Selected HW2 Solutions

Math 158, Complex Analysis
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(1) Text, §18, Exercise 1.

Solution:

(a) Let $\epsilon > 0$ and set $\delta = \epsilon$. Then for any $z \in \mathbb{C}$ such that
$$0 < |z - z_0| < \delta$$
we have
$$|\mathrm{Re}(z) - \mathrm{Re}(z_0)| = |\mathrm{Re}(z - z_0)| \leq |z - z_0| < \delta = \epsilon$$
since ϵ is arbitrary, this proves that $\lim_{z \to z_0} \mathrm{Re}(z) = \mathrm{Re}(z_0)$.

(b) Let $\epsilon > 0$ and set $\delta = \epsilon$. Then for any $z \in \mathbb{C}$ such that
$$0 < |z - z_0| < \delta$$
we have
$$|\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon$$
since ϵ is arbitrary, this proves that $\lim_{z \to z_0} \overline{z} = \overline{z_0}$.

(c) Let $\epsilon > 0$ and set $\delta = \epsilon$. Then for any $z \in \mathbb{C}$ such that
$$0 < |z| < \delta$$
we have
$$\left| \frac{\overline{z}^2}{z} - 0 \right| = \left| \frac{\overline{z}\,\overline{z}}{z} \right| = \frac{|\overline{z}||\overline{z}|}{|z|} = \frac{|z||z|}{|z|} = |z| < \delta = \epsilon$$
since ϵ is arbitrary, this proves that $\lim_{z \to 0} \frac{\overline{z}^2}{z} = 0$.

(2) Text, §18, Exercise 4.

Solution: (Please note that the question asks you to use mathematical induction and property (9) of §16 of text.)

Let $n = 1$, let $\epsilon > 0$ and set $\delta = \epsilon$. Then for any $z \in \mathbb{C}$ such that
$$0 < |z - z_0| < \delta$$
we have
$$|z - z_0| < \epsilon$$
since ϵ is arbitrary, this proves that $\lim_{z \to z_0} z = z_0$.

Now, suppose the claim holds for n. We will show that the claim holds for $n + 1$.
$$\begin{aligned} \lim_{z \to z_0} z^{n+1} &= \lim_{z \to z_0} (z^n) z \\ &= z_0^n z_0 \\ &= z_0^{n+1} \end{aligned}$$
where the second line uses property (9) since $\lim_{z \to z_0} z^n = z_0^n$ and $\lim_{z \to z_0} z = z_0$. Therefore, the claim holds by induction.

(3) Text, §18, Exercise 13.

Relevant definitions:

(a) An ϵ-**neighborhood** of infinity is a set of $z \in \mathbb{C}$ such that

$$|z| > \frac{1}{\epsilon}$$

(b) A set S is **unbounded** if and only if for any $R > 0$, there exists $z \in S$ such that $|z| > R$.

Solution:

($\Longrightarrow$) Suppose that S is unbounded. Let $\epsilon > 0$ and set $R = 1/\epsilon$. As S is unbounded, there exists a $z \in S$ such that $|z| > R = 1/\epsilon$, so this z is in the ϵ-neighborhood of infinity. Since ϵ is arbirary, there exists an element of S in *every* neighborhood of infinity.

($\Longleftarrow$) Suppose now that there exists an element of S in every neighborhood of infinity. Let $R > 0$ and set $\epsilon = 1/R$. As $\epsilon > 0$, there exists a $z \in S$ such that $|z| > 1/\epsilon = R$. As R is arbitrary, S is unbounded.