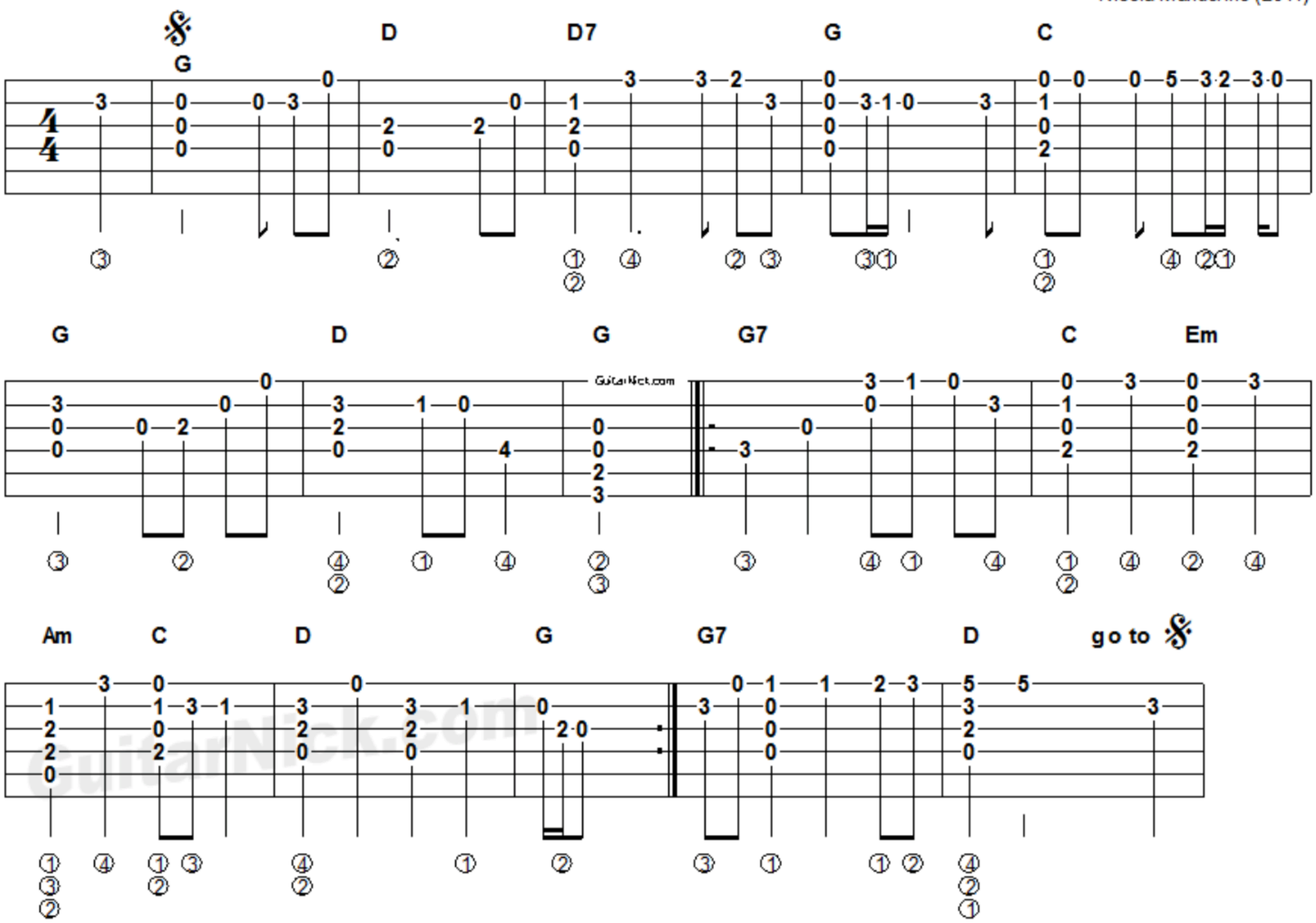
Go to **www.guitarnick.com**
lots of free guitar tabs & video lessons

Hey Jude

Beatles

TablEdited by
Nicola Mandorino (2011)

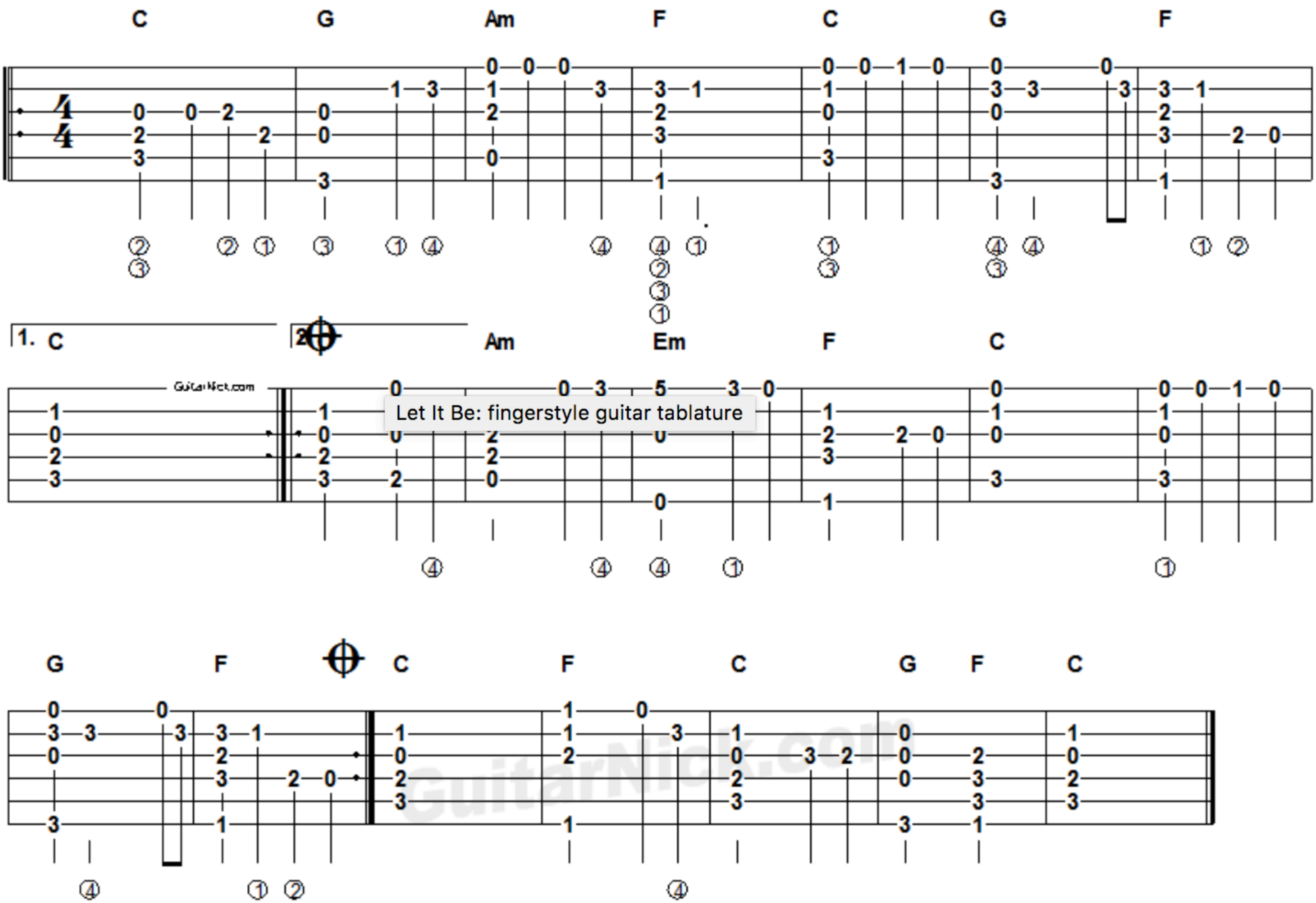


Go to **www.guitarnick.com**
lots of free guitar tabs & video lessons

Let It Be

Lennon/McCartney

Arranged & tabledited by
Nicola Mandorino (2010)



MAT 367S – Midterm Exam #2
1:10 – 2:00, March 23, 2015

`No tools allowed.`

Problem #1: [4+5=9 points]
a) Find the flow $\Phi_t(x)$ of the vector field
$$-x\frac{\partial}{\partial x}$$
on $\mathbb{R}$.

The corresponding differential equation is
$$\frac{dx}{\mathrm{d}t} = -x.$$
Its solution $x(t)$ with initial condition $x(0) = x_0$ reads as
$$x(t) = e^{-t}x_0.$$
That is, the flow is $\Phi_t(x_0) = e^{-t}x_0$. Dropping the zero from the notation,
$$\Phi_t(x) = e^{-t}x.$$

b) Find the flow $\Phi_t(x, y)$ of the vector field
$$x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$
on $\mathbb{R}^2$.

Similar to part a, the ODE is
$$\frac{dx}{\mathrm{d}t} = x, \quad \frac{dy}{\mathrm{d}t} = -y.$$
Solution $\gamma(t) = (x(t), y(t))$ with initial condition (x_0, y_0) is
$$x(t) = e^t x_0, \quad y(t) = e^{-t}y_0,$$
from which we read off the flow $\Phi_t(x_0, y_0) = (e^t x_0,\ e^{-t}y_0)$, or $\Phi_t(x, y) = (e^t x,\ e^{-t}y)$.

Problem #2: [4+5+5=14 points]
a) Show that
$$\Phi_t(x) = x + tx$$
cannot possibly be the flow of a vector field X on $\mathbb{R}$.

Doesn't have the flow property $\Phi_{t_1+t_2}(x) = \Phi_{t_1}(\Phi_{t_2}(x))$.

b) Show that the function

$$\Phi(t,x) = \left(\sqrt{x} + t\right)^2,$$

is the flow $\Phi_t(x) = \Phi(t,x)$ of a vector field on $\{x|\ x > 0\} \subset \mathbb{R}$. (The vector field is not complete; please don't worry about the domain of definition of the flow.)

Check the flow property $\Phi_{t_1+t_2}(x) = \Phi_{t_1}(\Phi_{t_2}(x))$.

c) Find the vector field on $\{x|\ x > 0\} \subset \mathbb{R}$ having the flow described in part b).

We calculate, for a smooth function f

$$\begin{aligned}
X(f)(x) &= \frac{d}{dt}|_{t=0} f(\Phi_t(x)) \\
&= \frac{d}{dt}|_{t=0} f\left(\left(\sqrt{x}+t\right)^2\right) \\
&= \frac{\partial f}{\partial x}\frac{d}{dt}|_{t=0}\left(\sqrt{x}+t\right)^2 \ \textit{(by chain rule)} \\
&= \frac{\partial f}{\partial x}\left(2(\sqrt{x}+t)\right)\Big|_{t=0} \\
&= 2\sqrt{x}\frac{\partial f}{\partial x}.
\end{aligned}$$

Hence $X = 2\sqrt{x}\frac{\partial}{\partial x}$.

Problem #3: [6 points]

Consider the following coordinate transformation on $\mathbb{R}^2$,

$$u = x + 2y, \quad v = y - 3x.$$

Express the coordinate vector fields

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

for the $x - y$-coordinates in terms of the coordinate vector fields

$$\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$$

for the $u - v$ coordinates.

Use

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v},$$

and similar.

Problem #4: [5 points] Compute the Lie bracket $[X, Y]$ of the following two vector fields on $\mathbb{R}^3$.

$$X = x\frac{\partial}{\partial y} + z\frac{\partial}{\partial x}, \quad Y = x\frac{\partial}{\partial z} + y\frac{\partial}{\partial x}.$$

The answer is

$$[X, Y] = z\frac{\partial}{\partial z} - y\frac{\partial}{\partial y}.$$

Problem #5: [6 points]

Let $S \subset M$ be a submanifold. A vector field $X \in \mathfrak{X}(M)$ is said to *vanish along* S if $X_p = 0$ for all $p \in S$.

Show that if $X, Y \in \mathfrak{X}(M)$ are two vector fields such that X vanishes along S, and Y is tangent to S, then $[X, Y]$ vanishes along S.

Use 'related vector fields': Let $i \colon S \to M$ *be the inclusion map. Then* Y *being tangent to* S *means that there exists a vector field* Z *on* S *with* $Z \sim_i X$*, while* X *vanishing along* S *means that* $0 \sim_i X$*. We have*

$$0 \sim_i X, \quad Z \sim_i Y \qquad \Rightarrow \qquad 0 = [0, Z] \sim_i [X, Y],$$

which means that $[X, Y]$ *vanishes along* S*.*