Merve Nur Öztürk	Atakan Süslü	Betül Rana Kuran
2311322	*2311371*	*2311173*

1 Introduction

It is practical to use numerical methods in order to solve differential equations when the analytical solution is time-consuming or an analytical solution does not exist, which was the case for our homework problem. In this homework, we are given an aircraft which rolls on the ground and takes off some time later, and we are asked to find the minimum time required for this aircraft to take off at different airport altitudes. We know that the aircraft will take off when the lift force is greater than the weight of the aircraft. Since the lift force is a function of velocity;

$$L = C_L \cdot \frac{1}{2} \cdot \rho_\infty \cdot V_\infty^2 \cdot S \tag{1}$$

where the lift coefficient C_L is assumed to be constant and equal to the maximum lift coefficient ($C_{L,max} = 1.792$), first, we should calculate velocity which is described in a differential form for this problem using the force balance equation in the horizontal direction:

$$\frac{W}{g} \cdot \frac{dV}{dt} = T - D - \mu \cdot (W - L) \tag{2}$$

where the lift and drag forces are initially zero. The next values of drag force are calculated by this formula:

$$D = C_D \cdot \frac{1}{2} \cdot \rho_\infty \cdot V_\infty^2 \cdot S \tag{3}$$

The drag coefficient C_D is also assumed to be constant ($C_D = 0.215$) and its value is calculated by this equation;

$$C_D = 0.0207 + 0.0605 C_L^2 \tag{4}$$

Therefore, we are asked to use Euler's and RK2 methods so that we can approach the solution with the help of numerical methods.

2 Method

Euler's method can be defined as the first-order Taylor Series Expansion. In this method, the space between the initial point of the independent variable and the desired next point is divided into discrete points. The difference between respective discrete points is named as step size, Δx. Then, a first-order ordinary differential equation at the initial point is calculated and treated as slope.

$$\frac{dy}{dx} \equiv y\prime \equiv f(x, y) \equiv SLOPE \tag{5}$$

The slope is multiplied by the stepsize and added to the initial value of the dependent variable. The result gives the next value of the dependent variable.

$$\text{Next Value} = \text{Previous Value} + \text{Step Size} \times \text{Slope}$$

$$y(x + \Delta x) = y(x) + \Delta x f(x, y) \tag{6}$$

$$y_{i+1} = y_i + \Delta x f(x_i, y_i) \tag{7}$$

This equation is repeated until the x_i exceeds the desired independent point.

Second-Order Runge-Kutta method has a similar equation form with Euler's method:

$$y_{i+1} = y_i + \Delta x \phi(x_i, y_i, \Delta x) \tag{8}$$

However, in second order RK method, instead of $y\prime$, ϕ, which is called the increment function, is used. ϕ is weighted slope function over the interval.

$$\begin{aligned} \phi &= a_1 k_1 + a_2 k_2 \\ \frac{dy}{dx} &= f(x, y) \\ k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + p_i \Delta x, y_i + p_i \Delta x k_1) \end{aligned} \tag{9}$$

where $p_i < 1$. From the second-order Taylor Series expansion about (x_i, y_i), two equations are determined:

$$a_1 + a_2 = 1$$
$$a_2 p_1 = \frac{1}{2}$$

We chose $p_i = 2/3$. Therefore, we found $a_1 = 1/4$ and $a_2 = 3/4$ as our weights.

Trapezoidal integration rule is a basic version of calculating the area under a curve. It is based on dividing the area under the curve $y(x)$ into small trapezoids. Then, areas of trapezoids which is equal to $\frac{1}{2}(V_i + V_{i+1})\Delta t$ is summed up:

$$\int_0^{N_p \Delta t} V(t) dt = \sum_{i=0}^{N_p - 1} \frac{1}{2}(V_i + V_{i+1})\Delta t \tag{10}$$

3 Results and Discussion

3.1 Calculation of velocity at sea level with Euler's Method

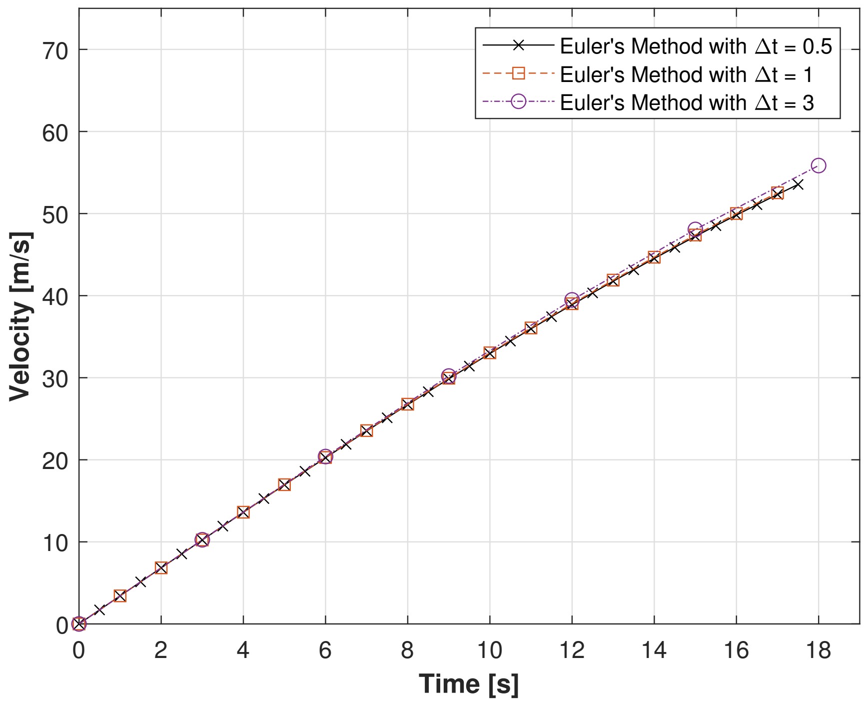

Figure 1: Calculated velocities at sea level using Euler's Method with three different step sizes.

As can be seen in Figure 1, by using different time step sizes, we obtain different results. Initially, the results were approximately the same for all time step sizes, but as time progresses, the margin of error increases. This increase occurs at different rates for each time step size. In order to illustrate, when $\Delta t = 3$, the error grows faster than other cases. If we assume that we have the least error when $\Delta t = 0.5$, since it is the smallest step size among all, we can observe that the curves of the other step sizes deviate more as the time step size gets higher. Thus, when $\Delta t = 3$, the result is the least accurate.

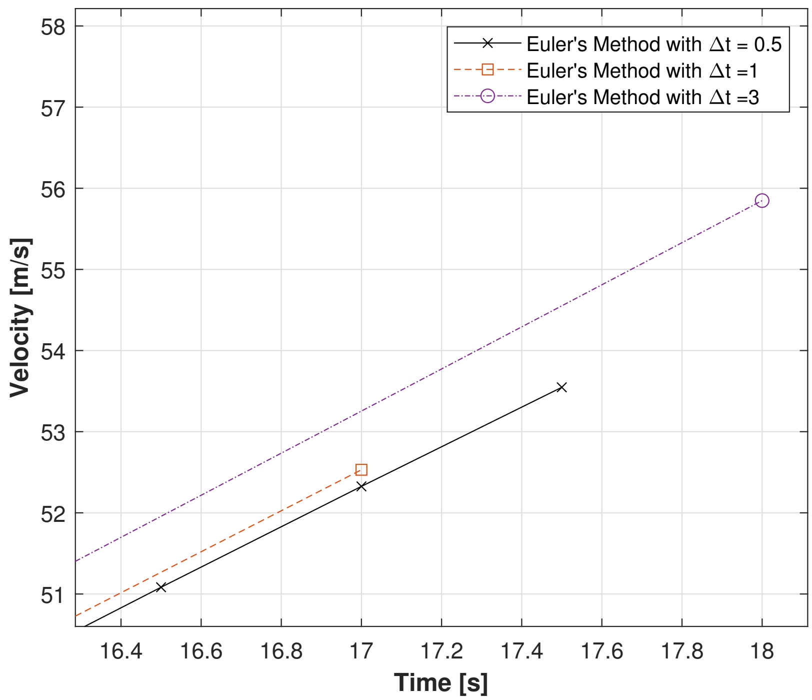

Figure 2: Detailed version of Figure 1 focused on the endpoints.

The effects of using different time step sizes on the result can be clearly seen in Figure 2. In the first problem of the homework, we were asked to find the minimum time needed for the given aircraft to lift off at sea level. When we take a closer look at the curves as we did in Figure 2, the endpoints, which gives the lift-off time and velocity, are different from each other, and as we mentioned before, we approach the result more precisely for $\Delta t = 0.5$. That means if we want to obtain a more precise result, we must decrease the step size.

3.2 Calculation of velocity at different altitudes with Euler's Method

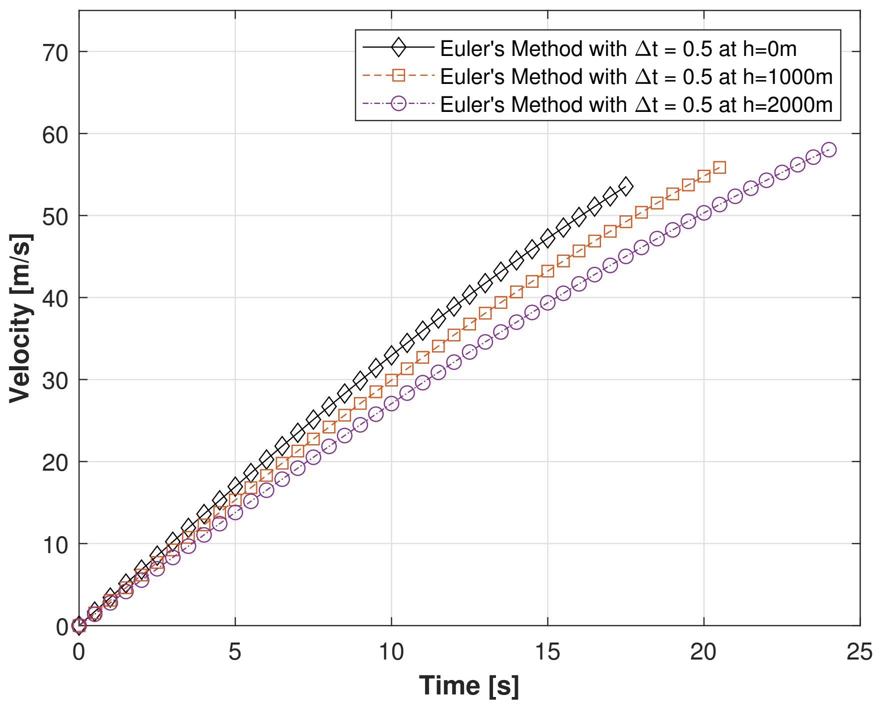

Figure 3: Calculated velocities at different altitudes using Euler's method with $\Delta t = 0.5$.

It can be obtained from Figure 3 that it takes longer for the given aircraft to take off at higher altitudes. This is because as the altitude gets higher, the air density decreases, which results in lower maximum available thrust ($T_{A,max} = T_{A,max,SL}\frac{\rho_\infty}{\rho_{\infty,SL}}$). In addition to that, lift and drag forces also decrease since they are functions of the free stream air density. The decrease in drag should generally result in shorter take-off time while the decrease in thrust and lift force acts exactly the opposite way. However, the effect of both the lift force and the thrust overcomes the effect of the drag force. Therefore, take off time increases at higher altitudes.

3.3 Calculation of velocity at sea level with second order RK Method

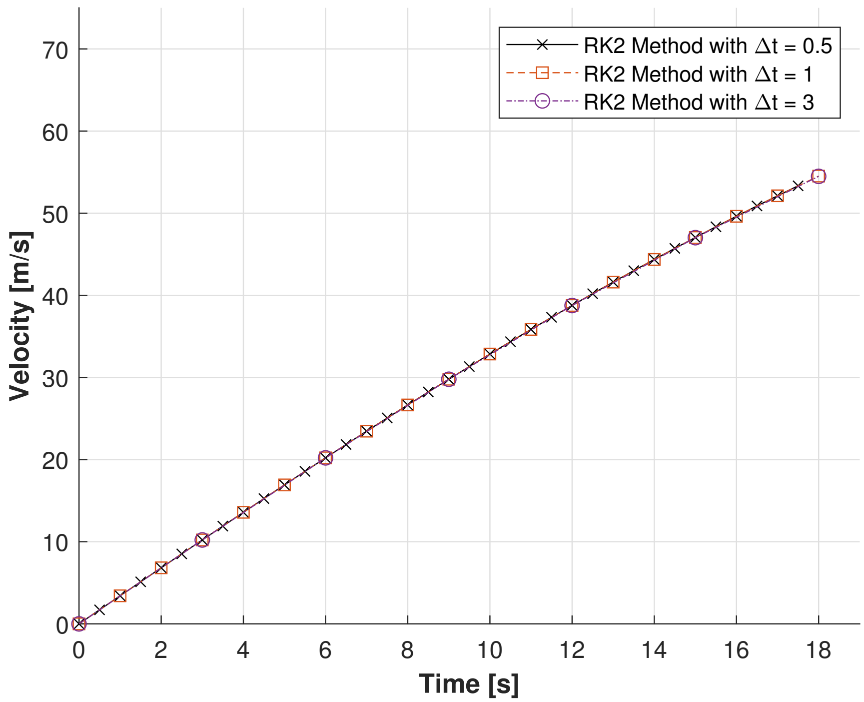

Figure 4: Calculated velocities at sea level using second order RK method with different step sizes.

The results in Figure 4 are more similar to each other than Figure 1. This is because RK2 method with $p_i = 2/3$ is more accurate than Euler's method. In RK2 method with $p_i = 2/3$, average slopes at t_i and $t_{i+p_i\Delta t}$ are used instead of using only one slope at t_i in every iteration. This leads to more stable results than Euler's method even with a larger step size. In our case, error became significant only at the endpoints. That error has occured because algorithm jumps to the next time step even if the true result is closer to the current evaluation point than the next evaluation point. To decrease this error at the endpoints, a smaller time step must be used.

3.4 Comparison of Euler's method and RK2 method at sea level with different step sizes

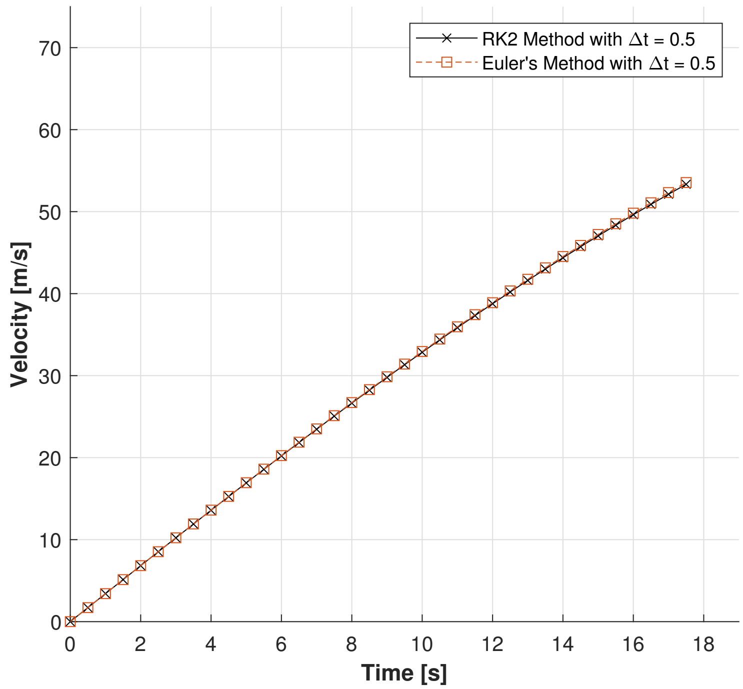

Figure 5: Calculated velocities at sea level using second order RK method and Euler's method for $\Delta t = 0.5$.

As can be seen from the Figure 5, for $\Delta t = 0.5$, results of RK2 method with $p_i = 2/3$ and Euler's method are approximately the same. This is not surprising since the time step size is so small that the margin of error does not change significantly for different numerical methods, and we gain accurate results for both methods.

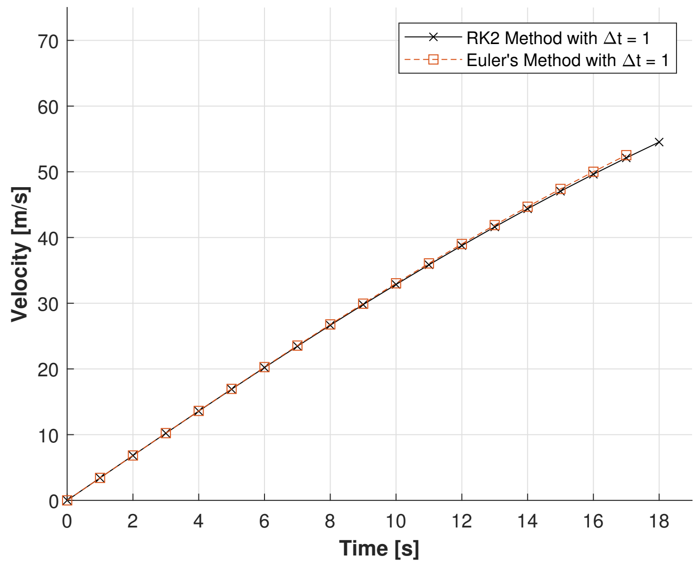

Figure 6: Calculated velocities at sea level using second order RK method and Euler's method for $\Delta t = 1$.

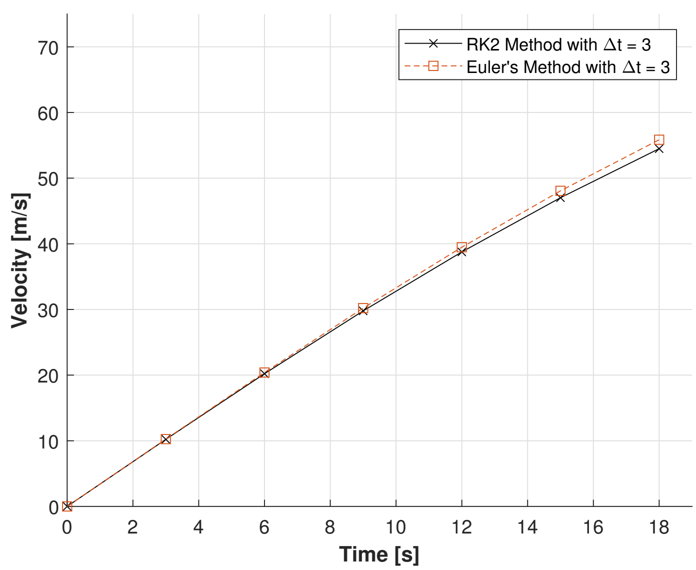

Figure 7: Calculated velocities at sea level using second order RK method and Euler's method for $\Delta t = 3$.

The Figures 6 and 7 show that increasing step size caused the increase in the difference between the results of Euler's method and RK2. This is another proof of larger step sizes leads to an increase in error. The RK2 method has less truncation error than the Euler's method; therefore, it is more accurate than Euler's method with bigger time steps.

3.5 Calculation of ground roll distance at different altitudes

Table 1: Take-off distance x for different altitudes.

Altitude (m)	$\boldsymbol{x}$ **(m)**
0	493.686
1000	607.273
2000	772.654

First, velocities were calculated using RK2 Method with $\Delta t = 0.5$, which is the smallest timestep used in this problem. Then, integral was taken for each interval using Trapezoidal Integration Rule.

As can be seen in Table 1, at higher altitudes, take off distance increases dramatically.

4 Conclusion

We have calculated the minimum take-off time using Euler's Method and second-order RK Method at different altitudes. We used different time steps for each method and compared the effect of different time step sizes.

Our computations demonstrated that second-order RK Method gives more reliable results than Euler's Method when computed with bigger step sizes. We observed that using smaller time step sizes minimizes the effect of truncation error caused by the usage of different numerical methods. Therefore, if small time step sizes are used, there is no significant difference between these two numerical methods.

Also, we have shown that at higher altitudes, both minimum take-off time and ground roll distance are longer. That is because airplane has to reach a higher velocity to take off.