(1) Text, §39, Exercise 6.

Solution: The function z is given by $z(x) = x + iy(x)$, where

$$y(x) = \begin{cases} x^3 \sin\left(\frac{\pi}{x}\right) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

(a) We will show in part (b) that $y(x)$ is continuous on $[0,1]$ and so $z(x)$ is an arc. $z(x)$ intersects the x-axis at the points x where $y(x) = 0$. We have $y(0) = 0$. And for $x \in (0,1]$, $y(x) = 0 \iff \sin(\pi/x) = 0 \iff \pi/x = n\pi \iff x = 1/n$, where $n = 1, 2, 3 \ldots$.

(b) We will show that $z(x)$ is a smooth arc by showing (i) $z(x)$ is differentiable on $[0,1]$, hence $z(x)$ is continuous on $[0,1]$, (ii) $z'(x)$ is continuous $[0,1]$, and (iii) $z'(x)$ is nonzero on $(0,1)$.

(i) When $x > 0$, $y(x)$ is clearly differentiable with derivative $y'(x) = 3x^2 \sin(\pi/x) - \pi x \cos(\pi/x)$. When $x = 0$

$$\begin{aligned} y'(0) &= \lim_{x\to 0} \frac{y(x) - y(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x^3 \sin(\pi/x)}{x} \\ &= \lim_{x\to 0} x^2 \sin(\pi/x) \\ &= 0, \end{aligned}$$

where the last line is by the Squeeze Theorem, since $0 < |x^2 \sin(\pi/x)| \leq x^2$. Therefore $y(x)$ is differentiable on $[0,1]$. This shows that $z(x)$ is differentiable on $[0,1]$ with derivative $z'(x) = 1 + iy'(x)$ and so $z(x)$ is also continuous on $[0,1]$.

(ii) Clearly $y'(x)$ is continuous on $(0,1]$. Now, at $x = 0$, again by the Squeeze Theorem

$$\lim_{x\to 0} y'(x) = \lim_{x\to 0}(3x^2 \sin(\pi/x) - \pi x \cos(\pi/x)) = 0$$

since for $x > 0$ we have

$$0 < |3x^2 \sin(\pi/x) - \pi x \cos(\pi/x)| \leq 3x^2 + \pi|x|$$

Therefore $y'(x)$ is continuous on $[0,1]$.

(iii) Since $z'(x) = 1 + iy'(x)$, $z'(x)$ does not vanish on $(0,1)$.

(2) Exercise M.

Solution:

(a) $s_j = z_j - z_{j-1} = e^{\frac{2\pi i}{n} j}(1 - e^{-\frac{2\pi i}{n}})$

(b) (Note that m_j is the midpoint *on the arc.*) $m_j = e^{\frac{2\pi i}{n}(j - \frac{1}{2})}$

(c) The image of the unit circle under $f(z) = \frac{1}{z}$ is the unit circle reflected about the x-axis.

(d) $f(m_j) = e^{-\frac{2\pi i}{n}(j-\frac{1}{2})}$

(e)

$$\begin{aligned}
\mathrm{RS}(n) = \sum_{j=1}^{n} s_j f(m_j) &= \sum_{j=1}^{n} \left(e^{\frac{\pi i}{n}} - e^{-\frac{\pi i}{n}}\right) \\
&= \sum_{j=1}^{n} 2i \sin(\pi/n) \\
&= 2in \sin(\pi/n)
\end{aligned}$$

(f) $\mathrm{RS}(2) = 4i \sin(\pi/2) = 4i$
$\mathrm{RS}(4) = 8i \sin(\pi/4) = 4\sqrt{2}i$

(g) $\mathrm{RS}(16) = 32i \sin(\pi/16) \approx (6.24289)i$
$\mathrm{RS}(100) = 200i \sin(\pi/100) \approx (6.28215)i$
(It looks like the integral is converging to $2\pi i$.)

(h)

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{RS}(n) &= \lim_{n\to\infty} 2in \left(\frac{\sin(\pi/n)}{\pi/n}\right)(\pi/n) \\
&= (2\pi i)\left(\lim_{m\to 0} \frac{\sin(m\pi)}{m\pi}\right) \\
&= 2\pi i
\end{aligned}$$

(i) The parametrized integral is

$$\int_0^{2\pi} f(\gamma(t))\gamma'(t)dt = \int_0^{2\pi} e^{-it} i e^{it} dt = 2\pi i$$

which is consistent with the above calculation using the Riemann sum.