(1) Text, §18, Exercise 11.

Solution: Let a, b, c, and d be complex numbers satisfying $ad - bc \neq 0$ and let

$$T(z) = \frac{az+b}{cz+d}.$$

(a) Suppose that $c = 0$. Then $ad \neq 0$, so in particular $a \neq 0$. This implies that

$$\begin{aligned}\lim_{z\to\infty} T(z) &= \lim_{z\to\infty} \frac{az+b}{d}\\ &= \infty,\end{aligned}$$

since

$$\begin{aligned}\lim_{z\to 0} \frac{1}{T(\frac{1}{z})} &= \lim_{z\to 0} \frac{d}{\frac{a}{z}+b}\\ &= \lim_{z\to 0} \frac{dz}{a+bz}\\ &= 0.\end{aligned}$$

(b) Suppose that $c \neq 0$. Then

$$\begin{aligned}\lim_{z\to\infty} T(z) &= \lim_{z\to 0} T\left(\frac{1}{z}\right)\\ &= \lim_{z\to 0} \frac{\frac{a}{z}+b}{\frac{c}{z}+d}\\ &= \lim_{z\to 0} \frac{a+bz}{c+dz}\\ &= \frac{a}{c}.\end{aligned}$$

In addition,

$$\begin{aligned}\lim_{z\to\frac{-d}{c}} \frac{1}{T(z)} &= \lim_{z\to\frac{-d}{c}} \frac{cz+d}{az+b}\\ &= \frac{0}{a(\frac{-d}{c})+b}\\ &= 0.\end{aligned}$$

where in the second line above we used the fact that

$$a\left(\frac{-d}{c}\right) + b = \frac{(bc-ad)}{c} \neq 0$$

From this we conclude that

$$\lim_{z\to\frac{-d}{c}} T(z) = \infty.$$

(2) Text, §94, Exercise 6.

Solution: Suppose that the linear fractional transformation $T(z) = \dfrac{az+b}{cz+d}$ fixes three distinct points z_1, z_2, and z_3 in the finite plane. Then

(1) $$z_k = \frac{az_k + b}{cz_k + d}, \qquad (k = 1, 2, 3)$$

which gives the equation

(2) $$cz_k^2 + (d-a)z_k - b = 0 \qquad (k = 1, 2, 3)$$

If $c \neq 0$, this implies that the quadratic equation above has three distinct solutions, which is impossible. Thus $c = 0$, and so we have

(3) $$(d-a)z_k = b \qquad (k = 1, 2, 3)$$

Now the only way this linear equation can have more than one solution is if $d - a = b = 0$, which gives $d = a$ and $b = 0$. Thus

(4) $$T(z) = \frac{a \cdot z + 0}{0 \cdot z + a} = z,$$

so T is the identity transformation.

Next let us consider the case in which one of the fixed points is ∞. Without loss of generality, we may assume $z_1 = \infty$. Then

$$\infty = T(\infty) = \frac{a}{c}$$

so $c = 0$. Equations (1) and (2) will still hold for z_2 and z_3. And because $c = 0$, equation (3) also holds for z_2 and z_3. Since z_2 and z_3 are distinct solutions of the linear equation (3), we see again that $d = a$ and $b = 0$. Hence $T(z) = z$ in this case as well by (4).

(3) Exercise E. Suppose there are two linear fractional transformations f, g and three distinct values $a, b, c \in \hat{\mathbb{C}}$ such that $f(a) = g(a)$, $f(b) = g(b)$ and $f(c) = g(c)$. Show that $f = g$. (One approach: reduce this to showing that any map fixing three distinct points is the identity.)

Relevant Theorem from class: Given a set of three distinct points, z_1, z_2, z_3 in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and a second set of distinct points w_1, w_2, w_3 in $\hat{\mathbb{C}}$ there exists exactly one linear fractional transformation $f(z)$ such that

$$f(z_i) = w_i \quad (\text{for } i = 1, 2, 3).$$

Solution: Let f and g be linear fractional transformations such that for the distinct values $a, b, c \in \hat{\mathbb{C}}$ we have $f(a) = g(a)$, $f(b) = g(b)$ and $f(c) = g(c)$. We know g^{-1} exists and is also a linear fractional transformation (see text, §93 equation (8)). So, $g^{-1} \circ f$ is a linear fractional transformation (see text, §93) fixing a, b and c and is therefore the identity map (by §94, Exercise 6). Then for any $z \in \hat{\mathbb{C}}$

$$\begin{aligned}(g^{-1} \circ f)(z) &= z \\ (g \circ g^{-1} \circ f)(z) &= g(z) \\ f(z) &= g(z)\end{aligned}$$

This shows that $f = g$.