ASSIGNMENT 6

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Download all python codes from

https://github.com/Atlakeerthana/Assignment6/tree/main/Assignment6

and latex-tikz codes from

https://github.com/Atlakeerthana/Assignment6/tree/main/Assignment6

1 Question No 2.74(e)

In each of the following find the equation for the ellipse that satisfies the given conditions:

a. Conjugate axis length= 24, foci= $\begin{pmatrix} 0 \\ \pm 13 \end{pmatrix}$

2 Solution

Given that,

$$\text{Conjugate axis length } = 2b = 24 \tag{2.0.1}$$

$$\text{Foci} = \mathbf{F} = \begin{pmatrix} 0 \\ \pm 13 \end{pmatrix} \tag{2.0.2}$$

Lemma 2.1. *The standard equation of an ellipse is given by:*

$$\frac{\mathbf{y}^\top D\mathbf{y}}{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f} = 1 \tag{2.0.3}$$

$$\text{where, } D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.4}$$

Lemma 2.2. *The coordinates of foci of ellipse* $\mathbf{F}$ *with y-axis as major axis are:*

$$\mathbf{F} = \begin{pmatrix} 0 \\ \pm\left(\sqrt{\frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}}\right) \end{pmatrix} \tag{2.0.5}$$

Also,the length of semi major axis, a is

$$a = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \tag{2.0.6}$$

and the length of semi minor axis, b is

$$b = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} \tag{2.0.7}$$

1) Using (2.0.1) length of conjugate axis is:

$$2b = 24 \tag{2.0.8}$$

$$\implies b = 12 \tag{2.0.9}$$

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = 12 \tag{2.0.10}$$

$$\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2} = 144 \tag{2.0.11}$$

$$\implies \lambda_2 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{144} \tag{2.0.12}$$

2) Using (2.0.5), the focus of ellipse is given as:

$$\mathbf{F} = \begin{pmatrix} 0 \\ \pm\left(\sqrt{\frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}}\right) \end{pmatrix} \tag{2.0.13}$$

or

$$\|\mathbf{F}\|^2 = \frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2} \tag{2.0.14}$$

$$\|\mathbf{F}\|^2 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1} - \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2} \tag{2.0.15}$$

3) Using (2.0.2) we get:

$$\implies 169 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1} - \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2} \tag{2.0.16}$$

4) Putting (2.0.11) in above equation we get:

$$169 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1} - 144 \tag{2.0.17}$$

$$25 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1} \tag{2.0.18}$$

$$\implies \lambda_1 = \frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{25} \tag{2.0.19}$$

5) Using lemma (2.1),the standard equation of ellipse is given by :

$$\frac{\mathbf{y}^\top D\mathbf{y}}{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f} = 1 \tag{2.0.20}$$

$$\implies \frac{\mathbf{y}^\top \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{y}}{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f} = 1 \tag{2.0.21}$$

6) Putting (2.0.12) and (2.0.19) in above equation we get:

$$\implies \mathbf{y}^\top \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{144} \end{pmatrix} \mathbf{y} = 1 \tag{2.0.22}$$

7) So,the equation of ellipse is:

$$\mathbf{y}^\top \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{144} \end{pmatrix} \mathbf{y} = 1 \tag{2.0.23}$$

8) The Plot of ellipse is:

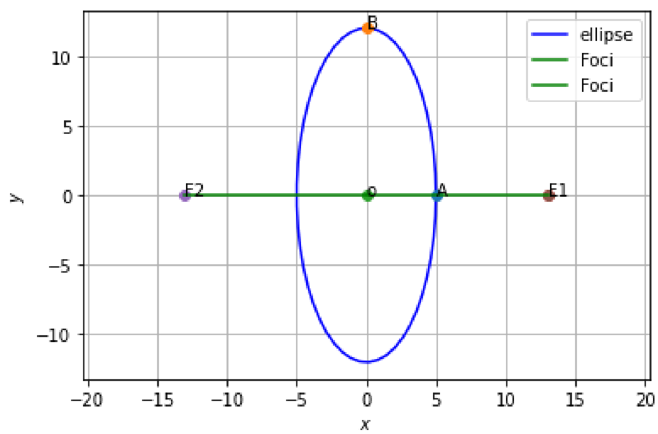

Fig. 2.1: Ellipse $\frac{x^2}{25} + \frac{y^2}{144} = 1$