(1) Text, §34, Exercise 12. Use the reflection principle (§28) to show that for all z, (a) $\overline{\sin z} = \sin \overline{z}$; (b) $\overline{\cos z} = \cos \overline{z}$.

Theorem[Reflection Principle] Suppose that a function f is analytic in some domain D which contains a segment of the x-axis and is symmetric with respect to this segment. Then

$$\overline{f(z)} = f(\overline{z})$$

for each point $z \in D$ if and only if $f(x)$ is real for each point x on the segment.

Solution:

(a) We know that $\sin z$ is entire and the complex plane is symmetric with respect to the x-axis. Since

$$\begin{aligned} \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\ &= \frac{2i \sin x}{2i} \in \mathbb{R} \end{aligned}$$

by the reflection principle $\overline{\sin z} = \sin \overline{z}$.
Note this can be shown computationally by §34 equation (13) as follows, let $z = x + iy$, then

$$\begin{aligned} \sin(\overline{z}) = \sin(x - iy) &= \sin x \cosh(-y) + i \cos(x) \sinh(-y) \\ &= \sin x \cosh y - i \cos x \sinh y \\ &= \mathrm{Re}(\sin(x + iy)) - i\mathrm{Im}(\sin(x + iy)) \\ &= \overline{\sin(z)} \end{aligned}$$

(b) Similarly, we know that $\cos z$ is entire and the complex plane is symmetric with respect to the x-axis. Since

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ &= \frac{2 \cos x}{2} \in \mathbb{R} \end{aligned}$$

by the reflection principle, $\overline{\cos z} = \cos \overline{z}$.

(2) Text, §36, Exercise 3. Solve the equation $\cos z = \sqrt{2}$ for z.

Solution: Since

$$\cos z = \sqrt{2} \Longleftrightarrow \cos^{-1} \sqrt{2} = z$$

we must solve $\cos^{-1}\sqrt{2} = z$. We have

$$
\begin{aligned}
z = \cos^{-1}\sqrt{2} &= -i\log[\sqrt{2} + i(1-(\sqrt{2})^2)^{1/2}] && \text{(by §36 equation (3))}\\
&= -i\log[\sqrt{2} + i(-1)^{1/2}]\\
&= -i\log[\sqrt{2} \pm 1]\\
&= -i\left(\ln(\sqrt{2}\pm 1) + 2n\pi i\right) && \text{(since } \log z = \mathrm{Log} z + 2n\pi i \text{ for } n \in \mathbb{Z}\text{)}\\
&= -i\ln(\sqrt{2}\pm 1) + 2n\pi && \text{for } n \in \mathbb{Z}
\end{aligned}
$$

(3) Exercise J. Directly from the defining equations, verify that $\cos(z) = \cosh(iz)$ and $\sin(z) = -i\sinh(iz)$. Explain why this means that cos is obtained by the composition of a rotation with cosh. Now explain how to write the sine function as a composition of several functions.

Solution: By equation (1) in §35 and equation (1) in §34 we have

$$
\begin{aligned}
\sinh(iz) &= \frac{e^{iz} - e^{-iz}}{2}\\
&= i\left(\frac{e^{iz} - e^{-iz}}{2i}\right)\\
&= i\sin(z).
\end{aligned}
$$

Hence $\sin(z) = -i\sinh(iz)$. Similarly,

$$
\begin{aligned}
\cosh(iz) &= \frac{e^{iz} + e^{-iz}}{2}\\
&= \cos(z).
\end{aligned}
$$

Recalling that multiplication by i means rotation by $\frac{\pi}{2}$, the relation $\cos(z) = \cosh(iz)$ implies that $\cos(z)$ is obtained by first rotating z by $\frac{\pi}{2}$ and then applying cosh, hence it is a composition of this rotation and cosh. Similarly, the relation $\sin(z) = -i\sinh(iz)$ implies that $\sin(z)$ is a composition of a rotation by $\frac{\pi}{2}$ with sinh and finally rotation by $-\frac{\pi}{2}$.