

THE SPRAWL CONJECTURE FOR CONVEX BODIES

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ABSTRACT. What is the average distance between two points on the sphere of radius n in a normed space? There is a natural choice of measure making this average-distance statistic into an affine invariant, and we explore the conjecture that the ℓ^2 and ℓ^∞ norms provide the two extreme values of this invariant on \mathbb{R}^d for every d .

1. INTRODUCING THE SPRAWL

Consider the following asymptotic geometric statistic on a metric space: take the average distance between two points on the sphere of radius r , measured relative to r , as r gets large. To make sense of averaging, one needs a measure. For a metric space (X, d) with basepoint x_0 and measures μ_r on the spheres $S_r = S_r(x_0)$, let the *sprawl* of X be

$$E(X) = E(X, d, x_0, \{\mu_r\}) := \lim_{r \rightarrow \infty} \frac{1}{r} \int_{S_r \times S_r} d(x, y) d\mu_r(x) d\mu_r(y),$$

if the limit exists. Note that since $0 \leq d(x, y) \leq 2r$, the value is always between 0 and 2. If $E = 2$, this means that one can typically pass through the origin when traveling between any two points on the sphere without taking a significant detour. (The name is intended to invoke urban sprawl: a higher value means a lack of significant shortcuts between points on the periphery of the “city.”)

In locally finite graphs (and therefore in finitely generated groups with their Cayley graphs), we can just use the uniform measure (i.e., counting measure), since the sphere of radius n is a finite set. A second setting, and the focus of this note, is finite-dimensional normed spaces: take \mathbb{R}^d with a choice of a convex, centrally symmetric polytope L . Then there is an intrinsically defined norm on \mathbb{R}^d and there are natural probability measures on its metric spheres, defined below in §2—for instance, if L is the round circle in \mathbb{R}^2 , we recover Euclidean distance and measure proportional to arclength. Some first examples: $E(\mathbb{R}) = 1$ with any norm (because the distance between points on the sphere is equally likely to be 0 or $2r$). For the Euclidean norm on the plane, we do a simple integral to find that $E(\mathbb{R}^2, \ell^2) = \frac{4}{\pi} \approx 1.273$, while for the sup-norm we get $E(\mathbb{R}^2, \ell^\infty) = \frac{4}{3} \approx 1.333$. The main conjecture would imply that all norms on the plane have sprawls pinched between these two values (see Figure 5).

Sprawl was introduced in [6], and this note should be read as a close companion to that article. This asymptotic statistic is also discussed in [5, 4, 3] in the context of curvature conditions: negative curvature is associated with high sprawl. Indeed, outside of convex geometry, it is easy to find examples with maximal sprawl. In the hyperbolic plane with its natural measure, if two rays make an angle θ , then their

time- r points are at most $2r - c(\theta)$ apart; this means that $E(\mathbb{H}^2) = 2$. Similarly, in a regular tree of finite degree, $E(T) = 2$.

1.1. Motivation and connections to other ideas. For us, the original question for general groups was suggested by ergodic techniques for studying statistical geometry in lattices, where asymptotic averages over large metric spheres are a key tool (see, for just one instance, [7, §3.3]). Similar geometric invariants have been introduced by other authors in quite different contexts. For instance, in a paper developing analytic techniques for general metric spaces [9], Yann Ollivier considers an L^2 average-distance statistic on small balls (as opposed to our L^1 average-distance on large spheres), which he calls the *spread*. Independently, and with motivations from category theory and biodiversity, Simon Willerton has defined his own *spread* in [10], which turns out to be a kind of L^{-1} asymptotic distance average. (The bibliography of that paper points to other metric notions from the same family of ideas.)

1.2. The Sprawl Conjecture. Below, we explore the values that the sprawl can take in finite-dimensional normed spaces in connection with the conjecture that the Euclidean norm and the sup norm achieve the extreme values among norms on \mathbb{R}^d , for every d . There is a continuous one-parameter family of perimeters interpolating from the sphere to the cube, and the sprawl is continuous in L , so all of the values between those two endpoints are achieved as $E(L)$ for some L . Conjecturally, this is everything.

Sprawl Conjecture. *The sprawl is an affine isoperimetric invariant. That is,*

$$\{E(L) : \text{perimeters } L \subset \mathbb{R}^d\} = [E(\mathbb{R}^d, \ell^2), E(\mathbb{R}^d, \ell^\infty)].$$

If true, this would place the sprawl in good company among a large family of affine isoperimetric invariants of convex bodies, which are by definition geometric statistics whose extremes are achieved (in the centrally symmetric case) by cubes and round balls. (In §3, it will be explained that this conjecture also implies bounds on the sprawl of free abelian groups with any finite generating sets.) The current paper proves some results cited in [6] and gives an algorithm, some key calculations, and some empirical evidence for exploring this Sprawl Conjecture.

One of the elements of interest in studying this conjecture is that it is (at least apparently) immune to the main tools for proving affine isoperimetric inequalities: rearrangements. For instance, it is known that the Mahler volume (defined in §6) is maximized by round balls; this is proved by applying a sequence of *Steiner symmetrizations* under which a convex body converges to a ball and using the fact that Mahler volume is monotonic under these transformations. Another example of an affine isoperimetric invariant is the average distance between points in a *solid* body rather than on its perimeter; here, one can apply the powerful *Brascamp-Lieb-Luttinger inequality* to quickly deduce that a round ball realizes the minimum ([8], using [1]). Neither of these approaches yields results in our case—sprawl is not monotonic under any of the usual kinds of symmetrization, and it is not in the right form to apply any of the well-known analytic rearrangement inequalities—so we undertake experimental methods.

1.3. Acknowledgments. We thank the referees for careful reading and numerous helpful suggestions.

2. NORMS AND MEASURES INDUCED BY POLYTOPES

Definition 1. A *convex body* is a compact convex set in \mathbb{R}^d with nonempty interior. We will use the word *perimeter* to mean the boundary of a centrally symmetric convex body in \mathbb{R}^d . The *induced norm* on \mathbb{R}^d for a perimeter L is the unique norm for which L is the unit sphere. (And these recover all possible norms on \mathbb{R}^d , as L varies.) The *cone* \hat{A} on a set $A \subset \mathbb{R}^d$ is $\{ta : 0 \leq t \leq 1, a \in A\}$. The *cone measure* μ_L on a perimeter is given by $\mu_L(A) = \frac{\text{Vol}(\hat{A})}{\text{Vol}(\hat{L})}$ for measurable subsets $A \subset L$.

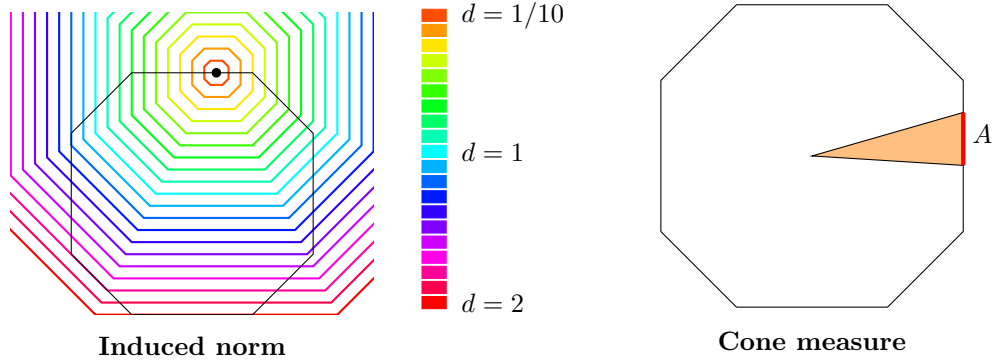


FIGURE 1. In the pictures above, L is an octagon (shown in black). On the left, the induced distance function $d(x, y) = \|x - y\|_L$ from a fixed point x is shown by its level sets. Every point on L has a distance between 0 and 2 from the marked point. On the right, the shaded triangle has area $7/10$ while the entire octagon encloses area 14, so the arc A has measure $\mu_L(A) = 1/20$.

Because of the homogeneity of the norm, one can define the sprawl for any perimeter $L \subset \mathbb{R}^d$ by

$$E(L) := E(\mathbb{R}^d, \|\cdot\|_L, 0, \mu_L) = \int_{L \times L} \|x - y\|_L d\mu_L(x) d\mu_L(y),$$

measuring average distance between points of L in its intrinsic geometry. We remark that $E(L) = E(TL)$ for any linear transformation $T \in GL_d(\mathbb{R})$, since both the norm and the measure push forward appropriately: that is, $\|Tx - Ty\|_{TL} = \|x - y\|_L$ and $d\mu_{TL}(Tx) = d\mu_L(x)$. That means in particular that the intrinsic geometry comes with a natural scale, so dilating a shape L does not change its sprawl, since both the norm and the measure are rescaled accordingly (in particular, if L is a polygon, then the distance from any point on a side σ to any point on the opposite side $-\sigma$ equals 2, as in Figure 1.)

3. CONNECTION TO GROUP THEORY

The geometry of a finitely generated group is studied via the Cayley graph that models the word metric with respect to particular generators. In geometric group theory, we frequently study groups at large scale, or asymptotically, focusing on properties invariant under *quasi-isometry* in order to remove the dependence on the choice of generating set and produce invariants of the group itself. However,

interesting geometric statistics need not be invariant in this way. Even for the approachable class of free abelian groups \mathbb{Z}^d , statistics as basic as the degree- d polynomial encoding the growth function do depend nontrivially on a choice of generators.

Burago showed in [2] that word metrics on \mathbb{Z}^d (and in fact, any suitably periodic metrics on \mathbb{R}^d) are boundedly close to norms:

$$|d_S(x, y) - \|x - y\|| \leq K = K(S)$$

for a certain norm $\|\cdot\|$. In [5], we study the distribution of points in spheres for \mathbb{Z}^d with various finite generating sets S , and we find that the counting measure on the discrete spheres S_n converges to cone measure (strongly, in an appropriate sense). Putting this information together, we obtain the following description.

Theorem 2 (Geometry of spheres in \mathbb{Z}^d [2, 5]). *Consider the word metric on \mathbb{Z}^d given by a finite symmetric generating set S . Let L be the boundary polytope of the convex hull of S in \mathbb{R}^d .*

Then the word metric is asymptotic to the norm induced by L , the spheres satisfy $\frac{1}{n}S_n \rightarrow L$, and the counting measure on $\frac{1}{n}S_n$ converges to μ_L .

In particular,

$$E(\mathbb{Z}^d, S) = \lim_{n \rightarrow \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y) = \int_{L \times L} \|x - y\|_L d\mu_L(x) d\mu_L(y) = E(L)$$

for this perimeter L .

Thus sprawls for \mathbb{Z}^d are readily computable, since the integrand in $E(L)$ is piecewise linear. Furthermore, this shows that in addition to being intrinsic to L and natural with respect to affine transformations, the cone measure is motivated by an application from group theory.

Let us call L the *limit shape* of (\mathbb{Z}^d, S) . For instance, in the case of the *chess-knight* generators $S_{\text{ch}} = \{(\pm 2, \pm 1), (\pm 1, \pm 2)\}$ which move around \mathbb{Z}^2 by knight's moves, the limit shape L is the (non-regular) octagon seen in Figures 1 and 2.

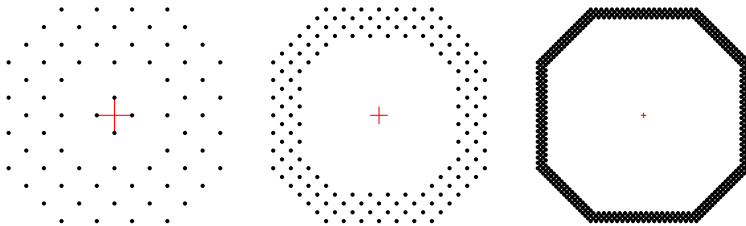


FIGURE 2. This figure shows the spheres of radius 3, 6, and 20 in the chess-knight metric, with the plane rescaled by $1/3$, $1/6$, and $1/20$. This illustrates that $S_n \approx nL$.

4. SPRAWL IN THE PLANE

In this section we study the sprawl of a convex body in dimension $d = 2$, by first introducing an algorithm for evaluating $E(L)$ when L is a polygon. (We note that sprawl is continuous in L and any perimeter can be approximated arbitrarily closely by a polygon.)

This algorithm can be given to a computer (we implemented it in Sage) but can also be used to produce precise formulas, such as those given below for the regular polygons.

4.1. *Cutline algorithm.* To compute the sprawl of a polygon, we will average the distances between pairs of sides. Pick two sides σ and τ of L and parametrize each of them (say clockwise) over $[0, 1]$; then the distance in the L -norm from $\sigma(s)$ to $\tau(t)$ is piecewise linear in s and t . For an appropriate triangulation of the square $[0, 1] \times [0, 1]$ (which will be explained below), average-distance is a linear function on each triangle. We outline here a method for triangulating, which we call the *cutline* algorithm for computing the sprawl of a polygon. We note that the algorithm generalizes straightforwardly to higher dimensions.

Let $\mathbf{c}(s, t) := \tau(t) - \sigma(s)$. To compute the L -norm of this vector, we translate it to the origin and find the dilate of L at which this vector ends. The first picture in Figure 3 shows the vectors $\mathbf{c}(0, 0)$ and $\mathbf{c}(1, 1)$ in red. When based at the origin, all the vectors $\mathbf{c}(s, t)$ lie in a sector between these two. This sector (and its opposite) are shown in the second picture. The distance $d_{st} := \|\mathbf{c}(s, t)\|_L$ is linear in s and t as long as $\mathbf{c}(s, t)$ points towards the interior of a side of L , with a transition when $\mathbf{c}(s, t)$ points towards a vertex (see the dashed line in the second picture). The vectors pointing in this direction are shown in the third picture, and the corresponding set of times $(s, t) \in [0, 1] \times [0, 1]$ is a line in the parameter square (shown in red in the fourth picture) with both endpoints on the boundary. We call this a “cutline.” The vertex set for the triangulation is the vertices of the square together with the endpoints of the cutlines coming from these transitional directions. We will only need to compute the values of d_{st} at these vertices.

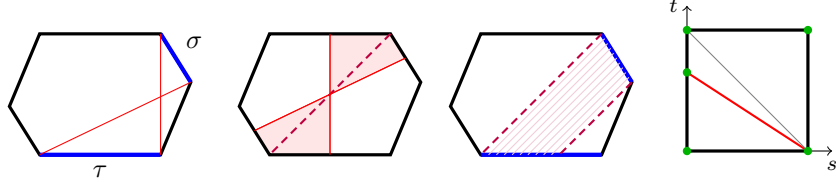


FIGURE 3. A depiction of the algorithm for finding the average distance between sides σ and τ . Values of $d_{st} = \|\tau(t) - \sigma(s)\|_L$ need only be computed at the five vertices shown here in order to work out the average value over the whole parameter square, since the function is linear on each triangle.

If the cutlines do not triangulate the square, add lines between the marked vertices as needed to complete a triangulation. (One such dummy cutline is shown in grey.) By linearity, we find the average value of d_{st} on a triangle by averaging the three values at the vertices. Weight this by the area of the triangle, and sum up over all the triangles to get the average over the square. This is the average distance E_{ij} between sides σ_i and σ_j . Finally, letting $w_i := \mu_L(\sigma_i)$, the sprawl is computed as a weighted average of averages:

$$E(L) = \frac{\sum_{i,j} w_i w_j E_{ij}}{\sum_{i,j} w_i w_j}.$$

4.2. Regular polygons. By applying the cutline algorithm, we find formulas for the sprawls of regular polygons.

Proposition 3. *Let P_n be the regular n -gon and let S^1 be the round circle. Then*

$$\begin{aligned} E(P_4) &= \frac{4}{3}; \\ E(P_6) &= \frac{23}{18} < \frac{4}{3}; \\ E(P_8) &= \frac{1+2\sqrt{2}}{3} < \frac{23}{18}; \\ E(P_n) &= \begin{cases} \frac{4}{\pi} \cdot \left(\frac{\pi/n}{\tan(\pi/n)} + \frac{1}{3} \frac{\pi}{n} \tan(\frac{\pi}{n}) \right), & n \in 4\mathbb{N}, \\ \frac{4}{\pi} \cdot \left(\frac{\pi/n}{\sin(\pi/n)} - \frac{1}{6} \frac{\pi}{n} \sin(\frac{\pi}{n}) \right), & n \in 4\mathbb{N} + 2, \end{cases} \\ E(S^1) &= \frac{4}{\pi}. \end{aligned}$$

To prove the formula for regular polygons, one can write E_{1j} for the average distance from σ_1 to σ_j and re-express that in terms of the L -lengths of the chords of the polygon P_n . These lengths themselves can then be written as trigonometric functions of π/n . Trigonometric identities finish the proof, since $E(P_n)$ is the weighted average of the E_{1j} .

We note that the formulas for regular polygons each converge quickly to $4/\pi$, and track close together. We write $f \sim g$ to mean that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.

Corollary 4. *Writing $E_{4\mathbb{N}}(x)$ for a function whose values agree with $E(P_x)$ when $x \in 4\mathbb{N}$, and likewise $E_{4\mathbb{N}+2}(x)$, we have:*

$$E_{4\mathbb{N}}(x) - \frac{4}{\pi} \sim \frac{16\pi^3}{45x^4}, \quad E_{4\mathbb{N}+2}(x) - \frac{4}{\pi} \sim \frac{17\pi^3}{90x^4}, \quad E_{4\mathbb{N}}(x) - E_{4\mathbb{N}+2}(x) \sim \frac{\pi^3}{6x^4}.$$

We will discuss these facts later in connection with the data shown in Figure 6.

4.3. Hexagons. With a bit of careful parametrization, we can completely describe the possible sprawls of hexagons.

Let $H_{x,y}$ be the hexagon with vertices $\mathbf{v}_1 = (x, y)$, $\mathbf{v}_2 = (1, 1)$, $\mathbf{v}_3 = (-1, 1)$, where $x \geq 1$, $y \geq 0$, and $x + y \leq 2$. Thus $H_{1,0}$ is a square (realized as a degenerate hexagon) and $H_{2,0}$ is a linear transform of the regular hexagon, giving

$$E(H_{1,0}) = \frac{4}{3}, \quad E(H_{2,0}) = \frac{23}{18}.$$

Lemma 5. *Every convex, centrally symmetric hexagon is equivalent by a linear transformation to some $H_{x,y}$.*

Proof. Take a hexagon with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, -\mathbf{v}_1, -\mathbf{v}_2, -\mathbf{v}_3$. We can always find a linear transformation sending $\mathbf{v}_2 \mapsto (1, 1)$ and $\mathbf{v}_3 \mapsto (-1, 1)$. This reduces the parameter space to $\{\mathbf{v}_1 = (x, y) : x \geq 1, -1 \leq y \leq 1\}$. Also, without loss of generality, we have $x + |y| \leq 2$; otherwise, change the choice of $\mathbf{v}_2, \mathbf{v}_3$, as in Figure 4. Finally, up to reflection in one of the coordinate axes, we can assume $y \geq 0$. \square

Applying the algorithm sketched above, we can compute the side-pair averages, and obtain the following formula for a hexagon parametrized as above.

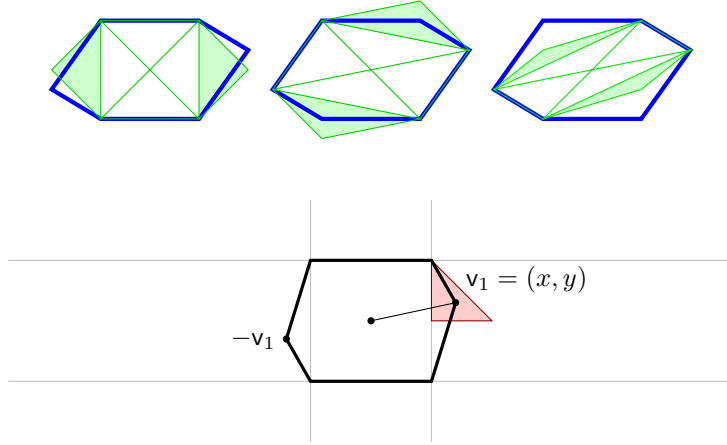


FIGURE 4. Hexagon reduction. On top we have shown the choices of which pair of sides to map to horizontal; the middle figure has v_1 in the desired position.

$$E(H_{x,y}) = \frac{x^2y^2 + xy^3 + 4x^3 + 7x^2 + 4x^2y - y^3 + 7xy - y^2 + 4x + 5y + 1}{3x^3 + 3x^2y + 6x^2 + 6xy + 3x + 3y}.$$

Thus we have reduced the task of bounding the sprawl of hexagons to a calculus exercise (which we omit): verifying that in the domain defined by $x \geq 1$, $y \geq 0$, and $x + y \leq 2$, this quantity takes values between $23/18$ and $4/3$.

This establishes the following statement:

Theorem 6 (Sprawls of hexagons).

$$\{E(H) : \text{hexagons } H\} = \left[\frac{23}{18}, \frac{4}{3} \right].$$

The hexagon case provides evidence, taken together with the fast convergence for sprawls of regular polygons towards $4/\pi$, for the main conjecture.

Sprawl Conjecture ($d = 2$ case). *The circle and the square are the extreme cases for all perimeters in \mathbb{R}^2 . That is,*

$$\{E(L) : \text{perimeters } L \subset \mathbb{R}^2\} = \left[\frac{4}{\pi}, \frac{4}{3} \right].$$

Some empirical evidence is shown in Figure 6, where sprawls of 100,000 random polygons are displayed.

5. HIGHER DIMENSION

In this section we give exact calculations for the sprawl of the ℓ^2 norm (sphere), the ℓ^∞ norm (cube), and the ℓ^1 norm (*orthoplex*, or cross-polytope), in every dimension d . We can then analyze the asymptotics as $d \rightarrow \infty$. We will use Sphere_d to denote the sphere in \mathbb{R}^d (that is, a copy of S^{d-1}), and likewise use Cube_d and Orth_d to refer to perimeters in \mathbb{R}^d . We will obtain the following somewhat surprising picture, with the ℓ^1 norm looking much more flat (Euclidean) than cubical.

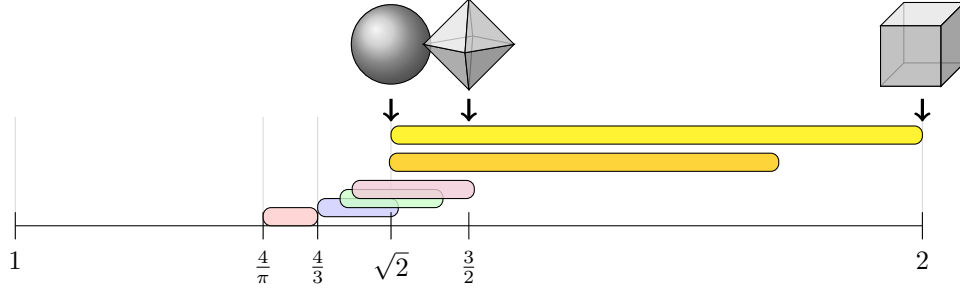


FIGURE 5. In this figure, reproduced from [6], the range $[E(\text{Sphere}_d), E(\text{Cube}_d)]$ is shown for $d = 2, 3, 4, 5, 100, \infty$.

Although the sprawl conjecture is open in every dimension d , it is remarked in [6] that it holds asymptotically as $d \rightarrow \infty$, in the sense that

$$\lim_{d \rightarrow \infty} E(\text{Sphere}_d) = \sqrt{2} \leq \liminf_{d \rightarrow \infty} E(L_d)$$

holds for all sequences of perimeters $L_d \subset \mathbb{R}^d$, and $E(L) < 2 = \lim_{d \rightarrow \infty} E(\text{Cube}_d)$ for all perimeters L in any dimension.

In the computations below, recall that for natural numbers n , the double factorial $n!!$ denotes the product of all the natural numbers up to n that have the same parity:

$$n!! = \prod_{i; 0 \leq 2i < n} (n - 2i).$$

Double factorials will occur in the calculations, but they can be re-expressed in two cases:

$$(2n)!! = 2^n \cdot n! ; \quad (2n+1)!! = \frac{(2n+1)!}{2^n \cdot n!}.$$

Here Γ stands for Euler's gamma function, which has the property that $\Gamma(n) = (n-1)!$ for natural numbers.

To get rates of approach, we use an approximation for $n!$ that goes one term beyond Stirling's formula:

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right).$$

5.1. The sphere.

Proposition 7.

$$E(\text{Sphere}_d) = \frac{2^{d-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}d)^2}{\Gamma(d - \frac{1}{2})}.$$

Thus, $E(\text{Sphere}_d) \rightarrow \sqrt{2}$ as $d \rightarrow \infty$, with

$$\sqrt{2} - E(\text{Sphere}_d) \sim \frac{1}{8d}.$$

Proof. Recall that, where A_k denotes the surface area of S^k (so that $A_1 = 2\pi$ and $A_2 = 4\pi$), there is a recursive formula given by $A_k = \int_0^\pi A_{k-1} \sin^{k-1}(\theta) d\theta$. The

distance between two points on the sphere that subtend an angle θ at the origin is $\sqrt{2 - 2 \cos \theta}$. Then we find that the A_k terms cancel out, giving

$$E(\text{Sphere}_d) = \frac{\int_0^\pi \sqrt{2 - 2 \cos \theta} \cdot \sin^{d-2}(\theta) d\theta}{\int_0^\pi \sin^{d-2}(\theta) d\theta},$$

which can be computed explicitly.

Let

$$a_n = \int_0^\pi \sqrt{2 - 2 \cos \theta} \sin^n \theta d\theta, \quad b_n = \int_0^\pi \sin^n \theta d\theta,$$

so that $E(\text{Sphere}_d) = a_{d-2}/b_{d-2}$.

Integrating by parts gives $b_{n+2} = \frac{n+1}{n+2} b_n$, so since $b_0 = \pi$ and $b_1 = 2$, we get $b_n = c_n \frac{(n-1)!!}{n!!}$, with $c_n = \pi$ if n is even and 2 if n is odd.

Change of variables and integration by parts gives the recursion $a_{n+1} = \frac{2n+2}{2n+3} a_n$. Since $a_0 = 4$, this gives $a_n = 4 \frac{(2n)!!}{(2n+1)!!}$.

Combining and re-indexing with $d = n + 2$, we get

$$E(\text{Sphere}_d) = \frac{a_{d-2}}{b_{d-2}} = e_d \frac{(2d-4)!! (d-2)!!}{(2d-3)!! (d-3)!!}$$

where e_d is $4/\pi$ if d is even, and 2 if d is odd. Re-expressing the double factorials completes the proof. \square

As a remark, recall that the gamma function $\Gamma(z)$ is an integer for whole numbers z but has $\sqrt{\pi}$ in the denominator for half-integers z . This property allows us to express the formula above without dependence on the parity of d .

5.2. The cube.

Proposition 8.

$$E(\text{Cube}_d) = \frac{2d+2}{d} - \left(\frac{2d+1}{2d^2} \right) \left(\frac{4^d d!^2}{(2d)!} \right).$$

Thus, $E(\text{Cube}_d) \rightarrow 2$ as $d \rightarrow \infty$, with

$$2 - E(\text{Cube}_d) \sim \frac{\sqrt{\pi}}{\sqrt{d}}.$$

Proof. Let x_i and y_i be independently distributed uniformly on the interval $I = [-1, 1]$. We will use these random variables to compute the sprawl for Cube_d , which we identify with the $(d-1)$ -complex in \mathbb{R}^d with vertices $(\pm 1, \dots, \pm 1)$. To fix notation: Cube_1 is a pair of points on the line and Cube_2 is a square in the plane. Cube_d has $2d$ top-dimensional facets, each a copy of I^{d-1} . Note that each facet is the locus of points satisfying $x_i = c$ for $c = \pm 1$. It has exactly one opposite face ($x_i = -c$), and all the others are adjacent since the defining equations can be simultaneously satisfied. For a point in \mathbb{R}^d to be in Cube_d , all coordinates must be in I , and at least one of its coordinates must be ± 1 .

We compute

$$P(|x_i - y_i| < r) = \frac{4r - r^2}{4}; \quad P(|1 - y_i| < r) = \frac{r}{2}$$

by considering the uniform measure on the square I^2 and calculating the portion of the area between the lines $x - y = r$ and $y - x = r$ in the first case, and above the line $y = r$ in the second. From this we get cumulative distribution functions

$$F_{\text{same}}(r) = P(d_\infty(x, y) < r : x, y \text{ on same face}) = \frac{(4r - r^2)^{d-1}}{4^{d-1}};$$

$$F_{\text{adj}}(r) = P(d_\infty(x, y) < r : x, y \text{ on adjacent faces}) = \frac{(4r - r^2)^{d-2}}{4^{d-2}} \left(\frac{r}{2}\right)^2.$$

To find expectations, we integrate $\int_0^2 r F'(r) dr$.

The d -cube has $2d$ faces, so if x is placed randomly, then the probability that y is on the same face or on the opposite face is $1/2d$ in each case, while all of the other $2d - 2$ faces are in the adjacent case. Recalling that the distance between any two points on opposite faces is 2, we get

$$E(\text{Cube}_d) = \frac{1 \cdot 2 + 1 \cdot \int_0^2 r F'_{\text{same}}(r) dr + (2d - 2) \cdot \int_0^2 r F'_{\text{adj}}(r) dr}{2d}.$$

From this and some algebraic manipulation we derive

$$\begin{aligned} E(\text{Cube}_d) = \frac{1}{d} + \frac{d-1}{4^{d-1}d} & \left[2 \int_0^2 r^{d-1} (4-r)^{d-2} dr + (d-1) \int_0^2 r^d (4-r)^{d-2} dr \right. \\ & \left. + (2-d) \int_0^2 r^{d+1} (4-r)^{d-3} dr \right]. \end{aligned}$$

Let's let $I_{m,n} = \int_0^2 r^m (4-r)^n dr$. Integration by parts and some further manipulations will give recursive formulas, for instance

$$I_{n,n} = 2^2 \cdot \frac{2n}{2n+1} I_{n-1,n-1},$$

which simplifies to $I_{n,n} = 2^{2n+1} \frac{(2n)!!}{(2n+1)!!}$ since $I_{0,0} = 2$.

The $I_{n+1,n}$, $I_{n+2,n}$, and $I_{n+4,n}$ are derived similarly, from which we find

$$E(\text{Cube}_d) = 2 - \left(2 + \frac{1}{d}\right) \frac{(2d-2)!!}{(2d-1)!!} + \frac{2}{d}.$$

Re-expressing the double factorials once again completes the proof. \square

5.3. The orthoplex.

Proposition 9.

$$E(\text{Orth}_d) = \frac{3d-2}{2d-1}.$$

Thus, $E(\text{Orth}_d) \rightarrow \frac{3}{2}$ as $d \rightarrow \infty$, with

$$\frac{3}{2} - E(\text{Orth}_d) \sim \frac{1}{4d}.$$

Proof. First note that by symmetry, the expectation of $\|x - y\|_1$ is equal to d times the expectation of $|x_1 - y_1|$. Thus

$$E(\text{Orth}_d) = d \int_{I^2} |x_1 - y_1| d\mu(x_1) d\mu(y_1),$$

where $d\mu$ is the measure induced by μ on a single coordinate axis of \mathbb{R}^d . That measure is given by

$$d\mu(x_1) = \frac{(1 - |x_1|)^{d-2}}{(d-2)!} dx_1,$$

as can be verified by considering how much volume the orthoplex has at height x_1 . We can renormalize to a probability measure by taking $\nu = \frac{(d-1)!}{2}\mu$, so that $\int_{\text{Orth}_d^2} d\nu^2 = 1$. Thus we are calculating

$$E(\text{Orth}_d) = d \int_{I^2} |x_1 - y_1| d\nu^2 = \frac{d(d-1)^2}{4} \int_{I^2} |x - y| \cdot (1 - |x|)^{d-2} (1 - |y|)^{d-2} dx dy.$$

But again by symmetry, this is just

$$2d(d-1)^2 \int_{x=0}^1 x(1-x)^{d-2} \int_{y=0}^x (1-y)^{d-2} dy dx.$$

Evaluating in y and then performing light manipulation gives us

$$\begin{aligned} & 2d(d-1) \left[\int_0^1 x(1-x)^{d-2} dx - \int_0^1 x(1-x)^{2d-3} dx \right] \\ &= 2d(d-1) \left[\frac{1}{(d-1)d} - \frac{1}{(2d-2)(2d-1)} \right] = \frac{3d-2}{2d-1}, \end{aligned}$$

as desired. \square

6. EXPERIMENTAL EVIDENCE

In closing, we present some empirical evidence that sprawl measures the “relative roundness” of polygons, with extremes at the square and circle, in dimension 2. We will do this by comparing it with Mahler volume, which is known to take extremes at the circle and the square, and to be monotonic with respect to certain kinds of symmetrization. This data set contains a hundred thousand random centrally-symmetric polygons, each generated by dropping 50 points at random in a 10×10 square, symmetrizing, and taking the convex hull.

Let us recall the definition of *Mahler volume* of a convex body. For a convex centrally-symmetric body Ω , define its *polar body* by

$$\Omega^\circ := \{x \in \mathbb{R}^d : x \cdot y \leq 1 \quad \forall y \in \Omega\}.$$

Thus for instance, the round ball is its own polar body in every dimension, $(\Omega^\circ)^\circ = \Omega$, and the orthoplex and cube solids are duals. The *Mahler volume* of Ω is defined to be

$$M(\Omega) := \text{Vol}(\Omega) \cdot \text{Vol}(\Omega^\circ).$$

Let us also say that for any perimeter L , we write $M(L)$ for the Mahler volume of the convex hull of L , to unify notation. Just as for the sprawl, this is a statistic that is continuous in L and invariant under linear transformations. Mahler conjectured in 1939 that M was an affine isoperimetric invariant; it was proved by Santaló in 1949 that the spheres did indeed realize the upper bound, but the lower bound for cubes is established only in dimension 2 and remains an open problem for $d > 2$. The Mahler volume of a regular polygon is given by $M(P_n) = n^2 \sin^2(\pi/n)$ (compare Proposition 3).

Here are a few observations to be made from the results above and the random-polygon data; some are visible in the figure.

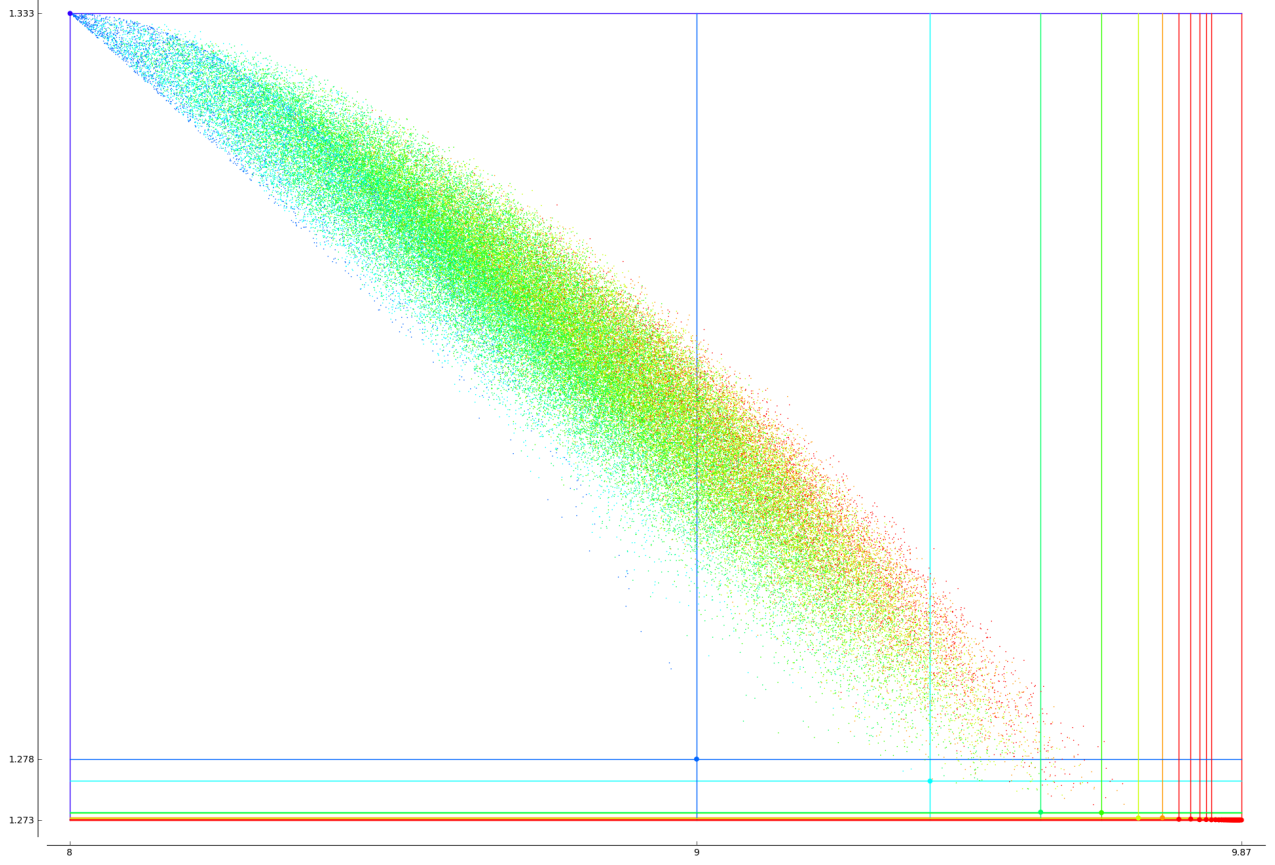


FIGURE 6. One hundred thousand data points $(M(P), E(P))$ for random polygons P , showing high correlation between sprawl and Mahler volume and supporting the conjecture that $\frac{4}{\pi} < E(P) \leq \frac{4}{3}$. Colors range by number of sides, from blue for quadrilaterals and hexagons to red for polygons with at least 20 sides. The horizontal lines $E = E(P_n)$ and vertical lines $M = M(P_n)$ have been plotted for the regular polygons ($n = 4, 6, 8, \dots, 26$).

- $E(P_n) \rightarrow E(S^1)$ much faster than $M(P_n) \rightarrow M(S^1)$.
- The hexagons in the figure (seen in blue) all lie in a region bounded by two curves, which correspond to two sides of the triangle in the parameter space for hexagons shown in Figure 4. (The third side of the triangle consists of quadrilaterals with a dummy vertex, so it collapses to the point $(8, 4/3)$ in the plot.) This suggests that E is bounded above and below by functions of M in the hexagon case.
- The regular n -gon is the roundest among all n -gons for each n , as measured by M . However, in contrast with Mahler volume, the average distance between points on a regular $2k$ -gon *never* becomes monotonic as a function of k , but eventually always alternates between increasing and decreasing: $E(P_{4k}) > E(P_{4k+4}) > E(P_{4k+2}) > E(S^1)$ for sufficiently large k . This is

clear from inspecting the asymptotics in Corollary 4. For instance, sprawl sees the 16-gon to be slightly less round than the 14-gon (so that the roundest possible 16-gon is not the regular one). This seems to indicate that sprawl is sensitive to different kinds of symmetries than Mahler volume.

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