5 Consider the three-dimensional normal distribution $p(\mathbf{x}|\omega) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 5 \end{pmatrix}$.

(a) Find the probability density at the point $\mathbf{x}_0 = (.5, 0, 1)^t$.

(b) Construct the whitening transformation $\mathbf{A}_w$. Show your $\boldsymbol{\Lambda}$ and $\boldsymbol{\Phi}$ matrices. Next, convert the distribution to one centered on the origin with covariance matrix equal to the identity matrix, $p(\mathbf{x}|\omega) \sim N(\mathbf{0}, \mathbf{I})$.

(c) Apply the same overall transformation to $\mathbf{x}_0$ to yield a transformed point $\mathbf{x}_w$.

(d) By explicit calculation, confirm that the Mahalanobis distance from $\mathbf{x}_0$ to the mean $\boldsymbol{\mu}$ in the original distribution is the same as for $\mathbf{x}_w$ to $\mathbf{0}$ in the transformed distribution.

(e) Does the probability density remain unchanged under a general linear transformation? In other words, is $p(\mathbf{x}_0|N(\boldsymbol{\mu}, \boldsymbol{\Sigma})) = p(\mathbf{T}^t\mathbf{x}_0|N(\mathbf{T}^t\boldsymbol{\mu}, \mathbf{T}^t\boldsymbol{\Sigma}\mathbf{T}))$ for some linear transform $\mathbf{T}$? Explain.

(f) Prove that a general whitening transform $\mathbf{A}_w = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-1/2}$ when applied to a Gaussian distribution insures that the final distribution has covariance proportional to the identity matrix $\mathbf{I}$. Check whether normalization is preserved by the transformation.

6 Suppose we have three categories with $P(\omega_1) = 1/2, P(\omega_2) = P(\omega_3) = 1/4$ and the following distributions

- $p(x|\omega_1) \sim N(0, 1)$
- $p(x|\omega_2) \sim N(.5, 1)$
- $p(x|\omega_3) \sim N(1, 1)$,

and that we sample the following four points: $x = 0.6, 0.1, 0.9, 1.1$.

(a) Calculate explicitly the probability that the sequence actually came from $\omega_1, \omega_3, \omega_3$ Be careful to consider normalization.

(b) Repeat for the sequence $\omega_1, \omega_2, \omega_2, \omega_3$.

(c) Find the sequence having the maximum probability.