(1) Text, §54, Exercise 3.

Solution: Let R be a closed bounded region such that $f(z)$ is continuous on R, $f(z)$ is analytic and not constant on the interior of R, and $f(z)$ is never 0 on R. Then the function $g(z) = \frac{1}{f(z)}$ is well-defined and satisfies these same properties on R. Therefore, by the maximum modulus principle, $|g(z)|$ attains its maximum at some point z_0 on the boundary of R and never in the interior of R, that is

$$\left|\frac{1}{f(z)}\right| \leq \left|\frac{1}{f(z_0)}\right| \quad \text{for all } z \in \mathbb{C}.$$

Hence $|f(z)|$ satisfies

$$|f(z)| \geq |f(z_0)| \quad \text{for all } z \in \mathbb{C}$$

that is, $|f(z)|$ attains its minimum at the point z_0 which is in the boundary and never in the interior of R.

(2) Text, §54, Exercise 4.
The problem asks us to provide a counterexample to the following statement: *If $f(z)$ is continuous on a closed bounded region R and analytic and not constant throughout the interior of R then $|f(z)|$ has a minimum value in R which occurs on the boundary of R and never in the interior of R.*

Note we have removed the hypothesis from Problem 3 that $f(z)$ is never 0 on R. This will show that the condition that $f(z)$ is never 0 on R is necessary in Problem 3.

Solution: Let $f(z) = z$ and let R be the closed unit disk in the complex plane. Clearly f is analytic and not constant through the interior of R and f is continuous on R. But the minimum of $|f(z)|$ occurs at $z = 0$, which is in the interior of R.

(3) Exercise P.

Solution:

(a) Using partial fractions we see that

$$\frac{2}{z^2+1} = \frac{i}{z+i} + \frac{-i}{z-i}.$$

Hence for γ a contour in $\mathbb{C}$ we have

$$\int_\gamma \frac{2}{z^2+1}dz = \int_\gamma \frac{i}{z+i}dz + \int_\gamma \frac{-i}{z-i}dz \tag{1}$$

(b) $f(z)$ is analytic except at $z = i$ and $z = -i$. We will consider the cases where γ is a simple closed contour.

(i) Suppose neither i nor $-i$ are on or in γ. Then by the Cauchy-Goursat Theorem the integral in (1) is 0.

(ii) Suppose i is on or in γ but $-i$ is not. Then $\frac{i}{z+i}$ is analytic on and in γ hence by the Cauchy Grousat Theorem and the Cauchy integral formula for $g(z) = -i$, which is analytic on and in γ, we have

$$\begin{aligned}\int_\gamma \frac{2}{z^2+1}dz &= \int_\gamma \frac{i}{z+i}dz + \int_\gamma \frac{-i}{z-i}dz \\ &= 0 + 2\pi i(-i) \\ &= 2\pi.\end{aligned}$$

(iii) Suppose $-i$ is on or in γ but i is not. Then $\frac{-i}{z-i}$ is analytic on and in γ hence by the Cauchy Grousat Theorem and the Cauchy integral formula for $g(z) = i$, which is analytic on and in γ, we have

$$\begin{aligned}\int_\gamma \frac{2}{z^2+1}dz &= \int_\gamma \frac{i}{z+i}dz + \int_\gamma \frac{-i}{z-i}dz \\ &= 2\pi i(i) + 0 \\ &= -2\pi.\end{aligned}$$

(iv) Suppose both i and $-i$ are on or in γ. Then by the Cauchy integral formula for $g_1(z) = i$ and $g_2(z) = -i$, which are analytic on and in γ, we have

$$\begin{aligned}\int_\gamma \frac{2}{z^2+1}dz &= \int_\gamma \frac{i}{z+i}dz + \int_\gamma \frac{-i}{z-i}dz \\ &= 2\pi i(i) + 2\pi i(-i) \\ &= 0.\end{aligned}$$

Note: You can also easily show that if you have a closed contour that winds around $-i$ n-times and around i k-times the integral is

$$\begin{aligned}\int_\gamma \frac{2}{z^2+1}dz &= \int_\gamma \frac{i}{z+i}dz + \int_\gamma \frac{-i}{z-i}dz \\ &= 2n\pi i(i) + 2k\pi i(-i) \\ &= 2\pi(k-n).\end{aligned}$$