(1) Text, §43, Exercise 5.

Solution: Let $R > 1$ and let C_R be the circle $|z| = R$, oriented in the counterclockwise direction. For all $z \in C_R$ except the point $z = -R$ (i.e. $-\pi < \operatorname{Arg} z < \pi$), we have

$$|\operatorname{Log} z| = |\ln R + i\operatorname{Arg} z| \leq |\ln R| + |\operatorname{Arg} z| < \ln R + \pi$$

and $|z^2| = R^2$. Thus, for such z

$$\left|\frac{\operatorname{Log} z}{z^2}\right| < \frac{\ln R + \pi}{R^2}$$

so, since the function $\frac{\operatorname{Log} z}{z^2}$ is continuous on C_R for $-\pi < \operatorname{Arg} z < \pi$, by the Theorem from §43 of the Text,

$$\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz\right| < \frac{\ln R + \pi}{R^2}\, 2\pi R = \frac{2\pi(\ln R + \pi)}{R}.$$

Next, by L'Hospital's Rule, since $\lim_{R\to\infty} (\pi + \ln R) = \infty$ and $\lim_{R\to\infty} R = \infty$

$$\lim_{R\to\infty} \frac{\pi + \ln R}{R} = \lim_{R\to\infty} \frac{1}{R} = 0.$$

So, since by above estimate

$$-\left(\frac{2\pi(\ln R + \pi)}{R}\right) < \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz < \frac{2\pi(\ln R + \pi)}{R}$$

then by the Squeeze Theorem

$$\lim_{R\to\infty} \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz = 0.$$

(2) Text, §49, Exercise 1.

Solution: In all parts we use the fact that the unit circle $C : |z| = 1$ oriented in either direction is a simple closed contour.

(a) The function $f(z) = \dfrac{z^2}{z-3}$ is analytic at all z except $z = 3$. Thus, $f(z)$ is analytic on and inside the unit circle C (oriented in either direction). By the Cauchy–Goursat Theorem, it follows that $\int_C f(z)\, dz = 0$.

(b) The function $f(z) = z\, e^{-z}$ is entire, so the Cauchy–Goursat Theorem applies to the integral of f along all closed contours, and in particular, $\int_C z\, e^{-z}\, dz = 0$.

(c) Let $f(z) = \dfrac{1}{z^2 + 2z + 2}$. Since $z^2 + 2z + 2 = 0$ if and only if $z = -1 \pm i$. $f(z)$ is analytic at all z except $z = -1 \pm i$. These two points lie outside C since $|-1 \pm i| = \sqrt{2}$, so again by the Cauchy–Goursat Theorem, $\int_C f(z)\, dz = 0$.

(d) $f(z) = \text{sech} z = \frac{1}{\cosh z}$ is analytic except where $\cosh z = 0$ and

$$\begin{aligned} \cosh z = 0 \iff \frac{e^z + e^{-z}}{2} &= 0 \\ e^z &= -e^{-z} \\ e^{2z} &= -1 \\ 2z &= \pi i + 2\pi i n \quad \text{for } n = 0, \pm 1, \pm 2, \dots \\ z &= \frac{\pi i}{2} + \pi i n \quad \text{for } n = 0, \pm 1, \pm 2, \dots. \end{aligned}$$

Thus, $f(z)$ is analytic at all points except $z = \pm\pi i/2, \pm 3\pi i/2, \dots$. Since $\pi/2 > 1$, we see that $f(z)$ is analytic at all points on and interior to the unit circle C. Hence by the Cauchy–Goursat Theorem, $\int_C \text{sech}\, dz = 0$.

(e) $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ is analytic at all z except when

$$\cos z = 0 \iff z = \frac{\pi}{2} + n\pi, \text{for } n = 0, \pm 1, \pm 2. \dots.$$

These points all lie outside C, so again by the Cauchy–Goursat Theorem, $\int_C \tan z\, dz = 0$.

(f) $f(z) = \text{Log}\,(z+2)$ is analytic at all points z except along the ray $x \leq -2$ on the negative x-axis. Hence $f(z)$ is analytic at all points inside and on C, and by the Cauchy–Goursat Theorem $\int_C \text{Log}\,(z+2)\, dz = 0$.