MAT 357

Assignment 4

Rui Ji

1000340918

March 23, 2016

1. Assume that Σ is a σ-algebra. on a set M and μ: $\Sigma \to [0, \infty]$ is a map that satisfies:

   - $\mu(\emptyset) = 0$;
   - μ is finitely additive;
   - μ is upwards continuous.

   In order to show μ is a measure, it suffices to show that μ satisfies monotonic and countably additive.

   (a) First let's show μ is monotonic.
   For any sets A, B in Σ such that $A \subseteq B$, we have $B = A \cup (B \backslash A)$. We know that $A \cap (B \backslash A) = \emptyset$. Notice that $B \backslash A = B \cap A^c = (B^c \cup A)^c$ and Σ is closed under countable union and complement; hence, $B \setminus A \in \Sigma$. Then by finite additive we have

   $$\mu(B) = \mu(A) + \mu(B \setminus A).$$

   Since $0 \leq \mu(B \setminus A)$ then $\mu(A) \leq \mu(B)$.

   (b) Now let's show μ is countably additive.
   Suppose $(E_k)_{k=1}^{\infty}$ are countable many pairwise disjoint sets in Σ. Since Σ is closed under countable union then $\bigcup_{k=1}^{\infty} E_k$ and $\bigcup_{k=1}^{n} E_k$ are still in Σ. Define $E = \bigcup_{k=1}^{\infty} E_k$ and $E'_n = \bigcup_{k=1}^{n} E_k$, then $E'_1 \subseteq E'_2 \subseteq ... \subseteq E'_n$ and we have

   $$\bigcup_{n=1}^{\infty} E'_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{\infty} E_k = E.$$

   By upwards continuity we have

   $$\begin{aligned} \mu(E) &= \mu(\bigcup_{n=1}^{\infty} E'_n) \\ &= \lim_{n\to\infty} \mu(E'_n). \end{aligned}$$

   Since set $(E_k)_{k=1}^{n}$ are disjoint then

   $$\mu(E'_n) = \sum_{k=1}^{n} \mu(E_k).$$

   Therefore, we have

   $$\mu(E) = \lim_{n\to\infty} \mu(E'_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$

2. (a) Let P be the collection of all σ-algebra containing $\mathcal{A}$, and define Σ_A as follow

   $$\Sigma_A = \bigcap_{\Sigma \in P} \Sigma.$$

   Claim: Σ_A is the σ-algebra generated by $\mathcal{A}$.

i. First we show that P is nonempty by proving that 2^M is a σ-algebra containing $\mathcal{A}$. We already know that $\mathcal{A} \subseteq 2^M$, remaining to show that 2^M is a σ-algebra.
   - $\emptyset \in 2^M$, since $\emptyset$ is a subset of any set which means $\emptyset \subset M$.
   - Suppose $A \in 2^M$, which means $A \subseteq M$, then $A^c = (M \setminus A) \subseteq M$; hence $A^c \in 2^M$.
   - Suppose $(A_n)_{n=1}^{\infty}$ where $A_n \in 2^M$ that is $A_n \subseteq M$, then $(\bigcup_{n=1}^{\infty} A_n) \subseteq M$.

   Follows that 2^M is a σ-algebra containing $\mathcal{A}$.

ii. Then we show that Σ_A is a σ-algebra containing $\mathcal{A}$.
   Notice that by definition of P, we know $A \in \Sigma$ for all $\Sigma \in P$, hence, A is in their intersection Σ_A. Remaining to show that Σ_A is a σ-algebra.
   - Any $\Sigma \in P$, by definition of σ-algebra we know that $\emptyset \in \Sigma$; hence, $\emptyset \in \bigcap_{\Sigma \in P} \Sigma = \Sigma_A$.
   - Any $A \in \Sigma_A$ by definition of Σ_A we know that $A \in \Sigma$ for all $\Sigma \in P$ which implies $A^c \in \Sigma$ for all $\Sigma \in P$; hence, A^c is in Σ_A.
   - Suppose $(A_n)_{n=1}^{\infty}$ where $A_n \in \Sigma_A$, then by definition of Σ_A we know that each $A_n \in \Sigma$ for all $\Sigma \in P$, which means $(\bigcup_{n=1}^{\infty} A_n) \in \Sigma$ for all $\Sigma \in P$; hence, $(\bigcup_{n=1}^{\infty} A_n)$ is in Σ_A.

iii. Finally let's show that Σ_A is the smallest σ-algebra containing $\mathcal{A}$.
   Suppose there exist a σ-algebra Σ'_A containing $\mathcal{A}$ which is strictly contained in Σ_A, by defination of P we know that $\Sigma'_A \in P$, which contradicting the fact that $\Sigma_A = \bigcap_{\Sigma \in P} \Sigma$. Therefore, Σ_A is a smallest σ-algebra containing $\mathcal{A}$.

(b) i. First let's show (1), (2) are the same (i.e. the σ-algebra generated by the standard topology and the σ-algebra generated by the open intervals are the same). Let the σ-algebra generated by the standard topology be Σ_A, and Σ_B be the σ-algebra generated by the open intervals.
   Note: let B be the collection of all open intervals in real line, and the topology generated by B is called standard topology.
   Claim: every open set in standard topology is at most countable union of disjoint open intervals.
   - Suppose G is an open set in standard topology, then any $a \in G$, there exist an open interval (x, y) such that $a \in (x, y) \subseteq G$. Now define,

   $$a' = \inf\{x : a \in (x, y) \subseteq G\}, \qquad a'' = \sup\{y : a \in (x, y) \subseteq G\}$$

   - Now, let's show $(a', a'') \subseteq G$. For any point $z \in (a', a'')$, suppose $a' < z \leq a$ then by definition of infimum we know there exist interval $(x, y) \subseteq G$ such that $a' < x < z \leq a < y$. Similarly, if $a \leq z < a''$ then by definition of supremum we know there exist interval $(x, y) \subseteq G$ such that $x < a \leq z < y < a''$. Therefore, $(a', a'') \subseteq G$ denoted by I_a.
   - For any $a \in G$ we construct such I_a. Now let's show that for any $a, b \in G$ and $a \neq b$, suppose $a < b$, if $I_a \neq I_b$ then $I_a \cap I_b = \emptyset$. Suppose $I_a \cap I_b \neq \emptyset$, then let $min(a', b') = x$ and $max(a'', b'') = y$ and we have $a, b \in (x, y) = (a', a'') \cup (b', b'') \subseteq G$, contradicting the definition of I_a, I_b. Hence, $I_a \cap I_b = \emptyset$ and G can be written as disjoint union of such open intervals. Pick a rational number in one of those intervals, then we know that there are at most countable many such intervals.

   As we desired every open set in standard topology is at most countable union of disjoint open intervals, which implies $\Sigma_A \subseteq \Sigma_B$.
   Also, by definition of standard topology we know open intervals are contained in it, which means $\Sigma_B \subseteq \Sigma_A$. Therefore, as we desired $\Sigma_A = \Sigma_B$.

ii. Now let's show (2), (3) are the same (i.e. the σ-algebra generated by the open intervals and the σ-algebra generated by the compact intervals are the same). Notice that the set B of compact intervals on $\mathbb{R}$ is $\{[a,b] \mid a \leq b\}$.
Suppose Σ_A is the σ-algebra generated by the open intervals and Σ_B is the σ-algebra generated by the compact intervals.
First, notice that for any compact interval $[a,b]$ we have

$$[a,b] = ((-\infty,a) \cup (b,\infty))^c$$

then Σ_A contains B, which means the $\Sigma_B \subseteq \Sigma_A$.
Also notice that for any open interval $(-\infty,a)$, (a,b), (a,∞) or $(-\infty,\infty)$ we have

$$(\bigcup_{n=1}^{\infty}[a,a+n])^c = [a,\infty)^c = (-\infty,a)$$

$$(\bigcup_{n=1}^{\infty}[a-n,a] \cup [b,b+n])^c = ((-\infty,a] \cup [b,\infty))^c = (a,b)$$

$$(\bigcup_{n=1}^{\infty}[a-n,a])^c = (-\infty,a]^c = (a,\infty)$$

$$\bigcup_{n=1}^{\infty}[a-n,a+n] = (-\infty,\infty)$$

Hence, Σ_B contains all open intervals, which means the $\Sigma_A \subseteq \Sigma_B$. Then $\Sigma_A = \Sigma_B$.

iii. Now let's show (3), (4) are the same (i.e. the σ-algebra generated by the compact intervals and σ-algebra generated by the collection of intervals of the form (a,∞) for all $a \in \mathbb{R}$ are the same).
Suppose Σ_A is the σ-algebra generated by the compact intervals and Σ_B is the σ-algebra generated by the collection of intervals of the form (a,∞) for all $a \in \mathbb{R}$.
For any open interval (a,∞) we have

$$(\bigcup_{n=1}^{\infty}[a-n,a])^c = (-\infty,a]^c = (a,\infty)$$

then $\Sigma_B \subseteq \Sigma_A$.
Notice that for any compact interval $[a,b]$ we have

$$\bigcap_{n=1}^{\infty}((a-\frac{1}{n},\infty) \cap (b,\infty)^c) = \bigcap_{n=1}^{\infty}((a-\frac{1}{n},\infty) \cap (-\infty,b]) = \bigcap_{n=1}^{\infty}(a-\frac{1}{n},b] = [a,b]$$

Hence, Σ_B contains A, which means the $\Sigma_B \subseteq \Sigma_A$. Then $\Sigma_A = \Sigma_B$.

(c) First from textbook Section 6.4 we know that every open interval is Lebesgue measurable; hence, Lebesgue σ-algebra $\mathcal{M}$ contains all open intervals. Also notice that Borel σ-algebra $\mathcal{B}$ is generated by the open intervals; hence, $\mathcal{B} \subseteq \mathcal{M}$.

(d) First lets show the cardinality of $\mathcal{M}$ is $|\mathcal{P}(\mathbb{R})|$.

We will show this using Cantor set $C = \cap_{n=0}^{\infty} C^n$ (defined in textbook chapter 2 section 8), where

$$C^0 = [0,1]$$
$$C^1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$$
$$C^2 = [0,\frac{1}{9}] \cup [\frac{2}{9},[\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$$
$$...$$

and so on.

Proved in textbook C is uncountable, closed and outer measure is zero. Since it is closed then it is measurable and the measure of C is it's outer measure which is zero. Also, since it is uncountable and it is subset of $\mathbb{R}$, then $|C| = |\mathbb{R}|$, which implies $|\mathcal{P}(C)| = |\mathcal{P}(\mathbb{R})|$. Notice that any subset of C is also zero set (monotonic) and all zero sets are measurable. Therefore, $|\mathcal{M}| \geq |\mathcal{P}(\mathbb{R})|$. Because $\mathcal{M} \subseteq P(\mathbb{R})$, $|\mathcal{M}| \leq |\mathcal{P}(\mathbb{R})|$.

As we desired, $|\mathcal{M}| = |\mathcal{P}(\mathbb{R})|$.

3. Let (M, Σ, μ) be a measure space and $\mathcal{Z} = \{Z \subseteq M : \exists A \in \Sigma \textit{ such that } Z \subseteq A, \mu(A) = 0\}$

(a) We will show the following two properties are equivalent.
(i)$\mathcal{Z} \subseteq \Sigma$.
(ii) If $A \subseteq B \subseteq C, A \in \Sigma, C \in \Sigma$, and $\mu(C \setminus A) = 0$, then $B \in \Sigma$.
First we show that $(i) \Rightarrow (ii)$.

Suppose $A \subseteq B \subseteq C, A \in \Sigma, C \in \Sigma$, and $\mu(C \setminus A) = 0$. Notice $B = A \cup (B \setminus A)$ and $(B \setminus A) \subseteq (C \setminus A)$ where $\mu(C \setminus A) = 0$. By (i) we know that $B \setminus A$ is in Σ; hence, $B \in \Sigma$.

Now we show that $(ii) \Rightarrow (i)$.

Notice for any $Z \in \mathcal{Z}$, we have $\emptyset \subseteq Z \subseteq A$ and $\mu(A \setminus \emptyset) = \mu(A) = 0$; hence by (ii) we have $Z \in \Sigma$ which implies $\mathcal{Z} \subseteq \Sigma$.

As we desired, $(i), (ii)$ are equivalent.

(b) Define $\Sigma' = \{ E \subseteq M :$ there exist measurable sets $A, B \in \Sigma$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0 \}$.
First lets show that Σ' is a σ-algebra.

- $\emptyset \in \Sigma'$. Since $\emptyset \in \Sigma$ and $\emptyset \subseteq \emptyset \subseteq \emptyset$; hence, $\emptyset \in \Sigma'$.
- Closed under complement. Suppose $E \in \Sigma'$, by definition of Σ' we know there exist $A, B \in \Sigma$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. Then $B^c \subseteq E^c \subseteq A^c$, and $A^c \setminus B^c = A^c \cap (B^c)^c = A^c \cap B = B \setminus A$; hence, $\mu(A^c \setminus B^c) = 0$.
- Closed under countable union. Suppose $(E_i)_{i=1}^{\infty}$ where $E_i \in \Sigma'$, by definition of Σ' we know there exist $A_i, B_i \in \Sigma$ such that $A_i \subseteq E_i \subseteq B_i$ and $\mu(B_i \setminus A_i) = 0$. Define $A = \bigcup_{i=1}^{\infty} A_i$, $E = \bigcup_{i=1}^{\infty} E_i$ and $B = \bigcup_{i=1}^{\infty} B_i$. Then, $A \subseteq E \subseteq B$ and we have

$$B \setminus A = \bigcup_{i=1}^{\infty} (B_i \setminus A) \subseteq \bigcup_{i=1}^{\infty} (B_i \setminus A_i);$$

hence,

$$\mu(B \setminus A) \leq \mu(\bigcup_{i=1}^{\infty} (B_i \setminus A_i)) = \sum_{i=1}^{\infty} \mu(B_i \setminus A_i) = 0.$$

Therefore, $E = \bigcup_{i=1}^{\infty} E_i$ is in Σ'.

Now let's show Σ' is the σ-algebra generated by $\Sigma \cup \mathcal{Z}$.

- Σ' contains $\Sigma \cup \mathcal{Z}$. Any $E \in \Sigma$ we have $E \subseteq E \subseteq E$ and $\mu(E \setminus E) = \mu(\emptyset) = 0$, by definition of Σ' we have $E \in \Sigma'$ which implies that Σ' contains Σ. Any $E \in \mathcal{Z}$ by definition of $\mathcal{Z}$, there exist $A \in \Sigma$ such that $E \subseteq A$ and $\mu(A) = 0$; hence, $\emptyset \subseteq E \subseteq A$ and $\mu(A \setminus \emptyset) = \mu(A) = 0$. Then by definition of Σ' we have $E \in \Sigma'$ which implies that Σ' contains $\mathcal{Z}$. Then as we desired Σ' contains $\Sigma \cup \mathcal{Z}$.
- Σ' smallest σ-algebra contains $\Sigma \cup \mathcal{Z}$. It suffices to show that Σ' is contained in any σ-algebra which contains $\Sigma \cup \mathcal{Z}$. Suppose Σ_A is any σ-algebra contains $\Sigma \cup \mathcal{Z}$, any $E \in \Sigma'$, by definition of Σ' there exist $A, B \in \Sigma$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. Then $E = A \cup (E \setminus A)$, notice that $(E \setminus A) \subset (B \setminus A)$ which implies $E \setminus A \in \mathcal{Z}$. Since Σ_A contains $\Sigma \cup \mathcal{Z}$, then $A, (E \setminus A)$ are both in Σ_A, which implies $E \in \Sigma_A$. Follow that Σ' is contained in Σ_A. Since Σ_A is arbitrary, then Σ' smallest σ-algebra contains $\Sigma \cup \mathcal{Z}$.

Then let's define μ' on Σ'. For any $E \in \Sigma'$, exist $A, B \in \Sigma$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$ we define $\mu'(E) = \mu(A)$. Now let's show μ' is a measure.

- μ' is well defined. It suffices to show that $\mu'(E)$ is independent with the choice of A, B. Suppose there is another pair $A', B' \in \Sigma$ such that $A' \subseteq E \subseteq B'$ and $\mu(B' \setminus A') = 0$. Notice that $A' \subseteq E \subseteq B$ and $A' \setminus A \subseteq E \setminus A \subseteq B \setminus A$; hence, $\mu(A' \setminus A) = 0$ (i.e $\mu(A' \cap A^c) = 0$). Similarly we could get $\mu(A \setminus A') = 0$ (i.e $\mu(A \cap A'^c) = 0$). Then we have

$$\mu'(A) = \mu'(A \cup (A' \cap A^c)) = \mu'((A \cup A') \cap (A \cup A^c)) = \mu'(A \cup A')$$

$$\mu'(A') = \mu'(A' \cup (A \cap A'^c)) = \mu'((A' \cup A) \cap (A' \cup A'^c)) = \mu'(A \cup A')$$

  Hence, $\mu'(A) = \mu'(A')$. Follows $\mu'(E)$ is independent with the choice of A, B.
- μ' is a measure.
  - $\mu'(\emptyset) = 0$. Notice that $\emptyset \subseteq \emptyset \subseteq \emptyset$; hence, $\mu'(\emptyset) = \mu(\emptyset) = 0$.
  - monotonicity. Suppose $E_1, E_2 \in \Sigma'$ where $E_1 \subseteq E_2$. Choose $A_1, B_1, A_2, B_2 \in \Sigma$ such that $A_1 \subseteq E_1 \subseteq B_1$ and $A_2 \subseteq E_2 \subseteq B_2$ where $\mu(B_1 \setminus A_1) = \mu(B_2 \setminus A_2) = 0$. Notice that $\mu'(E_2) = \mu(A_2) = \mu(A_2) + \mu(B_2 \setminus A_2) = \mu(B_2)$ and $A_1 \subseteq E_2 \subseteq B_2$. Then $\mu(B_2) \geq \mu(A_1)$; hence, $\mu'(E_2) = \mu(B_2) \geq \mu(A_1) = \mu'(E_1)$.
  - countable additive. Let $E_i \in \Sigma'$ for all $i \in \mathbb{N}$ be a sequence of pairwise disjoint sets and by definition there exist sequences $A_i, B_i \in \Sigma$ such that $A_i \subseteq E_i \subseteq B_i$ for all i, $\mu(B_i \setminus A_i) = 0$ and $\mu'(E_i) = \mu(A_i)$. Define $A = \bigcup_{i=1}^\infty A_i$, $E = \bigcup_{i=1}^\infty E_i$ and $B = \bigcup_{i=1}^\infty B_i$. Then we have

$$\mu(B \setminus A) = \bigcup_{i=1}^{\infty}(B_i \setminus A) \subseteq \bigcup_{i=1}^{\infty}(B_i \setminus A_i);$$

    hence,

$$\mu(B \setminus A) \leq \mu(\bigcup_{i=1}^{\infty}(B_i \setminus A_i)) = \sum_{i=1}^{\infty} \mu(B_i \setminus A_i) = 0.$$

    Hence $\mu'(E) = \mu(A) = \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \mu'(E_i)$.
- Now let's show (M, Σ', μ') is complete. Let $E \in \Sigma'$ such that $\mu'(E) = 0$ and let $E' \subseteq E$. Choose $A, B \in \Sigma$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. Since $\mu(A) = \mu'(E) = 0$, then $\mu(B) = \mu(A) + \mu(B \setminus A) = 0$. Notice $\emptyset \subseteq E' \subseteq E \subseteq B$ and $\mu(B \setminus \emptyset) = \mu(B) = 0$; therefore, $E' \in \Sigma'$.

(c) Lebesgue outer measure m is a measure when restricted to Borel σ-algebra $\mathcal{B}$, then $(M, \mathcal{B}, m)$ is a measure space, now we claim it's completion is Lebesgue σ-algebra.

- We already know that the completion of $(M, \mathcal{B}, m)$ could be defined as $(M, \bar{\mathcal{B}}, \bar{m})$, where $\bar{\mathcal{B}} =\{ E \subseteq M$: there exist measurable sets $A, B \in \mathcal{B}$ such that $A \subseteq E \subseteq B$ and $m(B \setminus A) = 0 \}$.
- First we will show that any $E \in \bar{\mathcal{B}}$ is Lebesgue measurable. Notice that $E = A \coprod (E \setminus A)$. Since $(E \setminus A) \subseteq (B \setminus A)$, by monotonic we have $m^*(E \setminus A) \leq m^*(B \setminus A) = 0$. Then $m^*(E \setminus A) = 0$, which means $E \setminus A$ is Lebesgue measurable. Then by countable additivity we know that E is Lebesgue measurable. Therefore $\bar{\mathcal{B}} \subseteq$ Lebesgue σ-algebra
- Now we show that every Lebesgue measurable set E is in $\bar{\mathcal{B}}$. By regularity, we have for any Lebesgue measurable set E there exist a F_α set and a G_δ set such that $F \subset E \subset G$ and $m(G \setminus F) = 0$. From question 2, we know that $\mathcal{B}$ can be generated by open intervals. Since σ-algebra is closed under complement and countable union, then F_α set and G_δ are in $\mathcal{B}$. Therefore $E \in \bar{\mathcal{B}}$, which means Lebesgue σ-algebra $\subseteq \bar{\mathcal{B}}$. Then as desired we have Lebesgue σ-algebra $= \bar{\mathcal{B}}$.

Then, Lebesgue σ-algebra is a completion of Borel σ-algebra.

4. (a) In order to show that $\sim$ is a equivalence relation, we have to show that $\sim$ satisfies reflexivity, symmetry and transitivity.

- Reflexivity: any $x \in S^1$, notice that $R_0 x = x$; hence, $x \sim x$.
- Symmetry: given $x, y \in S^1$, suppose $x \sim y$ then there exist $q \in \mathbb{Q}$ such that $R_q x = y$, which means if we rotate x by an angle $2\pi q$ with respect to the origin we could get y. Therefore, if we rotate y by an angle $-2\pi q$ with respect to the origin we could get x that is $R_{-q} y = x$; hence, $y \sim x$.
- Transitivity: given $x, y, z \in S^1$, suppose $x \sim y$ and $y \sim z$. Then there exist $q_1, q_2 \in \mathbb{Q}$ such that $R_{q_1} x = y$ and $R_{q_2} y = z$. Then $R_{q_2}(R_{q_1} x) = R_{q_2} y = z$. Notice, that $R_{q_2}(R_{q_1} x)$ means first rotates x by $2\pi q_1$ then by $2\pi q_2$ that is rotates x by $2\pi(q_1 + q_2)$ and since $\mathbb{Q}$ is closed under addition; hence, $x \sim z$.

(b) Let $A = [1, 0)$.

- First we will show $\coprod_{q \in A} R_q(E)$ is a disjoint union. Suppose there exist $q_1, q_2 \in A$ such that $q_1 \neq q_2$ and $R_{q_1}(E) \cap R_{q_2}(E) \neq \emptyset$. Then we could pick $y \in R_{q_1} \cap R_{q_2}$. Then there exist x_1, x_2 in E such that $R_{q_1}(x_1) = y$ and $R_{q_2}(x_2) = y$ that is $x_1 \sim y$ and $x_2 \sim y$. By our definition of E we know $x_1 = x_2 = x$. Also notice that $q_1, q_2 \in A = [0, 1)$ which implies $0 \leq 2\pi q_1 < 2\pi$ and $0 \leq 2\pi q_2 < 2\pi$. Since rotation restrict to $[0, 2\pi)$ is injection then $R_{q_1}(x) \neq R_{q_2}(x)$ for $q_1 \neq q_2$, contradiction. Therefore, $\coprod_{q \in A} R_q(E)$ is a disjoint union.
- Now, we will show that $S^1 = \coprod_{q \in A} R_q(E)$. For any $y \in S^1$, suppose that $y \in [x]$, then by definition of E we know there exist $y' \in E$ such that $y' \in [x]$ as well. By transitivity, we know that $y' \sim y$. Then exist $q \in \mathbb{Q}$ such that $R_q y' = y$. We could find $q' \in [0, 2\pi)$ such that $q = 2\pi k + q'$ where $k \in \mathbb{Z}$ and by periodicity we have $y = R_q(y') = R_{q'}(y') \in R_{q'}(E) \subset \coprod_{q \in A} R_q(E)$; hence, $S^1 \subseteq \coprod_{q \in A} R_q(E)$. Also, notice that $R_t : S^1 \to S^1$ for $t \in R$; therefore, $\coprod_{q \in A} R_q(E) \subseteq S^1$. Then as we desired $S^1 = \coprod_{q \in A} R_q(E)$.

(c) Notice that rotation is orthogonal transformation, and as proved in textbook(pg 396) orthogonal transformation on R^n preserved outer measure; hence, $m^*(E) = m^*(R_q(E))$, for all $q \in \mathbb{Q}$.

(d) Notice we could identify S^1 with $[0, 1)$ preserving measure as mentioned in the question, then $m^*(S^1) = m^*([0, 1)) = 1$. Suppose $m^*(E) = 0$, since $S^1 = \coprod_{q \in A} R_q(E)$, then by subadditivity we have

$$1 = m^*(S^1) = m^*(\coprod_{q\in A} R_q(E)) \leq \sum_{q\in A} m^*(R_q(E)) = \sum_{q\in A} m^*(E).$$

If we enumerate all rational number in $[0,1)$, we have

$$\sum_{q\in A} m^*(E) = \sum_{i=1}^{\infty} m^*(E) = \sum_{i=1}^{\infty} 0 = 0$$

contradiction. Hence, $m^*(E) > 0$.

(e) Suppose m^* satisfies countable additivity, notice from equation (1) and $m^*(E) = m^*(R_q(E))$ we have,

$$m^*(S^1) = m^*(\coprod_{q\in A} R_q(E)) = \sum_{q\in A} m^*(R_q(E)) = \sum_{q\in A} m^*(E).$$

Notice we could identify S^1 with $[0,1)$ preserving measure as mentioned in the question, then

$$1 = m^*(S^1) = \sum_{q\in A} m^*(R_q(E)).$$

Since $m^*(E) > 0$ and if we enumerate all rational number in $[0,1)$, we would have

$$\sum_{q\in A} m^*(R_q(E)) = \sum_{i=1}^{\infty} m^*(E) = \infty$$

contradiction. Then m^* does not satisfy countable additivity.

(f) The same way, we could show that m^* does not satisfy finite additivity. Suppose $m^*(E) = k$ and let $n = 1 + \lceil \frac{1}{k} \rceil$. Using the previous argument if we enumerate all rational numbers in $[0,1)$ we would have,

$$m^*(S^1) = m^*(\coprod_{q\in A} R_q(E)) = m^*(\coprod_{i=1}^{\infty} R_{q_i}(E)).$$

Notice that,

$$\coprod_{i=1}^{n} R_{q_i}(E) \subseteq \coprod_{i=1}^{\infty} R_{q_i}(E)$$

and by monotonic we have,

$$m^*(\coprod_{i=1}^{n} R_{q_i}(E)) \leq m^*(\coprod_{i=1}^{\infty} R_{q_i}(E)) = m^*(S^1) = 1$$

But if m^* satisfies finite additivity, since $m^*(E) = m^*(R_q(E))$ we would get,

$$1 > (1 + \left\lceil \frac{1}{k} \right\rceil)k = \sum_{i=1}^{n} m^*(E) = \sum_{i=1}^{n} m^*(R_{q_i}(E)) = m^*(\coprod_{i=1}^{n} R_{q_i}(E)) \leq m^*(S^1) = 1$$

contradiction; hence, does not satisfies finite additivity.

(g) Suppose E is Lebesgue-measurable, then by countable additivity we have

$$m(S^1) = m(\coprod_{q \in A} R_q(E)) = m(\coprod_{i=1}^{\infty} R_{q_i}(E)) = \sum_{i=1}^{\infty} m^*(R_{q_i}(E)).$$

And $m(E) = m(R_q(E))$, since R_q is meseometry. Then if $m(E) = 0$ we will get,

$$1 = m(S^1) = \sum_{i=1}^{\infty} m^*(R_{q_i}(E)) = \sum_{i=1}^{\infty} m^*(E) = \sum_{i=1}^{\infty} 0 = 0.$$

If $m(E) = k > 0$ then we will get,

$$1 = m(S^1) = \sum_{i=1}^{\infty} m^*(R_{q_i}(E)) = \sum_{i=1}^{\infty} m^*(E) = \sum_{i=1}^{\infty} k = \infty.$$

Therefore, either case we get a contradiction, which means E is not Lebesgue-measurable.