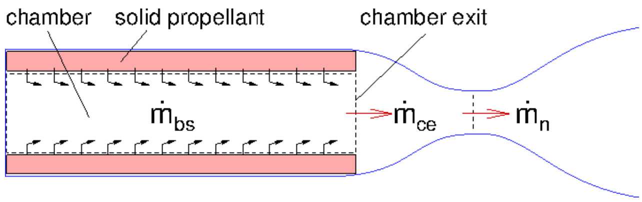
METU Department of Aerospace Eng
AE305 Numerical Methods
HW#2, Fall 2020

The gas pressure p_c inside a solid propellant rocket's propellant cavity (chamber, c) with constant burn temperature T_c may be modeled in a lumped manner by the following equation:

$$\frac{dp_c}{dt} = \frac{RT_c}{V_c}\left(\dot{m}_{\text{bs}} - \dot{m}_{\text{ce}} - \rho_c \dot{V}_c\right)$$

where p_c is the pressure of the lumped gas in the cavity, V_c is the cavity volume which changes in time due to the propellant burn, $\dot{m}_{\text{bs}}$ is the mass flow rate of the gas supplied from the burning surface (bs) of the propellant into the cavity,, $\dot{m}_{\text{ce}}$ is the gas mass flow rate exiting the cavity toward the nozzle, and ρ_c is the density of the lumped gas.



The propellant inner surface and hence the cavity shapes are generally complex (see the right figure above). For simplicity assume the cavity cross-sectional area or the propellant burning surface remains perfectly circular all along the propellant with an instantaneous radius of r during the burn. Also assume that between the chamber exit and nozzle throat (*) no gas mass accumulates, and hence $\dot{m}_{\text{ce}} = \dot{m}_n$. With these assumptions and writing $\dot{m}_{\text{bs}} = \rho_p 2\pi r L \dot{r}$, where ρ_p is the propellant density to be burned to generate the gas, L is the length of the propellant along the chamber, and $V_c = \pi r^2 L$, we may write the following equation for the chamber pressure:

$$\frac{dp_c}{dt} = RT_c\left[\frac{2\dot{r}}{r}(\rho_p - \rho_c) - \frac{\dot{m}_n}{\pi r^2 L}\right]$$

where the burn rate $\dot{r}$ is related to the chamber pressure by the relation:

$$\dot{r} = a p_c^n$$

with a and n being two constants that characterize the propellant's burn with pressure which are generally obtained by experiments. Also, rocket nozzles rapidly get chocked after firing, and hence from ideal gas flow relations we may express the mass flow rate in terms of the chamber pressure p_c, temperature T_c, and nozzle throat area A^*:

$$\dot{m}_n = p_c A^* \sqrt{\frac{\gamma}{RT_c}}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

where R is the gas constant, and γ is the ratio of specific heats.

Now we can sum up all the relations as follows:

$$\begin{aligned}\frac{dp_c}{dt} &= RT_c\left[\frac{2ap_c^n}{r}\left(\rho_p - \frac{p_c}{RT_c}\right) - \frac{p_c A^*}{\pi r^2 L}\sqrt{\frac{\gamma}{RT_c}}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}\right],\\ \frac{dr}{dt} &= ap_c^n\end{aligned}$$

Unfortunately though, a circular cross sectional solid propellant does not provide the desired characteristics. Therefore, the inner surface is not designed with a perfectly circular shape. To account for these effects, consider Eq.(1) is corrected for the ragged surface by a propellant design dependent factor $f_{\text{cor}}(r)$, in which case the abpve differential equations are rewritten as

$$\boxed{\frac{dp_c}{dt} = RT_c\left[f_{\text{cor}}(r)\frac{2ap_c^n}{r}\left(\rho_p - \frac{p_c}{RT_c}\right) - \frac{p_c A^*}{\pi r^2 L}\sqrt{\frac{\gamma}{RT_c}}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}\right]} \qquad (1)$$

$$\boxed{\frac{dr}{dt} = ap_c^n} \tag{2}$$

where r is now considered as an "equivalent" radius for simplicity.

By solving these two ordinary differential equations (Eqs(1) and (2)), one may obtain the chamber pressure which is considered as the stagnation pressure because the Mach number in the chamber is very small. Hence, assuming an ideal nozzle exists expanding to ambient pressure p_a, one can write the important performance indicator specific impulse I_{sp} as

$$I_{\text{sp}} = \frac{1}{g}\sqrt{\frac{2\gamma RT_c}{\gamma - 1}\left[1 - \left(\frac{p_a}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{3}$$

Now consider a small solid propellant rocket engine with a non-circular chamber cross section with a crudely approximated correction factor of

$$\begin{aligned} \eta &= \frac{r_{\text{final}} - r}{r_{\text{final}} - r_{\text{init}}}, \\ f_{\text{cor}} &= 1, \quad \eta > 0.15, \\ &= 1 - \exp(-7\eta), \quad 0 \le \eta \le 0.15, \\ &= 0, \quad \eta < 0 \end{aligned} \tag{4}$$

and the following charactersitics:

$$\begin{aligned} \textbf{Propellant density}, &\quad \rho_p = 1140\,\text{kg/m}^3 \\ \textbf{Propallent burn rate prop. constant}, &\quad a = 5.55 \times 10^{-5}\,\text{m/s} \times \text{ unit of } [p]^{-n}, \\ \textbf{Propallent burn rate exponent}, &\quad n = 0.305, \\ \textbf{Propellant initial inner radius}, &\quad r_0 = 5.00\,\text{cm} \\ \textbf{Propellant outer (final) radius}, &\quad r_f = 15.00\,\text{cm} \\ \textbf{Propellant length}, &\quad L = 1.25\,\text{m} \\ \textbf{Burn temperature}, &\quad T_c = 2810\,\text{K} \\ \textbf{Gas constant}, &\quad R = 365\,\text{J/kgK} \\ \textbf{Ratio of specific heats}, &\quad \gamma = 1.25 \\ \textbf{Nozzle throat radius}, &\quad r^* = 3.00\,\text{cm} \end{aligned}$$

For this rocket with a perfectly expanding nozzle at sea level ($p_a = 101325$ Pa), and the initial chamber pressure equal to the ambient pressure p_a,

(1) Until p_c becomes equal to p_a following the propellant burn diminishes (f_{cor} becomes zero), calculate the chamber pressure p_c, burn rate $\dot{r}$, and the specific impulse I_{sp}, all as a function of time, using the 4-th order Runge-Kutta method with 3 different time steps. For the three time steps plot *(i)* the chamber pressure vs. time, *(ii)* the burn rate $\dot{r}$ vs. time, *(iv)* the specific impulse vs. time, on separate graphs (on each graph 3 curves, one for each Δt). Compare the results for the three time steps, and discuss.

(2) Now assume you want to observe the nozzle throat area effect. Change the throat radius from 2.0 cm to 6.0 cm with increments of 1.0 cm, and calculate the same variables as in Part **(1)** but using the smallest time step you used there. Plot again *(i)* the chamber pressure vs. time, *(ii)* the burn rate $\dot{r}$ vs. time, *(iv)* the specific impulse vs. time curves on separate graphs (on each graph 5 curves, one for each throat radius), and discuss the comparisons.

(3) (BONUS) Repeat the solution of *(i)* the chamber pressure vs. time, *(ii)* the burn rate $\dot{r}$ vs. time, *(iv)* the specific impulse vs. time curves. by solving the ODEs with a constant time step and then adaptive stepping. Record the total numbers of time intervals needed for both computations, and compare. Which approach, use of constant time step or adaptive time steps, required less computational effort? For the two time stepping approaches plot the results. Discuss the impact of the time steps on the quality of the results.