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|---|---|---|
| **HW 1** | Jan 24 | §2/1 §3/1 §4/1,4,5,6 §5/1,2,13,A §8/1,6,9,10 §10/B,7,8 §11/4 |
| **HW 2** | Jan 31 | §14/1,C §18/1,4,5,13 §90/2,4 §92/2,7,8 |
| **HW 3** | Feb 7 | §18/11, §94/2,6,9,D,E,F |
| **HW 4** | Feb 14 | §20/2,3,4,G, §23/1,2,3,4,9,H |
| **HW 5** | Feb 21 | §25/1,2,3,4,6,I, §29/1,2,3,4,11 |
| **HW 6** | Feb 28 | §31/1,3,6, §32/1,2, §33/2,9, §34/12, §35/1,8,J, §36/3 |
| Exam | Mar 6 | See Practice Problems |
| **HW 7** | Mar 27 | §38/2,3, §39/2,6,K,L, §41/M, §42/3,9,10 |
| **HW 8** | Apr 3 | §43/1,2,3,5, §45/2, §49/1 |
| **HW 9** | Apr 10 | §49/2c,5,6,N, §52/1,3,10 |
| **HW 10** | Apr 19 | §54/3,4,5,9,O,P,Q |

**A :** Show that $f(\bar{z}) = \overline{f(z)}$ for polynomials with real coefficients. Conclude that $f(z) = 0 \iff f(\bar{z}) = 0$ for these polynomials. In other words, complex roots of real polynomials come in conjugate pairs.

**B :** Choose three complex numbers and find the square root(s) and cube root(s) of each. Include pictures!

**C :** Choose two real numbers $r$ and $s$ and illustrate $e^r$ and $e^s$ using the power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

by measuring out the first few terms of the partial sum along the $x$ axis. Now illustrate $e^{ir}$ and $e^{is}$ using the same series expansion.

**D :** Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, all the complex numbers without zero. Show that $\rho(a+bi) = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ defines a homomorphism $\mathbb{C}^* \to GL_2(\mathbb{R})$. (That is, show that the map has output in the general linear group and that $\rho(zw) = \rho(z) \cdot \rho(w)$ for all $z, w \in \mathbb{C}^*$.) Is it a bijection?

**E :** Suppose there are two Möbius transformations $f, g$, and three distinct values $a, b, c \in \hat{\mathbb{C}}$ such that $f(a) = g(a), f(b) = g(b), f(c) = g(c)$. Show that $f = g$. (One approach: reduce this to showing that any map fixing three distinct points is the identity.)

**F :** The *cross-ratio* of $p, q, r, s \in \hat{\mathbb{C}}$ is given by $[p, q, r, s] = \frac{(p-q)(r-s)}{(p-s)(r-q)}$. It is easy to check that $[z, q, r, s]$ sends $q, r, s$ to $0, 1, \infty$.

Thus to send $a, b, c$ to $\alpha, \beta, \gamma$, we use the cross-ratio construction

$$[z, \alpha, \beta, \gamma]^{-1} \circ [z, a, b, c].$$

(a) Suppose $q, r, s$ lie on a circle $C$, and suppose that reading from $q$ to $r$ to $s$ goes counterclockwise around $C$. Show that $p$ is inside $C$, on $C$, or outside $C$ if $\mathrm{Im}[p, q, r, s]$ is positive, zero, or negative, respectively.

(b) Suppose again that $q, r, s$ lie on a circle $C$. What is the effect of $f(z) = [z, q, s, r]^{-1} \circ [z, q, r, s]$? (For starters, what is $f(C)$?)

(c) Find two different FLTs $f, g$ that carry $\mathbb{D}$ to $\mathbb{H}$. Find $f \circ g^{-1}$.

**G :** Consider the function $f(z) = z^3$. Suppose that it stretches a certain tangent vector by 5 and rotates it by $\pi$. Where could that vector have been based?

**H :** We showed in class that for constant functions $f(z) \equiv \lambda$, $f'(z) \equiv 0$.

(a) Give a geometric explanation for why

$$f'(z) \equiv 0 \implies f(z) \equiv \lambda.$$

(That is, constant functions are *exactly* those with zero derivative.)

(b) Show that if any differentiable function $f\colon\ \mathbb{C} \to \mathbb{C}$ satisfies $|f(z)| = 2$ for all $z$, then $f$ is constant.

**I :** Consider $f(x + iy) = (x^2 + y^2) + i(y/x)$.

(a) Show that this is not analytic on any domain using the Cauchy-Riemann equations.

(b) Show that this is not conformal by finding two curves whose angle of intersection is not preserved.

(c) For what functions $v(x, y)$ can $f(x + iy) = (x^2 + y^2) + iv$ be analytic?

**J :** Directly from the defining equations, verify that $\cos(z) = \cosh(iz)$ and $\sin(z) = -i \cdot \sinh(iz)$. Explain why this means that cos is obtained by the composition of a rotation with cosh. Now explain how to write the sine function as a composition of several functions.

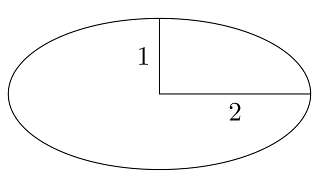
**K :** For each of the following parametrized curves, sketch the curve. Then find its tangent vector $\gamma'(t)$ at the start point and end point. Using the arc length formula (p124 or notes from class), find the length of the curve.

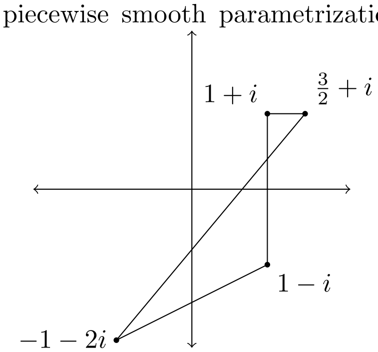
(a) $\gamma(t) = e^{it}$, $t \in [1, 4]$

(b) $\gamma(t) = e^{it^2}$, $t \in [1, 2]$

(c) $\gamma(t) = \begin{cases} (t+1) + i(t^2 + 2t) & t < 0 \\ 1 - t & t \geq 0, \end{cases} \qquad t \in [-2, 2]$

**L :** For each of the following curves, give a piecewise smooth parametrization.



**M :** We will study $\int_C 1/z\ dz$ using the Riemann sum interpretation of contour integrals. Here, $C$ is the unit circle, parametrized as $\gamma(t) = e^{it}$ over $t \in [0, 2\pi]$.

Divide $C$ into $n$ equally spaced points, so that $z_j = e^{\frac{2\pi i}{n} j}$. Sketch some of them. Let $s_j$ be the secant vector $s_j = z_j - z_{j-1}$. Compute its value and sketch some of them. Let $m_j$ be the MIDPOINT on $\gamma$ of the arc between $z_{j-1}$ and $z_j$. Compute its value and sketch some of them. In a separate plane, draw the image of $\gamma$ under $f(z) = 1/z$. Compute the images $f(m_j)$ and sketch some of them.

The Riemann sum is defined to be $RS(n) = \sum_{j=1}^{n} s_j \cdot f(m_j)$. Compute a formula for $RS(n)$ in terms of $n$ and compute $RS(2)$ and $RS(4)$ by hand, and $RS(16)$ and $RS(100)$ using a calculator. Now find the limit as $n \to \infty$, and check that you get the same answer as you would by using the parametrized definition of a contour integral: $\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\ dt$.

**N :** Suppose a function $f(z)$ is analytic on a contour $\gamma$ and in a small open neighborhood $\mathcal{N}$ of $\gamma$. A *perturbation* of the contour is a nearby contour (meaning that it is still in $\mathcal{N}$). Give some basic conditions on $\gamma$, $\mathcal{N}$, or the perturbation that would ensure that the value of the integral $\int_\gamma f$ is unchanged. (There is more than one right answer.)

Now use Green's Theorem (p150-151) to prove that

$$\oint_\gamma \bar{z}\ dz = 2iA,$$

where $A$ is the area enlcosed by a simple closed curve $\gamma$. (Be sure to check the hypotheses.)

Conclude that the above statement about perturbations is false without the assumption of analyticity.

**O :** Use contour integration to derive the formula for the area of a triangle in the following steps.

(a) Suppose $A$ and $B$ are two complex numbers. Give a parametrization for $[AB]$, the straight line from $A$ to $B$.

(b) Now find a formula for $\displaystyle\int_{[AB]} \bar{z}\ dz$ in terms of $A$ and $B$.

(c) Using this, find a formula for the area of the triangle with vertices at 0, $b$, and $c + hi$, where $b, c, h \in \mathbb{R}$. (And use usual Euclidean geometry to check that you are right!)

**P :** Recall from real calculus that the method of partial fractions is used to integrate rational functions. In it, the polynomial denominator is factored into linear terms and irreducible quadratic terms. In complex analysis, there are no irreducible quadratics. (Why?)

Consider the integral

$$\oint_\gamma \frac{2}{z^2 + 1}\ dz.$$

(a) Use partial fractions to split up the integrand.

(b) Using this expression, find all the possible values of the integral, for all closed curves $\gamma$. Draw some pictures of some of these curves $\gamma$ to illustrate, and explain what results you are using.

**Q :** As discussed in class: the Maximum Modulus Principle rules out any dark areas surrounded by color on colormaps of entire functions. (Black can only appear gradually as the modulus goes to infinity.) For an analytic function, dark areas must be centered at a black spot which represents a singularity.

For each of the following, state the result and explain how to see it on a colormap. (Or what kind of behavior it rules out on a colormap.)

(a) Gauss Mean Value Theorem

(b) Fundamental Theorem of Algebra

(c) Liouville's Theorem

Recommended: Go to the colormap notebook and make yourself a few examples, corroborating your observations here. Functions to try: $f(z) = 4z^3 + z - i$ (find the zeros!); $\cos z$; some Möbius transformation. Compare to non-analytic functions. Describe your findings.