Recall that the *cross-ratio* of $p, q, r, s \in \hat{\mathbb{C}}$ is given by $[p, q, r, s] = \frac{(p-q)(r-s)}{(p-s)(r-q)}$.

Recall that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}; \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}; \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}; \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

GENERAL

(1) *Given a circle centered at p of radius r, let us say that* inversion in the circle *is the continuous map $I : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the following property: given any $z \in \mathbb{C}$, its image $I(z)$ is the unique point on the ray $\overrightarrow{pz}$ such that $|z - p| \cdot |I(z) - p| = r^2$. As a consequence of continuity and this property, I must exchange p and ∞. The defining property also ensures that I must fix each point in the circle.*

(a) *Prove that $f(z) = 1/\bar{z}$ is inversion in the unit circle by showing that it has the property listed above.*

The center of the unit circle is $p = 0$ and the radius is $r = 1$, so I need to show that $|z| = |1\bar{z}|$ if and only if $z \in C$. Well, $|1/\bar{z}| = 1/|\bar{z}| = 1/|z|$, so

$$|z| = |1/\bar{z}| \iff |z|^2 = 1 \iff |z| = 1 \iff z \in C. \qquad \checkmark$$

(b) *Suppose $g(z) = z + b$ for some complex number b. Explain* geometrically *the effect of $g^{-1} \circ f \circ g$. Begin by finding its fixed points.*

Well, what does this map do? It first moves things by adding b, then does inversion in the unit circle, and then subtracts off b again. Thus, what happens to the circle of radius 1 centered at $-b$? It gets moved to be the unit circle C, and then it gets fixed by f, and then it gets moved back to be centered at $-b$ again. So the fixed set of $g^{-1} \circ f \circ g$ is the circle $C - b$, which has radius 1 and is centered at $-b$. In fact, the composition is just inversion in the circle $C - b$, because the inside of that circle is mapped by the first step to the inside of C, which is mapped by f to the outside of C, which is mapped back by translation to the outside of $C - b$. And vice versa!

(c) *Now let $h(z) = kz$ for some real number $k > 0$ and do the same for $h^{-1} \circ f \circ h$.*

Similarly, the circle of radius $1/k$ is first sent to the unit circle, then we invert in C, then we send its radius back from 1 to $1/k$. So the fixed set is $\frac{1}{k}C$ and the map is inversion in the circle $\frac{1}{k}C$.

(d) *Using the previous parts, give a formula for inversion in a circle of radius r centered at $p \in \mathbb{C}$.*

Well, first we can move it to the origin by subtracting p, then we can scale it by $1/r$ to make it into the unit circle C, then we do inversion, then we scale by r, then we add p. That is:

$(z \mapsto z + p) \circ (z \mapsto rz) \circ (z \mapsto \frac{1}{\bar{z}}) \circ (z \mapsto \frac{1}{r}z) \circ (z \mapsto z - p)$, which is

$$F(z) = r\left(\frac{1}{\overline{\left(\frac{z-p}{r}\right)}}\right) + p = r\left(\frac{1}{\left(\frac{\bar{z}-\bar{p}}{r}\right)}\right) + p = \frac{r^2}{\bar{z} - \bar{p}} + p.$$

MÖBIUS TRANSFORMATIONS AND CROSS-RATIOS

(2) *Write the fractional linear transformation $f(z) = \frac{iz+1}{2z-1}$ as a cross-ratio map. (That is, $f(z) = [z, q, r, s]$ for some $q, r, s \in \hat{\mathbb{C}}$. Find those values.)*

Well, $[z, q, r, s]$ sends (q, r, s) to $(0, 1, \infty)$. So what is sent to $(0, 1, \infty)$ by f? It's easy to see that $f(i) = 0$, and it's easy to see that $f(1/2) = \infty$, so we've found q and s. The last one takes a computation:

$$\frac{iz+1}{2z-1} = 1 \implies iz+1 = 2z-1 \implies (i-2)z = -2 \implies z = \frac{-2}{i-2} = \frac{2}{2-i} = \frac{4+2i}{5}.$$

So the answer is $f(z) = \left[z, i, \frac{4}{5} + \frac{2}{5}i, \frac{1}{2}\right]$. (The answer is unique because we showed in class that a FLT is determined by where it takes three points.)

(3) *What is the derivative of $f(z) = [z, -2, -1, \infty]$? Derive this computationally and give a geometric description of f.*

This one is very easy geometrically. f sends $-2 \mapsto 0$, $-1 \mapsto 1$, and $\infty \mapsto \infty$. Drawing this, it's clear that this is performed by $f(z) = z + 2$, and once again since FLTs are uniquely determined by the image of three points, this is then the only possible answer, so $f'(z) = 1$.

On the other hand, we can use the fact that

$$[z, q, r, s] = \frac{(r-s)z + (sq - qr)}{(r-q)z + (sq - sr)} = \left(\frac{r-s}{r-q}\right)\frac{z-q}{z-s}$$

to find that its derivative, in general, is

$$\left(\frac{r-s}{r-q}\right)\left(\frac{(z-s)-(z-q)}{(z-s)^2}\right) = \left(\frac{r-s}{r-q}\right)\frac{q-s}{(z-s)^2}.$$

For us, since the coefficient of s^2 on the top and bottom is 1, this equals 1 since $s = \infty$. (Think of taking a limit as $s \to \infty$: all that matters are the coefficients of the highest power of s.) So this gives the same answer.

(4) *What is the FLT that sends $1 \mapsto i$, $i \mapsto -1$, and $-1 \mapsto -i$? What is its derivative? (Note: this can be done with blind computation but it is much cleaner to think geometrically.)*

Drawing a picture makes it clear that the map is counterclockwise rotation by $\pi/2$; in other words, $f(z) = iz$. This is a FLT because it can be written as $f(z) = \frac{iz+0}{0z+1}$. Clearly its derivative is i at every point.

DERIVATIVES

(5) *Does there exist an entire map that rotates every tangent vector by the same amount but does not amplify all tangent vectors by the same amount? If so, give an example. If not, prove it.*

There does not exist such a map. Here's a proof using the Cauchy-Riemann equations. Write $f(z) = u + iv$, where $u(x, y)$ and $v(x, y)$ are both real, as usual. Then the CR equations are: $u_x = v_y$, $u_y = -v_x$. And the derivative is given by $f'(z) = u_x + iv_x$.

Now without loss of generality we can try to find a function $f(z)$ that rotates all tangent vectors by ZERO and has a non-constant stretch factor. (Because if we could find one with constant rotation by θ, we could compose it with the rotation by $-\theta$ to get one with constant rotation by 0.)

Thus for our map, $f'(z)$ would have to be REAL at every point, because multiplication by anything with a nonzero imaginary part has some nonzero argument. But since $f'(z) = u_x + iv_x$, this means that $v_x = 0$, and u_x is not constant (by hypothesis). Since we are supposing this function to be entire, the CR equations are satisfied everywhere, so $v_y = u_x$ and $u_y = -v_x = 0$. The latter means that $u(x, y)$ does not depend on y, and $v(x, y)$ does not depend on x. So since u depends only on x, the derivative u_x is a function of x as well; on the other hand v depends only on y, so v_y is a function of y. We've assumed that u_x is not constant, so it must have nontrivial dependence on x, but then it can't also depend only on y. We have found a contradiction, so the function we are seeking must not exist.

(6) *On a previous homework you showed that there is* NO *choice of* $v\colon \mathbb{R}^2 \to \mathbb{R}$ *such that* $f(z) = u + iv$ *is analytic, if* $u(x, y) = x^2 + y^2$*. How about if* $u(x, y) = ax^2 + bxy + cy^2$*? (That is, find the conditions on* $a, b, c \in \mathbb{R}$ *that make it possible for a function with this real part to be analytic anywhere.)*

We compute $u_x = 2ax + by$ and $u_y = bx + 2cy$. Thus the CR equations tell us that $v_y = 2ax + by$ and $v_x = -bx - 2cy$. Integrating the first one, we get $v(x, y) = 2axy + \frac{1}{2}by^2 + g(x)$, where g is some function that depends only on x. On the other hand, $v(x, y) = -\frac{1}{2}bx^2 - 2cxy + h(y)$, where h is some function that depends only on y. Rearranging, we get

$$h(y) = 2(a + c)xy + \frac{1}{2}b(x^2 + y^2) + g(x).$$

What needs to be true to make the right-hand side independent of x? We need $g(x)$ to be constant and the other two terms to be zero. This means $a = -c$ and $b = 0$. So the most general u that allows for analyticity is $u(x, y) = ax^2 - ay^2$.

And you already knew this was a possible real part of an analytic function, because it's the real part of $f(z) = az^2$!

(7) *What is the effect of* $f(z) = -1/z$ *on tangent vectors based at* i*? at* $2 + 3i$*?*

We will use the fact that $f_*(p, v) = \big(f(p),\ f'(p) \cdot v\big)$.

Well, $f'(z) = 1/z^2$. Thus, $f'(i) = -1$, so the effect of f on tangent vectors based at i is to send them to their negatives (i.e., rotate them by π).

Now $2 + 3i = (\sqrt{13})e^{i \arctan(3/2)}$. So

$$f'(2 + 3i) = \frac{1}{13e^{2i \arctan(3/2)}} = \frac{1}{13}e^{-2i \arctan(3/2)}.$$

So what f does to these tangent vectors is to shrink their magnitude by a factor of 13 and rotate them by an angle of $-2 \arctan(3/2)$.

(8) *Reread the proof of the reflection principle, either from class or from the book. Where does it use the hypothesis that the domain* D *on which the function is defined is symmetric over the* x*–axis?*

For starters, for the equation $f(\bar{z}) = \overline{f(z)}$ to make sense, we need to know both the value of z and of $\bar{z}$.

The proof defines a function $F(z) = \overline{f(\bar{z})}$, and shows that it is analytic by showing that the CR equations are satisfied. But where is F defined and where is it analytic? z is in the domain of F only if $\bar{z}$ is in the domain of f, and in fact we need to know that f satisfies the CR equations at $\bar{z}$ to conclude that F satisfies the CR equations at z. So if D is the domain of analyticity of f, then $\bar{D}$ is the domain of analyticity of F. If we want these to match, we need $D = \bar{D}$.

Or, looking line-by-line through the book proof: he writes "now, because $f(x+it)$ is an analytic function of $x+it$" and uses it to conclude something about $z = x+iy \in D$, where $t = -y$. But how do you know that f is analytic at $x+it = x-iy = \bar{z}$? You need to know that f is analytic on $\bar{D}$ and that only follows from analyticity on D if $D = \bar{D}$.

Exponentials, logs, and trig

(9) Computations.

(a) *Compute all values of the log base i of 5.*

OK, what does it mean for z to be a value of the log base i of 5? It means that $i^z = 5$, or really since i^z is multivalued we will seek z such that 5 is the principal value of i^z.

$i^z = e^{z \log i}$ and the principal value of $\log i$ is $\frac{\pi}{2}i$, so we have to solve $e^{\frac{\pi}{2}zi} = 5$. If $z = x+iy$, then $\frac{\pi}{2}zi = -\frac{\pi}{2}y + \frac{\pi}{2}xi$, so raising e to this power gives $(e^{-\pi y/2})(e^{i(\pi x/2)})$. For this to equal 5 we must have $-\pi y/2 = \ln 5$ to get the right modulus, and that means $y = -\frac{2}{\pi}\ln 5$. To get the right argument we need $\pi x/2 = 2\pi n$ for some $n \in \mathbb{Z}$. Solving, we get $x = 4n$, or in other words $x \in 4\mathbb{Z}$.

Thus the full set of solutions is $4\mathbb{Z} - i\frac{2}{\pi}\ln 5$. (They are evenly spaced points on a horizontal line.)

(b) *Derive the trig identity $\sin(2z) = 2\sin z \cos z$.*

We have

$$\sin(2z) = \frac{e^{2iz} - e^{2-iz}}{2} = \frac{(e^{iz}+e^{-iz})(e^{iz}-e^{-iz})}{2},$$

whereas

$$\sin z \cos z = \frac{(e^{iz}+e^{-iz})(e^{iz}-e^{-iz})}{4}.$$

They clearly differ by a factor of 2.

(c) *Show that the derivative of $\tanh$ is sech^2. At what points is the hyperbolic tangent function angle-preserving?*

We have $\tanh = \sinh / \cosh$, so

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}\ , \qquad \operatorname{sech}^2(z) = \frac{4}{(e^z+e^{-z})^2}.$$

Using the quotient rule,

$$\frac{d}{dz}\tanh(z) = \frac{(e^z+e^{-z})^2 - (e^z-e^{-z})^2}{(e^z+e^{-z})^2} = \frac{2e^z \cdot 2e^{-z}}{(e^z+e^{-z})^2},$$

so since $e^z \cdot e^{-z} = 1$, we are done.

(10) *Find the mistake in this argument from class:*

We will find the image under $\cos z$ of a vertical line. Such a line has the form $z \in \{c+yi\}$ for some real constant c. As y varies, iz and $-iz$ travel along parallel (horizontal) lines, $iz \in \{-y+ic\}$ and $-iz \in \{y-ic\}$. To find the cosine of the original values z, we must take the average of e^{iz} with e^{-iz}. But applying the exponential to a vertical line gives a ray based at the origin, and in this case we will get two rays that are mirror images over the x–axis. Thus the average value is a point on the real axis.

(Note that if that argument were true, every point from any vertical line in $\mathbb{C}$ would have a real output when you apply cosine—so cosine would take all of $\mathbb{C}$ to the real line!)

After finding the mistake, fix it. Sketch by hand the image of a particular vertical line with $c \neq 0$. Use a computer or graphing utility to draw a more precise picture.

The mistake is that the two lines have opposite parametrizations, so the average of the two is not a point on the real axis. Rather, as you go far out on one of the rays (say the red ray in my picture below), you get close to zero on the other, so the average point is very nearly half the point on the red ray, and $1/2$ times a point on the red ray is still on the red ray. So as you go out to infinity on either ray, the curve is asymptotic to that ray. The picture below was drawn in Grapher as $x + iy = \cos(2 + it), t = -10...10$.