You may use any true fact from class or the book, but say what you are using.

(1) Find all solutions to the equation $iz^5 = 1 + i$.

(2) Find all solutions to the equation $e^z = 0$.

(3) Consider $f(z) = e^z$, which is an entire function $\mathbb{C} \to \mathbb{C}$. Can it be extended continuously to a function on $\hat{\mathbb{C}}$?

(4) Find the inverse of the function $f(z) = \frac{3z+i}{2z+2}$. Regarding f as a function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, what is $f(\infty)$ and why? What is the image of f?

(5) Use Cauchy's formula to evaluate the following integrals, where S is a closed square with positive orientation and the four vertices $\pm 2 \pm 2i$.
(a) $\oint_S \frac{e^{-z}}{z - \frac{\pi i}{2}}\, dz$
(b) $\oint_S \frac{\cos z}{z(z^2-8)}\, dz$
(c) $\oint_S \frac{z}{2z+1}\, dz$

(6) Let $f(z) = e^z - \bar{z}$. Now let L be the contour that travels the straight line from 0 to πi at unit speed.
(a) What is $f(L)$?
(b) Give the formula for the 3-term midpoint Riemann sum approximation to $\int_L f$. Give the associated picture.
(c) Give the definition of derivative. At what points does $f'(z)$ exist? At what points if $f(z)$ analytic?
(d) Suppose γ is some other contour between 0 and πi. What can you say about $\int_\gamma f$?

(7) Let L be a straight line in $\mathbb{C}$ which makes an angle of ϕ with the real axis. Let M be the (unique) line segment that connects L to the origin while making a right angle with L. Let m be the length of M. (So $m = 0$ if and only if L goes through 0.)
(a) In general, what is the geometric effect on $\mathbb{C}$ of multiplication by a complex number $a + bi$?
(b) What is the geometric effect on $\mathbb{C}$ of multiplication by the complex number $e^{-i\phi}$?
(c) What is $e^{-i\phi}L$?
(d) Let p be any point on L. Give a geometric argument that $m = |\mathrm{Im}(e^{-i\phi}p)|$. (You can also give a computational argument as a double-check, but you must give a geometric argument for full credit.)

(8) This question concerns the following true statement:

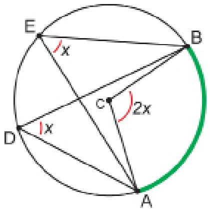
$$|z| = 1 \implies \operatorname{Im}\left(\frac{z}{(z+1)^2}\right) = 0.$$

(a) Prove this by letting $z = x + yi$ and verifying the calculation with respect to those real variables x, y.

(b) Prove this without real coordinates by first proving that for a complex number w,

$$\operatorname{Im}(w) = 0 \iff w = \bar{w}.$$

(c) It is a beautiful theorem of Euclidean geometry that if you take any arc of a circle, the angle subtended at the origin is two times the angle subtended at any point on the circle.



Prove the statement geometrically with the following steps:

- Show geometrically that if $\arg w_1 = \arg w_2$, then $\operatorname{Im}(w_1/w_2) = 0$.
- Explain why $z + 1$ can be represented as the vector pointing from -1 to z in the complex plane.
- Show that $\arg z = 2\arg(z+1)$.
- Deduce the statement.

(9) $f(z) = e^z$ is a continuous function, so for any closed and bounded set R, there must be a point in R where f achieves its maximum value.

Let R be the rectangular region $0 \le x \le 1$, $0 \le y \le \pi$. What does the maximum modulus principle say about the location of any point $z_0 \in R$ such that $|f(z)| \le |f(z_0)|$ for all $z \in R$?

Using your knowledge of e^z, find all such points z_0.

(10) Show that if f is entire, and $f(z) = f(1/z)$ for all $z \neq 0$, then f is constant. Here are the steps:

- Explain why there is some real number M such that $|z| \le 1 \implies |f(z)| \le M$.
- Explain why for every complex number $z \neq 0$, either z or $1/z$ (or both!) is in the closed unit disk $\overline{\mathbb{D}}$.
- Conclude that $|f(z)| \le M$ for every $z \in \mathbb{C}$.
- Finally conclude that f is constant.