Selected HW1 Solutions

Math 158, Complex Analysis
TA: Christine Offerman

(1) Text, §4, Exercise 4.

Solution: Let $z = x + iy$, then z satisfies

$$\begin{aligned} &(|x| - |y|)^2 \geq 0 \\ \iff &|x|^2 + |y|^2 - 2|x||y| \geq 0 \\ \iff &2x^2 + 2y^2 \geq |x|^2 + 2|xy| + |y|^2 \\ \iff &2x^2 + 2y^2 \geq (|x| + |y|)^2 \end{aligned}$$

Since both sides of the last inequality are nonnegative, it is equivalent to

$$\sqrt{2(x^2 + y^2)} \geq |x| + |y|.$$

which proves the desired result. $\square$

(2) Text, §8, Exercise 9.

Solution: Let $z \in \mathbb{C}$, then z satisfies

$$\begin{aligned} (1 + z + z^2 + \cdots + z^n)(1 - z) &= (1 + z + z^2 + \cdots + z^n) - (z + z^2 + \cdots + z^n + z^{n+1}) \\ &= 1 - z^{n+1} \end{aligned}$$

So, if $z \neq 1$ then z satisfies

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

This proves the first equation.

Now let $z = e^{i\theta}$, so that

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}$$

Equating the real parts of both sides we have

$$\begin{aligned} \mathrm{Re}(1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta}) &= \mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right) \\ 1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta &= \mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right) \end{aligned}$$

Then the real part of fraction on the right hand side is

$$\begin{aligned} \mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right) &= \mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \cdot \frac{ie^{-i\theta/2}}{ie^{-i\theta/2}}\right) \\ &= \mathrm{Re}\left(\frac{ie^{-i\theta/2} - ie^{-i\theta(2n+1)/2}}{ie^{-i\theta/2} - ie^{i\theta/2}}\right) \\ &= \mathrm{Re}\left(\frac{i(\cos(\theta/2) - \cos(\theta(2n+1)/2)) + \sin(\theta/2) + \sin(\theta(2n+1)/2)}{2\sin(\theta/2)}\right) \\ &= \frac{1}{2} + \frac{\sin(\theta(2n+1)/2)}{2\sin(\theta/2)} \end{aligned}$$

This proves the result. $\square$

(3) Exercise A: Show that $f(\bar{z}) = \overline{f(z)}$ for polynomials with real coefficients. Conclude that $f(z) = 0 \iff f(\bar{z}) = 0$ for these polynomials. In other words, complex roots of real polynomials come in conjugate pairs.

Solution: Let $f(z)$ be a polynomial with real coefficients, write

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_i \in \mathbb{R} \text{ for all } i, z \in \mathbb{C})$$

Then

$$f(\bar{z}) = a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \cdots + a_n \bar{z}^n$$

We will first prove the claim that $\bar{z}^n = \overline{z^n}$ for $n \geq 1$ by induction on n. Let $n = 1$ then the result is trivial.

Now suppose the result holds for n, we will show the result holds for $n + 1$.

$$\begin{aligned} \bar{z}^{n+1} &= \bar{z}^n \bar{z} \\ &= \overline{z^n} \bar{z} \quad \text{(since the result holds for } n\text{)} \\ &= \overline{z^n z} \quad \text{(by text p.13 equation (4))} \\ &= \overline{z^{n+1}} \end{aligned}$$

Therefore the claim holds by induction.

Similarly using equation (2) from p.13 of the text we can show that

$$\bar{z_1} + \bar{z_2} + \cdots + \bar{z_n} = \overline{z_1 + \cdots + z_n} \quad (\text{for } z_i \in \mathbb{C})$$

Therefore,

$$\begin{aligned} f(\bar{z}) &= a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \cdots + a_n \bar{z}^n \\ &= a_0 + a_1 \bar{z} + a_2 \overline{z^2} + \cdots + a_n \overline{z^n} \\ &= \bar{a_0} + \overline{a_1 z} + \overline{a_2 z^2} + \cdots + \overline{a_n z^n} \quad \text{(since all } a_i \in \mathbb{R}\text{)} \\ &= \overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} \\ &= \overline{f(z)} \end{aligned}$$

This proves the first statement.

Finally, $f(z) = 0$ implies $f(\bar{z}) = \overline{f(z)} = \bar{0} = 0$, and $f(\bar{z}) = 0$ implies $f(z) = \overline{\overline{f(z)}} = \overline{f(\bar{z})} = \bar{0} = 0$, which proves the second statement. $\square$