(1) Text, §25, Exercise 6.

Solution: First we will verify that $g(z)$ is analytic on the given domain with derivative $g'(z) = \frac{1}{z}$. For $z = re^{i\theta}$, $g(z) = \ln r + i\theta$ with

$$u(r, \theta) = \ln r \qquad v(r, \theta) = \theta$$

By the Cauchy-Riemann equations, we have

$$ru_r = v_\theta \Longleftrightarrow r\left(\frac{1}{r}\right) = 1$$
$$u_\theta = -rv_r \Longleftrightarrow 0 = 0$$

Since u and v have continuous partial derivatives for $r > 0$ and $0 < \theta < 2\pi$, the derivative $g'(z)$ will exist on this domain, hence g is analytic on this domain. At those points

$$g'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{z}$$

Next we will show that $G(z) = g(z^2+1)$ is analytic for $x > 0, y > 0$ with derivative $G'(z) = \frac{2z}{z^2+1}$. First we will check that if $x > 0, y > 0$ then z^2+1 is in the domain of g. If $z = x + iy$ then $z^2 + 1 = (x^2 - y^2 + 1) + i(2xy)$. So, if $x > 0$ and $y > 0$ then $\mathrm{Im}(z^2 + 1) = 2xy > 0$. This implies

$$|z^2 + 1| = \sqrt{(x^2 - y^2 + 1)^2 + (2xy)^2} > 0$$

Writing $z = re^{i\theta}$ we have $0 < \theta < \frac{\pi}{2}$, so $z^2 + 1 = (re^{i\theta})^2 + 1 = r'e^{i\theta'}$ where $0 < \theta' < 2\theta < \pi < 2\pi$. Therefore $z^2 + 1$ is in the given domain for g.

Since the polynomial $z^2 + 1$ is analytic for $x > 0$ and $y > 0$ and g is analytic on the range of $z^2 + 1$ for $x > 0, y > 0$, their composition $G(z)$ is analytic on the domain $x > 0, y > 0$. By the chain rule, for $x > 0, y > 0$ we have

$$G'(z) = g'(z^2 + 1)\frac{d}{dz}(z^2 + 1) = \frac{2z}{z^2 + 1}$$

(2) Text, §29, Exercise 4. Show in two ways that $f(z) = e^{z^2}$ is entire. What is its derivative?

Solution: The following two arguments each show why $f(z) = e^{z^2}$ is entire.

(i) Let $z = x + iy$, then $f(z) = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$, with

$$u(x, y) = e^{x^2-y^2}\cos(2xy) \qquad v(x, y) = e^{x^2-y^2}\sin(2xy)$$

Taking partial derivatives we have

$$u_x = 2x\,e^{x^2-y^2}\cos(2xy) - 2y\,e^{x^2-y^2}\sin(2xy) = v_y$$
$$u_y = -2y\,e^{x^2-y^2}\cos(2xy) - 2x\,e^{x^2-y^2}\sin(2xy) = -v_x$$

Since the Cauchy-Riemann equations hold for all points and both u and v have continuous partial derivatives at all points, f is entire.

(ii) We can write $f(z) = g \circ h(z)$, where $h(z) = z^2$ and $g(w) = e^w$. Since both g and h are entire, so if f.

From method (ii) we can use the chain rule to deduce that at all points

$$f'(z) = g'(h(z))h'(z) = e^{z^2}(2z).$$