D : OK, so the first thing to show is that $\rho(a+bi)$ is indeed in the set $GL_2(\mathbb{R})$. It is a two-by-two matrix, so we just need to check that its determinant is nonzero, since that's what it means to be in $GL_2(\mathbb{R})$. The determinant is a^2+b^2, and we know that a and b are not both zero (since $a+bi \in \mathbb{C}^*$, so $a+bi \neq 0+0i$), so it is strictly greater than zero.

Next we verify that ρ is a homomorphism. To do this, let $z = a+bi$ and $w = c+di$. Then $zw = (ac-bd)+(ad+bc)i$, so

$$\rho(zw) = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}.$$

On the other hand,

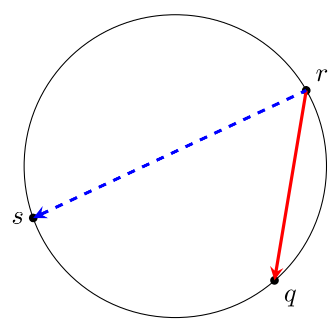
$$\rho(z)\rho(w) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix},$$

and this verifies the needed equality.

A bijection is both injective (1-1) and surjective (onto), but this map is clearly not onto since there are many elements of $GL_2(\mathbb{R})$ not in the image of ρ, such as $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 55 \end{smallmatrix}\right)$.

F : (a) This one stumped a lot of people. We want to show that if q, r, s are arranged counterclockwise around a circle C, then

$$p \text{ is } \begin{cases} \text{inside } C, & \operatorname{Im}[p,q,r,s] > 0 \\ \text{on } C, & \operatorname{Im}[p,q,r,s] = 0 \\ \text{outside } C, & \operatorname{Im}[p,q,r,s] < 0. \end{cases}$$



Now $[p,q,r,s] = \frac{(p-q)(r-s)}{(p-s)(r-q)}$. We know that this map sends the circle C to the real axis, because it sends $q \mapsto 0$, $r \mapsto 1$, and $s \mapsto \infty$ (as is easily verified by plugging in!). That means it is clear that only points on the circle have images on the real axis, i.e., $p \in C \iff \operatorname{Im}[p,q,r,s] = 0$, as desired. We also know that the FLT sends $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, so that means it must send the inside and outside of the circle to the upper and lower half-plane, in some order. It only remains to show which is which, so it suffices to work out where one single point is sent. I shall choose the only point that I know to be outside C, namely $p = \infty$. If I can show that $\operatorname{Im}[\infty,q,r,s] < 0$, I will be done.

We have $[p,q,r,s] = \frac{(p-q)(r-s)}{(p-s)(r-q)} = \frac{(r-s)p+(sq-qr)}{(r-q)p+(sq-sr)}$, so

$$[\infty,q,r,s] = \frac{r-s}{r-q} = \frac{s-r}{q-r}.$$

The numerator and denominator are complex numbers: these are the vectors represented in the picture. To decide whether the ratio is above or below the real axis, we just need to figure out whether its principal argument is positive or negative. Now $\operatorname{Arg}\left(\frac{s-r}{q-r}\right) = \arg(s-r) - \arg(q-r) \cap (-\pi, \pi]$. That means we just have to decide whether the rotation from the dashed vector to the solid vector is positive or negative. That clearly only depends on whether the points are arranged CW or CCW, and in the picture it is a negative rotation. Thus we have shown $[\infty, q, r, s]$ has a negative principal argument, so it's in the lower half-plane, as desired.

(b) What is the effect of $f(z) = [z, q, s, r]^{-1} \circ [z, q, r, s]$? Well, this sends the ordered triple (q, r, s) to the triple (q, s, r). So it must preserve the circle C as a set, since that is the only circle/line containing those three points. (So $f(C) = C$.) But let's suppose that the original triple is oriented CCW. Then the inside of the circle is mapped to the upper half-plane $\mathbb{H}$ by the first transformation. However, $[z, q, s, r]$ maps the inside of the circle to the LOWER half-plane, so it maps the outside of the circle to $\mathbb{H}$. Thus its inverse maps $\mathbb{H}$ to the outside of the circle, and therefore the composition sends the inside of the circle to the outside of the circle, and vice versa.

(c) We want two different FLTs that send $\mathbb{D}$ (the unit disk) to $\mathbb{H}$ (the upper half-plane). Easy enough: we just need to send the unit circle to the real line, since that is the boundary of each shape. The geometry of Möbius transformations does the rest. But of course $[z, q, r, s]$ sends (q, r, s) to $(0, 1, \infty)$. So my two maps are

$$f(z) = [z, 1, i, -1] \; ; \qquad g(z) = [z, i, -1, 1].$$

Note I've taken care to choose my triples oriented CCW so that the image is $\mathbb{H}$ and not the lower half-plane. The composition $f \circ g^{-1}$ is a map $\mathbb{H} \to \mathbb{H}$. In my case, this composition takes $0 \mapsto i \mapsto 1$, $1 \mapsto -1 \mapsto \infty$, and $\infty \mapsto 1 \mapsto 0$, so it must be equal to $[z, \infty, 0, 1]$.

H : (a) Why does identically zero derivative imply a constant function? Because it means that all tangent vectors are zero. Suppose the image of the map contained at least two distinct points, so that there are some $p, q \in \mathbb{C}$ with $f(p) \neq f(q)$. Then consider a path γ connecting p and q. Consider vectors tangent to γ. The map f sends γ to some contour in $\mathbb{C}$ connecting $f(p)$ with $f(q)$. But any path with all zero derivatives has zero length (because the speed of travel is just the magnitude of the tangent vector, as you learned in calculus). Therefore the distance from $f(p)$ to $f(q)$ is zero, so they are the same point after all. We have proved by contradiction that the function is constant.

(b) If the image of a function has constant norm, then the entire complex plane is mapped to a circle centered at the origin; in this case, a circle of radius two. But then all tangent vectors at a point are in the directions tangent to the circle, which means there are only two possible directions of tangent vectors at any point. But at an input point, tangent vectors can point in any direction. So it is impossible that the derivative exists anywhere: if $f'(p)$ existed for any $p \in \mathbb{C}$, then all tangent vectors based at p would be rotated by the same amount; if $f'(p) = re^{i\theta}$, then *all* vectors

$\vec{v}$ based at p would be rotated by θ, making all directions appear in the output, since $\vec{v}$ was arbitrary.

I : (a) Let $f(x+iy) = (x^2+y^2) + i(y/x)$, so $u = x^2+y^2$ and $v = y/x$. Then $u_x = 2x$, $u_y = 2y$, $v_x = -y/x^2$, $v_y = 1/x$. The CR equations require that $u_x = v_y$, so for them to be satisfied requires $2x = 1/x$, or $x = 1/\sqrt{2}$. The other equation says $u_y = -v_x$, which requires $2y = y/x^2$, which says that either $y = 0$ or $x = 1/\sqrt{2}$. Thus the solution set to the simultaneous equations is the line $x = 1/\sqrt{2}$, so there is no open set on which they are satisfied, which means the function can't be analytic anywhere.

(b) The Jacobian at $z = 1$ is $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which preserves $\binom{0}{1}$ but sends $\binom{1}{1}$ to $\binom{2}{1}$, so it does not preserve the angle between those vectors. Thus two curves that do the trick are the lines $\mathrm{Im} z = 0$ (because it goes through 1 in the direction $\binom{0}{1}$ and $\mathrm{Im} z = \mathrm{Re} z - 1$, because it goes through 1 in the direction $\binom{1}{1}$.

J : Recall that

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}; \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}; \qquad \cosh z = \frac{e^{z}+e^{-z}}{2}; \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}.$$

Well, $\cosh(iz) = \frac{e^{iz}+e^{-iz}}{2} = \cos z$ is immediate.

Now, $\sinh(iz) = \frac{e^{iz}-e^{-iz}}{2}$, and $-i = 1/i$, so so $-i\sinh(iz) = \frac{e^{iz}-e^{-iz}}{2i} = \sin z$.

That means that, geometrically, cos can be obtained by first rotating counterclockwise by a right angle and then performing cosh.

Likewise, a recipe for sin is to first rotate CCW by a right angle, then do sinh, and then undo the first part by rotating CW by a right angle.