(1) Text, §49, Exercise 6.

Solution: Define

$$f(z) = \begin{cases} \sqrt{r}\, e^{i\theta/2} & \text{if } z = r\, e^{i\theta},\ -\pi/2 < \theta < 3\pi/2 \\ 0 & \text{if } z = 0 \end{cases}$$

Since $f(z)$ is *not* analytic at $z = 0$, we cannot use the Cauchy–Goursat Theorem to conclude that $\int_C f(z)\, dz = 0$.

However, by direct calculation we can show that the contour integral of f over C is zero. Let

- C_1 be the circular arc $z = e^{i\theta},\ 0 \le \theta \le \pi$, oriented counterclockwise,
- C_2 be the directed line segment from $z = -1$ to $z = 0$, and
- C_3 the directed line segment from $z = 0$ to $z = 1$.

Then

$$\begin{aligned} \int_{C_1} f(z)\, dz &= \int_0^{\pi} e^{i\theta/2}\, ie^{i\theta}\, d\theta \\ &= i \int_0^{\pi} e^{i(3\theta/2)}\, d\theta \\ &= \frac{2}{3}\ e^{i(3\theta/2)}\Big|_0^{\pi} \\ &= \frac{2}{3}\,(-i-1). \end{aligned}$$

For $z \in C_2$, then $f(z) = \sqrt{|z|}e^{i\pi/2} = i\sqrt{|z|}$. Now we parametrize C_2 by $z = -r$, for $0 \le r \le 1$, and so

$$\begin{aligned} \int_{C_2} f(z)\, dz &= \int_1^0 i\sqrt{r} \cdot (-dr) \\ &= \frac{2i}{3}. \end{aligned}$$

Finally, for $z \in C_3,\ f(z) = \sqrt{|z|}$. Parametrizing C_3 by $z = r,\ 0 \le r \le 1$, we obtain

$$\begin{aligned} \int_{C_3} f(z)\, dz &= \int_0^1 \sqrt{r}\, dr \\ &= \frac{2}{3}. \end{aligned}$$

From the above, it follows that

$$\int_C f(z)\, dz = \sum_{j=1}^{3} \int_{C_j} f(z)\, dz = 0.$$

(2) Exercise N.

Relevant Theorem (p.157 Text) If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)dz = 0$$

for every closed contour C lying in D.

Solution:

(a) **CLAIM:** Let γ be a contour and $\mathcal{N}$ a tubular neighborhood of γ in which the function $f(z)$ is analytic. If γ' is a contour in $\mathcal{N}$ with the same endpoints and orientation as γ then

$$\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz$$

Proof. Let C be the path given by $\gamma - \gamma'$. Then C is a closed contour. By the Theorem on p.157 of the Text (stated above), since f is analytic throughout the simply connected domain enclosed by γ and γ' we have

$$\int_\gamma f(z)dz - \int_{\gamma'} f(z)dz = \int_C f(z)dz = 0$$

Therefore,

$$\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz$$

$\square$

(b) Let $z = x + iy$ and γ a simple closed contour where A is the area enclosed by γ. Then

$$\begin{aligned}
\int_\gamma \bar{z}dz &= \int_\gamma (x - iy)(dx + idy) \\
&= \int_\gamma (x - iy)dx + i(x - iy)dy \\
&= \int_\gamma (i - (-i))dxdy \\
&= 2iA
\end{aligned}$$

where in the second line we applied Green's Theorem where

$$P = x - iy, \quad Q = i(x - iy)$$

so we can see that P, Q, and their first order partial derivatives are continuous throughout the closed region consisting of all points interior to and on γ.

(c) Consider the function $f(z) = \bar{z}$. Note that this function is nowhere analytic. Let γ be a simple closed contour enclosing the area A. Let γ' be a perturbation of γ such that γ' is a simple closed contour enclosing area A' where $A' \neq A$ (such a perturbation can be easily constructed). Then

$$2iA = \int_\gamma f(z)dz \neq \int_{\gamma'} f(z)dz = 2iA'$$

So, our claim in part(a) is not true if the function $f(z)$ is not analytic in $\mathcal{N}$.