(1) Text, §20, Exercise 2.

Solution:

(a) Let $P(z)$ be the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \neq 0)$$

of degree $n, (n \geq 1)$. We will show by induction that $P'(z)$ exists for all $n \geq 1$ and has the form given in the problem statement.

Let $n = 1$ then $P(z) = a_0 + a_1 z$ where $a_1 \neq 0$. Then, since we know constant functions are differentiable and $a_1 z$ is differentiable, by equations (1) and (3) in §20 of the text we have

$$\frac{d}{dz}(a_0 + a_1 z) = \frac{d}{dz}(a_0) + \frac{d}{dz}(a_1 z) = a_1$$

Hence $P'(z) = a_1$.

Now assume the result holds for $n-1$. We will show the result for n. Write $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ for $a_n \neq 0$. Then by our inductive assumption and since $a_n z^n$ is differentiable, by equations (1), (2), and (3) in §20 of the text we have

$$\begin{aligned}
&\frac{d}{dz}\left(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\right) \\
&\quad = \frac{d}{dz}\left(a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}\right) + \frac{d}{dz}\left(a_n z^n\right) \\
&\quad = a_1 + a_2 2 z_0 + \cdots + a_n n z_0^{n-1}
\end{aligned}$$

This proves the result.

(b) Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ of degree $n \geq 1$. By part (a), all derivatives $P^{(n)}(z), n \geq 1$ exist and have the form shown in (a). Therefore, we have

$$\begin{aligned}
P(0) &= a_0 + a_1(0) + a_2(0)^2 + \cdots + a_n(0)^n = a_0 \\
P'(0) &= \left(\frac{d}{dz}P(z)\right)\bigg|_{z=0} = \left(a_1 + a_2 2z + \cdots + a_n n z^{n-1}\right)|_{z=0} = a_1 + a_2 2(0) + \cdots + a_n n(0)^{n-1} = a_1 \\
P''(0) &= \left(\frac{d}{dz}P'(z)\right)\bigg|_{z=0} = \left(a_2 2 + \cdots + a_n n(n-1) z^{n-2}\right)|_{z=0} = a_2 2 + \cdots + a_n(0)^{n-1} = 2a_2 \\
&\vdots \\
P^{(n)}(0) &= \left(\frac{d}{dz}P^{n-1}(z)\right)\bigg|_{z=0} = \left(a_n n(n-1)(n-2)\cdots(2)(1)\right)|_{z=0} = n! a_n
\end{aligned}$$

which proves the result.

(2) Text, §23, Exercise 3.

Solution:

(a) For $f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$ where $z \neq 0$ we have

$$u(r,\theta) = \frac{1}{r}\cos\theta \qquad v(r,\theta) = -\frac{1}{r}\sin\theta$$

By the Cauchy-Riemann equations, we have

$$\begin{aligned} ru_r = v_\theta &\iff -\frac{1}{r}\cos\theta = -\frac{1}{r}\cos\theta \\ u_\theta = -rv_r &\iff -\frac{1}{r}\sin\theta = -\frac{1}{r}\sin\theta \end{aligned}$$

Since u and v have continuous partial derivatives for $z \neq 0$, the derivative $f'(z)$ will exist for all $z \neq 0$. At those points

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta}\left(-\frac{1}{r^2}\cos\theta + i\frac{1}{r^2}\sin\theta\right) \\ &= -\frac{1}{r^2}e^{-i2\theta} \\ &= -\frac{1}{z^2} \end{aligned}$$

(b) For $f(z) = x^2 + iy^2$ we have

$$u(x,y) = x^2 \qquad v(x,y) = y^2$$

By the Cauchy-Riemann equations, we have

$$\begin{aligned} u_x = v_y &\iff 2x = 2y \implies x = y \\ u_y = -v_x &\iff 0 = 0 \end{aligned}$$

Since u and v are polynomials, they have continuous partial derivatives, and hence the derivative $f'(z)$ will exist only at points in the line $y = x$. At those points $z = x + ix$, and

$$f'(z) = u_x + iv_x = 2x$$

(c) For $f(z) = z\operatorname{Im} z = xy + iy^2$ we have

$$u(x,y) = xy \qquad v(x,y) = y^2$$

Then

$$\begin{aligned} u_x = v_y &\iff y = 2y \implies y = 0 \\ u_y = -v_x &\iff x = 0. \end{aligned}$$

Since u and v are polynomials, they have continuous partial derivatives, and hence $f'(z)$ exists only at $z = 0$. At $z = 0$, the derivative is

$$f'(0) = u_x|_{(x,y)=(0,0)} + iv_x|_{(x,y)=(0,0)} = 0.$$

(3) Exercise G. Consider the function $f(z) = z^3$. Suppose that it stretches a certain tangent vector by 5 and rotates it by π. Where could that vector have been based?

Solution: Since $f'(z) = 3z^2$ we have

$$\begin{aligned}
3z^2 &= 5e^{i\pi} \\
3z^2 &= -5 \\
z^2 &= \frac{-5}{3} \\
z^2 &= \frac{5}{3}e^{i\pi} \\
z &= \sqrt{\frac{5}{3}}e^{i\pi/2}, \sqrt{\frac{5}{3}}e^{-i\pi/2}
\end{aligned}$$

Therefore, we have the two solutions $\boxed{z = i\sqrt{\tfrac{5}{3}}, -i\sqrt{\tfrac{5}{3}}}$.