✓

Introducing $\mathbb{C}$

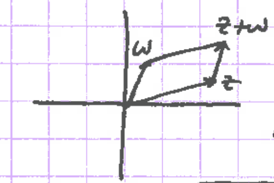
Let's meet $\mathbb{C}$ the usual way, and then re-mystify it.

$\mathbb{C} \cong \mathbb{R}^2$ as a vector space

$z \leftrightarrow (x,y)$

$\overset{\shortparallel}{x+yi}$

- usual $\mathbb{R}$ arithmetic
- $i^2 = -1$
- i commutes w/ $\mathbb{R}$



This identification is "natural" wrt addition, scalar mult.

$\boxed{\text{Norm / modulus}} \longrightarrow$ $\boxed{\begin{array}{l} \mathbb{C} \longrightarrow [0,\infty) \\ z \longmapsto |z| := \sqrt{x^2+y^2} \end{array}}$

Euclidean length of vector

$\boxed{\begin{array}{l} Im, Re : \mathbb{C} \longrightarrow \mathbb{R} \\ \text{Imaginary, Real parts} \end{array}}$

$\boxed{\begin{array}{l} \mathbb{C} \to \mathbb{C} \\ z \longmapsto \bar{z} \\ \text{Conjugate} \end{array}}$

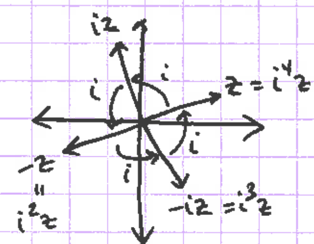
Note for $\begin{array}{l} z = x+yi \\ w = u+vi \end{array}$, have $zw = (xu-yv) + (yu+xv)i$

multiplication not (apparently) "natural" in $\mathbb{R}^2$ coordinates

$|z| + |w| \geq |z+w|$; $|zw| = |z| \cdot |w|$

(norm better adapted to mult.)

How shall we geometrize $\mathbb{C}$ multiplication?



i rotates CCW by $\frac{\pi}{2}$,

so let $\mathbb{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

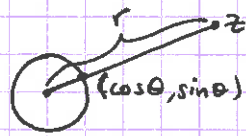
So is i a vector in the plane or a rotation of the plane?

(think wave-particle duality of light!)

A move to geometrize multiplication:

introduce POLAR COORDINATES

$$\underset{\underset{re^{i\theta}}{\shortparallel}}{z} \longleftrightarrow (\overset{\text{modulus}}{r}, \overset{\text{argument}}{\theta})$$



$$z = r\cos\theta + ir\sin\theta = r\underbrace{(\cos\theta + i\sin\theta)}_{\substack{\text{also called } \operatorname{cis}\theta \\ \text{or } e^{i\theta}}}$$

Major Aside: <u>why</u> define $e^{i\theta} := \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$?

- "operator overloading": it will obey the same defining rules as e^x, $x \in \mathbb{R}$
- "rigidity": it turns out to be the <u>only</u> way to define a function on $\mathbb{C}$ that is differentiable and specializes to familiar e^x on $\mathbb{R}$.

CHECK: $e^{i\theta} \cdot e^{i\alpha} \overset{?}{=} e^{i(\theta+\alpha)}$

$$(\cos\theta + i\sin\theta)(\cos\alpha + i\sin\alpha) = \underbrace{(\cos\theta\cos\alpha - \sin\theta\sin\alpha)}_{\cos(\theta+\alpha)} + \underbrace{(\cos\theta\sin\alpha + \sin\theta\cos\alpha)}_{\sin(\theta+\alpha)}i$$

✓

In fact, this "explains" these trig identities.

$$(re^{i\theta})(se^{i\alpha}) = \underbrace{(rs)}_{\substack{\text{modulus} \\ \text{multiplies}}} e^{i(\theta+\alpha)} \leftarrow \text{argument adds}$$

HISTORICAL SIDEBAR

arithmetic of $\sqrt{-1}$ is as old as the Italian cubic-wranglers (Bombelli et al)

Here is a fascinating attempt to draw $\mathbb{C}$ due to Wallis 1673:

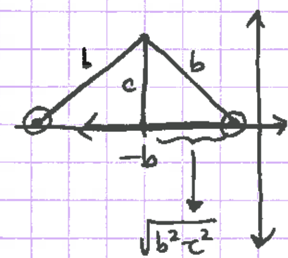
$$x^2+2bx+c^2=0 \implies x=-b\pm\sqrt{b^2-c^2}$$

$(b, c \geq 0)$

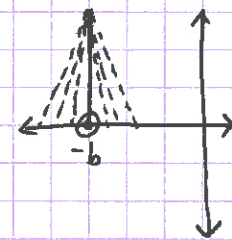
or

$$-b \pm (\sqrt{-1})\sqrt{c^2-b^2}$$

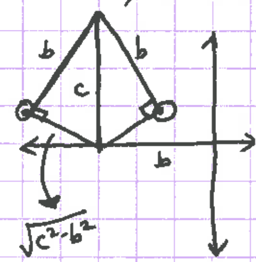
$(b>c)$



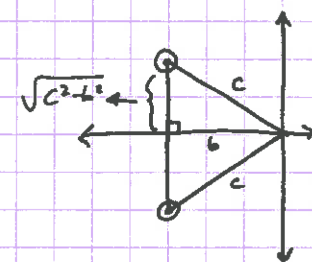
$(b=c)$

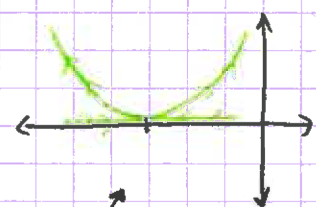


$(c>b$, Wallis$)$



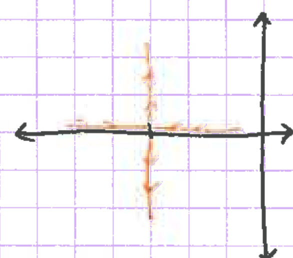
$(c>b$, Modern$)$





Wallis's movie, fixed b

$$\left\{\left(-b\pm\frac{b}{t}\sqrt{t^2-b^2},\ \frac{t^2-b^2}{t}\right)\right\}$$



Modern movie, fixed b

de Moivre's Formula ←

in recap from last time:

- equality for rect. coords. is equality of x, y
- equality for polar coords. is $\begin{cases} \text{equality of } r \\ \theta \text{ differ by } 2\pi\mathbb{Z} \end{cases}$

term PRINCIPAL ARGUMENT of z

$-\pi < \Theta \leq \pi$

arg (multival.) vs Arg

branch of fcn: (multi-val)
single-valued fcn
locally compatible w/ f

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$|z_1 z_2| = |z_1| \cdot |z_2|$

Powers + Roots.

Easy to see that $z = re^{i\theta} \Rightarrow z^n = (r^n)e^{in\theta}$.
(so can derive double, triple angle formulas painlessly!)

How about n^{th} roots?

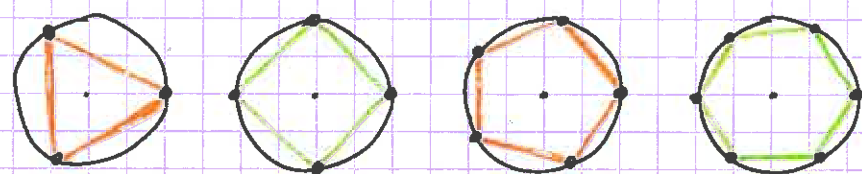
If $z^n = w$ $(z = re^{i\theta}, w = se^{i\alpha})$ then $r^n = s$
and $n\theta = \alpha$ up to $2\pi\mathbb{Z}$
$(n\theta - \alpha \in 2\pi\mathbb{Z})$

so $w^{1/n}$ is multivalued:

$r = \sqrt[n]{s}$ well-def, but $\theta = \frac{\alpha}{n} + \frac{2\pi k}{n}$ $k \in \mathbb{Z}$
(positive real) are all valid arguments.

Abuse of notation: $w^{1/n}$ denotes this set. $\{ \sqrt[n]{s} \cdot e^{i(\frac{\alpha}{n} + \frac{2\pi k}{n})} : k \in \mathbb{Z} \}$

In particular, $e^{2\pi i \frac{k}{n}} = \omega_n^k$ are the ROOTS OF UNITY. ω_n^1 called PRINCIPAL ROOT.



Duality of position and motion:
rotational symmetries of regular n-gon
form a regular n-gon

So for arbitrary $w^{1/n}$, find principal root,
then draw circle at that modulus and inscribe n-gon.

ways of visualizing $f : \mathbb{C} \to \mathbb{C}$

(A) graph living in $\mathbb{R}^4$

(B) modulus graph in $\mathbb{R}^3$

(C) argument graph in $\mathbb{R}^3$

(D) pair of planes : input + output curves

(E) pair of planes : input + output domains

(F) homotopy / sliding image

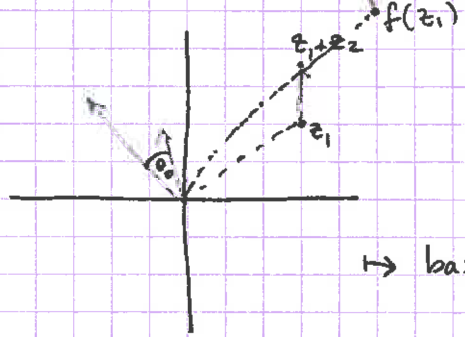
(G) vector field

(H) colormap

Visualizing Functions, Part 1 of Many

$\boxed{\text{Example 0}}$ Fix a $z_0 \in \mathbb{C}$, consider $z \mapsto z_0 z$

<u>Complex multiplication by z_0</u> $= r_0 e^{i\theta_0}$.



effect: dilation by r_0,
rotation about 0 by θ_0.

base z_1, tip $z_1 + z_2$
(difference z_2)

$\mapsto$ base $z_0 z_1$, tip $z_0(z_1+z_2)$
(difference $z_0 z_2$)

so vectors in arbitrary locations of arbitrary lengths are also stretched by r_0, rotated by θ_0
(ie, argument incremented by θ_0)

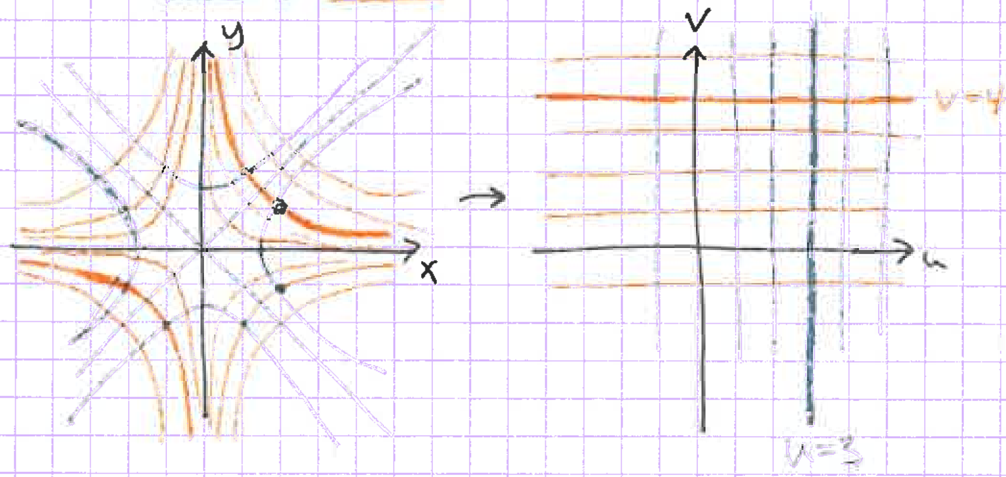
This is a <u>globally</u> <u>uniform</u> <u>similarity</u>.

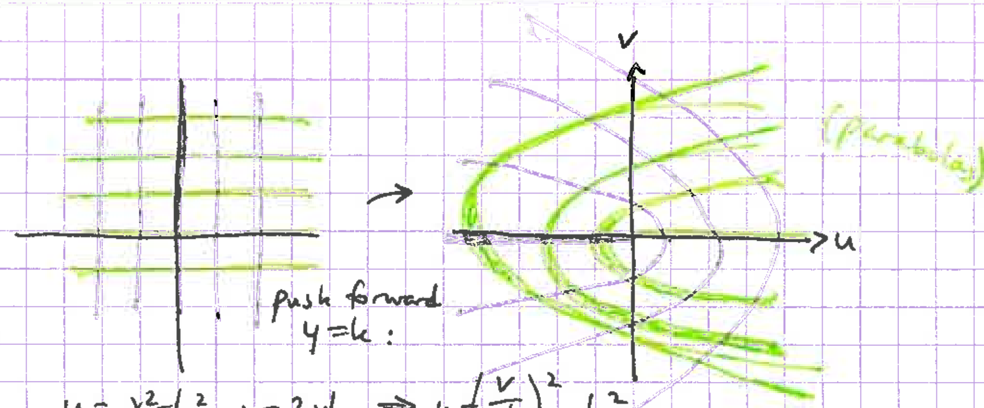
In fact, if $z_0 = a+bi$, then $1+0i \mapsto a+bi$, $0+1i \mapsto -b+ai$ ie $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

linear transf!
(det $= r^2$)

$\boxed{\text{Ex 1}}$ $z \mapsto z^2$. Put $w = z^2$, $w = u + iv$

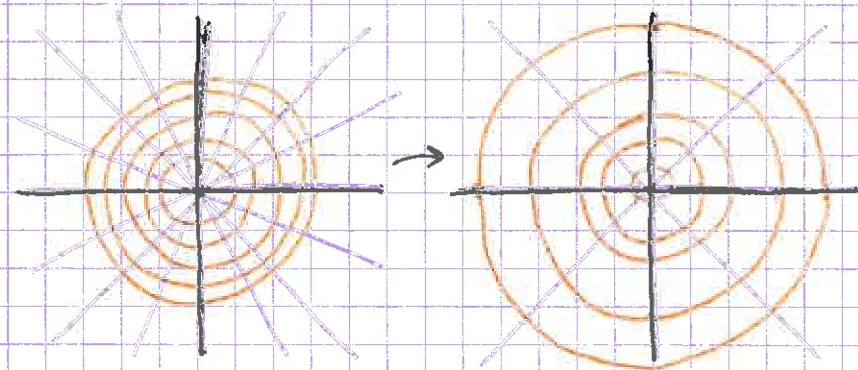
Then $\boxed{u = x^2 - y^2}$, $\boxed{v = 2xy}$.





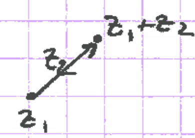
push forward
$y=k$:

$$u = x^2 - k^2, \quad v = 2xk \Rightarrow u = \left(\frac{v}{2k}\right)^2 - k^2$$



So a vector based at 0 is complex-multiplied by a factor of <u>itself</u>;
how about other vectors?

$$f(z_1) = z_1^2, \quad f(z_2) = (z_1 + z_2)^2$$
$$= z_1^2 + \underbrace{2z_1z_2 + z_2^2}$$



$$f(z_2) - f(z_1) = z_2(2z_1 + z_2)$$

So this multiplicative factor $\rightarrow 2z_1$
as $z_2 \rightarrow 0$ in any way.

Return to Exp

$\exp(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$
$= e^x(\cos y + i \sin y)$

modulus e^x, argument y

First let's note that arithmetic identities carry over, as in

$$(z_1+z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \quad \text{(binomial thm)}$$

Now, why is our def. of e^z reasonable?

① defining property of e^x : $e^a e^b = e^{a+b}$ $\mathbb{C}$ ✓

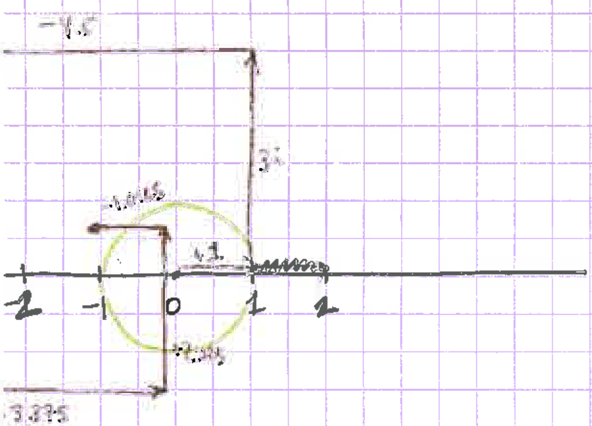
② defining property : $\frac{d}{dx} e^x = e^x$

We don't know how to take $\mathbb{C}$-derivs. yet, but can consider $\frac{d}{dt} e^{it} \overset{?}{=} i e^{it}$

$(\cos t, \sin t)$ is unit-speed motion w/ tangent vector $(-\sin t, \cos t) = i \cdot (\cos t, \sin t)$ ✓

③ power series! $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

eg $e^3 = 1 + 3 + \frac{9}{2} + \frac{27}{6} + \frac{81}{24} + \frac{243}{120} + \frac{729}{720} + \dots$
$= 1 + 3 + 4.5 + 4.5 + 3.375 + 2.025 + 1.0125 + \dots$



④ $e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$

postponed!

$e^{\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1/4}{2} + \frac{1/8}{6} + \dots$
$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \dots$

$e^{\frac{3}{4}} = 1 + \frac{3}{4} + \frac{9}{32} + \frac{27/64}{6}$

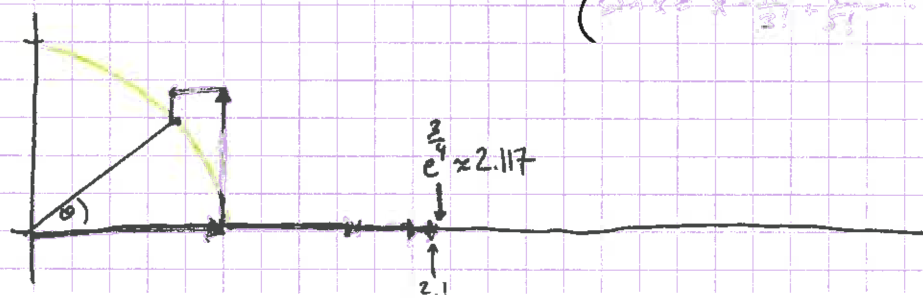
$1 + .75 + .28 + .07 + \dots = 2.117\dots$

2.1 already

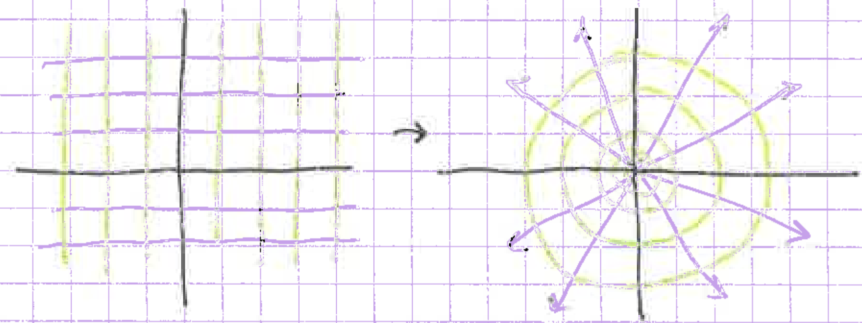
$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{i\theta^4}{4!} + \dots$

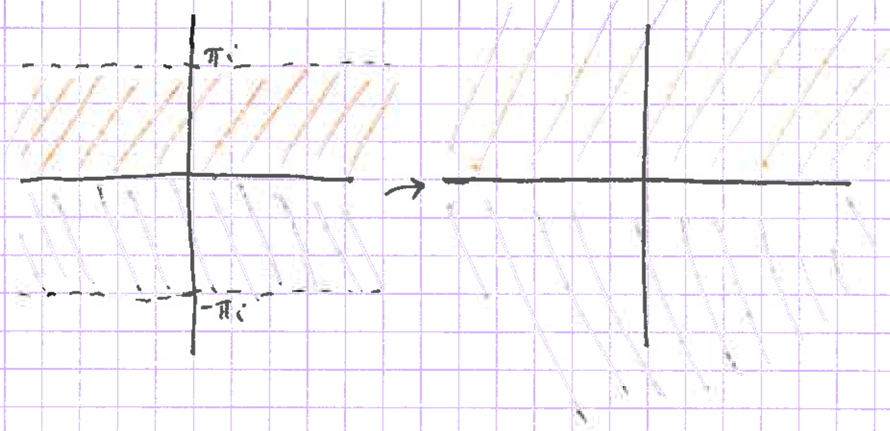
$= \cos\theta + i \sin\theta$

$\begin{cases} \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \end{cases}$



$\boxed{\text{Ex 2}}$ exp as a transformation





Note $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \begin{cases} \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} \end{cases}$

Next: basic topology notions; Stereographic Projection

basic nbhd (open) $B_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$

punctured $B_\varepsilon^*(z_0) = \{ z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon \}$

interior, open, closed, bounded, <u>domain</u>, region (not nec. open)

open: $S = S^\circ$

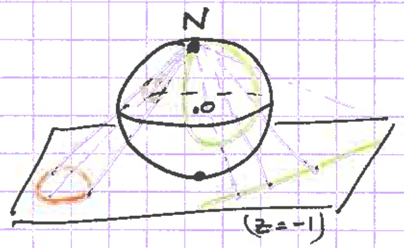
closed: S^c open ($\Leftrightarrow$ contains all accum pts)

bounded: contained in some $B_r(z)$

$S^\circ := \{ z \in S \mid \exists\, \varepsilon > 0 \text{ st } B_\varepsilon(z) \subseteq S \}$

domain: [nonempty, path-conn. open set.]

$\boxed{\text{stereographic projection}}$



gives a map from $S^2 \setminus \{N\} \to \mathbb{R}^2 \cong \mathbb{C}$

that is
- bijective
- preserves convergence, maps open $\leftrightarrow$ open

This is called a HOMEOMORPHISM.

if we add one extra point to $\mathbb{C}$, called ∞, then $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ corresponds bijectively to S^2 setwise.

How about convergence? It's easy to see that $p_n \to N$ on S^2
iff p_n eventually enter every $B_\varepsilon(N)$
iff p_n eventually leave every $B_r(S)$.

In fact $B_\varepsilon(N) = B_r(S)^c$ for some $r = r(\varepsilon)$.

So basic neighborhoods of ∞ in <u>Riemann sphere</u> $\hat{\mathbb{C}}$ are exteriors of $\overline{B_r(0)}$.

lines in $\mathbb{C}$ $\rightsquigarrow$ circles through N on S^2

Limit example + nonexample

yes

$$\lim_{z\to 1} \frac{i\bar{z}}{2}$$

let p be small $\mathbb{C}$ number

$$1+p \mapsto \frac{i(1+\bar{p})}{2} = \frac{i}{2} + \frac{i\bar{p}}{2}$$

half the length of p

no

$$\lim_{z\to 0} \frac{z}{\bar{z}}$$

b/c $\frac{z}{\bar{z}} \in \mathbb{C}$

but angle depends on $\arg(z)$.

$$\frac{1}{z} = \frac{1}{r} e^{-i\theta}$$

Limits and Continuity

$\lim_{z\to z_0} f(z) = w$ means for any seq $z_n \to z_0$, $f(z_n) \to w$.

where converging means entering all nbhds.
($\forall \varepsilon>0\ \exists N$ st $n>N \Rightarrow z_n \in B_\varepsilon(z_0)$)

i.e., $f(z)\to w$ means $\forall \varepsilon\ \exists \delta$ st $z\in B_\delta(z_0) \Rightarrow f(z) \in B_\varepsilon(w)$.

Easy Obs: seq conv in $\mathbb{C} \Longleftrightarrow$ Re, Im parts conv in $\mathbb{R}$.

limit laws carry over; new versions for ∞.

① $\lim_{z\to z_0} f(z) = \infty \Longleftrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0. \Leftrightarrow |f(z)| \to \infty$

② $\lim_{z\to\infty} f(z) = w \Longleftrightarrow \lim_{z\to 0} f(\frac{1}{z}) = w$

suggests nothing's special about ∞ and $z \mapsto \frac{1}{z}$ makes it work/look just like 0

Continuity:

$$\lim_{z\to z_0} f(z) = f(z_0).$$

Easy Obs: nonzeroness is an open condition:
$f(z_0)\neq 0$, f cont $\Rightarrow \exists \varepsilon>0$ st $z\in B_\varepsilon(z_0) \Downarrow f(z)\neq 0$.

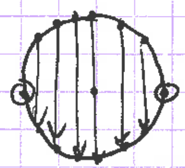
$\boxed{\text{Ex 3}}$ $z \mapsto 1/z$. $\quad 0 \leftrightarrows \infty$

$z\bar{z} = |z|^2$, so $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ • orientation-~~reversing~~ preserving!!

$|\frac{1}{z}| = \frac{1}{|z|}$ multiplies length by $\frac{1}{|z|^2}$, sending previous length to reciprocal.

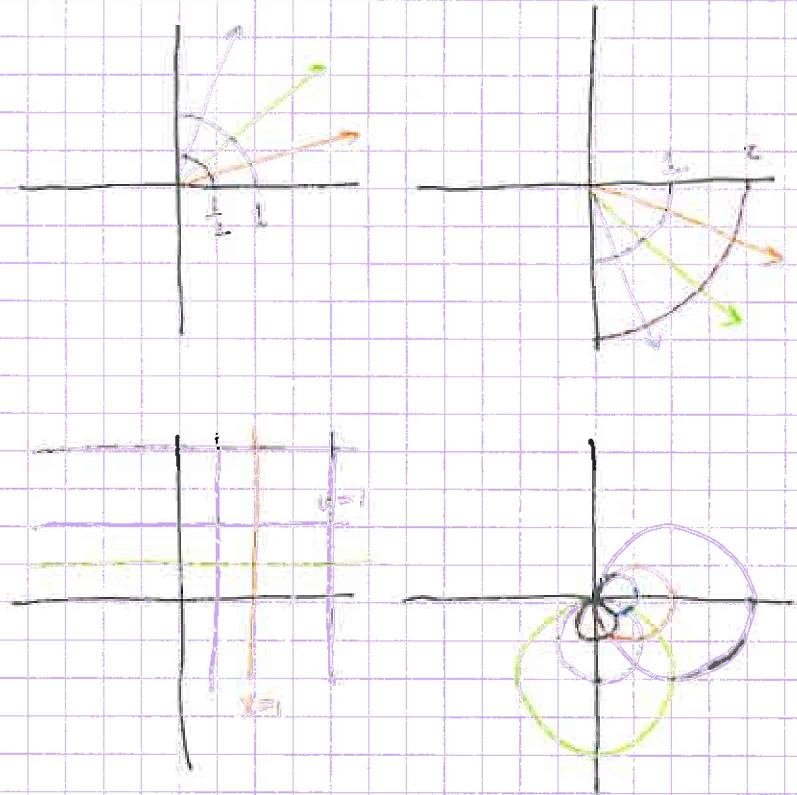
(and argument to negative)

• swaps inside of $\mathbb{D}$ for out.



unit circle.

$1/z$, continued.



Can decompose
$$\begin{tikzcd} z \arrow[rr, mapsto] \arrow[dr, mapsto, "\text{inv.}"'] & & \frac{1}{z} \\ & \frac{1}{\bar{z}} \arrow[ur, "\text{conj.}"'] & \end{tikzcd}$$

$\mathcal{I}_C : z \mapsto \frac{1}{\bar{z}}$
is inversion in unit circle.

$\begin{pmatrix} A=0 \\ B,C \text{ not both } 0 \end{pmatrix}$: line

$\begin{pmatrix} B,C=0 \\ \frac{D}{A}<0 \end{pmatrix}$: circle of radius $\sqrt{-D/A}$ centered at 0

$\begin{pmatrix} A\neq 0 \\ B^2+C^2>4AD \end{pmatrix}$: circle centered at $\left(\frac{-B}{2A}, \frac{-C}{2A}\right)$

let's see that $z \mapsto \frac{1}{z}$, $z \mapsto \frac{1}{\bar z}$ preserve(s) complex circles (circles + lines).

General form:

$$A(x^2+y^2)+Bx+Cy+D=0, \quad A,B,C,D \in \mathbb{R}, \quad B^2+C^2>4AD$$

$$A|z|^2 + B\cdot \mathrm{Re}(z) + C\cdot \mathrm{Im}(z) + D = 0$$

$$A z\bar z + B'(z+\bar z) + C'(z-\bar z) + D = 0$$

$$A z\bar z + B'' z + C'' \bar z + D = 0$$

if $w = \frac{1}{z}$, then $w\bar w = \frac{1}{z\bar z}$

$$\Updownarrow$$

$$A + B''\frac{1}{\bar z} + C''\frac{1}{z} + D\frac{1}{z\bar z} = A + B''\bar w + C'' w + D w\bar w = 0$$

Note: same clearly true of affine maps $z \mapsto z_0 z + w_0$ (these also preserve ∞)

Lecture 5

MÖBIUS TRANSFORMATIONS

$f(z) = \frac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$

- if $ad=bc$, this is $f(z) \equiv \frac{a}{c}$.
- a tuple (a,b,c,d) and (ka,kb,kc,kd) give the same map.
- by factoring out $\frac{1}{\sqrt{ad-bc}}$, can assume wlog $ad-bc=1$

(then there's 1-1 corr btw nondeg. maps and tuples)

$$w = \frac{az+b}{cz+d} \implies czw + dw - az - b = 0$$

$$Azw + Bw + Cz + D = 0 \quad \text{suggestive!}$$

- $w = \frac{a}{c} + \frac{bc-ad}{c}\cdot\frac{1}{cz+d}$ so $f = \mathrm{aff}_2 \circ \mathrm{inv} \circ \mathrm{aff}_1$

so preserves $\mathbb{C}$-circles.

i.e., continuous on $\hat{\mathbb{C}}$

- well-def on ∞ : $f(\infty) = \frac{a}{c}$, $f(\frac{-d}{c}) = \infty$

- preserves $\hat{\mathbb{R}}$ if $a,b,c,d \in \mathbb{R}$

- composition is nice! note identity is $\frac{1z+0}{0z+1}$

note $w = \frac{az+b}{cz+d} \Rightarrow czw + dw = az+b$

$\Rightarrow (cw-a)z = b-dw \Rightarrow z = \frac{-dw+b}{cw-a}$

one easily verifies that matrix mult is "faithful"

very nice subgroup w/ $\mathbb{R}$-coeffs, $\det > 0$

called $PSL_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}$

These preserve $\hat{\mathbb{R}}$, $\mathbb{H}$, and contain dilations, rotations, axial translations.

generally can identify Möb with $PSL_2\mathbb{C} = SL_2\mathbb{C}/_{A \sim \lambda A}$

$= SL_2\mathbb{C}/_{\pm I}$

- transitivity
  - $PSL_2\mathbb{C}$ is triply-transitive on $\hat{\mathbb{C}}$
  - $PSL_2\mathbb{R}$ is triply-transitive on $\hat{\mathbb{R}}$

Proof ① $f(z) = [z,q,r,s] := \frac{(z-q)(r-s)}{(z-s)(r-q)}$ $\quad q \mapsto 0$, $r \mapsto 1$, $s \mapsto \infty$

So $[z,a',b',c']^{-1} \circ [z,a,b,c]$ does the trick.

② Fixing $0,1,\infty \Rightarrow$ Id. straightforward algebraically.

geometrically, any z is on 3 circles (w/ $0,1$; $0,\infty$; $1,\infty$)

so [oops - needs angle-preserving]