Recall that the *cross-ratio* of $p, q, r, s \in \hat{\mathbb{C}}$ is given by $[p,q,r,s] = \frac{(p-q)(r-s)}{(p-s)(r-q)}$.

Recall that

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}; \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}; \qquad \cosh z = \frac{e^{z}+e^{-z}}{2}; \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}.$$

## General

(1) Given a circle centered at $p$ of radius $r$, let us say that *inversion in the circle* is the continuous map $I : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the following property: given any $z \in \mathbb{C}$, its image $I(z)$ is the unique point on the ray $\overrightarrow{pz}$ such that $|z-p| \cdot |I(z)-p| = r^2$. As a consequence of continuity and this property, $I$ must exchange $p$ and $\infty$. The defining property also ensures that $I$ must fix each point in the circle.

(a) Prove that $f(z) = 1/\bar{z}$ is inversion in the unit circle by showing that it has the property listed above.

(b) Suppose $g(z) = z + b$ for some complex number $b$. Explain *geometrically* the effect of $g^{-1} \circ f \circ g$. Begin by finding its fixed points.

(c) Now let $h(z) = kz$ for some real number $k > 0$ and do the same for $h^{-1} \circ f \circ h$.

(d) Using the previous parts, give a formula for inversion in a circle of radius $r$ centered at $p \in \mathbb{C}$.

## Möbius transformations and cross-ratios

(2) Write the fractional linear transformation $f(z) = \frac{iz+1}{2z-1}$ as a cross-ratio map. (That is, $f(z) = [z, q, r, s]$ for some $q, r, s \in \hat{\mathbb{C}}$. Find those values.)

(3) What is the derivative of $f(z) = [z, -2, -1, \infty]$? Derive this computationally and give a geometric description of $f$.

(4) What is the FLT that sends $1 \mapsto i$, $i \mapsto -1$, and $-1 \mapsto -i$? What is its derivative? (Note: this can be done with blind computation but it is much cleaner to think geometrically.)

## Derivatives

(5) Does there exist an entire map that rotates every tangent vector by the same amount but does not amplify all tangent vectors by the same amount? If so, give an example. If not, prove it.

(6) On a previous homework you showed that there is no choice of $v \colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(z) = u + iv$ is analytic, if $u(x,y) = x^2 + y^2$. How about if $u(x,y) = ax^2 + bxy + cy^2$? (That is, find the conditions on $a, b, c \in \mathbb{R}$ that make it possible for a function with this real part to be analytic anywhere.)

(7) What is the effect of $f(z) = -1/z$ on tangent vectors based at $i$? at $2+3i$?

(8) Reread the proof of the reflection principle, either from class or from the book. Where does it use the hypothesis that the domain $D$ on which the function is defined is symmetric over the $x$–axis?

## Exponentials, logs, and trig

(9) Computations.

(a) Compute all values of the log base $i$ of 5.

(b) Derive the trig identity $\sin(2z) = 2\sin z \cos z$.

(c) Show that the derivative of tanh is $\text{sech}^2$. At what points is the hyperbolic tangent function angle-preserving?

(10) Find the mistake in this argument from class:

*We will find the image under* $\cos z$ *of a vertical line. Such a line has the form* $z \in \{c + yi\}$ *for some real constant* $c$. *As* $y$ *varies,* $iz$ *and* $-iz$ *travel along parallel (vertical) lines,* $iz \in \{-y + ic\}$ *and* $-iz \in \{y - ic\}$. *To find the cosine of the original values* $z$, *we must take the average of* $e^{iz}$ *with* $e^{-iz}$. *But applying the exponential to a vertical line gives a ray based at the origin, and in this case we will get two rays that are mirror images over the* $x$*–axis. Thus the average value is a point on the real axis.*

(Note that if that argument were true, every point from any vertical line in $\mathbb{C}$ would have a real output when you apply cosine—so cosine would take all of $\mathbb{C}$ to the real line!)

After finding the mistake, fix it. Sketch by hand the image of a particular vertical line with $c \neq 0$. Use a computer or graphing utility to draw a more precise picture.