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1 Introduction

Elliptic partial differential equations can be solved by using finite difference equations, and the most common FDE for the solution of an elliptic PDE is obtained by second-order central difference approximations of the derivatives. Afterwards, the FDE can be solved either by direct solution methods or iterative methods. In this homework, a 2D heat equation is given:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

It is requested to compute the steady state temperature distribution on a given 2D model of a room having a width of 10m and a height of 6m by solving the above equation for two different cases, one is with a radiator and the other is without a radiator. First, the iterative methods are used; Point Jacobi, Gauss-Seidel, SOR and Line Gauss-Seidel methods, then the solution is obtained by the direct solution methods. For the final solution, it is asked to plot the heat flux distribution in the room, and the heat flux vector is given as follows:

$$\vec{q} = -k\nabla T \tag{2}$$

It is also asked to compare the convergence rates of the iterative methods.

2 Method

FDE of the governing equation of this homework (Equation 1) is an elliptic partial differential equation,and it is obtained by second-order central difference approximations of the derivatives.

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} \tag{3}$$

When i and j coincide with the boundries of the room or the radiator, they are excluded from the solution of the heat equation since at these points the temperatures are equal to temperatures of the boundries.
When $\beta = \frac{\Delta x}{\Delta y}$ substituted into Equation 3, equation becomes:

$$\begin{aligned} T_{i+1,j} - 2T_{i,j} + T_{i-1,j} + \beta^2(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}) &= 0 \\ \beta^2 T_{i,j-1} + T_{i-1,j} - 2(1+\beta^2)T_{i,j} + T_{i+1,j} + \beta^2 T_{i,j+1} &= 0 \end{aligned} \tag{4}$$

This five-point formula gives the system of linear algebraic equations, which can be solved by two methods, namely direct solution methods and iterative solution methods.

2.1 Direct Solution Method

2.1.1 Gauss Elimination

In this homework, Gauss elimination method is used as direct solution method. The system of linear equations formed by five-point formula (Equation 4).

$$aT_{i-1,j} + bT_{i,j} + cT_{i+1,j} + dT_{i,j-1} + eT_{i,j+1} = 0 \tag{5}$$

with Neumann type BC for insulated wall and Dirichlet type BCs for other walls, the system of equations can be writen in $\underline{A}\ \underline{T} = \underline{f}$ form where $\underline{A}$ is coefficient matrix. This equation is solved by Gauss elimination method.

2.2 Iterative Solution Methods

2.2.1 Point Iterative Methods

Point Jacobi Iteration In this method, every terms in Equation 4 are kept at k where k is denoted as iteration level except $T_{i,j}$. $T_{i,j}$ which is kept at new iteration level $k+1$. Then equation becomes:

$$T_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}[T_{i-1,j}^{k} + T_{i+1,j}^{k} + \beta^2(T_{i,j-1}^{k} + T_{i,j+1}^{k})] \tag{6}$$

where k corresponds previous calculated values or initial guess at the start of iteration process of T values over the room except boundary conditions.

Gauss-Seidel Iteration Differently from Point Jacobi method, in Gauss-Seidel method, the unknown dependent variable values of Equation 4 are used as soon as they become available at the $k+1$ iteration level.

$$T_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}[T_{i-1,j}^{k+1} + T_{i+1,j}^{k} + \beta^2(T_{i,j-1}^{k+1} + T_{i,j+1}^{k})] \tag{7}$$

Successive Over-Relaxation Method In this method, the convergence of solution is can be accelerated by multiplying the relaxation parameter, ω, with the amount of the change in each step. First, $T_{i,j}^{k}$ is substracted from the Gauss-Seidel iteration method to obtain ΔT.

$$T_{i,j}^{k+1}|_{GS} - T_{i,j}^{k} = \Delta T|GS = \frac{1}{2(1+\beta^2)}[T_{i-1,j}^{k+1} + T_{i+1,j}^{k} + \beta^2(T_{i,j-1}^{k+1} + T_{i,j+1}^{k})] - T_{i,j}^{k} \tag{8}$$

Then, ΔT is multiplied by ω

$$\begin{aligned} T_{i,j}^{k+1}|_{SOR} &= T_{i,j}^{k} + \omega\Delta T|GS \\ &= (1-\omega)T_{i,j}^{k} + \frac{\omega}{2(1+\beta^2)}[T_{i-1,j}^{k+1} + T_{i+1,j}^{k} + \beta^2(T_{i,j-1}^{k+1} + T_{i,j+1}^{k})] \end{aligned} \tag{9}$$

To obtain convergent solutions, relaxatation parameter,ω, is chosen in (0,2) interval. The optimum value of ω can be determined by numerical experimentations. If ω is chosen in (0,1) interval, under-relaxation occurs, which prevents the divegence by slowing the convergence for the solution of certain non-linear partial derivative equations.

2.2.2 Line Iterations

Line Gauss-Seidel Method In this method, iterations are made line by line, in one direction. One more term of Equation 7 is expressed at $k+1$ iteration level. Then, equation becomes:

$$T_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}[T_{i-1,j}^{k+1} + T_{i+1,j}^{k+1} + \beta^2(T_{i,j-1}^{k+1} + T_{i,j+1}^{k})] \tag{10}$$

To make iteration line by line, j is kept constant. Therefore, every $T_{,j}$ term should be in left hand side of the equation.

$$T_{i-1,j}^{k+1} - 2(1+\beta^2)T_{i,j}^{k+1} + T_{i+1,j}^{k+1} = -\beta^2(T_{i,j+1}^{k} + T_{i,j-1}^{k+1})] \tag{11}$$

For constant j grid lines, Equation 11 is applied to all i's. The system of linear equations which includes tridiagonal coefficient matrix is formed. Its convergence rate is higher than the Gauss-Seidel method. On the other hand, a system of equations is solved every iterations;hence, in each iteration, it requires more computations. The effect of BCs at i=0 and i=imax is instantly visible in the solution.

3 Results and Discussion

3.1 Solution of the Heat Equation in the Absence of a Radiator

Figures 1, 2, 3, 4, and 5 illustrates the solution of the heat equation by using Point Jacobi, Gauss-Seidel, SOR, Line Gauss-Seidel, and Direct methods, respectively. As can be seen from these figures, all the solutions with different methods converged to approximately the same heat flux distributions. In all solutions, flux vectors have the same directions, which are from the walls with higher temperature to the wall with the lower temperature perpendicularly. Moreover, near the insulated wall, these flux vectors are parallel and the temperature contours are perpendicular to the wall since the temperature gradient is zero at the insulated wall. Therefore, there is no heat transfer across the insulated wall.

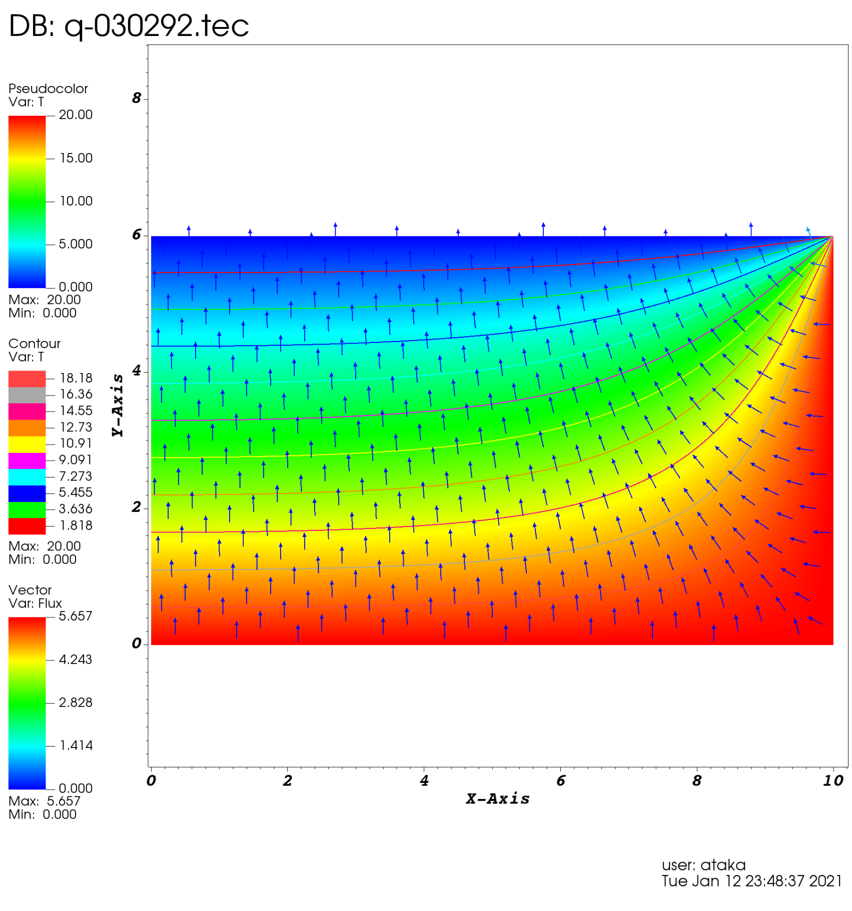

Figure 1: Solution of the heat equation by Point Jacobi method.

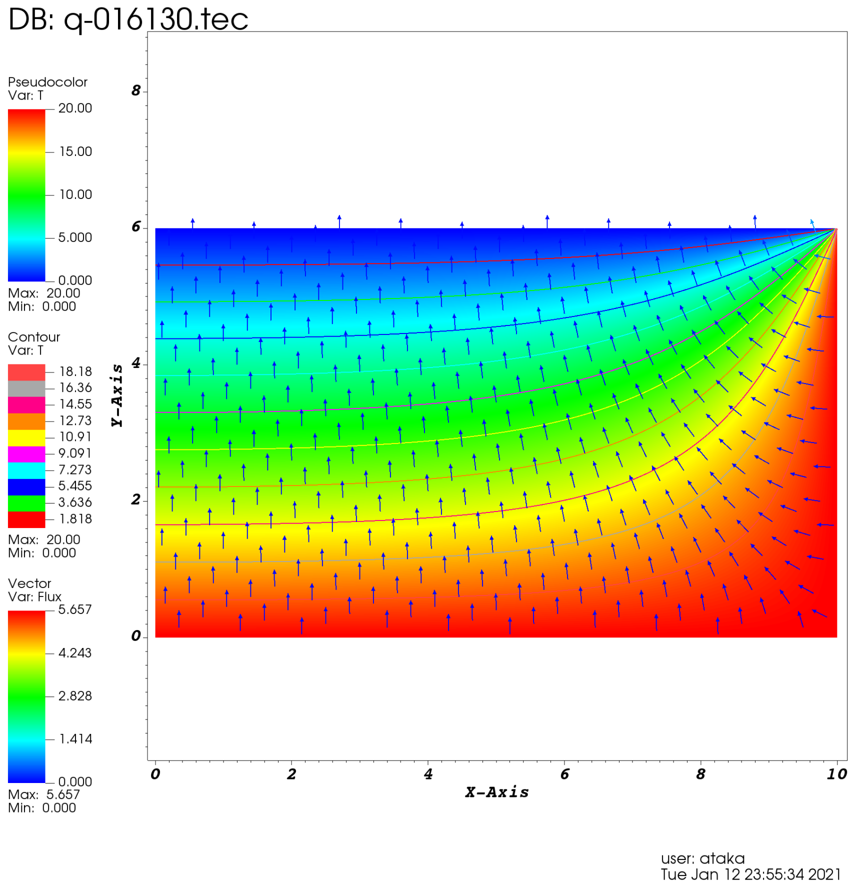

Figure 2: Solution of the heat equation by Gauss-Seidel method.

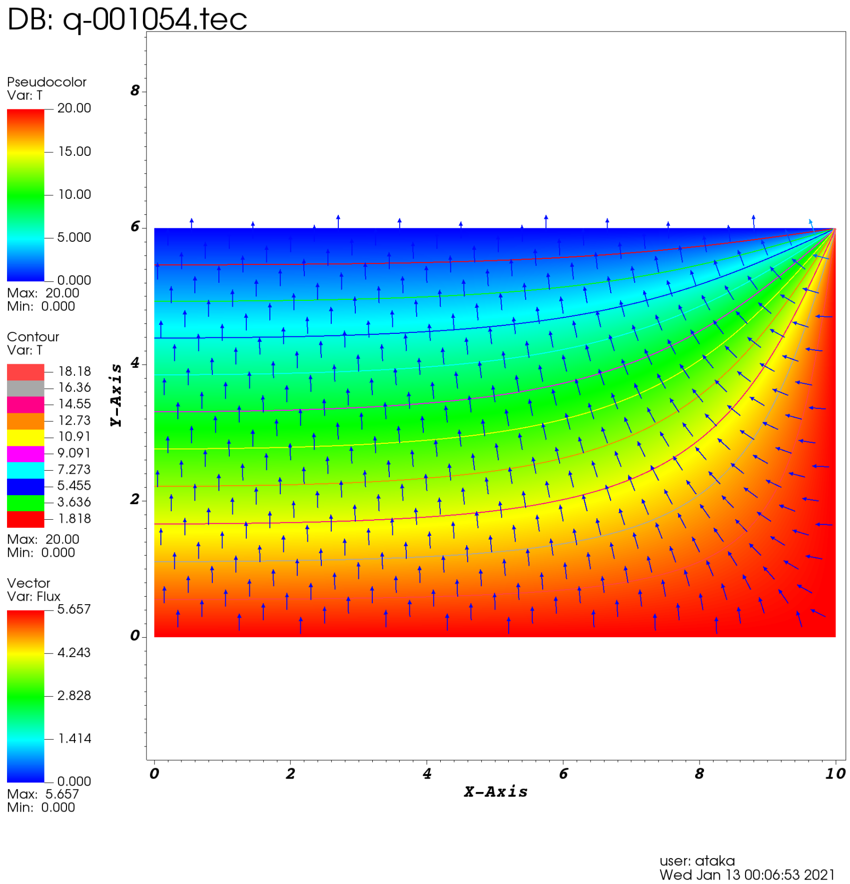

Figure 3: Solution of the heat equation by SOR method.

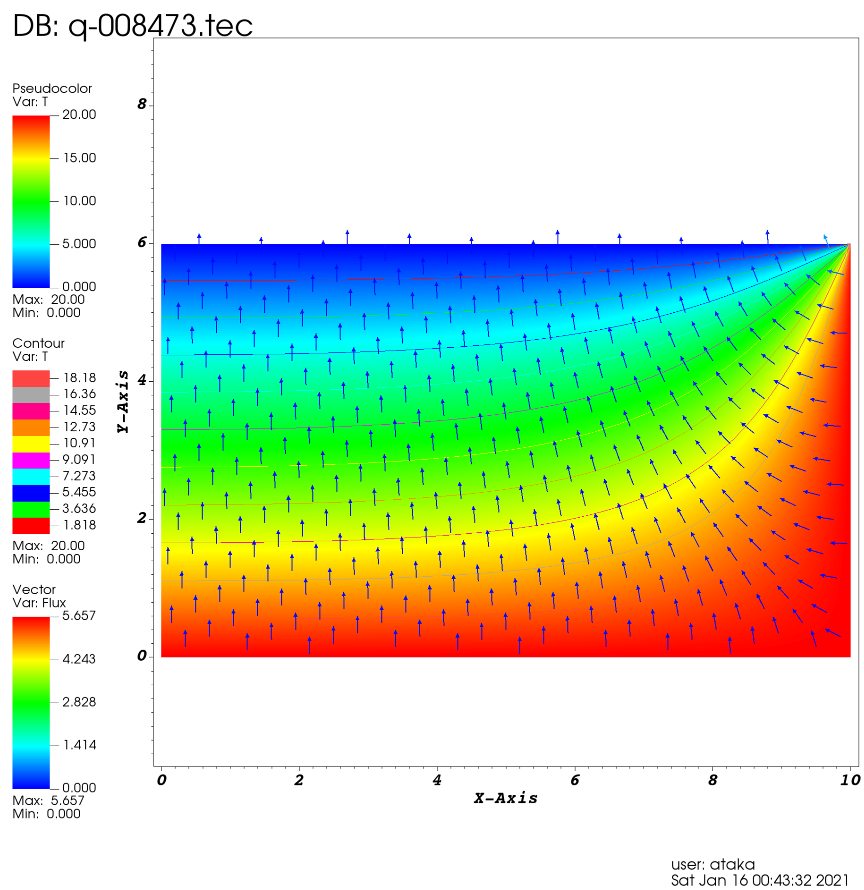

Figure 4: Solution of the heat equation by Line Gauss-Seidel method.

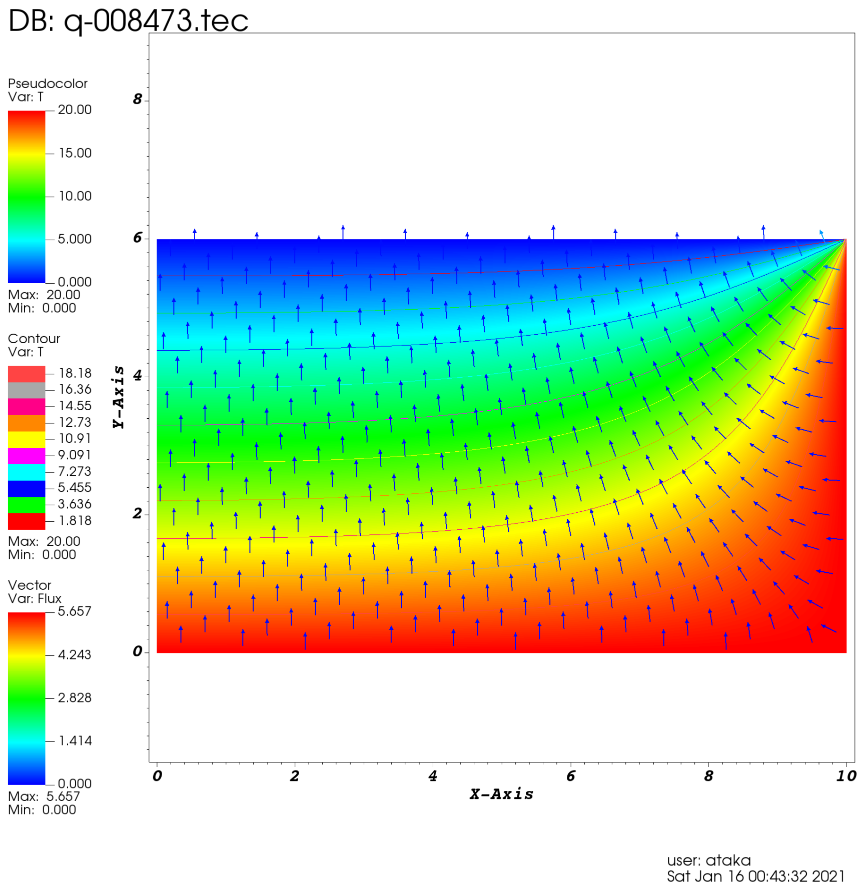

Figure 5: Solution of the heat equation by Direct method.

3.1.1 Temperature Distributions along x=5m and y=3m

Figures 6 and 7 shows the temperature distributions along x=5 and y=3 lines respectively. It can be oberved that all methods gave the same result with a negligible difference among them although their convergence rates differ.

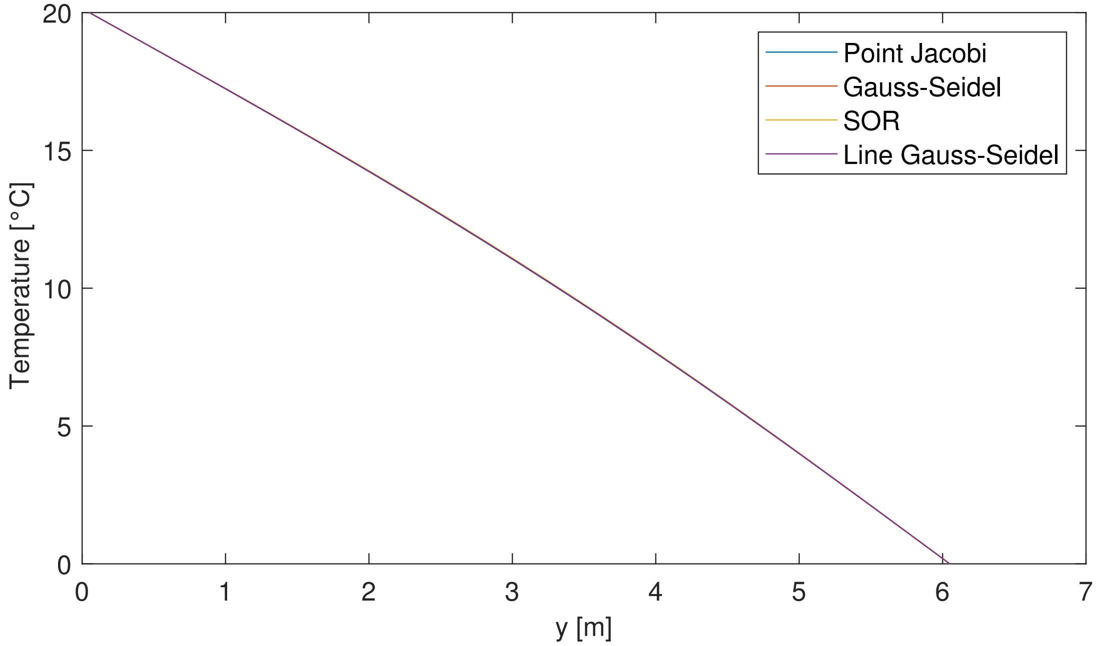

Figure 6: Temperature distributions along x=5m

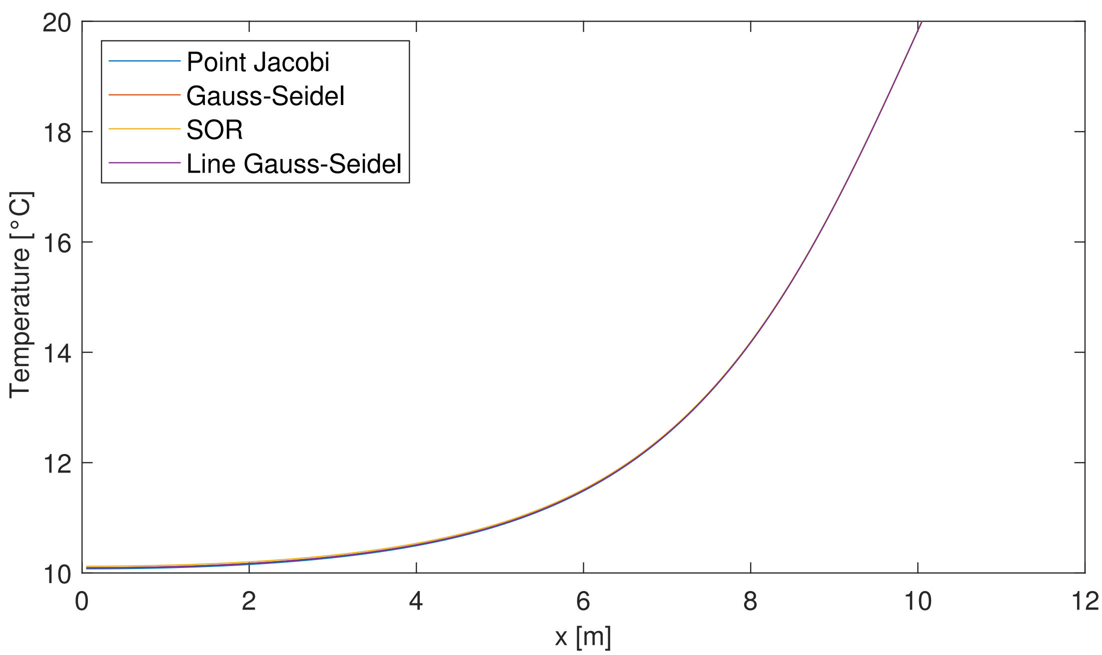

Figure 7: Temperature distributions along y=3m

3.2 Solution of the Heat Equation with Existence of a Radiator

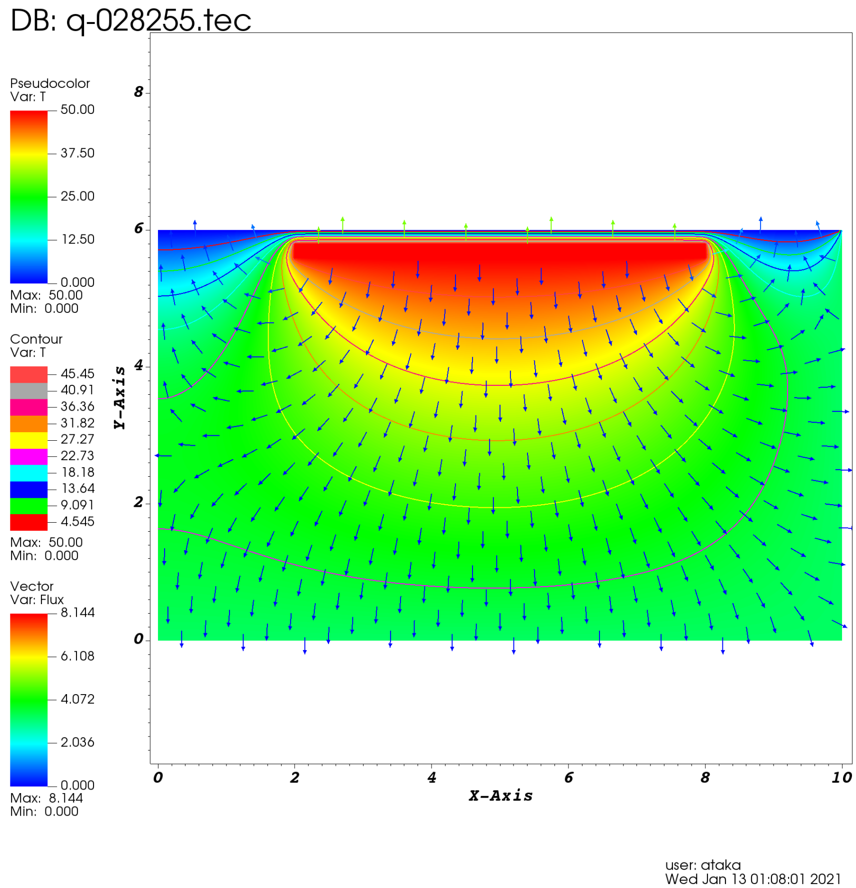

Figure 8: Solution of the heat equation by Point Jacobi method.

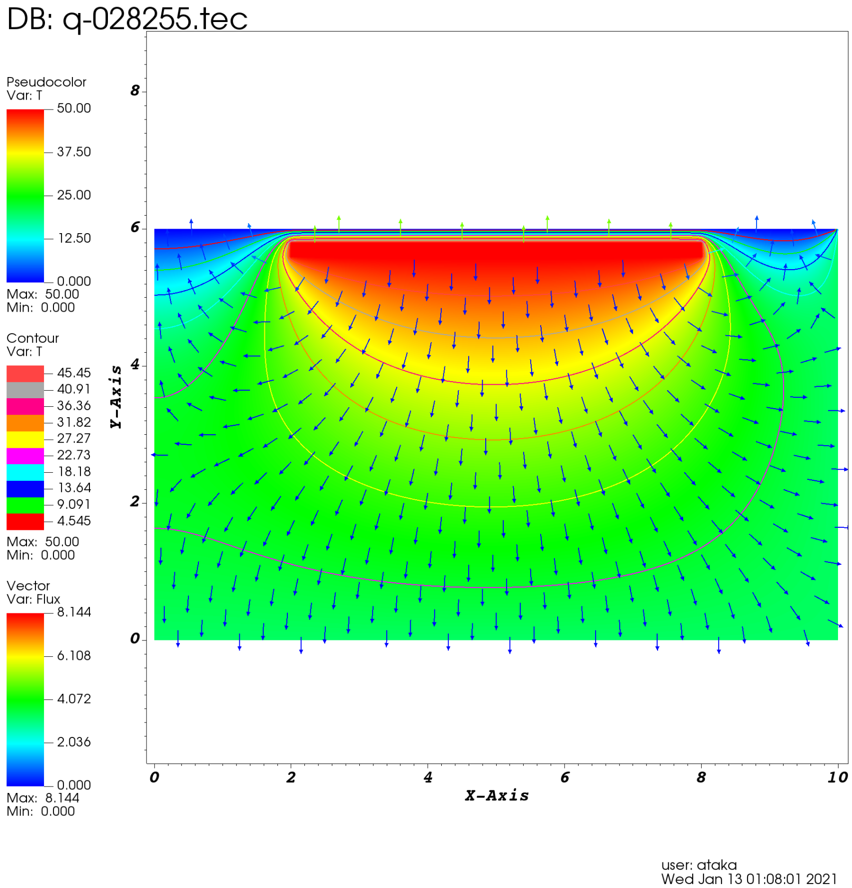

Figure 9: Solution of the heat equation by Gauss-Seidel method.

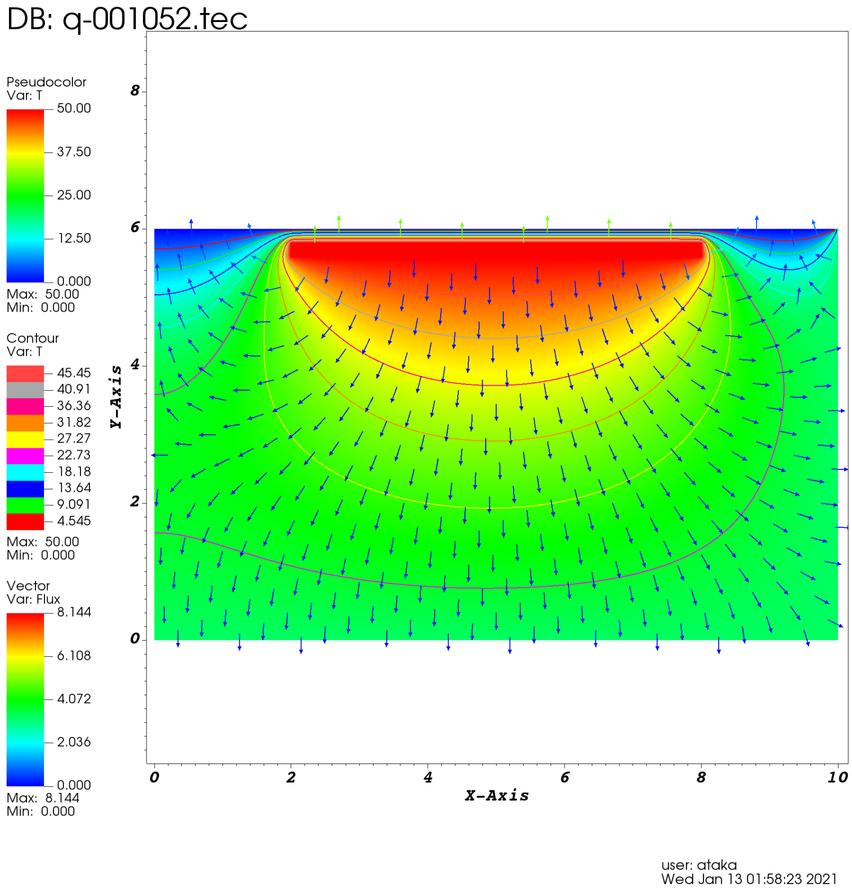

Figure 10: Solution of the heat equation by SOR method.

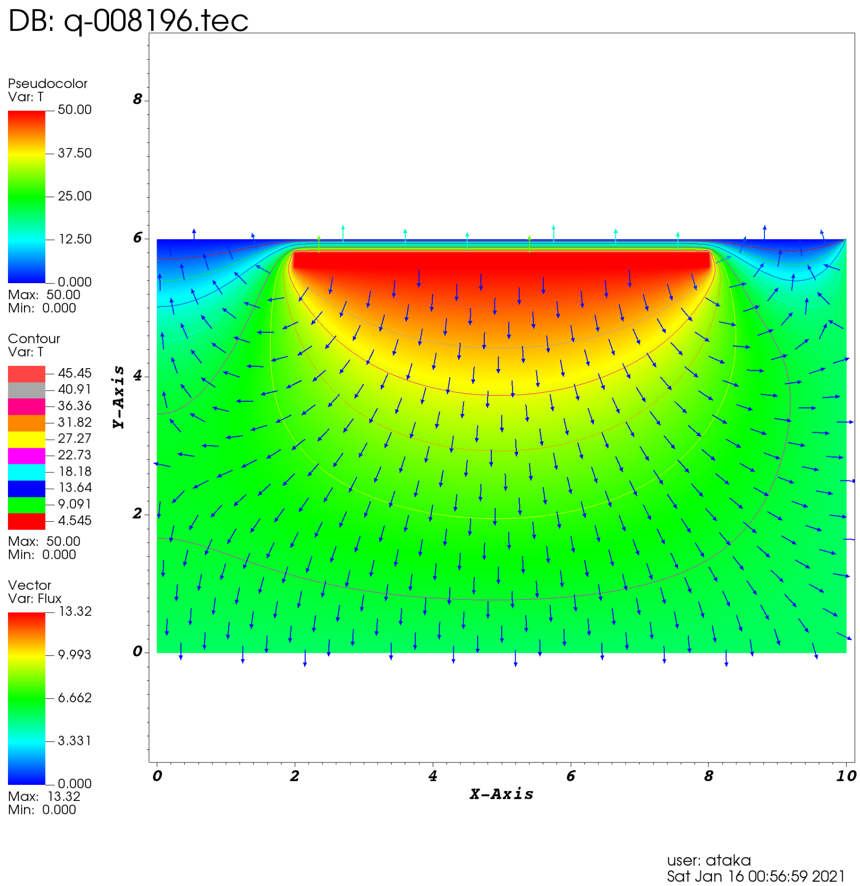

Figure 11: Solution of the heat equation by Line Gauss-Seidel method.

Figures 8, 9, 10, and 11 demonstrates the solution of the heat equation by using Point Jacobi, Gauss-Seidel, SOR,and Line Gauss-Seidel methods, respectively.From these figures, it can be observed that all the solutions with different methods converged to nearly the same heat flux distributions. In all solutions, the directions of the flux vectors are from the highest temperature surface ,which is the radiator, to the lower temperature surfaces, which is the walls of the room, perpendicularly, except the insulated wall. Since the wall is insulated, the temperature gradient across the wall is zero; hence the flux vectors are parallel near the surface. Also, from the figures, it can be seen that all temperature contours which are intersect with the insulated wall are perpendicular to the wall. Thus, it can said that heat transfer does not occur from the room to the wall or reverse of it.

3.2.1 Temperature Distributions along x=5m and y=3m

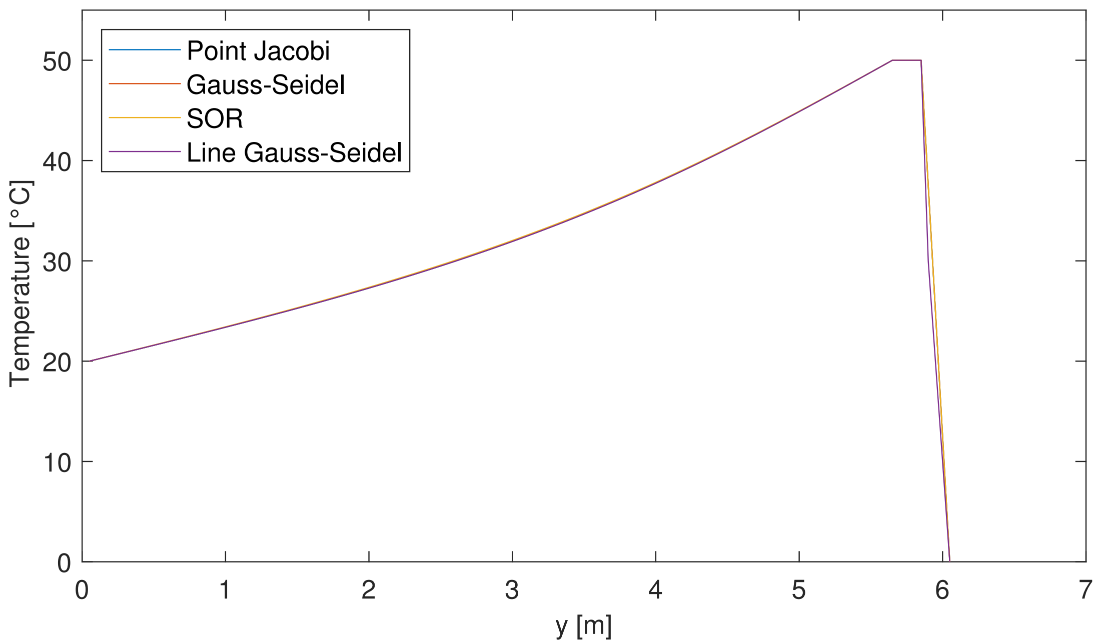

Figure 12: Temperature distributions along x=5m

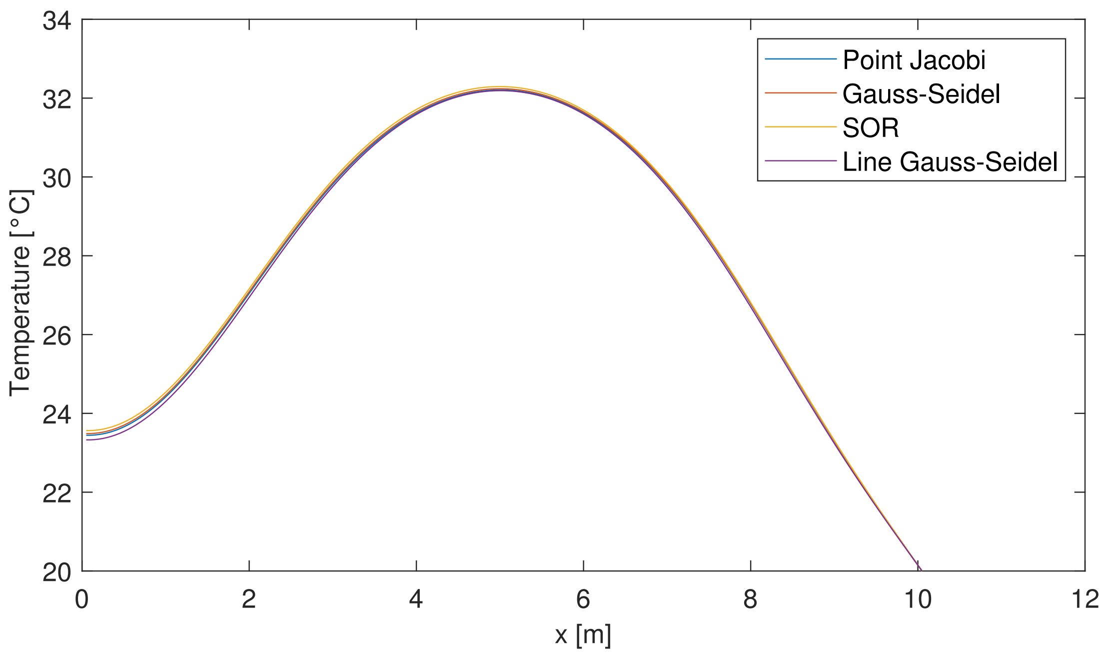

Figure 13: Temperature distributions along y=3m

Figures 12 and 13 illustrates the temperature distributions with the existence of radiator along x=5m and y=3m lines, respectively. In Figure 12, sudden change is observed near the y = 6m. The reason for this is that there is radiator between y = 5.6m and 5.8m. Therefore, a sudden rise in temperature is seen before y=5.8 meters. This causes distortion in both figures. However, in Figure 13, due to the the length of the radiator is longer in x-axis, its effect along y = 3m is less sudden.

3.3 Comparison of the Convergence Rates

Figures 14 and 15 demonstrate the convergence rates of Point Jacobi, Gauss-Seidel,SOR, and Line Gauss-Seidel methods for the solutions of the heat equation without and with the radiator, respectively. As can be observed from these figure, the responses of Point Jacobi method are the slowest. On the other hand, Gauss-Seidel method converges faster for both. This is because it uses newly computed values of the unknowns as they become available as mentioned in Subsection 2.2. Furthermore, Line Gauss-Seidel method gives the faster convergence rate than Gauss-Seidel method. Because as mentioned in Subsection 2.2, in Line-Gauss Seidel method, system of equations are solved simultaneously and it includes one more unknown variable than Gaus-Seidel method. .However, among all methods, SOR method computed the solutions with minimum number of iterations which is a result of the use of the relaxation parameter. This is also mentioned in Subsection 2.2. The relaxation parameter is taken 1.9 for both since this is its largest possible value which is found experimentally. For higher relaxation parameter values, the solutions did not converge.

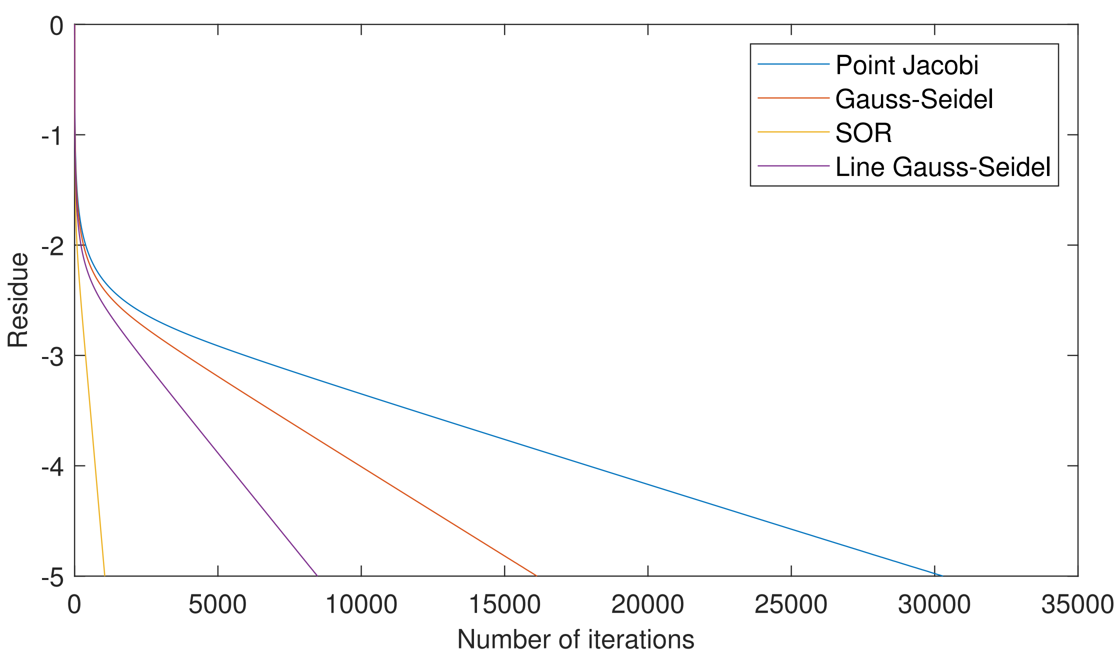

Figure 14: Convergence rates of Point Jacobi, Gauss-Seidel and SOR methods in the absence of the radiator.

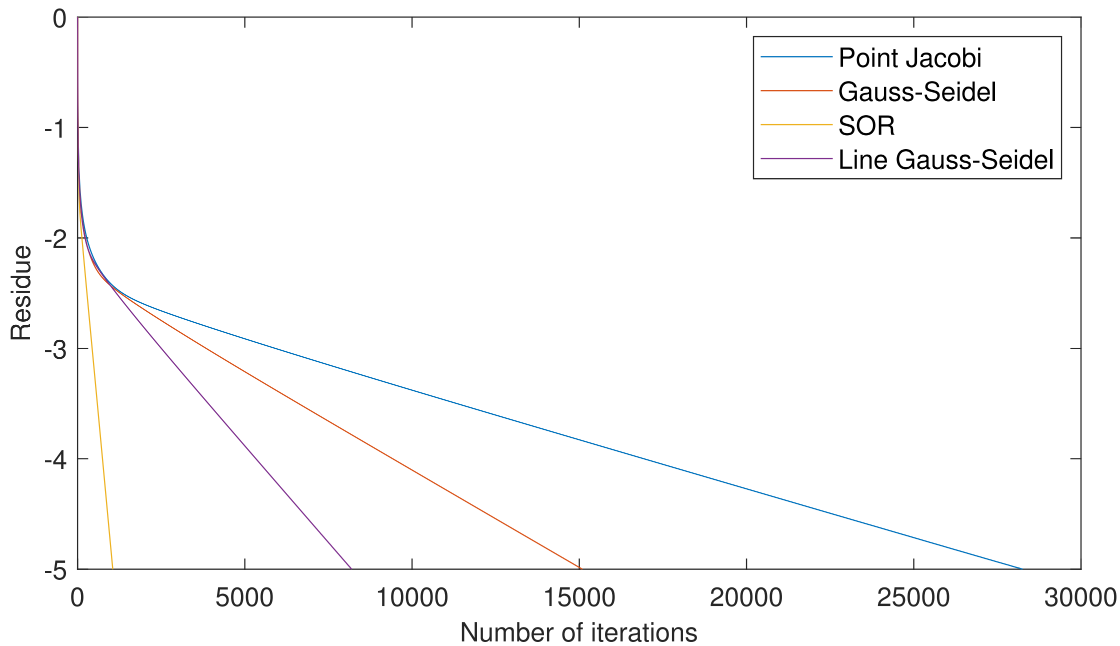

Figure 15: Convergence rates of Point Jacobi, Gauss-Seidel and SOR methods in the presence of the radiator.

3.4 Temperature Distribution on a 2D Model of a Room with Radiator with SOR Method

3.4.1 Comparison of Different values of Δx and Δy

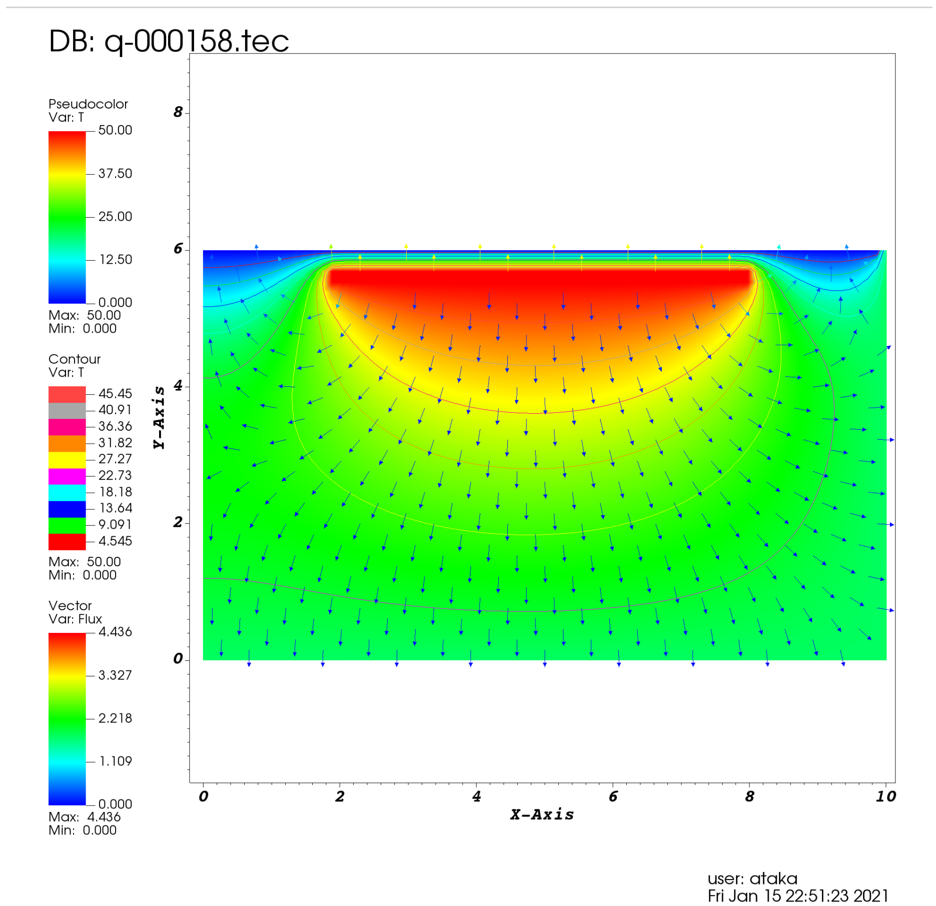

Figure 16: Temperature distribution in the room with $\Delta x = 0.135$ and $\Delta y = 0.154$

Figure 10, 16, and 17 illustrates temperature distribution in the room with SOR method with different Δx and Δy values. It can be seen that although low Δx and Δy values give a more precise result, if Δx and Δy values are reasonably seleced, differences in the result would be negligible. On the other hand, when Δx and Δy values are lower, convergence rate of the solution is slower, which can be seen from Figure 18. It can be concluded that, too low values of Δx and Δy should be avoided because they would increase solution time and they would not increase the accuracy of result significantly.

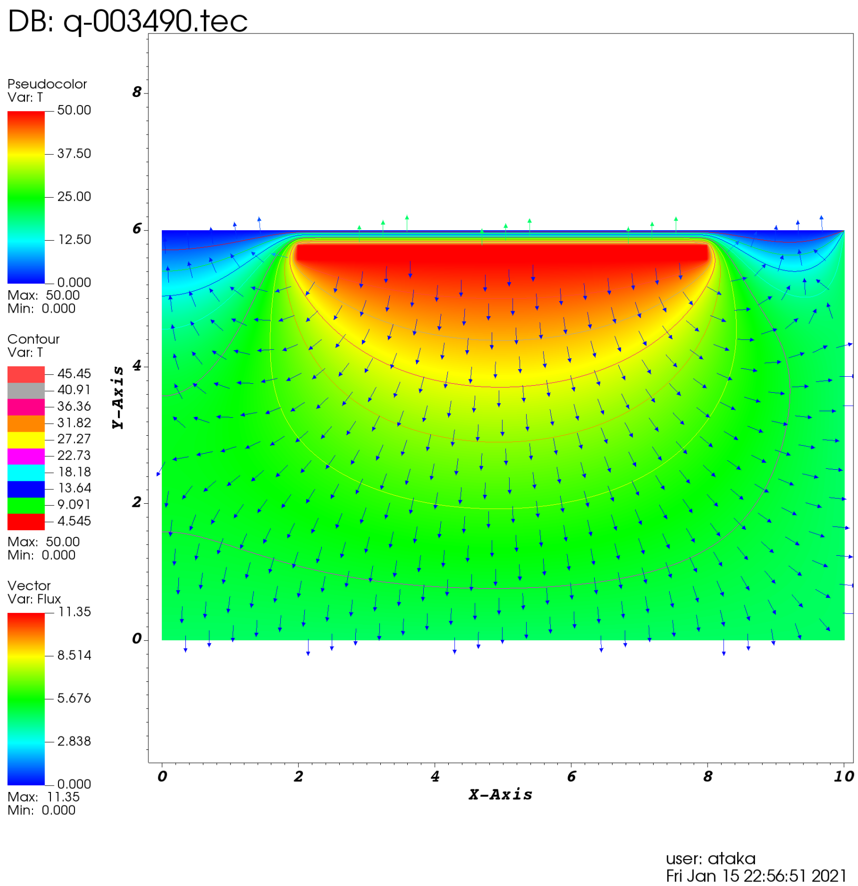

Figure 17: Temperature distribution on the room with $\Delta x = 0.025$ and $\Delta y = 0.025$

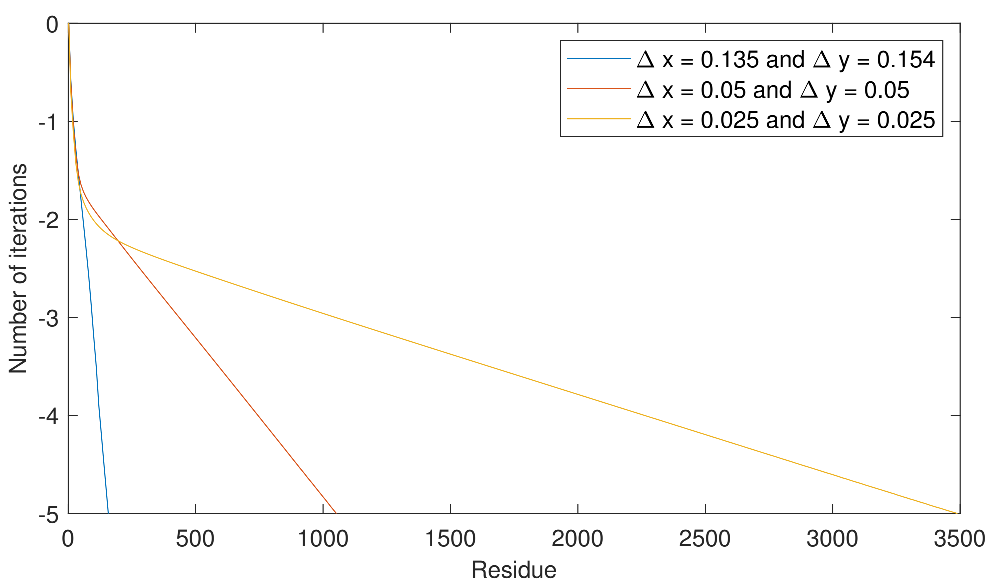

Figure 18: Convergence rates of SOR method with different Δx and Δy values.

3.4.2 Comparison of Different Sizes and Locations of the Radiator

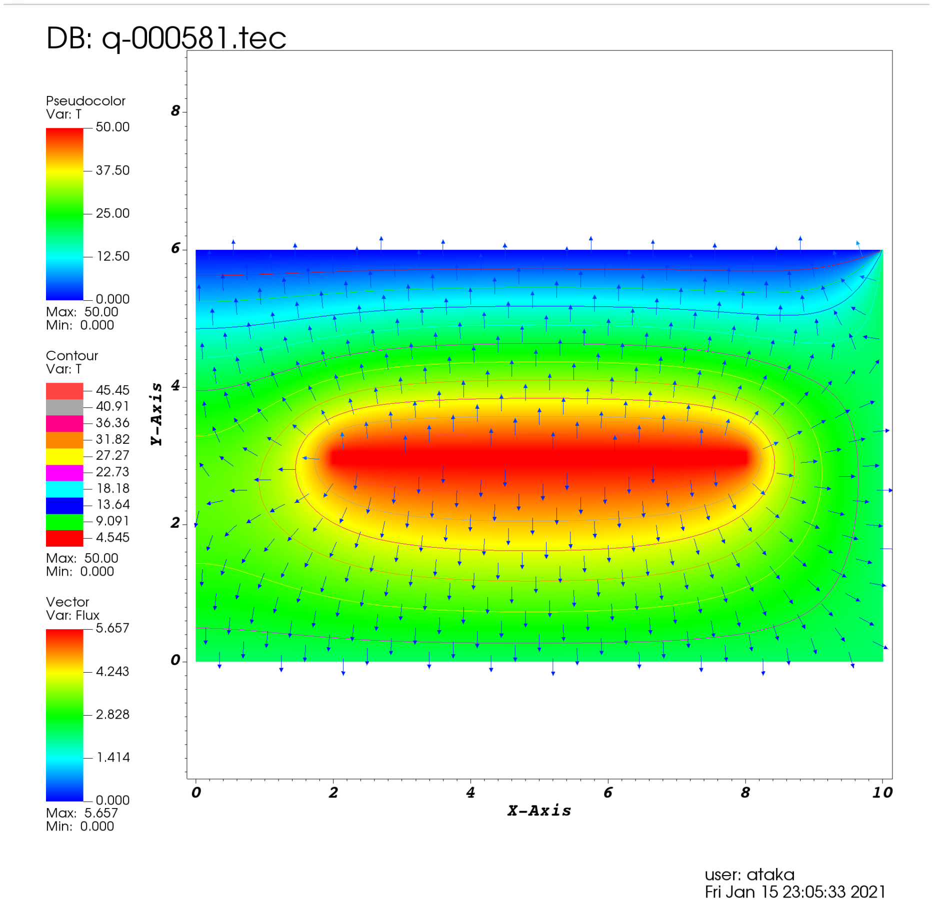

Figure 19: Temperature distribution on the room with radiator at middle of the room

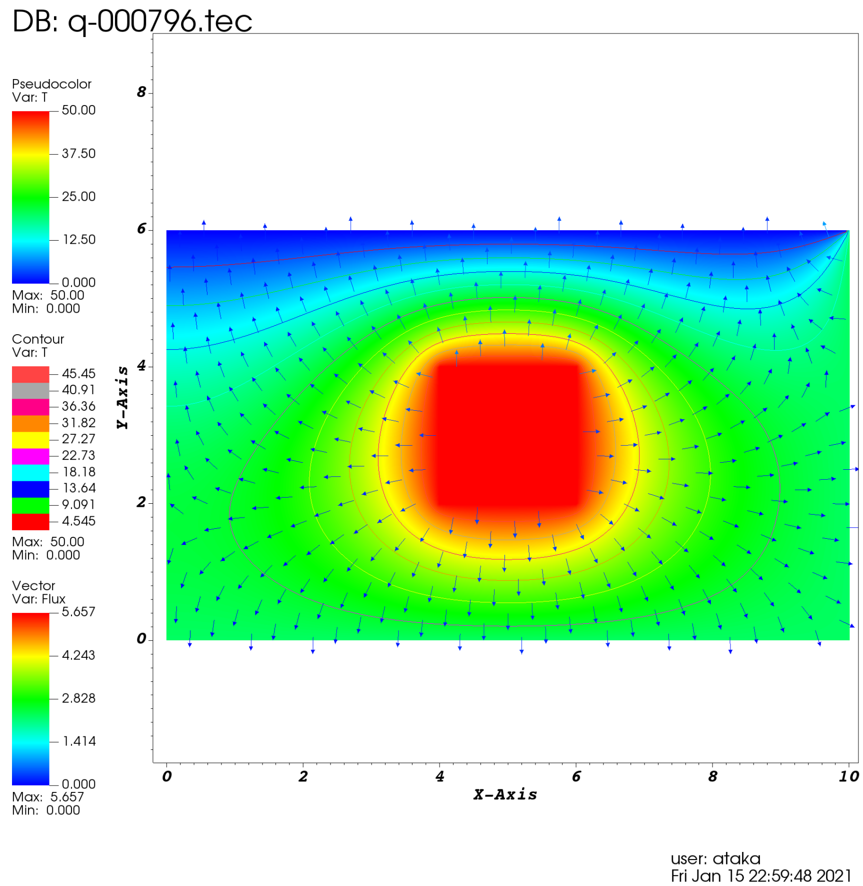

Figure 20: Temperature distribution on the room with radiator at middle of the room with different size

Figure 10 demonstrates the temperature distribution on the room with radiator with size $6m \times 0.2m$, which is neighbor to the wall with $0°C$. Figure 20 illustrates the temperature distribution on the room with radiator with the same size but at the middle of the room. Figure 19 shows the temperature distribution with radiator at the middle of the room with size $2m \times 2m$. It can be obtained from Figure 10 and 20, changing the location of the radiator from the coldest wall to the middle of the room causes inefficiency to evenly heat the entire room. Since all walls do not have the same temperature, putting the source of heat at the middle of the room is not reasonable to heat everywhere evenly. Moreover, from Figure 20 and 19, it can be observed that changing the size of the radiator is not logical. Since horizontal length of the radiator is shortened, it can not be sufficient to heat area near to the edges of the coldest wall. But, vertical length of the raditor is extended, it is more effective to heat the middle of the coldest wall than in Figure 20. Overall, its effectiveness less than the other radiator which is longer in horizontal direction.

4 Conclusion

Given a 2D model of a room, with a size of 10m x 6m, it is requested to solve heat conduction equation for the room. For this purpose, various numerical methods for the solution of an elliptic partial differential equation are used. First, Point Jacobi, Gauss-Seidel, Line Gauss-Seidel and SOR methods are used for two cases, one in the existence of a radiator and the other is in the absence of a radiator. It was concluded that Point Jacobi method is the slowest responding method since it does not uses the newly computed values of unknowns in the k+1 iteration level as they become available whereas Gauss-Seidel, Line Gauss-Seidel and SOR methods use them. As a result of this, Gauss-Seidel, Line Gauss-Seidel and SOR methods converged faster than Point Jacobi method. Comparing Gauss-Seidel and Line Gauss-Seidel methods, Line Gauss-Seidel method converges faster as one more unknown variable is expressed at the k + 1 iteration level and the influence of the BC at $i = 1$ and $i = imax$ are reflected into the solution immediately. Furthermore, it was obvious from Figures 14 and 15, convergence of SOR method occurs earlier then that of all methods. This is because SOR method makes use of the amount of change in each step to estimate the unknown variable at the new iteration level to accelerate the convergence of the solution, and it multiplies the change by the relaxation parameter.

According to the experimentations made, the value of Δx and Δy had an impact on the solution. For the calculations mentioned above, $\Delta x = 0.05$ and $\Delta y = 0.05$. Then the calculations are repeated for higher and smaller values of Δx and Δy such that for the former $\Delta x = 0.135$ and $\Delta y = 0.154$ and for the latter $\Delta x = 0.025$ and $\Delta y = 0.025$. It was observed that the medium and higher values yield to more accurate results whereas smaller values lead to distortions in the temperature distribution.

The location and the size of the radiator also changed the solution. For the first calculations, the radiator was placed near the upper wall and its size was 6m x 0.2m. When it is moved to the middle of the room, the upper wall is less heated. While it is still in the middle of the room, but changed in size (2m x 2m), the middle of the upper wall is more heated since the radiator is longer vertically, but the edges of the upper wall is less heated since the radiator is shorter horizontally.