{13} + L_1 s_1 - d_2 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{ee}J = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial d_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial d_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial d_2} & \frac{\partial z}{\partial \theta_3} \end{pmatrix} \qquad \begin{aligned} x &= L_3 c_{13} + L_1 c_1 + d_2 s_1 \\ y &= L_3 s_{13} + L_1 s_1 - d_2 c_1 \\ z &= 0 \end{aligned} $$

$$ \frac{\partial x}{\partial \theta_1} = -L_3 s_{13} - L_1 s_1 + d_2 c_1 $$

$$ \frac{\partial x}{\partial d_2} = s_1 $$

$$ \frac{\partial x}{\partial \theta_3} = -L_3 s_{13} + 0 + 0 $$

$$ \frac{\partial y}{\partial \theta_1} = L_3 c_{13} + L_1 c_1 + d_2 s_1 $$

$$ \frac{\partial y}{\partial d_2} = -c_1 $$

$$ \frac{\partial y}{\partial \theta_3} = L_3 c_{13} + 0 - 0 $$

$$ \frac{\partial z}{\partial \theta_{1,2,3}} = 0 $$

$$ {}^{0}J = \begin{pmatrix} -L_3 s_{13} - L_1 s_1 + d_2 c_1 & s_1 & -L_3 s_{13} \\ L_3 c_{13} + L_1 c_1 + d_2 s_1 & -c_1 & L_3 c_{13} \\ 0 & 0 & 0 \end{pmatrix} $$

6b) ${}^0v_0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$; ${}^0\omega_0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$

R) i=0 $\quad {}^1\omega_1 = {}^1_0R\,\cancel{{}^0\omega_0} + \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix} = \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}$

$${}^1v_1 = {}^1_0R\left(\cancel{{}^0v_0} + \cancel{{}^0\omega_0} \times {}^0P_1\right) = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

P) i=1 $\quad {}^2\omega_2 = {}^2_1R\,{}^1\omega_1 = \begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}\begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix} = \begin{pmatrix}0\\\dot\theta_1\\0\end{pmatrix} = {}^2\omega_2$

$${}^2v_2 = {}^2_1R\left({}^1v_1 + {}^1\omega_1 \times {}^2P_1\right) = \begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}\left[\begin{pmatrix}0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix} \times \begin{pmatrix}L_1\\-d_2\\0\end{pmatrix}\right]$$

$${}^2v_2 = \begin{pmatrix}d_2\dot\theta_1\\0\\-L_1\dot\theta_1 + \dot d_2\end{pmatrix}$$

R) i=2 $\quad {}^3\omega_3 = {}^3_2R\,{}^2\omega_2 + \begin{pmatrix}0\\0\\\dot\theta_3\end{pmatrix} = \begin{pmatrix}c_3&0&-s_3\\-s_3&0&-c_3\\0&1&0\end{pmatrix}\begin{pmatrix}0\\\dot\theta_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_3\end{pmatrix} = \begin{pmatrix}0\\0\\\dot\theta_1+\dot\theta_3\end{pmatrix}$

$${}^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right) = {}^3_2R\left[\begin{pmatrix}d_2\dot\theta_1\\0\\-L_1\dot\theta_1\end{pmatrix} + \cancel{\begin{pmatrix}0\\\dot\theta_1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\0\end{pmatrix}}\right]$$

$${}^3v_3 = \begin{pmatrix}c_3&0&-s_3\\-s_3&0&-c_3\\0&1&0\end{pmatrix}\begin{pmatrix}d_2\dot\theta_1\\\dot\theta_1\\-L_1\dot\theta_1+\dot d_2\end{pmatrix} = \begin{pmatrix}d_2c_3\dot\theta_1 + L_1s_3\dot\theta_1 - s_3\dot d_2\\L_1c_3\dot\theta_1 - d_2s_3\dot\theta_1 - c_3\dot d_2\\\dot\theta_1\end{pmatrix}$$

$${}^{ee}v_{ee} = {}^4_3R\left({}^3v_3 + {}^3\omega_3 \times {}^3P_4\right) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\left(\begin{pmatrix}d_2c_3\dot\theta_1+L_1s_3\dot\theta_1-s_3\dot d_2\\L_1c_3\dot\theta_1-d_2s_3\dot\theta_1-c_3\dot d_2\\\dot\theta_1\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_1+\dot\theta_3\end{pmatrix} \times \begin{pmatrix}L_3\\0\\0\end{pmatrix}\right)$$

$${}^{ee}J = \begin{pmatrix}d_2c_3\dot\theta_1 + L_1s_1\dot\theta_1 - s_3\dot d_2\\L_1c_3\dot\theta_1 - d_2s_3\dot\theta_1 - c_3\dot d_2 + L_3(\dot\theta_1+\dot\theta_3)\\0\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}d_2c_3+L_1s_3 & -s_3 & 0\\L_1c_3-d_2s_3+L_3 & -c_3 & L_3\\1 & 0 & 0\end{pmatrix}}_{{}^{ee}J}\begin{pmatrix}\dot\theta_1\\\dot d_2\\\dot\theta_3\end{pmatrix}$$

$${}^0J = {}^0_{ee}R \cdot {}^{ee}J = \begin{pmatrix}-L_3s_{13}-L_1s_1+d_2c_1 & s_1 & -L_3s_{13}\\L_3c_{13}+L_1c_1+d_2s_1 & -c_1 & L_3c_{13}\end{pmatrix}$$

6c $\quad {}^{i}f_i = {}^{i}_{i+1}R \, {}^{i+1}f_{i+1}$

$${}^{i}n_i = {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$$

R) $i=3$ $\quad {}^{3}f_3 = {}^{3}_{ee}R \, {}^{ee}f_{ee}$; ${}^{ee}f_{ee} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$; ${}^{ee}n_{ee} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$${}^{3}n_3 = \cancel{{}^{3}_{ee}R \, {}^{ee}n_{ee}}^{\,0} + {}^{3}P_{ee} \times {}^{3}f_3 = \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ L_3 f_y \end{pmatrix}$$

P) $i=2$ $\quad {}^{2}f_2 = {}^{2}_{3}R \, {}^{3}f_3 = \begin{pmatrix} c_3 & -s_3 & 0 \\ 0 & 0 & 1 \\ -s_3 & -c_3 & 0 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix} = \begin{pmatrix} f_x c_3 - f_y s_3 \\ 0 \\ -f_y c_3 - f_x s_3 \end{pmatrix}$

$${}^{2}n_2 = {}^{2}_{3}R \, {}^{3}n_3 + {}^{2}P_3 \times {}^{2}f_2 = \begin{pmatrix} c_3 & -s_3 & 0 \\ 0 & 0 & 1 \\ -s_3 & -c_3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ L_3 f_y \end{pmatrix} + \cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix}}^{\,0} = \begin{pmatrix} 0 \\ L_3 f_y \\ 0 \end{pmatrix}$$

R) $i=1$ $\quad {}^{1}f_1 = {}^{1}_{2}R \, {}^{2}f_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_x c_3 - f_y s_3 \\ 0 \\ -f_y c_3 - f_x s_3 \end{pmatrix} = \begin{pmatrix} f_x c_3 - f_y s_3 \\ f_y c_3 + f_x s_3 \\ 0 \end{pmatrix}$

$${}^{1}n_1 = {}^{1}_{2}R \, {}^{2}n_2 + {}^{1}P_2 \times {}^{1}f_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ L_3 f_y \\ 0 \end{pmatrix} + \begin{pmatrix} L_1 \\ -d_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} f_x c_3 - f_y s_3 \\ f_y c_3 + f_x s_3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ L_3 f_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f_x d_2 c_3 - f_y d_2 s_3 + f_y L_1 c_3 + f_x L_1 s_3 \end{pmatrix}$$

$${}^{1}n_1 = \begin{pmatrix} 0 \\ 0 \\ f_y L_3 + f_x c_3 d_2 - f_y s_3 d_2 + f_y c_3 L_1 + f_x s_3 L_1 \end{pmatrix}$$

6c $\tau_1 = {}^1n_1^T \, {}^1\hat{z}_1 = f_x(c_3 d_2 + s_3 L_1) + f_y(L_3 - s_3 d_2 + c_3 L_1)$

$\tau_2 = {}^2\vec{f}_2^T \, {}^2\hat{z}_2 = f_x(-s_3) + f_y(-c_3)$

$\tau_3 = {}^3n_3^T \, {}^3\hat{z}_3 = f_y(L_3)$

$$\vec{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \underbrace{\begin{pmatrix} c_3 d_2 + s_3 L_1 & -s_3 & 0 \\ L_3 - s_3 d_2 + c_3 L_1 & -c_3 & L_3 \\ 0 & 0 & 0 \end{pmatrix}}_{{}^{ee}J} \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix}$$

$${}^0J = {}^0_{ee}R \, {}^{ee}J$$

$${}^0J = \begin{matrix} -L_3 s_{13} - L_1 s_1 + d_2 c_1 & s_1 & -L_3 s_{13} \\ L_3 c_{13} + L_1 c_1 + d_2 s_1 & -c_1 & L_3 c_{13} \\ 0 & 0 & 0 \end{matrix}$$

```
clc;
clear;
close all;

%Problem 2, Jacobian. Aly Khater

syms l1 l2 t1 t2 t3;

x = l1*cos(t1)+l2*cos(t1)*cos(t2);
y = l1*sin(t1)+l2*sin(t1)*cos(t2);
z = l2*sin(t2);

jacobian([x,y,z],[t1,t2,t3])
```

```
ans =

[- l1*sin(t1) - l2*cos(t2)*sin(t1), -l2*cos(t1)*sin(t2), 0]
[  l1*cos(t1) + l2*cos(t1)*cos(t2), -l2*sin(t1)*sin(t2), 0]
[                                0,         l2*cos(t2), 0]
```

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```
%PRoblem 3 work
clc;
clear;
close all;

syms l1 l2 l3 t1 t2 t3

M11 = -l1*sin(t1)-l2*sin(t1)*cos(t2)-l3*sin(t1)*cos(t2+t3);
M12 = -l2*cos(t1)*sin(t2)-l3*cos(t1)*sin(t2+t3);
M13 = -l3*cos(t1)*sin(t2+t3);
M21 = l1*cos(t1)+l2*cos(t1)*cos(t2)+l3*cos(t1)*cos(t2+t3);
M22 = l2*sin(t1)*sin(t2)-l3*sin(t1)*sin(t2+t3);
M23 = -l3*sin(t1)*sin(t2+t3);
M31 = 0;
M32 = l2*cos(t2)+l3*cos(t2+t3);
M33 = l3*cos(t2+t3);

A = [M11, M12, M13;
     M21, M22, M23;
     M31, M32, M33];

d = simplify(det(A))
```

```
d =

-l2*l3*(l1 + l3*cos(t2 + t3) + l2*cos(t2))*(sin(t2 + t3)*cos(t1)^2*cos(t2) - cos(t2 + t3)*cos(t1)^2*sin(t2) + cos(t2 + t3)*sin(t1)^2*sin(t2) + sin(t2 + t3)*cos(t2)*
```

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```
%Problem 6c computations

clc;
clear;
close all;

syms t3 fx fy l3 l1 d2 t1

r23 = [cos(t3) -sin(t3) 0; 0 0 1; -sin(t3) -cos(t3) 0];
f33 = [fx;fy;0];

f22 = r23*f33
```

f22 =

$$\begin{pmatrix} \mathrm{fx}\cos(t_3) - \mathrm{fy}\sin(t_3) \\ 0 \\ -\mathrm{fy}\cos(t_3) - \mathrm{fx}\sin(t_3) \end{pmatrix}$$

```
n33 = [0;0;l3*fy];
n22 = r23*n33
```

n22 =

$$\begin{pmatrix} 0 \\ \mathrm{fy}\, l_3 \\ 0 \end{pmatrix}$$

```
r12 = [1 0 0; 0 0 -1; 0 1 0];
f11 = r12*f22
```

f11 =

$$\begin{pmatrix} \mathrm{fx}\cos(t_3) - \mathrm{fy}\sin(t_3) \\ \mathrm{fy}\cos(t_3) + \mathrm{fx}\sin(t_3) \\ 0 \end{pmatrix}$$

```
p12 = [l1;-d2;0];
cr11 = cross(p12,f11)
```

cr11 =

$$\begin{pmatrix} 0 \\ 0 \\ d_2\,(\mathrm{fx}\cos(t_3) - \mathrm{fy}\sin(t_3)) + l_1\,(\mathrm{fy}\cos(t_3) + \mathrm{fx}\sin(t_3)) \end{pmatrix}$$

```
n11 = r12*n22+cr11
```

n11 =

$$\begin{pmatrix} 0 \\ 0 \\ \mathrm{fy}\, l_3 + d_2\ (\mathrm{fx}\cos(t_3) - \mathrm{fy}\sin(t_3)) + l_1\ (\mathrm{fy}\cos(t_3) + \mathrm{fx}\sin(t_3)) \end{pmatrix}$$

```
r0ee = [cos(t1+t3) -sin(t1+t3) 0;
        sin(t1+t3) cos(t1+t3) 0;
        0 0 1];
T = [cos(t3)*d2+sin(t3)*l1 -sin(t3) 0;
     l3-sin(t3)*d2+cos(t3)*l1 -cos(t3) l3;
     0 0 0];
jac = simplify(r0ee*T)
```

jac =

$$\begin{pmatrix} d_2\cos(t_1) - l_3\sin(t_1+t_3) - l_1\sin(t_1) & \sin(t_1) & -l_3\sin(t_1+t_3) \\ l_3\cos(t_1+t_3) + l_1\cos(t_1) + d_2\sin(t_1) & -\cos(t_1) & l_3\cos(t_1+t_3) \\ 0 & 0 & 0 \end{pmatrix}$$

```
%Problem 6 Velocity Propagation work
clc;
clear;
close all;

syms t11


mat = [1 0 0; 0 0 -1; 0 1 0];
a = inv(mat);

matb = [0;0;t11];

a*matb
```

ans =

$$\begin{pmatrix} 0 \\ t_{11} \\ 0 \end{pmatrix}$$

```
syms l1 d2

matc = [0;0;t11];
matd = [l1;-d2;0];

cr = cross(matc,matd)
```

cr =

$$\begin{pmatrix} d_2\, t_{11} \\ l_1\, t_{11} \\ 0 \end{pmatrix}$$

```
v22 = a*cr
```

v22 =

$$\begin{pmatrix} d_2\, t_{11} \\ 0 \\ -l_1\, t_{11} \end{pmatrix}$$

```
syms t3 t33 d22
r23 = [cos(t3) -sin(t3) 0;0 0 1; -sin(t3) -cos(t3) 0];
r32 = transpose(r23)
```

r32 =

$$\begin{pmatrix} \cos(t_3) & 0 & -\sin(t_3) \\ -\sin(t_3) & 0 & -\cos(t_3) \\ 0 & 1 & 0 \end{pmatrix}$$

```
w33a = r32*[0;t11;0];
w33 = w33a+[0;0;t33]
```

w33 =

$$\begin{pmatrix} 0 \\ 0 \\ t_{11} + t_{33} \end{pmatrix}$$

```
v33 = r32*[d2*t11;t11;-l1*t11+d22]
```

v33 =

$$\begin{pmatrix} d_2\, t_{11} \cos(t_3) - \sin(t_3)\ (d_{22} - l_1\, t_{11}) \\ -\cos(t_3)\ (d_{22} - l_1\, t_{11}) - d_2\, t_{11} \sin(t_3) \\ t_{11} \end{pmatrix}$$

```
syms l3
cr3ee = cross(w33,[l3;0;0])
```

cr3ee =

$$\begin{pmatrix} 0 \\ l_3\ (t_{11} + t_{33}) \\ 0 \end{pmatrix}$$